

Sample Complexity of Variance-Reduced Distributionally Robust Q-Learning

Shengbo Wang

*Department of Management Science and Engineering
Stanford University*

SHENGBO.WANG@STANFORD.EDU

Nian Si

*Department of Industrial Engineering and Decision Analytics
The Hong Kong University of Science and Technology*

NIANSI@UST.HK

Jose Blanchet

*Department of Management Science and Engineering
Stanford University*

JOSE.BLANCHET@STANFORD.EDU

Zhengyuan Zhou

*Stern School of Business
New York University*

ZZ26@STERN.NYU.EDU

Editor: Zhihua Zhang

Abstract

Dynamic decision-making under distributional shifts is of fundamental interest in theory and applications of reinforcement learning: The distribution of the environment in which the data is collected can differ from that of the environment in which the model is deployed. This paper presents two novel model-free algorithms, namely the distributionally robust Q-learning and its variance-reduced counterpart, that can effectively learn a robust policy despite distributional shifts. These algorithms are designed to efficiently approximate the q -function of an infinite-horizon γ -discounted robust Markov decision process with Kullback-Leibler ambiguity set to an entry-wise ϵ -degree of precision. Further, the variance-reduced distributionally robust Q-learning combines the synchronous Q-learning with variance-reduction techniques to enhance its performance. Consequently, we establish that it attains a minimax sample complexity upper bound of $\tilde{O}(|\mathbf{S}||\mathbf{A}|(1-\gamma)^{-4}\epsilon^{-2})$, where \mathbf{S} and \mathbf{A} denote the state and action spaces. This is the first complexity result that is independent of the ambiguity size δ , thereby providing new complexity theoretic insights. Additionally, a series of numerical experiments confirm the theoretical findings and the efficiency of the algorithms in handling distributional shifts.

Keywords: sample complexity, reinforcement learning, distributional robustness, Q-learning, stochastic approximation.

1. Introduction

Reinforcement learning (RL) (Sutton and Barto, 2018) focuses on how agents can learn to make optimal decisions in uncertain and dynamic environments. It is based on the principle of trial-and-error learning, where the agent interacts with the environment, receives rewards or penalties for its actions, and adjusts its behavior to maximize the expected long-term reward.

A significant obstacle in RL is the limited interaction between the agent and the environment, often due to factors such as data-collection costs or safety constraints. To overcome this, practitioners often rely on historical datasets or simulation environments to train the agent. However, this approach can suffer from distributional shifts (Quinonero-Candela et al., 2008) between the real-world environment and the data-collection/simulation environment, potentially leading to suboptimal learned policies when deployed in the actual environment. It is also observed in RL environments that an agent trained this way could be vulnerable to adversarial attacks (Lin et al., 2017; Pan et al., 2019).

To tackle these challenges, distributionally robust reinforcement learning (DR-RL) (Zhou et al., 2021; Yang et al., 2021; Liu et al., 2022; Shi and Chi, 2022; Wang et al., 2023b) has emerged as a promising approach. DR-RL seeks to learn policies that are robust to distributional shifts in the environment by explicitly considering a family of possible distributions that the agent may encounter during deployment. This approach allows the agent to learn a policy that performs well across a range of environments, rather than just the one it was trained on.

These benefits of distributionally robust policies motivate the exploration of a critical question: *Can we construct efficient reinforcement learning algorithms that achieve the desired robustness properties while also providing provable guarantees on their sample complexity?*

A growing body of literature aims to understand the sample complexities of distributionally robust reinforcement learning. Specifically, we are interested in a robust tabular Markov Decision Process (MDP) with state space \mathbf{S} and action space \mathbf{A} , in the discounted infinite-horizon setting with discount factor γ . To account for uncertainty, we use an ambiguity set based on Kullback-Leibler (KL) divergence with ambiguity size δ , which is arguably the most natural and challenging divergence in distributionally robust literature. Previous research has mainly focused on the *model-based* approach, where a specific model of the environment is estimated, and value iteration (VI) is run on the estimated model. Table 1 shows the worst-case sample complexity of model-based distributionally RL, with Shi and Chi (2022) proposing a method with state-of-the-art sample complexity in terms of $|\mathbf{S}|, |\mathbf{A}|, 1 - \gamma, \epsilon$.

Algorithm	Sample Complexity	Origin
DRV1	$\tilde{O}(\mathbf{S} ^2 \mathbf{A} e^{O(1-\gamma)^{-1}}(1-\gamma)^{-4}\epsilon^{-2}\delta^{-2})$	Zhou et al. (2021)
REVI/DRV1	$\tilde{O}(\mathbf{S} ^2 \mathbf{A} e^{O(1-\gamma)^{-1}}(1-\gamma)^{-4}\epsilon^{-2}\delta^{-2})$	Panaganti and Kalathil (2021)
DRV1	$\tilde{O}(\mathbf{S} ^2 \mathbf{A} (1-\gamma)^{-4}\epsilon^{-2}\mathbf{p}_\wedge^{-2}\delta^{-2})$	Yang et al. (2021)
DRV1-LCB	$\tilde{O}(\mathbf{S} \mathbf{A} (1-\gamma)^{-4}\epsilon^{-2}\mathbf{p}_\wedge^{-1}\delta^{-2})$	Shi and Chi (2022)

Table 1: Summary of sample complexity upper bounds for finding an ϵ -optimal robust policy in *model-based* distributionally robust RL (\mathbf{p}_\wedge is the minimal support probability of the nominal MDP; see, Def. 6).

1.1 Our Motivation

The emerging line of work mentioned above reflects the growing interest and fruitful results in the pursuit of sample-efficient distributionally robust reinforcement learning. At the same time, a closer scrutiny of the results suggests that two fundamental aspects of the problem are inadequately addressed.

For one thing, the complexity bounds of existing results exhibit $\tilde{O}(\delta^{-2})$ dependence as $\delta \downarrow 0$. This increase in the complexity bounds appears to reflect an increased need for learning the training environment as the training and adversarial environments become more alike. At the surface level, this makes sense: in the extreme case where δ is approaching ∞ , then (assuming known support of the distributions) no sample is needed to find an optimal distributionally robust policy. Nevertheless, such bounds have failed to align with the continuity property of the robust MDP: the robust value function should converge to the non-robust optimal cumulative reward as $\delta \downarrow 0$. Therefore, for all sufficiently small δ that may depend on the training environment and ϵ , the robust value function can be approximated by the output of a classical RL algorithm. Specifically, we expect an algorithm and analysis with a $\tilde{O}(1)$ dependence as $\delta \downarrow 0$. This is presently absent in the literature.

Additionally, with the exception of Wang et al. (2023b) (discussed in more detail in the next subsection), all the existing distributionally robust policy learning algorithms that have finite-sample guarantees (such as the ones mentioned above (Zhou et al., 2021; Panaganti and Kalathil, 2021; Yang et al., 2021; Shi and Chi, 2022)) are model-based, which estimates the underlying MDP first before provisioning some policy from it. Although model-based methods are often more sample-efficient and easier to analyze, their drawbacks are also well-understood (Sutton and Barto, 2018; François-Lavet et al., 2018): they are computationally intensive; they require more memory to store MDP models and often do not generalize well to non-tabular RL settings. These issues limit the practical applicability of model-based algorithms, which stand in contrast to model-free algorithms that learn to select actions without first learning an MDP model. Such methods are often more computationally efficient, have less storage overhead, and better generalize to RL with function approximation. In particular, *Q*-learning (Watkins and Dayan, 1992), as the prototypical model-free learning algorithm, has widely been both studied theoretically and deployed in practical applications. However, *Q*-learning is not robust (as demonstrated in our simulations), and the policy learned by *Q*-learning in one environment can perform poorly in another under a worst-case shift (with bounded magnitude).

As such, the above discussion naturally motivates the following research question:

*Can we design a variant of *Q*-Learning that is distributionally robust, where the sample complexity has the right scaling with δ ?*

1.2 Our Contributions

We answer the above question affirmatively and contribute to the existing literature on the worst-case sample complexity theory of *model-free* distributionally robust RL. We propose two distributionally robust variants of the *Q*-learning algorithm (Watkins and Dayan, 1992), namely DR *Q*-learning (Algorithm 1) and variance-reduced DR *Q*-learning (Algorithm 2), which effectively solve the DR-RL problem under the KL ambiguity set.

The proposed algorithms operate efficiently under the assumption of limited power of the adversary (as per Assumption 1), which is realistic in many real-world applications. We prove that both algorithms have near-optimal worst-case sample complexity guarantees in this regime. Additionally, the variance-reduced version exhibits superior complexity dependence on the effective horizon $(1 - \gamma)^{-1}$, as shown in Table 2. To the best of our knowledge, both algorithms and their worst-case sample complexity upper bounds represent state-of-the-art results in model-free distributionally robust RL. Moreover, our sample complexity upper bound for variance-reduced DR Q-learning matches the best-known upper bound for this DR-RL problem in Shi and Chi (2022) in terms of ϵ^{-2} and $(1 - \gamma)^{-4}$ dependence.

Algorithm	Sample Complexity	Origin
MLMC DR Q-learning	$\tilde{O}(\mathbf{S} \mathbf{A} (1 - \gamma)^{-5}\epsilon^{-2}\mathbf{p}_\wedge^{-6}\delta^{-4})$	Wang et al. (2023b)
DR Q-learning	$\tilde{O}(\mathbf{S} \mathbf{A} (1 - \gamma)^{-5}\epsilon^{-2}\mathbf{p}_\wedge^{-3})$	Theorem 9
Variance-reduced DR Q-learning	$\tilde{O}(\mathbf{S} \mathbf{A} (1 - \gamma)^{-4}\epsilon^{-2}\mathbf{p}_\wedge^{-3})$	Theorem 13

Table 2: Summary of sample complexity upper bounds for finding an ϵ -optimal robust policy in *model-free* distributionally robust RL (\mathbf{p}_\wedge is the minimal support probability of the nominal MDP; see, Def. 6).

The DR Q-learning Algorithm 1 is a direct extension of mini-batch Q-learning. Compared to the MLMC DR Q-learning method proposed by Wang et al. (2023b), Algorithm 1 is easier to implement in real-world applications. Additionally, this approach allows for the design of a more sophisticated variant, the variance-reduced DR Q-learning, which provides a provable enhancement of the worst-case sample complexity guarantee of DR Q-learning. To achieve this improvement, we leverage Wainwright’s variance reduction technique and algorithm structure (Wainwright, 2019a), adapting it to the DR-RL context and redesigning the variance reduction scheme accordingly.

Both the DR Q-learning and its variance-reduced version use a stochastic approximation (SA) step to iteratively update the estimator of the optimal DR q -function towards the fixed point of the population DR Bellman operator. However, both algorithms involve a bias that must be controlled at the algorithmic and iterative update levels. Our contribution to the literature lies in the near-optimal analysis of the biased SA resulting from DR Q-learning and its variance-reduced version. This analysis also generalizes to settings where the biased stochastic version of the contraction mapping is a monotonic contraction.

We highlight that these are the first algorithmic complexity results showing that the worst-case complexity dependence on the uncertainty set size δ is $O(1)$ as $\delta \rightarrow 0$ for the DR-RL problem with a KL ambiguity set. This resolves the issue of worst-case complexity bounds blowing up as δ approaches 0, a problem present in all previous works, including both model-based and model-free approaches (Yang et al., 2021; Panaganti and Kalathil, 2021; Shi and Chi, 2022; Wang et al., 2023b).

The significance of this characteristic lies in its theoretical illustration that as the adversary’s power δ approaches 0, not only does the solution to the DR-RL problem converge to that of the non-robust version, but so does the sample complexity required to solve it. This sheds light on the connection between robust and non-robust RL problems, indicating that

in more general settings and real-world applications, DR-RL problems with function approximation may be efficiently addressed by utilizing variants of the corresponding approach for non-robust RL problems.

1.3 Literature Review

This section is dedicated to reviewing the literature that is relevant to our work. The literature on RL and MDP is extensive. One major line of research focuses on developing algorithms that can efficiently learn policies to maximize cumulative discounted rewards. When discussing RL and MDP problems, we will concentrate on this infinite horizon discounted reward formulation.

Minimax Sample Complexity of Tabular RL: Recent years have seen significant developments in the worst-case sample complexity theory of tabular RL. Two principles, namely model-based and model-free, have motivated distinct algorithmic designs. In the model-based approach, the controller aims to gather a dataset so as to construct an empirical model of the underlying MDP and solving it using variations of the dynamic programming principle. Research (Azar et al., 2013; Sidford et al., 2018; Agarwal et al., 2020; Li et al., 2022) have proposed model-based algorithms and proven optimal upper bounds for achieving ϵ , with a matching lower bound $\tilde{\Omega}(|\mathbf{S}||\mathbf{A}|(1-\gamma)^{-3}\epsilon^{-2})$ proven in Azar et al. (2013). In contrast, the model-free approach involves maintaining only lower-dimensional statistics of the transition data, which are iteratively updated. As one of the most well known model-free algorithm, the sample complexity of Q-learning been extensively studied (Even-Dar et al., 2003; Wainwright, 2019b; Chen et al., 2020; Li et al., 2021). However, Li et al. (2021) have shown that the Q-learning has a minimax sample complexity of $\tilde{\Theta}(|\mathbf{S}||\mathbf{A}|(1-\gamma)^{-4}\epsilon^{-2})$, which doesn't match the lower bound $\tilde{\Omega}(|\mathbf{S}||\mathbf{A}|(1-\gamma)^{-3}\epsilon^{-2})$. Nevertheless, variance-reduced variants of the Q-learning, such as the one proposed in Wainwright (2019a), achieve the aforementioned sample complexity lower bound. Other algorithmic techniques such as Polyak-Ruppert averaging (Li et al., 2023) has been shown to result in optimal sample complexity.

Finite Analysis of SA: The classical theory of asymptotic convergence for SA has been extensively studied, as seen in Kushner and Yin (2013). Recent progress in the minimax and instant dependent sample complexity theory of Q-learning and its variants has been aided by advances in the finite-time analysis of SA. Traditional RL research focuses on settings where the random operator is unbiased. Wainwright (2019b) demonstrated a sample path bound for the SA recursion, which enables the use of variance reduction techniques to achieve optimal learning rates. In contrast, Chen et al. (2020, 2022) provided finite sample guarantees for SA only under a second moment bound on the martingale difference noise sequence. Additionally, research has been conducted on non-asymptotic analysis of SA procedures in the presence of bias, as documented in (Karimi et al., 2019; Wang, 2022).

Robust MDP and RL: Our work draws upon the theoretical framework of classical max-min control and robust MDPs, as established in previous works (González-Trejo et al., 2002; Iyengar, 2005; Nilim and El Ghaoui, 2005; Wiesemann et al., 2013; Xu and Mannor, 2010; Shapiro, 2022; Wang et al., 2024). These works have established the concept of distributional robustness in dynamic decision making. In particular, González-Trejo et al. (2002); Iyengar (2005); Nilim and El Ghaoui (2005) established the distributionally

robust dynamic programming principles for SA-rectangular adversaries under symmetric information structures, while Wiesemann et al. (2013); Wang et al. (2024) studies asymmetric settings, leading to the same the DR Bellman equation.

Recent research has shown great interests in learning DR policies from data (Si et al., 2020; Zhou et al., 2021; Yang et al., 2021; Liu et al., 2022; Shi and Chi, 2022; Wang et al., 2023b; Yang et al., 2023). For instance, (Si et al., 2020) studied the contextual bandit setting, while (Zhou et al., 2021; Panaganti and Kalathil, 2021; Yang et al., 2021; Shi and Chi, 2022) focused on the model-based tabular RL setting. On the other hand, (Liu et al., 2022; Wang et al., 2023b; Yang et al., 2023) tackled the DR-RL problem using a model-free approach¹. Before our work, the best worst-case sample complexity upper bound for DR-RL under the KL ambiguity set was established for the model-based DRVI-LCB algorithm, as proposed and analyzed by Shi and Chi (2022). Their analysis showed that the worst-case sample complexity has an upper bound of $\tilde{O}(|\mathbf{S}||\mathbf{A}|(1-\gamma)^{-4}\epsilon^{-2}\delta^{-2}\mathfrak{p}_\wedge^{-1})$.

2. Distributionally Robust Reinforcement Learning

2.1 Classical Tabular Reinforcement Learning

Let $\mathcal{M}_0 = (\mathbf{S}, \mathbf{A}, \mathbf{R}, P_0, N_0, \gamma)$ be a Markov decision process (MDP), where \mathbf{S} , \mathbf{A} , and $\mathbf{R} \subseteq \mathbb{R}_+$ are finite state, action, and reward spaces², respectively. Let $\mathcal{P}(\mathbf{U})$, where $\mathbf{U} = \mathbf{S}, \mathbf{A}, \mathbf{R}$, denote the set of probability measures on the power set $2^{\mathbf{U}}$. Then $P_0 = \{p_{s,a} \in \mathcal{P}(\mathbf{S}), s \in \mathbf{S}, a \in \mathbf{A}\}$ and $N_0 = \{\nu_{s,a} \in \mathcal{P}(\mathbf{R}), s \in \mathbf{S}, a \in \mathbf{A}\}$ are the sets of transition and reward distributions, respectively. $\gamma \in (0, 1)$ is the discount factor. Define $r_{\max} = \max\{r \in \mathbf{R}\}$ as the maximum reward.

At each time t , given the state process is at S_t and the decision maker takes action A_t , the subsequent state is determined by the conditional distribution $S_{t+1} \sim p_{S_t, A_t}$. Then, a randomized reward $R_t \sim \nu_{S_t, A_t}$ will be collected, independent of the history.

Let Π be the history-dependent policy class (see (Wang et al., 2024) for a rigorous construction). For $\pi \in \Pi$, the value function $v^\pi(s)$ is defined as:

$$v^\pi(s) := E \left[\sum_{t=0}^{\infty} \gamma^t R_t \middle| S_0 = s \right].$$

The optimal value function is

$$v^*(s) := \max_{\pi \in \Pi} v^\pi(s),$$

$\forall s \in \mathbf{S}$. It is well known that the optimal value function is the unique solution of the following Bellman equation:

$$v^*(s) = \max_{a \in \mathbf{A}} (E_{\nu_{s,a}}[R] + \gamma E_{p_{s,a}}[v^*(S)]).$$

where the expectations are taken over $R \sim \nu_{s,a}$ and $S \sim p_{s,a}$, respectively.

1. Liu et al. (2022)'s algorithm is infeasible: it requires an infinite number of samples in expectation for *each iteration*, and only asymptotic convergence is established with an infinite number of iterations.
2. We assume a finite reward space for simplicity. However, our results can be extended to continuous reward spaces by imposing a minimum density assumption, as described in Si et al. (2020).

An important implication of the Bellman equation is that it suffices to optimize within the stationary Markovian deterministic policy class.

We define the optimal q -function as

$$q^*(s, a) := E_{\nu_{s,a}}[R] + \gamma E_{p_{s,a}}[v^*(S)].$$

It is well-known that q^* satisfies its Bellman equation

$$q^*(s, a) = E_{\nu_{s,a}}[R] + \gamma E_{p_{s,a}} \left[\max_{b \in \mathbf{A}} q^*(S, b) \right].$$

An optimal policy can be constructed as $\pi^*(s) = \arg \max_{a \in \mathbf{A}} q^*(s, a)$. Therefore, policy learning in RL environments can be achieved if we can learn a good estimate of q^* .

2.2 Kullback-Leibler Divergence Constrained DR-RL

We consider a DR-RL setting where the adversary is constrained to perturb both transition probabilities and rewards within a KL divergence ball of radius δ . Specifically, for probability measures Q is absolutely continuous w.r.t. P on some measurable space (Ω, \mathcal{F}) , denoted by $Q \ll P$, define

$$D_{\text{KL}}(Q \| P) := \int_{\Omega} \log \left(\frac{dQ}{dP}(\omega) \right) P(d\omega), \quad (2.1)$$

where $\frac{dP}{dQ}$ is the Radon-Nikodym derivative.

For each $(s, a) \in \mathbf{S} \times \mathbf{A}$ and $\delta > 0$, we define KL ambiguity set that are centered at $p_{s,a} \in P_0$ and $\nu_{s,a} \in N_0$ of radius δ by

$$\begin{aligned} \mathcal{P}_{s,a}(\delta) &:= \{p : D_{\text{KL}}(p \| p_{s,a}) \leq \delta\}, \\ \mathcal{N}_{s,a}(\delta) &:= \{\nu : D_{\text{KL}}(\nu \| \nu_{s,a}) \leq \delta\}. \end{aligned} \quad (2.2)$$

These ambiguity sets represent the possible distributional shifts from the reference model P_0, N_0 . In particular, the parameter $\delta > 0$ controls the size of the ambiguity sets, quantifying the power of the adversary.

With these definitions in mind, we define the DR optimal value function as the solution to a fixed point equation—a.k.a. the DR Bellman equation—which serves as the learning objective of this paper.

Definition 1 *The DR Bellman operator \mathcal{B}_δ for the value function is defined as the mapping*

$$\mathcal{B}_\delta(v)(s) := \max_{a \in \mathbf{A}} \inf_{\substack{p \in \mathcal{P}_{s,a}(\delta), \\ \nu \in \mathcal{N}_{s,a}(\delta)}} (E_\nu[R] + \gamma E_p[v(S)]). \quad (2.3)$$

Define the DR optimal value function v_δ^* as the solution of the DR Bellman equation:

$$v_\delta^* = \mathcal{B}_\delta(v_\delta^*) \quad (2.4)$$

Moving forward, we will suppress the explicit dependence on δ .

The DR Bellman equation has a unique solution as the fixed point of \mathcal{B} , which is a consequence of \mathcal{B} being a contraction operator. Furthermore, the solution is equal to the

max-min control optimal value of a *SA-rectangular* distributionally robust MDP (DRMDP) (Iyengar, 2005; Nilim and El Ghaoui, 2005; Wiesemann et al., 2013). Specifically, this max-min optimal value is given by

$$u^*(s) := \sup_{\pi \in \Pi} \inf_{\kappa \in K} E^{\pi, \kappa} \left[\sum_{t=0}^{\infty} \gamma^t R_t \middle| s_0 = s \right] \quad (2.5)$$

where Π is the history-dependent policy class, and the adversary chooses a policy κ from an adversarial ambiguity set K that is induced by the KL ambiguity sets in (2.2).

Intuitively, this value represents the optimal reward in the following adversarial environment: When the controller selects a policy π , an adversary observes this policy and then chooses a counter-policy that determines the sequence of reward and transition distributions. The adversary's choice is constrained such that the reward and transition distributions induced by the counter-policy lie within the ambiguity set (2.2) of radius δ . The decisions made by both the controller and the adversary uniquely specify the law of the state-action-reward process, thereby determining the value of the policy pair (π, κ) .

The equivalence of the max-min control optimal value (2.5) and the solution to the DR Bellman equation (2.4) shows the optimality of stationary deterministic Markov control policies and stationary Markovian adversarial distribution choices. This equivalence, known as the *dynamic programming principle* (DPP), is explored in detail in Wang et al. (2024), where the adversary and controller can have asymmetric information structures. For those interested, we refer you to this paper.

We note that Wang et al. (2024) considers a setting where the reward is not randomized, i.e., $\mathcal{N}_{s,a} = \{\delta_{r(s,a)}\}$ for some reward function $r : \mathbf{S} \times \mathbf{A} \rightarrow [0, 1]$. However, it is straightforward to generalize the DPP to include randomized rewards in the SA-rectangular setting.

2.3 Dual and q -Function Formulations

The right-hand side of (2.3) can be challenging to work with because the measure underlying the expectations is not directly accessible. To address this, we use strong duality to reveal the dependence of the value on the reference transition and reward distributions, P_0 and N_0 . Specifically, we consider the dual representation:

Lemma 2 (Hu and Hong (2013), Theorem 1) *Let X be a random variable and μ_0 be a probability measure on (Ω, \mathcal{F}) s.t. X has a finite moment generating function in a neighborhood of zero. Then for any $\delta > 0$,*

$$\inf_{\mu: D_{\text{KL}}(\mu \| \mu_0) \leq \delta} E_\mu X = \sup_{\alpha \geq 0} \left\{ -\alpha \log E_{\mu_0} \left[e^{-X/\alpha} \right] - \alpha \delta \right\}.$$

Since the reward and values are bounded, directly apply Lemma 2 to the r.h.s. of (2.4), the DR value function v^* in fact satisfies the following *dual form* of the DR Bellman's equation.

$$v^*(s) = \max_{a \in \mathbf{A}} \left\{ \sup_{\alpha \geq 0} \left\{ -\alpha \log E_{\nu_{s,a}} \left[e^{-R/\alpha} \right] - \alpha \delta \right\} + \gamma \sup_{\beta \geq 0} \left\{ -\beta \log E_{p_{s,a}} \left[e^{-v^*(S)/\beta} \right] - \beta \delta \right\} \right\}.$$

Similar to the traditional RL policy learning approach, we utilize the optimal DR state-action value function, also known as the q -function, to address the DR-RL problem. The

q -function assigns real numbers to pairs of states and actions, and can be represented as a matrix $q \in \mathbb{R}^{\mathbf{S} \times \mathbf{A}}$. From now on, we will assume this representation. To simplify notation, let us define

$$v(q)(s) := \max_{b \in \mathbf{A}} q(s, b), \quad (2.6)$$

which is the value function induced by the q -function $q(\cdot, \cdot)$.

We proceed to rigorously define the optimal q -function and its Bellman equation.

Definition 3 *The optimal DR q -function is defined as*

$$q^*(s, a) := \inf_{\substack{p \in \mathcal{P}_{s,a}(\delta), \\ \nu \in \mathcal{N}_{s,a}(\delta)}} (E_\nu[R] + \gamma E_p[v^*(S)]) \quad (2.7)$$

where v^* is the DR optimal value function in Definition 1.

Similar to the Bellman operator, we can define the DR Bellman operator for the q -function as follows:

Definition 4 *Given $\delta > 0$ and $q \in \mathbb{R}^{\mathbf{S} \times \mathbf{A}}$, the primal form of the DR Bellman operator $\mathcal{T} : \mathbb{R}^{\mathbf{S} \times \mathbf{A}} \rightarrow \mathbb{R}^{\mathbf{S} \times \mathbf{A}}$ is defined as*

$$\mathcal{T}(q)(s, a) := \inf_{\substack{p \in \mathcal{P}_{s,a}(\delta), \\ \nu \in \mathcal{N}_{s,a}(\delta)}} (E_\nu[R] + \gamma E_p[v(q)(S)]) \quad (2.8)$$

The dual form of the DR Bellman operator is

$$\begin{aligned} \mathcal{T}(q)(s, a) = & \sup_{\alpha \geq 0} \left\{ -\alpha \log E_{\nu_{s,a}} \left[e^{-R/\alpha} \right] - \alpha \delta \right\} \\ & + \gamma \sup_{\beta \geq 0} \left\{ -\beta \log E_{p_{s,a}} \left[e^{-v(q)(S)/\beta} \right] - \beta \delta \right\}. \end{aligned} \quad (2.9)$$

The equivalence of the primal and dual form follows from Lemma 2. We remark that the dual form is usually easier to work with, as the outer supremum is a 1-d optimization problem and the dependence on the reference measures $\nu_{s,a}$ and $p_{s,a}$ are explicit.

Note that by definition (2.7) and the Bellman equation (2.4), we have $v(q^*) = v^*$. So, our definition implies that q^* is a fixed point of \mathcal{T} and the following Bellman equation for the q^* -function holds:

$$q^* = \mathcal{T}(q^*). \quad (2.10)$$

The uniqueness of the fixed point q^* of \mathcal{T} follows from the contraction property of the operator \mathcal{T} ; c.f. Lemma 19.

The optimal DR policy can be extracted from the optimal q -function by $\pi^*(s) = \arg \max_{a \in \mathbf{A}} q^*(s, a)$. Hence the goal the DR-RL paradigm is to learn the DR q -function and extract the corresponding robust policy.

2.4 Synchronous Q-Learning and Stochastic Approximation

The Q-learning estimates the optimal q -function by iteratively update the estimator $\{q_k : k \geq 0\}$ using samples generated by the reference measures. The classical synchronous Q-learning proceeds as follows. At iteration $k \in \mathbb{Z}_{\geq 0}$ and each $(s, a) \in \mathbf{S} \times \mathbf{A}$, we draw samples $R_{k+1} \sim \nu_{s,a}$ and $S_{k+1} \sim p_{s,a}$. Then perform the Q-learning update

$$q_{k+1}(s, a) = (1 - \lambda_k)q_k(s, a) + \lambda_k(R_{k+1} + \gamma v(q_k)(S_{k+1})) \quad (2.11)$$

for some chosen step-size sequence $\{\lambda_k\}$.

The synchronous Q-learning can be analyzed as a stochastic approximation (SA) algorithm. SA for the fixed point of a contraction operator \mathcal{L} refers to the class of algorithms using the update

$$X_{k+1} = (1 - \lambda_k)X_k + \lambda_k \mathcal{L}(X_k) + W_{k+1}. \quad (2.12)$$

$\{W_k\}$ is a sequence satisfying $E[W_k | W_{k-1}, \dots, W_1] = 0$ and some higher order moment conditions, thence is known as the martingale difference noise. The asymptotics of the above recursion are well-understood in the literature, as discussed in Kushner and Yin (2013). The recent developments of finite-time/sample behavior of SA is discussed in the literature review. The Q-learning recursion in (2.11) can be represented as an SA update if we notice that given any q -function, $R + \gamma v(q)(S)$ is an *unbiased* estimator of the population Bellman operator applied to q . However, the DR Q-learning and the variance-reduced version cannot be formulated in the same way as (2.12) with martingale difference noise, as there is bias present in the former algorithms. Consequently, to achieve the nearly optimal sample complexity bounds, we must conduct a tight analysis of these algorithms as biased SA, as we will explain in the subsequent sections.

3. The DR Q-Learning and Variance Reduction

This section introduces two model-free algorithms, the DR Q-learning (Section 3.1) and its variance-reduced version (Section 3.2), for learning the optimal q -function of a robust MDP. We also present the upper bounds on their worst-case sample complexity. In addition, we outline the fundamental ideas behind the proof of the sample complexity results in Section 3.3.

Prior to presenting the algorithms, we introduce several notations. Let $\nu_{s,a,n}$ and $p_{s,a,n}$ denote the empirical measure of $\mu_{s,a}$ and $p_{s,a}$ formed by n i.i.d. samples respectively; i.e. for $f : \mathbf{U} \rightarrow \mathbb{R}$, where \mathbf{U} could be the \mathbf{S} or \mathbf{R} ,

$$E_{\mu_{s,a,n}} f(U) := \frac{1}{n} \sum_{j=1}^n f(U_i) \quad (3.1)$$

for $\mu = \nu, p$ and $U_i = R_i, S_i$ are i.i.d. across i .

Assuming access to a simulator, we are able to draw samples and construct an empirical version of the DR Bellman operator.

Definition 5 Define the empirical DR Bellman operator on n i.i.d. samples by

$$\begin{aligned} \mathbf{T}(q)(s, a) &:= \sup_{\alpha \geq 0} \left\{ -\alpha \log E_{\nu_{s,a,n}} \left[e^{-R/\alpha} \right] - \alpha \delta \right\} \\ &\quad + \gamma \sup_{\beta \geq 0} \left\{ -\beta \log E_{p_{s,a,n}} \left[e^{-v(q)(S)/\beta} \right] - \beta \delta \right\}. \end{aligned} \tag{3.2}$$

Note that \mathbf{T} is a random operator whose randomness is coming from on the samples that we used to construct $\{\nu_{s,a,n}, p_{s,a,n} : (s, a) \in \mathbf{S} \times \mathbf{A}\}$.

Definition 5 presents the empirical DR Bellman operator in its dual form. Lemma 2 establishes that this definition is equivalent to the DR Bellman operator \mathcal{T} in (2.8) where the sets $\mathcal{P}_{s,a}(\delta)$ and $\mathcal{N}_{s,a}(\delta)$ are replaced with their empirical counterparts: $\{p : D_{\text{KL}}(p \| p_{s,a,n}) \leq \delta\}$ and $\{\nu : D_{\text{KL}}(\nu \| \nu_{s,a,n}) \leq \delta\}$.

The dual formulation of the empirical DR Bellman operator implies that it is generally a biased estimator of the population DR Bellman operator \mathcal{T} in the sense that $E[\mathbf{T}(q)] \neq \mathcal{T}(q)$ for a generic $q \in \mathbb{R}^{\mathbf{S} \times \mathbf{A}}$. This bias poses a significant challenge in the design of model-free algorithms and the analysis of sample complexities. Previous works Liu et al. (2022) and Wang et al. (2023b) eliminates this bias by using a randomized multilevel Monte Carlo estimator. However, the randomization procedure requires a random (and heavy-tailed) sample size. Therefore, the complexity bound is stated in terms of the expected number of samples. Also, this complex algorithmic design limits its generalizability. In contrast, this paper takes a different approach by directly analyzing the DR Q-learning and its variance-reduced version as biased SA. To achieve near-optimal sample complexity guarantees, the bias of the empirical DR Bellman operator and the propagation of the systematic error it causes are tightly controlled, and samples are optimally allocated so that the stochasticity is in balance with the cumulative bias. A detailed discussion of this approach is provided in Section 3.3.

To state the key assumption which constraint the operating regime of our algorithm, we introduce the following complexity metric parameter:

Definition 6 Define the minimum support probability as

$$\mathfrak{p}_\wedge := \min_{s,a \in \mathbf{S} \times \mathbf{A}} \min \left\{ \min_{r \in \mathbf{R} : \nu_{s,a}(r) > 0} \nu_{s,a}(r), \min_{s' \in \mathbf{S} : p_{s,a}(s') > 0} p_{s,a}(s') \right\}. \tag{3.3}$$

The intuition behind the dependence of the MDP complexity on the minimal support probability is that in order to estimate the DR Bellman operator with high accuracy in the worst case, it is necessary to know the entire support of the transition and reward distributions. As a result, at least $1/\mathfrak{p}_\wedge$ samples are required, as discussed in Wang et al. (2023b).

We are now prepared to present the main assumption that defines the operating regime for which our algorithms are optimized.

Assumption 1 (Limited Adversarial Power) Suppose the adversary's power δ satisfies $\delta < \frac{1}{24}\mathfrak{p}_\wedge$.

It should be noted that the constant $1/24$ is only for mathematical convenience and can potentially be improved.

Under this assumption, the adversary cannot collapse the support of the transition or reward distributions to a singleton, preventing them from completely restricting possible transition events under P_0 . This assumption regime is of practical significance because overly conservative policies can be produced if δ is large. Furthermore, the support of the reward and transition measures often encode physical constraints intrinsic to the real environment, which the adversary should not be allowed to violate.

We also make the following simplifying assumption.

Assumption 2 (Reward Bound) *The reward $\mathbf{R} \subset [0, 1]$.*

This assumption is straightforward to remove given that the results of the empirical Bellman operator hold for $\mathbf{R} \subset \mathbb{R}_{\geq 0}$. We assume it so as to clarify our presentation.

3.1 The Distributionally Robust Q-learning

First, we proposed the DR Q-learning Algorithm 1, a robust version of the classical Q-learning that is based on iteratively update the q -function by applying the n -sample empirical Bellman operator.

Algorithm 1 Distributionally Robust Q-Learning

Input: the total times of iteration k_0 and a batch size n_0 .

Initialization: $q_1 \equiv 0$; $k = 1$.

for $1 \leq k \leq k_0$ **do**

 Sample \mathbf{T}_{k+1} the n_0 -sample empirical DR Bellman operator as in Definition 5.

 Compute the Q-learning update

$$q_{k+1} = (1 - \lambda_k)q_k + \lambda_k \mathbf{T}_{k+1}(q_k) \quad (3.4)$$

 with stepsize $\lambda_k = 1/(1 + (1 - \gamma)k)$.

end for

return q_{k_0+1} .

Algorithm 1 can be viewed as a *biased* SA: We can rewrite the update (3.4) as

$$q_{k+1} = (1 - \lambda_k)q_k + \lambda_k \mathcal{T}(q_k) + \lambda_k (\mathbf{T}_{k+1}(q_k) - \mathcal{T}(q_k)).$$

This is in the form of (2.12). However, notice that $E[\mathbf{T}_{k+1}(q_k) - \mathcal{T}(q_k)|q_k] \neq 0$. Moreover, we note that the update (3.4) involves computing $\mathbf{T}_{k+1}(q_k)(s, a)$ for all $(s, a) \in \mathbf{S} \times \mathbf{A}$. Unlike a model-based algorithm, which requires storing the entire empirical kernel and reward distributions $\{p_{s,a,n}, \nu_{s,a,n} : (s, a) \in \mathbf{S} \times \mathbf{A}\}$, the update rule (3.4) can be implemented separately for each state-action pair. This allows $p_{s,a,n}$ and $\nu_{s,a,n}$ to be discarded immediately after the update, significantly reducing the memory requirements for running Algorithm 1 when the state space is large.

It turns out that, by leveraging the fact that the empirical Bellman operators are monotone contractions w.p.1 (as proven in Lemma 19), we can perform a stronger pathwise analysis of Algorithm 1 instead of treating it as a variant of the SA update in (2.12). As a result, we will prove in Section B.2 that the DR Q-learning algorithm satisfies the following error bound in Proposition 7.

To simplify notation, we define the dimensionality parameter $d := |\mathbf{S}||\mathbf{A}|(|\mathbf{S}| \vee |\mathbf{R}|)$. It will only show up inside the $\log(\cdot)$ term in our complexity bounds because of the use of union bound techniques.

Proposition 7 *Suppose that Assumptions 1 and 2 are satisfied. The output q_{k_0+1} of the distributionally robust Q-learning satisfies*

$$\|q_{k_0+1} - q^*\|_\infty \leq c \left(\frac{1}{(1-\gamma)^3 k_0} + \frac{1}{\mathfrak{p}_\wedge^3 (1-\gamma)^2 n_0} + \frac{1}{\mathfrak{p}_\wedge (1-\gamma)^{5/2} \sqrt{n_0 k_0}} \right) (\log(3dk_0/\eta))^2.$$

with probability at least $1 - \eta$, where c is an absolute constant.

By ‘‘absolute constant’’, we mean a constant that does not depend on the complexity metric parameters $\epsilon, \mathfrak{p}_\wedge, (1-\gamma)^{-1}, \eta, d$. Although the logarithmic term in the above proposition can be further improved, we will not focus on optimizing the logarithmic dependence in this paper. For clarity, we adjust the constant in the logarithmic factor using the inequality for $C_1 \geq 1, C_2 \geq e$, $\log(C_1 C_2) = \log(C_1) + \log(C_2) \leq C_1 \log(C_2)$, and incorporate C_1 into c . These adjustments are applied to all subsequent convergence results.

The proof of this Proposition, which is outlined in Section 3.3, will be postponed to Section B.2.

Proposition 7 provides an upper bound on the terminal error in the estimator after k_0 iterations of Algorithm 1. This bound is given by three terms that decay with rate $\tilde{O}(k_0^{-1})$, $\tilde{O}(n_0^{-1})$, and $\tilde{O}((k_0 n_0)^{-1/2})$, respectively, where the first and third terms resemble the upper bounds for standard Q-learning and the second term arises because of the bias. We optimize the algorithm parameters to balance these three terms and ensure that the right-hand side of the probability bound in Proposition 7 is less than ϵ . One way to achieve this is by selecting the parameters n_0 and k_0 as follows:

Corollary 8 *Assume Assumptions 1 and 2. Running Algorithm 1 with parameters*

$$k_0 = c_0 \frac{1}{(1-\gamma)^3 \epsilon} \log \left(\frac{3d}{(1-\gamma)\eta\epsilon} \right)^2 \quad \text{and} \quad n_0 = c_0 \frac{1}{\mathfrak{p}_\wedge^3 (1-\gamma)^2 \epsilon} \log(3dk_0/\eta)^2$$

will produce an output q_{k_0+1} s.t. $\|q_{k_0+1} - q^\|_\infty \leq \epsilon$ w.p. at least $1 - \eta$, where c is an absolute constant.*

An immediate consequence of Corollary 8 is the following the worst-case sample complexity upper bound of the robust Q-learning.

Theorem 9 *Assume Assumptions 1 and 2. Then the distributionally robust Q-learning Algorithm 1 with parameters specified in Corollary 8 computes a solution q_{k_0+1} s.t. $\|q_{k_0+1} - q^*\|_\infty$ w.p. at least $1 - \eta$ using*

$$\tilde{O} \left(\frac{|\mathbf{S}||\mathbf{A}|}{\mathfrak{p}_\wedge^3 (1-\gamma)^5 \epsilon^2} \right)$$

number of samples.

Proof The total number of samples used is $|\mathbf{S}||\mathbf{A}|n_0k_0$, implying the sample complexity upper bound. \blacksquare

Theorem 9 provides a near-optimal worst-case sample complexity guarantee that matches and beats the expected sample complexity upper bound in Wang et al. (2023b) in all parameter dependence. In particular, we have shown that the dependence on δ is $O(1)$ as $\delta \downarrow 0$. This resolves the issue of the worst-case complexity bound blowing up as $\delta \downarrow 0$ for KL divergence based DR-RL that present in all prior works (Yang et al., 2021; Panaganti and Kalathil, 2021; Shi and Chi, 2022; Wang et al., 2023b).

3.2 The Variance-Reduced Distributionally Robust Q-learning

We adapt Wainwright’s variance-reduced Q-learning (Wainwright, 2019a) to the robust RL setting. This is outlined in Algorithm 2.

Algorithm 2 Variance-Reduced Distributionally Robust Q-Learning

Input: the number of epochs l_{vr} , a sequence of recentering sample size $\{m_l\}_{l=1}^{l_{\text{vr}}}$, an epoch length k_{vr} and a batch size n_{vr} .

Initialization: $\hat{q}_0 \equiv 0$; $l = 1$; $k = 1$.

for $1 \leq l \leq l_{\text{vr}}$ **do**

 Compute $\tilde{\mathbf{T}}_l$, m_l -sample empirical DR Bellman operator as in Definition 5.

 Set $q_{l,1} = \hat{q}_{l-1}$.

for $1 \leq k \leq k_{\text{vr}}$ **do**

 Sample $\mathbf{T}_{l,k+1}$ an n_{vr} -sample empirical Bellman operator.

 Compute the recentered Q-learning update

$$q_{l,k+1} = (1 - \lambda_k)q_{l,k} + \lambda_k \left(\mathbf{T}_{l,k+1}(q_{l,k}) - \mathbf{T}_{l,k+1}(\hat{q}_{l-1}) + \tilde{\mathbf{T}}_l(\hat{q}_{l-1}) \right) \quad (3.5)$$

 with stepsize $\lambda_k = 1/(1 + (1 - \gamma)k)$.

end for

 Set $\hat{q}_l = q_{l,k_{\text{vr}}+1}$.

end for

return $\hat{q}_{l_{\text{vr}}}$

As in the Q-learning case, the update rule (3.5) can be implemented separately for each state-action pair. Thus, Algorithm 2 does not require storing or performing computations using the entire empirical kernel and reward distribution.

Before delving into the convergence rate theory of the DR variance-reduced Q-learning, we provide an intuitive description of this variance reduction scheme. The basic idea is to partition the algorithm into epochs. During each epoch, we perform a “recentered” version of stochastic approximation recursions with the aim of eliminating the variance component in the SA iteration ((2.12)). Specifically, instead of approximating q^* by one stochastic approximation, in each epoch, starting with an estimator \hat{q}_{l-1} , we recenter the SA procedure so that it approximates $\mathcal{T}(q_{l-1})$. However, since \mathcal{T} is not known, we use $\tilde{\mathbf{T}}_l(q_{l-1})$ as an natural estimator. By choosing a sequence of empirical DR Bellman operators

with exponentially increasing sample sizes, we expect that the errors $\|\hat{q}_l - q^*\|_\infty$ decrease exponentially with high probability.

This indeed holds true for Algorithm 2. The outer loop produces a sequence of estimators $\hat{q}_l, l \geq 1$. We will show that if \hat{q}_{l-1} is within some error from the optimal q^* , then \hat{q}_l will satisfy a better concentration bound by a geometric factor. This result is summarized in Proposition 10.

Denote the σ -field generated by the random samples used until the end of epoch l by \mathcal{F}_l . We define the conditional expectation $E_{l-1}[\cdot] := E[\cdot | \mathcal{F}_{l-1}]$ and probability measure $P_{l-1}(\cdot) := E_{l-1}[\mathbf{1}\{\cdot\}]$.

Proposition 10 *Assuming that Assumptions 1 and 2 are satisfied. On $\{\omega : \|\hat{q}_{l-1} - q^*\|_\infty \leq b\}$ for some $b \leq 1/(1-\gamma)$, under measure $P_{l-1}(\cdot)(\omega)$, we have that there exists numerical constant c s.t.*

$$\begin{aligned} \|\hat{q}_l - q^*\|_\infty &\leq c \left(\frac{b}{(1-\gamma)^2 k_{\text{vr}}} + \frac{b}{\mathfrak{p}_\wedge^{3/2} (1-\gamma)^{3/2} \sqrt{n_{\text{vr}} k_{\text{vr}}}} + \frac{b}{\mathfrak{p}_\wedge^{3/2} (1-\gamma) \sqrt{n_{\text{vr}}}} \right) \log(3dk_{\text{vr}}/\eta)^2 \\ &\quad + c \frac{1}{\mathfrak{p}_\wedge^{3/2} (1-\gamma)^2 \sqrt{m_l}} \sqrt{\log(3d/\eta)} \end{aligned}$$

w.p. at least $1 - \eta$, provided that $m_l \geq 8\mathfrak{p}_\wedge^{-2} \log(24d/\eta)$ and $n_{\text{vr}} \geq \mathfrak{p}_\wedge^{-1}$.

Proposition 10 implies that if the variance reduced algorithm finds an approximation of q^* with infinity norm b , then the error after one epoch is improved accordingly with high probability. This and the Markovian nature of the sequence $\{\hat{q}_l\}$ would imply a high probability bound for trajectories satisfying the pathwise property $\{\omega : \forall l \leq l_{\text{vr}} : \|\hat{q}_l - q^*\| \leq b_l\}$. This is formalized by the next theorem where we use $b_l = 2^{-l}(1-\gamma)^{-1}$.

Let us define the parameter choice: for sufficiently large c_{vr} absolute constant that doesn't depend on the complexity metric parameters $\epsilon, \mathfrak{p}_\wedge, (1-\gamma)^{-1}, \eta, d$, define

$$\begin{aligned} l_{\text{vr}} &= \left\lceil \log_2 \left(\frac{1}{\epsilon(1-\gamma)} \right) \right\rceil, \\ k_{\text{vr}} &= c_{\text{vr}} \frac{1}{(1-\gamma)^2} \log \left(\frac{3dl_{\text{vr}}}{(1-\gamma)\eta} \right)^2, \\ n_{\text{vr}} &= c_{\text{vr}} \frac{1}{\mathfrak{p}_\wedge^3 (1-\gamma)^2} \log(3dk_{\text{vr}}l_{\text{vr}}/\eta)^4, \\ m_l &= c_{\text{vr}} \frac{4^l}{\mathfrak{p}_\wedge^3 (1-\gamma)^2} \log(3dl_{\text{vr}}/\eta)^2. \end{aligned} \tag{3.6}$$

Notice that evidently $m_l \geq 8\mathfrak{p}_\wedge^{-2} \log(24d/\eta)$ and $n_{\text{vr}} \geq \mathfrak{p}_\wedge^{-1}$, satisfying the requirement of Proposition 10.

Proposition 11 *Assume Assumptions 1 and 2. For $\epsilon < (1-\gamma)^{-1}$, define parameters according to (3.6). Then, the sequence $\{\hat{q}_l, 0 \leq l \leq l_{\text{vr}}\}$ produced by Algorithm 2 satisfies the pathwise property that $\|\hat{q}_l - q^*\|_\infty \leq 2^{-l}(1-\gamma)^{-1}$ for all $0 \leq l \leq l_{\text{vr}}$ w.p. at least $1 - \eta$. In particular, the final estimator $\hat{q}_{l_{\text{vr}}}$ satisfies $\|\hat{q}_{l_{\text{vr}}} - q^*\|_\infty \leq 2^{-l_{\text{vr}}}(1-\gamma)^{-1}$ w.p. at least $1 - \eta$.*

Remark 12 The base of geometric growth in our choice of m_l in (3.6) can be modified. The same proof as in Proposition 11 suggests that with $m_l = \alpha^{2l}\Theta(\mathfrak{p}_\wedge^{-3}(1-\gamma)^{-2})$ and $l_{\text{vr}} = \lceil \log_\alpha(\epsilon^{-1}(1-\gamma)^{-1}) \rceil$ for some $\alpha > 1$, we have $\|\hat{q}_l - q^*\|_\infty \leq \alpha^{-l}(1-\gamma)^{-1}$ for all $0 \leq l \leq l_{\text{vr}}$ with probability at least $1 - \eta$. Running Algorithm 2 with this new parameter choice will yield the same sample complexity as in Theorem 13. The choice of base 4 in (3.6) was made only for clarity in our presentation.

Proposition 11 immediately implies the following worst-case sample complexity upper bound.

Theorem 13 Assume Assumptions 1 and 2. For $\epsilon < (1-\gamma)^{-1}$, the variance-reduced DR Q-learning Algorithm 2 with parameters specified in (3.6) computes a solution $\hat{q}_{l_{\text{vr}}}$ s.t. $\|\hat{q}_{l_{\text{vr}}} - q^*\|_\infty \leq \epsilon$ w.p. at least $1 - \eta$ using

$$\tilde{O}\left(\frac{|\mathbf{S}||\mathbf{A}|}{\mathfrak{p}_\wedge^3(1-\gamma)^4 \min(1, \epsilon^2)}\right)$$

number of samples.

Proof Given the specified parameters, the total number of samples used is

$$|\mathbf{S}||\mathbf{A}|\left(l_{\text{vr}}n_{\text{vr}}k_{\text{vr}} + \sum_{l=1}^{l_{\text{vr}}} m_l\right) = \tilde{O}\left(|\mathbf{S}||\mathbf{A}|\left(\frac{1}{\mathfrak{p}_\wedge^3(1-\gamma)^4} + \frac{4^{l_{\text{vr}}}}{\mathfrak{p}_\wedge^3(1-\gamma)^2}\right)\right)$$

This simplifies to the claimed result. ■

Theorem 13 establishes an upper bound of $\tilde{O}(|\mathbf{S}||\mathbf{A}|(1-\gamma)^{-4}\epsilon^{-2}\mathfrak{p}_\wedge^{-3})$ when $\epsilon \leq 1$, which is superior to the upper bound $\tilde{O}(|\mathbf{S}||\mathbf{A}|(1-\gamma)^{-5}\epsilon^{-2}\mathfrak{p}_\wedge^{-3})$ for Algorithm 1 (see Theorem 9) in terms of $1-\gamma$. This represents the best-known upper bound for DR-RL problems in the KL case, including both model-free and model-based algorithms (Shi and Chi, 2022). Although Shi and Chi (2022) achieve a similar rate of $\tilde{O}((1-\gamma)^{-4})$, their result suffers from a $\tilde{O}(\delta^{-2})$ dependence, which becomes problematic as $\delta \rightarrow 0$. In contrast, our upper bound is free from δ -dependence.

We recall that the information-theoretical lower bound for the sample complexity of the classical tabular RL problem is $\tilde{\Omega}(|\mathbf{S}||\mathbf{A}|(1-\gamma)^{-3}\epsilon^{-2})$ (Azar et al., 2013). In this setting, the variance-reduced Q-learning algorithm in Wainwright (2019a) is minimax optimal. For distributionally robust RL, Shi and Chi (2022) recently showed that the minimax lower bound dependence on $|\mathbf{S}||\mathbf{A}|$, $(1-\gamma)^{-1}$, and ϵ remains $\tilde{\Omega}(|\mathbf{S}||\mathbf{A}|(1-\gamma)^{-3}\epsilon^{-2})$ when δ is small. Furthermore, Shi et al. (2024) showed the information-theoretical lower bound may be $\tilde{\Omega}(|\mathbf{S}||\mathbf{A}|(1-\gamma)^{-4}\epsilon^{-2})$ when $\delta = O(1)$ for χ^2 -divergence uncertainty sets. However, their construction of hard instances violates our Assumption 1. It is currently unknown whether variance-reduced DR Q-learning can achieve those rates. Further refinement of this bound is left for future research.

Notice that the variance-reduced Algorithm 2 has the property that k_{vr} , n_{vr} , and m_l only depend on $\frac{1}{\epsilon}$ through $\log(l_{\text{vr}}) = \Theta(\log \log \frac{1}{\epsilon})$. Therefore, within a reasonable range of ϵ , the algorithm can operate with the sample complexity guarantee in Theorem 13 without needing to tune k_{vr} , n_{vr} , and m_l based on ϵ . This introduces significant versatility in application: for example, we can continue to run the algorithm beyond termination epoch l_{vr} without losing sample efficiency.

3.3 Overview of the Analysis of Algorithms

In this section, we provide a road map to proving the key results, Proposition 7 and 10.

Definition 14 *We say that \mathcal{L} is a monotonic γ -quasi-contraction with center q' if*

$$\|\mathcal{L}(q) - \mathcal{L}(q')\|_\infty \leq \gamma \|q - q'\|_\infty, \quad (3.7)$$

and entrywise

$$q_1 \geq q_2 \implies \mathcal{L}(q_1) \geq \mathcal{L}(q_2) \quad (3.8)$$

for all $q, q_1, q_2 \in \mathbb{R}^{|\mathbf{S}| \times |\mathbf{A}|}$. Moreover, a monotonic γ -contraction is such that the above identities hold for all $q' \in \mathbb{R}^{|\mathbf{S}| \times |\mathbf{A}|}$.

The term *quasi* refers to the fact that the relation 3.7 is only required for a single q' (Wainwright, 2019b). Therefore, a monotonic γ -contraction is a quasi-contraction with center q' for any $q' \in \mathbb{R}^{|\mathbf{S}| \times |\mathbf{A}|}$.

The successive application of monotonic γ -contractions under the rescaled linear stepsize $\lambda_k = \frac{1}{1+(1-\gamma)k}$ will satisfy the following deterministic bound:

Proposition 15 (Corollary 1, Wainwright (2019b)) *Let $\{\mathcal{L}_k, k \geq 2\}$ be a family of monotonic γ -quasi-contractions with center q' . Let $\mathcal{H}_k(q) = \mathcal{L}_k(q) - \mathcal{L}_k(q')$ the recentered operator. Then, for the sequence of step sizes $\{\lambda_k, k \geq 1\}$ the iterates of*

$$q_{k+1} - q' = (1 - \lambda_k)(q_k - q') + \lambda_k [\mathcal{H}_{k+1}(q_k) + w_{k+1}] \quad (3.9)$$

satisfies

$$\|q_{k+1} - q'\|_\infty \leq \lambda_k \left[\frac{\|q_1 - q'\|_\infty}{\lambda_1} + \gamma \sum_{j=1}^k \|p_j\|_\infty \right] + \|p_{k+1}\|_\infty$$

for all $k \geq 1$, where the sequence $\{p_k, k \geq 1\}$ is defined by $p_1 = 0$ and

$$p_{k+1} := (1 - \lambda_k)p_k + \lambda_k w_{k+1}.$$

A key observation is that the empirical robust Bellman operators $\mathbf{T}_k, \tilde{\mathbf{T}}_{l,k}$ used in the iterative updates of Algorithms 1 and 2 are monotonic γ -contractions (see Lemma 19).

In the proof of the main results, we apply the deterministic bound for contraction mappings from Proposition 15 to each sample path of the distributionally robust Q-learning and the inner loop of the variance-reduced version. We illustrate this by considering the distributionally robust Q-learning. Since $\{\mathbf{T}_{k+1}, k \geq 0\}$ are monotonic γ -contractions, they are quasi-contractions with center q^* . We can define $\mathbf{H}_{k+1}(q) := \mathbf{T}_{k+1}(q) - \mathbf{T}_{k+1}(q^*)$ for all $q \in \mathbb{R}^{|\mathbf{S}| \times |\mathbf{A}|}$. Then, the update rule of Algorithm 1 can be written as

$$\begin{aligned} q_{k+1} - q^* &= (1 - \lambda_k)(q_k - q^*) + \lambda_k [(\mathbf{T}_{k+1}(q_k) - \mathbf{T}_{k+1}(q^*)) + (\mathbf{T}_{k+1}(q^*) - \mathcal{T}(q^*))] \\ &= (1 - \lambda_k)(q_k - q^*) + \lambda_k [\mathbf{H}_{k+1}(q_k) + W_{k+1}]. \end{aligned}$$

where $W_{k+1} := \mathbf{T}_{k+1}(q^*) - \mathcal{T}(q^*)$ and we used the Bellman equation (2.10) that $q^* = \mathcal{T}(q^*)$.

This representation allows us to apply Proposition 15 to bound the error of the q -function estimation using the sequence $P_1 = 0$ and

$$P_{k+1} := (1 - \lambda_k)P_k + \lambda_k W_{k+1}.$$

Note that the only source of randomness in W_k is from $\mathbf{T}_{k+1}(q^*)$, which are i.i.d.. Therefore, the process P is a non-stationary auto-regressive (AR) process. It follows that the concentration properties of P_k can be derived from that of $\mathbf{T}_{k+1}(q^*)$.

While standard Q-learning updates utilize an unbiased empirical Bellman operator, the DR empirical Bellman operator is biased due to its non-linearity in the empirical measure (c.f. (3.2)), resulting in $E[W_k] \neq 0$. To achieve a canonical error rate of $O(n^{-1/2})$, it is necessary that both the bias and standard deviation of the n -sample DR empirical Bellman operator are $O(n^{-1/2})$. However, our DR Q-learning algorithms require an additional condition: the one-step bias must be of the order $O(n^{-1})$. This is because the final bias, which is the systematic error resulting from the repeated use of the DR Bellman estimator, is compounded by the one-step bias through the model-free Q-learning updates. We illustrate this algorithmic behavior using a simple biased SA instance in Appendix B.1. This imposes significant challenges on the design and analysis of our model-free algorithms.

Fortunately, we are able to establish tight bounds (in n_0 and δ) on the bias, c.f. Proposition 22, in the important regime when δ is small, as stated in Assumption 1. These bounds are central to our sample complexity analysis. We summarize the relevant bounds on the variance and bias of the empirical DR Bellman operator in Section A. By utilizing these variance and bias bounds, we can efficiently allocate samples such that the systematic error due to bias is balanced with the stochasticity in the estimator at the termination of the algorithm. With this optimal sample allocation, we can establish the worst-case sample complexity bounds as claimed.

The theory for the convergence rate of the variance-reduced DR Q-learning is more complex. In order to achieve the geometric convergence in Proposition 11, an $O(n^{-1})$ bias bound of the empirical DR Bellman operator is not enough. However, by introducing a recentered dynamics, a similar recursion can be derived in this context if we consider the *conditionally recentered noise* $\mathbf{H}_{l,k+1}(\hat{q}_{l-1}) - E[\mathbf{H}_{l,k+1}(\hat{q}_{l-1})|\hat{q}_{l-1}]$ and a “random bias” (denoted by D_l in Appendix B.3). For details, please refer to Appendix B.3.

4. Numerical Experiments

This section presents a numerical validation of our theoretical findings regarding the convergence properties of the proposed algorithms. We conduct a comparative analysis between our algorithms and MLMC DR Q-learning, as studied in Wang et al. (2023b). Additionally, we investigate the complexity of Algorithm 2 as the adversary’s power $\delta \downarrow 0$.

Section 4.1 demonstrates convergence and compares the proposed algorithms with multilevel Monte Carlo distributionally robust (MLMC DR) Q-learning. We use the hard MDP instances constructed in Li et al. (2021), where standard Q-learning performs at its worst-case complexity dependence of $\tilde{\Omega}((1 - \gamma)^{-4}\epsilon^{-2})$. Both algorithms in this paper show the canonical convergence rate of $O(\epsilon^{-2})$, with the variance-reduced version displaying superior performance.

In Section 4.2, we test the stability of sample complexity of the variance-reduced DR Q-learning Algorithm 2 as $\delta \downarrow 0$ using a simple DRMDP instance.

In the subsequent developments, we use $m_l = 2^l(1 - \gamma)^{-2}$ for the variance-reduced Algorithm 2. As explained in Remark 12, this choice (up to a log factor) yields the same complexity guarantee as stated in Theorem 13. An advantage of this parameter choice is that it allows us to run more epochs for the plots, thereby clarifying the convergence behavior.

4.1 Hard MDPs for the Q-learning

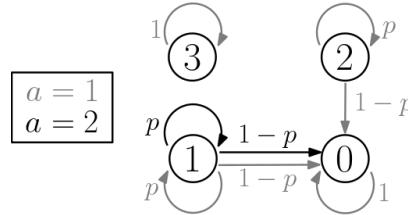


Figure 1: Hard MDP for the Q-learning transition diagram.

First, we demonstrate the convergence of the proposed algorithms using the MDP instance shown in Figure 1. This MDP has 4 states and 2 actions, with transition probabilities given for actions 1 and 2 labeled on the arrows between states. Constructed in Li et al. (2021), it is shown in that when $p = \frac{4\gamma-1}{3\gamma}$, standard non-robust Q-learning will have a sample complexity of $\tilde{\Theta}((1 - \gamma)^{-4}\epsilon^{-2})$.

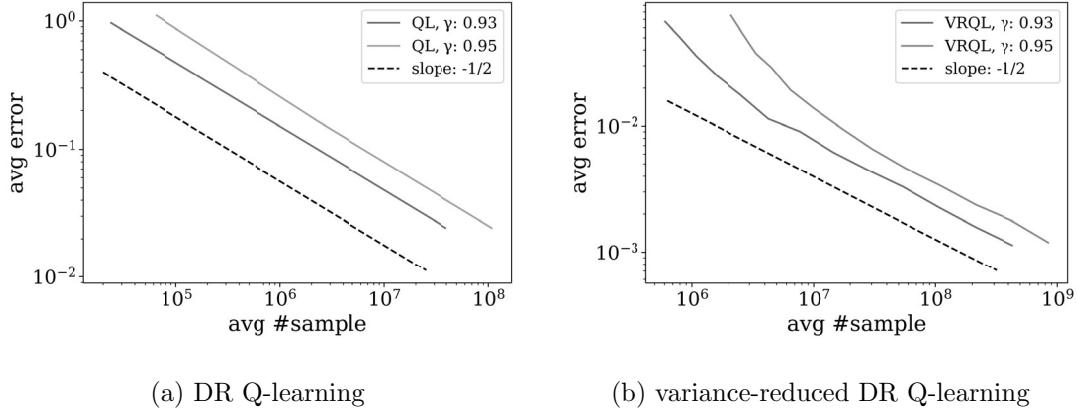


Figure 2: Convergence of Algorithm 1 and 2 on the MDP instance 1

Figures 2a and 2b depict the convergence properties of the two algorithms for $\gamma = \{0.93, 0.95\}$ and $\delta = 0.1$. These figures show the (4000 samples) averaged error of the output q -function in the infinity norm plotted against the (4000 samples) averaged number of samples used, both on a log-log scale. The parameters for DR Q-learning in Figure 2a are set according to 8. On the other hand, Figure 2b plots the averaged error achieved by the variance-reduced algorithm after each epoch against the total number of samples used.

The figures indicate that both algorithms converge to the optimal robust q^* , with the variance-reduced algorithm outperforming DR Q-learning. Additionally, when comparing the log-log error plot with a reference line having a slope of $-1/2$, we observe that the log error for both algorithms decays at a rate of $-1/2$ as the log of the samples increases. This behavior aligns with the ϵ^{-2} dependence of the sample complexity bounds in Theorems 9 and 13, corresponding to the canonical convergence rate of Monte Carlo estimations, which is $O(n^{-1/2})$.

Remark 16 *With $\delta = 0.1$ and $\gamma = 0.93$ or 0.95 , the DRMDP instances do not satisfy Assumption 1. However, the figures still show the canonical $n^{-1/2}$ convergence rate, suggesting that our proposed algorithms might perform well even outside the regime prescribed by Assumption 1.*

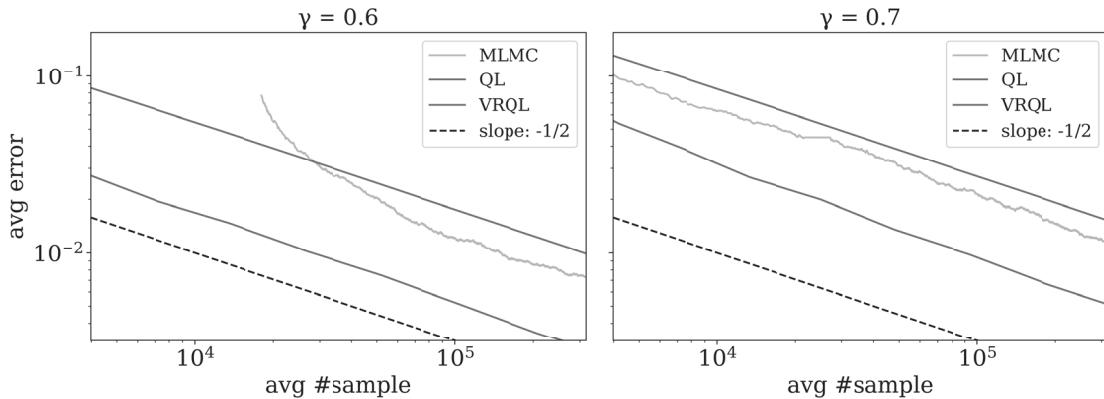


Figure 3: Comparing the performance of Algorithm 1, 2 and the MLMC DR Q-learning on the MDP 1.

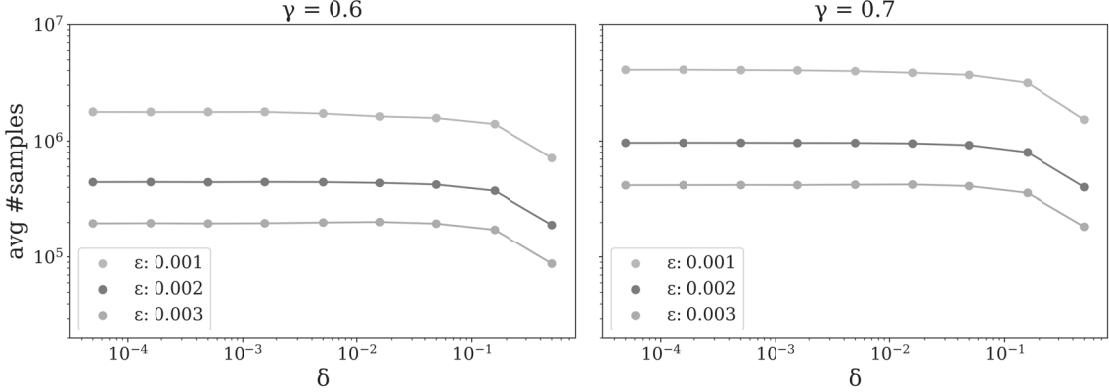
Figure 3 compares the performance of the algorithms proposed in this paper with the MLMC DR Q-learning in Wang et al. (2023b). We observe the performance comparison of three Q-learning methods: MLMC DR, DR, and DR-VR, for $\gamma \in \{0.6, 0.7\}$. The results indicate that the distributionally robust variance-reduced Q-learning approach achieves the smallest errors. Although our DR Q-learning method shows slightly lower expected performance than the MLMC DR Q-learning, it is worth noting that the line corresponding to MLMC DR Q-learning is considerably rougher. This suggests that the MLMC DR Q-learning approach has a higher degree of variability in terms of performance.

4.2 Testing the Small δ Regime

We proceed to empirically demonstrate the stability of the sample complexity of Algorithm 2 as $\delta \downarrow 0$.

First, we introduce a family of MDPs instance. Define reference MPDs with $S = \{1, 2\}$, $A = \{a_1, a_2\}$, transition kernel

$$P_{0,a_1} = P_{0,a_2} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}, \quad (4.1)$$


 Figure 4: Testing the sample complexity behavior as $\delta \downarrow 0$.

and deterministic reward function $r(1, \cdot) = 1$ and $r(2, \cdot) = 0$. For any positive adversarial power level δ , the worst-case transition kernel chosen by the adversary is

$$P_{\delta, a_1} = P_{\delta, a_2} = \begin{bmatrix} q(\delta) & 1 - q(\delta) \\ q(\delta) & 1 - q(\delta) \end{bmatrix}$$

where $q(\delta) < 1/2$ and $q(\delta) \uparrow 1/2$ as $\delta \downarrow 0$. In a classical tabular RL setting, this worst-case MDP ($\delta > 0$) should be easier to learn compared to (4.1), c.f. (Khamaru et al., 2021; Wang et al., 2023a).

Using this DRMDP instance, we plot the average number of samples required to achieve a fixed error ϵ while varying δ , as shown in Figure 4. We observe that the average number of samples increases as $\delta \downarrow 0$, because the worst-case MDP converges to the instance in (4.1), which is more challenging to learn. Additionally, the number of samples needed to reach the target error level becomes insensitive to increasingly small δ when $\delta \leq 10^{-2}$, confirming the theoretical results presented in this paper.

5. Extension: χ_2 Divergence Ambiguity Sets

We extend the variance-reduced version of the Q-learning Algorithm 2 to the setting where the adversary is constrained to perturbations within χ_2 divergence balls of radius δ . The χ_2 divergence is defined for $Q \ll P$ as

$$D_{\chi_2}(Q \| P) := \frac{1}{2} \int_{\Omega} \left(\frac{dQ}{dP}(\omega) - 1 \right)^2 P(d\omega). \quad (5.1)$$

Note that we follow the convention in Duchi and Namkoong (2021) to include an $1/2$ in (5.1).

We reuse the notation for the KL case in the discussion of this section. In particular, for each $(s, a) \in \mathbf{S} \times \mathbf{A}$ and $\delta > 0$, we define χ_2 ambiguity sets analogous to (2.2) as

$$\begin{aligned} \mathcal{P}_{s,a}(\delta) &:= \{p : D_{\chi_2}(p \| p_{s,a}) \leq \delta\}, \\ \mathcal{N}_{s,a}(\delta) &:= \{\nu : D_{\chi_2}(\nu \| \nu_{s,a}) \leq \delta\}. \end{aligned} \quad (5.2)$$

For χ_2 divergence defined in (5.1), we have the following strong duality.

Lemma 17 (Duchi and Namkoong (2021), Lemma 1) *Let X be a random variable and μ_0 a probability measure on (Ω, \mathcal{F}) . Then, for any $\delta > 0$,*

$$\inf_{\mu: D_{\chi_2}(\mu\|\mu_0) \leq \delta} E_\mu X = \sup_{\alpha \in \mathbb{R}} \left\{ \alpha - c(\delta) E_{\mu_0} [(\alpha - X)_+^2]^{1/2} \right\} \quad (5.3)$$

where $c(\delta) = \sqrt{1 + 2\delta}$ and $(\cdot)_+ := \max\{\cdot, 0\}$.

We note that the dual variable α can be optimized within $\alpha \geq \text{ess inf}_{\mu_0} X$.

We wish to learn the optimal q -function as defined in (2.7). To achieve this, we use the DR Bellman equation for the q -function (2.10) where the dual form of the Bellman operator $\mathcal{T}: \mathbb{R}^{\mathbf{S} \times \mathbf{A}} \rightarrow \mathbb{R}^{\mathbf{S} \times \mathbf{A}}$ in the χ_2 case is given by

$$\mathcal{T}(q)(s, a) := \sup_{\alpha \in \mathbb{R}} \left\{ \alpha - c(\delta) E_{\nu_{s,a}} [(\alpha - R)_+^2]^{1/2} \right\} + \gamma \sup_{\beta \in \mathbb{R}} \left\{ \beta - c(\delta) E_{p_{s,a}} [(\beta - v(q)(S))_+^2]^{1/2} \right\}. \quad (5.4)$$

Then, the empirical Bellman operator \mathbf{T} is similarly defined as in (3.2) using this dual representation as

$$\begin{aligned} \mathbf{T}(q)(s, a) &:= \sup_{\alpha \in \mathbb{R}} \left\{ \alpha - c(\delta) E_{\nu_{s,a,n}} [(\alpha - R)_+^2]^{1/2} \right\} \\ &\quad + \gamma \sup_{\beta \in \mathbb{R}} \left\{ \beta - c(\delta) E_{p_{s,a,n}} [(\beta - v(q)(S))_+^2]^{1/2} \right\}, \end{aligned} \quad (5.5)$$

where the empirical measures and expectations are defined in (3.1).

Recall the definition of the minimum support probability \mathbf{p}_\wedge in (3.3). As in the KL case, we also consider the regime $\delta = O(\delta)$:

Assumption 3 (Limited Adversarial Power) *Suppose the adversary's power $\delta < \frac{1}{2}\mathbf{p}_\wedge$.*

In this context, we will apply the variance-reduced Q-learning Algorithm 2 with the following parameters.

$$\begin{aligned} l_{\text{vr}} &= \left\lceil \log_2 \left(\frac{1}{\epsilon(1-\gamma)} \right) \right\rceil, \\ k_{\text{vr}} &= c_{\text{vr}} \frac{1}{(1-\gamma)^2} \log \left(\frac{3dl_{\text{vr}}}{(1-\gamma)\eta} \right)^2, \\ n_{\text{vr}} &= c_{\text{vr}} \frac{1}{\mathbf{p}_\wedge^2(1-\gamma)^2} \log(3dk_{\text{vr}}l_{\text{vr}}/\eta)^4, \\ m_l &= c_{\text{vr}} \frac{4^l}{\mathbf{p}_\wedge^2(1-\gamma)^2} \log(3dl_{\text{vr}}/\eta)^2. \end{aligned} \quad (5.6)$$

Notice that, compare to the specifications in (3.6), (5.6) has a \mathbf{p}_\wedge^{-2} dependence instead of \mathbf{p}_\wedge^{-3} . Running Algorithm 2 with these parameters will yield an estimate $\hat{q}_{l_{\text{vr}}}$ of q^* with an error of at most ϵ with high probability, leading to the following theorem.

Theorem 18 *Assume Assumptions 2 and 3. For $\epsilon < (1 - \gamma)^{-1}$, the variance-reduced DR Q-learning Algorithm 2 with parameters specified in (3.6) computes a solution $\hat{q}_{l_{\text{vr}}}$ s.t. $\|\hat{q}_{l_{\text{vr}}} - q^*\|_\infty \leq \epsilon$ w.p. at least $1 - \eta$ using*

$$\tilde{O}\left(\frac{|\mathbf{S}||\mathbf{A}|}{\mathfrak{p}_\wedge^2(1 - \gamma)^4 \min(1, \epsilon^2)}\right)$$

number of samples.

The proof of Theorem 18 closely follows the proof of Theorem 13. We first establish the analog of Proposition 10 and then apply it to achieve the statement in Proposition 11 using the parameters in (5.6) for the χ_2 divergence ambiguity set case. The sample complexity is then derived by summing the number of samples used in each epoch. This procedure is carried out in Appendix G.

Acknowledgments

The material in this paper is based upon work supported by the Air Force Office of Scientific Research under award number FA9550-20-1-0397. Additional support is gratefully acknowledged from NSF 2229012, 2312204, 2312205, 2419564; ONR 13983111, 13983263; and 2024 New York University Center for Global Economy and Business grant.

References

Alekh Agarwal, Sham Kakade, and Lin F. Yang. Model-based reinforcement learning with a generative model is minimax optimal. In Jacob Abernethy and Shivani Agarwal, editors, *Proceedings of Thirty Third Conference on Learning Theory*, volume 125 of *Proceedings of Machine Learning Research*, pages 67–83. PMLR, 09–12 Jul 2020. URL <https://proceedings.mlr.press/v125/agarwal20b.html>.

Mohammad Gheshlaghi Azar, Rémi Munos, and Hilbert J. Kappen. Minimax pac bounds on the sample complexity of reinforcement learning with a generative model. *Mach. Learn.*, 91(3):325–349, jun 2013. ISSN 0885-6125. doi: 10.1007/s10994-013-5368-1. URL <https://doi.org/10.1007/s10994-013-5368-1>.

Zaiwei Chen, Siva Theja Maguluri, Sanjay Shakkottai, and Karthikeyan Shanmugam. Finite-sample analysis of contractive stochastic approximation using smooth convex envelopes. In H. Larochelle, M. Ranzato, R. Hadsell, M.F. Balcan, and H. Lin, editors, *Advances in Neural Information Processing Systems*, volume 33, pages 8223–8234. Curran Associates, Inc., 2020. URL https://proceedings.neurips.cc/paper_files/paper/2020/file/5d44ee6f2c3f71b73125876103c8f6c4-Paper.pdf.

Zaiwei Chen, Sheng Zhang, Thinh T. Doan, John-Paul Clarke, and Siva Theja Maguluri. Finite-sample analysis of nonlinear stochastic approximation with applications in reinforcement learning. *Automatica*, 146:110623, 2022. ISSN 0005-1098. doi: <https://doi.org/10.1016/j.automatica.2022.110623>. URL <https://www.sciencedirect.com/science/article/pii/S0005109822004873>.

John C Duchi and Hongseok Namkoong. Learning models with uniform performance via distributionally robust optimization. *The Annals of Statistics*, 49(3):1378–1406, 2021.

Eyal Even-Dar, Yishay Mansour, and Peter Bartlett. Learning rates for q-learning. *Journal of machine learning Research*, 5(1), 2003.

Vincent François-Lavet, Peter Henderson, Riashat Islam, Marc G Bellemare, Joelle Pineau, et al. An introduction to deep reinforcement learning. *Foundations and Trends® in Machine Learning*, 11(3-4):219–354, 2018.

J. I. González-Trejo, O. Hernández-Lerma, and L. F. Hoyos-Reyes. Minimax control of discrete-time stochastic systems. *SIAM Journal on Control and Optimization*, 41(5):1626–1659, 2002. doi: 10.1137/S0363012901383837. URL <https://doi.org/10.1137/S0363012901383837>.

Zhaolin Hu and L Jeff Hong. Kullback-leibler divergence constrained distributionally robust optimization. *Available at Optimization Online*, 2013.

Garud Iyengar. Robust dynamic programming. *Math. Oper. Res.*, 30:257–280, 05 2005.

Belhal Karimi, Blazej Miasojedow, Eric Moulines, and Hoi-To Wai. Non-asymptotic analysis of biased stochastic approximation scheme, 2019.

Koulik Khamaru, Eric Xia, Martin J. Wainwright, and Michael I. Jordan. Instance-optimality in optimal value estimation: Adaptivity via variance-reduced q-learning, 2021. URL <https://arxiv.org/abs/2106.14352>.

H. Kushner and G.G. Yin. *Stochastic Approximation and Recursive Algorithms and Applications*. Stochastic Modelling and Applied Probability. Springer New York, 2013. ISBN 9781489926968. URL <https://books.google.com/books?id=sB0GCAAAQBAJ>.

Gen Li, Changxiao Cai, Yuxin Chen, Yuantao Gu, Yuting Wei, and Yuejie Chi. Is q-learning minimax optimal? a tight sample complexity analysis. *arXiv preprint arXiv:2102.06548*, 2021.

Gen Li, Laixi Shi, Yuxin Chen, Yuejie Chi, and Yuting Wei. Settling the sample complexity of model-based offline reinforcement learning, 2022. URL <https://arxiv.org/abs/2204.05275>.

Xiang Li, Wenhao Yang, Jiadong Liang, Zhihua Zhang, and Michael I Jordan. A statistical analysis of polyak-ruppert averaged Q-learning. In *International Conference on Artificial Intelligence and Statistics*, pages 2207–2261. PMLR, 2023.

Yen-Chen Lin, Zhang-Wei Hong, Yuan-Hong Liao, Meng-Li Shih, Ming-Yu Liu, and Min Sun. Tactics of adversarial attack on deep reinforcement learning agents. *arXiv preprint arXiv:1703.06748*, 2017.

Zijian Liu, Qinxun Bai, Jose Blanchet, Perry Dong, Wei Xu, Zhengqing Zhou, and Zhengyuan Zhou. Distributionally robust q -learning. In Kamalika Chaudhuri, Stefanie Jegelka, Le Song, Csaba Szepesvari, Gang Niu, and Sivan Sabato, editors, *Proceedings of the 39th International Conference on Machine Learning*, volume 162 of *Proceedings of Machine Learning Research*, pages 13623–13643. PMLR, 17–23 Jul 2022. URL <https://proceedings.mlr.press/v162/liu22a.html>.

Arnab Nelim and Laurent El Ghaoui. Robust control of Markov decision processes with uncertain transition matrices. *Operations Research*, 53(5):780–798, 2005.

Xinlei Pan, Chaowei Xiao, Warren He, Shuang Yang, Jian Peng, Mingjie Sun, Jinfeng Yi, Zijiang Yang, Mingyan Liu, Bo Li, et al. Characterizing attacks on deep reinforcement learning. *arXiv preprint arXiv:1907.09470*, 2019.

Kishan Panaganti and Dileep Kalathil. Sample complexity of robust reinforcement learning with a generative model, 2021. URL <https://arxiv.org/abs/2112.01506>.

Joaquin Quinonero-Candela, Masashi Sugiyama, Anton Schwaighofer, and Neil D Lawrence. *Dataset shift in machine learning*. Mit Press, 2008.

Philippe Rigollet. “18. s997: High dimensional statistics lecture notes, 2015.

Alexander Shapiro. Distributionally robust modeling of optimal control. *Operations Research Letters*, 50(5):561–567, 2022. ISSN 0167-6377. doi: <https://doi.org/10.1016/j.orl.2022.08.002>. URL <https://www.sciencedirect.com/science/article/pii/S0167637722000992>.

Alexander Shapiro, Darinka Dentcheva, and Andrzej Ruszczyński. *Lectures on Stochastic Programming: Modeling and Theory, Second Edition*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2014. doi: 10.1137/1.9781611973433. URL <https://epubs.siam.org/doi/abs/10.1137/1.9781611973433>.

Laixi Shi and Yuejie Chi. Distributionally robust model-based offline reinforcement learning with near-optimal sample complexity, 2022. URL <https://arxiv.org/abs/2208.05767>.

Laixi Shi, Gen Li, Yuting Wei, Yuxin Chen, Matthieu Geist, and Yuejie Chi. The curious price of distributional robustness in reinforcement learning with a generative model. *Advances in Neural Information Processing Systems*, 36, 2024.

Nian Si, Fan Zhang, Zhengyuan Zhou, and Jose Blanchet. Distributionally robust policy evaluation and learning in offline contextual bandits. In *International Conference on Machine Learning*, pages 8884–8894. PMLR, 2020.

Aaron Sidford, Mengdi Wang, Xian Wu, Lin Yang, and Yinyu Ye. Near-optimal time and sample complexities for solving markov decision processes with a generative model. In S. Bengio, H. Wallach, H. Larochelle, K. Grauman, N. Cesa-Bianchi, and R. Garnett, editors, *Advances in Neural Information Processing Systems*, volume 31. Curran Associates, Inc., 2018. URL <https://proceedings.neurips.cc/paper/2018/file/bb03e43ffe34eeb242a2ee4a4f125e56-Paper.pdf>.

Richard S Sutton and Andrew G Barto. *Reinforcement learning: An introduction*. MIT press, 2018.

Martin J. Wainwright. Variance-reduced q -learning is minimax optimal, 2019a. URL <https://arxiv.org/abs/1906.04697>.

Martin J. Wainwright. Stochastic approximation with cone-contractive operators: Sharp ℓ_∞ -bounds for q -learning, 2019b. URL <https://arxiv.org/abs/1905.06265>.

Gang Wang. Finite-time error bounds of biased stochastic approximation with application to td-learning. *IEEE Transactions on Signal Processing*, 70:950–962, 2022. doi: 10.1109/TSP.2021.3128723.

Shengbo Wang, Jose Blanchet, and Peter Glynn. Optimal sample complexity of reinforcement learning for uniformly ergodic discounted markov decision processes, 2023a.

Shengbo Wang, Nian Si, Jose Blanchet, and Zhengyuan Zhou. A finite sample complexity bound for distributionally robust Q-learning, 2023b. URL <https://arxiv.org/abs/2302.13203>.

Shengbo Wang, Nian Si, Jose Blanchet, and Zhengyuan Zhou. On the foundation of distributionally robust reinforcement learning, 2024.

Christopher JCH Watkins and Peter Dayan. Q-learning. *Machine learning*, 8:279–292, 1992.

Wolfram Wiesemann, Daniel Kuhn, and Breç Rustem. Robust Markov decision processes. *Mathematics of Operations Research*, 38(1):153–183, 2013. ISSN 0364765X, 15265471. URL <http://www.jstor.org/stable/23358653>.

Huan Xu and Shie Mannor. Distributionally robust Markov decision processes. In *Advances in Neural Information Processing Systems*, pages 2505–2513, 2010.

Wenhao Yang, Liangyu Zhang, and Zhihua Zhang. Towards theoretical understandings of robust Markov decision processes: Sample complexity and asymptotics, 2021. URL <https://arxiv.org/abs/2105.03863>.

Wenhao Yang, Han Wang, Tadashi Kozuno, Scott M. Jordan, and Zhihua Zhang. Avoiding model estimation in robust Markov decision processes with a generative model, 2023.

Zhengqing Zhou, Zhengyuan Zhou, Qinxun Bai, Linhai Qiu, Jose Blanchet, and Peter Glynn. Finite-sample regret bound for distributionally robust offline tabular reinforcement learning. In Arindam Banerjee and Kenji Fukumizu, editors, *Proceedings of The 24th International Conference on Artificial Intelligence and Statistics*, volume 130 of *Proceedings of Machine Learning Research*, pages 3331–3339. PMLR, 13–15 Apr 2021. URL <https://proceedings.mlr.press/v130/zhou21d.html>.

Appendices

Appendix A. The Empirical Robust Bellman Operator: KL Case

In this section, we establish the bias and concentration properties of the empirical DR Bellman operator. As pointed out in the previous sections, they are the key ingredients for proving our near-optimal sample complexity bounds. Let $\widehat{\mathbf{T}}_n$ be the empirical DR Bellman operator formed by n samples defined in Definition 5. To simplify the notation, we will omit the subscript n and only keep $\widehat{\mathbf{T}}$ when there is no confusion.

Even though the main results of this paper restrict $\mathbf{R} \subset [0, 1]$ to simplify notation and align with convention in the literature, in this section, we consider $\mathbf{R} \subset [0, r_{\max}]$. This allows our results to be directly applied to contexts beyond RL, such as supervised learning where r_{\max} may vary.

In order to employ the analysis outlined in the previous section, the empirical Bellman operators need to be contraction mappings. Indeed, we have

Lemma 19 $\widehat{\mathbf{T}}$ is a monotonic γ -contraction.

Direct consequences of $\widehat{\mathbf{T}}$ being a γ -contraction with $\gamma < 1$ are the following bounds:

Lemma 20 The following two bounds hold with probability 1:

$$\|\widehat{\mathbf{T}}(q)(s, a) - \mathcal{T}(q)(s, a)\|_{\infty} \leq 2(r_{\max} + \|q\|_{\infty});$$

and

$$\|q_*\|_{\infty} \leq \frac{r_{\max}}{1 - \gamma}.$$

As motivated in the paper, to obtain a desired complexity dependence on problem primitives, we need to develop good bounds on the bias and the variance of the empirical Bellman operator. We define the span seminorm of the q -function as $|q|_{\text{span}} := \max_{s, a} q(s, a) - \min_{s, a} q(s, a)$ and $|q|_{\text{span}} \leq (1 - \gamma)^{-1}$. The proofs of the following propositions are in Appendix C.

Proposition 21 The empirical DR Bellman operator satisfies the following variance bound:

$$\text{Var}(\widehat{\mathbf{T}}(q)(s, a)) \leq 104 \frac{r_{\max}^2 + \gamma^2 |q|_{\text{span}}^2}{\mathfrak{p}_{\wedge}^2 n} (\log(e(|\mathbf{R}| \vee |\mathbf{S}|))).$$

We note that here \mathfrak{p}_{\wedge} can be replaced by $\min_{s' \in S} \min \{p_{s, a}(s'), \nu_{s, a}(s')\}$. In particular, the variance upper bound can depend on the state and action. However, since we are interested in a minimax complexity bound, such distinction will not make a difference if we consider an example with only $O(1)$ number of states and actions.

We also have the following bound on the bias:

Proposition 22 Under Assumption 1, the empirical DR Bellman Operator satisfies the following bias bound:

$$|\text{Bias}(\widehat{\mathbf{T}}(q)(s, a))| := |E[\widehat{\mathbf{T}}(q)(s, a)] - \mathcal{T}(q)(s, a)| \leq 4480 \frac{r_{\max} + \gamma |q|_{\text{span}}}{\mathfrak{p}_{\wedge}^3 n} \log(e|\mathbf{S}| \vee |\mathbf{R}|).$$

Again, the dependence on \mathfrak{p}_\wedge can be replaced by $\min_{s' \in S} \min \{p_{s,a}(s'), \nu_{s,a}(s')\}$.

By Lemma 20, the DR empirical Bellman operator is bounded. This along with the uniform (across $s, a \in \mathbf{S} \times \mathbf{A}$) variance bound in Proposition 21 yields:

Proposition 23 *The empirical DR Bellman operator*

$$\|\widehat{\mathbf{T}}(q) - \mathcal{T}(q)\|_\infty \leq \frac{17(r_{\max} + \gamma|q|_{\text{span}})}{\mathfrak{p}_\wedge \sqrt{n}} \sqrt{\log(6|\mathbf{S}||\mathbf{A}|(|\mathbf{S}| \vee |\mathbf{R}|)/\eta)}$$

w.p. at least $1 - \eta$, provided that $n \geq 8\mathfrak{p}_\wedge^{-2} \log(12|\mathbf{S}||\mathbf{A}|(|\mathbf{S}| \vee |\mathbf{R}|)/\eta)$.

Recall that for fixed \hat{q} , we have defined the recentered DR Bellman operators

$$\widehat{\mathbf{H}}(\hat{q}) := \widehat{\mathbf{T}}(\hat{q}) - \widehat{\mathbf{T}}(q^*) \quad \text{and} \quad \mathcal{H}(\hat{q}) := \mathcal{T}(\hat{q}) - \mathcal{T}(q^*). \quad (\text{A.1})$$

For the variance-reduced algorithm, we instead consider the bias and concentration properties of the recentered operator $\widehat{\mathbf{H}}$. As we will observe, the recentering allows the concentration bounds to depend on the residual error in the q -function $\|\hat{q} - q^*\|_\infty$ instead of $\|\hat{q}\|_\infty$. As a consequence, one can imagine that as the algorithm progresses, $\|\hat{q}_t - q^*\|_\infty$ will progressively become smaller, making $\widehat{\mathbf{H}}$ having much better concentration properties than $\widehat{\mathbf{T}}$.

We start with bias and variance bounds.

Proposition 24 *Suppose Assumption 1 is enforced. Then*

$$|E[\widehat{\mathbf{H}}(\hat{q})(s, a) - \mathcal{H}(\hat{q})(s, a)]| \leq \frac{2^6 \|\hat{q} - q^*\|_\infty}{\mathfrak{p}_\wedge^{3/2} \sqrt{n}} \sqrt{\log(e|\mathbf{S}|)},$$

provided $n \geq \mathfrak{p}_\wedge^{-1}$, and

$$\text{Var}(\widehat{\mathbf{H}}(q^*))(s, a) \leq \frac{2^{12} \|\hat{q} - q^*\|_\infty^2}{\mathfrak{p}_\wedge^3 n} \log(e|\mathbf{S}|)$$

for all $n \geq 1$.

Similar to the extension from Proposition 21 to Proposition 23, we can obtain the following concentration bound for the recentered operator by extending the variance bound in Proposition 24.

Proposition 25 *Assume Assumption 1. Then w.p. at least $1 - \eta$*

$$\|\mathcal{H}(\hat{q}) - \widehat{\mathbf{H}}(\hat{q})\|_\infty \leq \frac{8 \|\hat{q} - q^*\|_\infty}{\mathfrak{p}_\wedge^{3/2} \sqrt{n}} \sqrt{\log(4|\mathbf{S}|^2|\mathbf{A}|/\eta)}$$

provided that $n \geq 8\mathfrak{p}_\wedge^{-2} \log(4|\mathbf{S}|^2|\mathbf{A}|/\eta)$

We emphasize that all of the propositions are $O(1)$ when $\delta \downarrow 0$. This is due to a more thorough analysis, which allows us to remove the dual variable α (see Lemma 2) in the bounds, as explained in Lemma 37 in detail.

Appendix B. Proofs for the Analysis of Algorithms: KL Case

Before we prove the main results in this paper, we illustrate the necessity of requiring the n -sample empirical Bellman operator to exhibit a bias of $\tilde{O}(n^{-1})$ to achieve an overall sample complexity of $\tilde{O}(\epsilon^{-2})$.

B.1 An Illustrative Example for the Necessity of the Bias Requirement

To understand our requirement that the bias of the empirical Bellman operator (3.2) decay like $O(n^{-1})$, we consider the following simple example. We use stochastic approximation to find the fixed point 0 of the equation $x = \gamma x$, $\gamma \in (0, 1)$. In the notation of (2.12), $\mathcal{L}(x) = \gamma x$. We generate biased noisy observation of $x \rightarrow \gamma x$; i.e. $\mathcal{L}_{n,k}(x) = \gamma x + U_{n,k} + \beta_n$, where $\{U_n, k : k = 1, \dots\}$ are i.i.d., each requires n -sample to compute, with $EU_{n,k} = 0$ and $\text{Var}U_{n,k} = \sigma_n^2$. The $U_{n,k}$ and β_n are analogous to the noise and bias associated with an n -sample empirical Bellman operator.

For simplicity, we consider a constant stepsize setting where the stepsize $\lambda = \tilde{\Theta}(\epsilon)$, ignoring the dependence on $1/(1-\gamma)$. We note that similar phenomena happen for discounted stepsize as well. We consider the stochastic approximation:

$$X_{t+1} = (1 - \lambda)X_t + \lambda(\gamma X_t + U_{n,t+1} + \beta_n)$$

with $X_0 = 1$. Then, it is not hard to see that

$$X_t = (1 - \lambda(1 - \gamma))^t + \lambda \sum_{k=0}^{t-1} (1 - \lambda(1 - \gamma))^k (U_{n,t-k} + \beta_n).$$

To achieve a statistical error of ϵ in mean squared, we need

$$t = \Omega\left(\frac{1}{\lambda(1 - \gamma)} \log\left(\frac{1}{\epsilon}\right)\right) = \tilde{\Omega}\left(\frac{1}{\epsilon}\right)$$

and

$$\text{Var}\left[\lambda \sum_{k=0}^{t-1} (1 - \lambda(1 - \gamma))^k U_{n,t-k}\right] = O(\epsilon^2), \quad \beta_n \lambda \sum_{k=0}^{t-1} (1 - \lambda(1 - \gamma))^k = O(\epsilon).$$

The variance requirement suggests

$$\frac{\lambda^2 (1 - (1 - \lambda(1 - \gamma))^{2t})}{\lambda(1 - \gamma)(2 - \lambda(1 - \gamma))} \sigma_n^2 = O(\epsilon^2).$$

So, with the choice of t earlier, we need $\sigma_n^2 = O(\epsilon^2/\lambda) = \tilde{O}(\epsilon)$. On the other hand, the bias requirement implies

$$\beta_n \frac{\lambda(1 - (1 - \lambda(1 - \gamma))^t)}{\lambda(1 - \gamma)} = O(\epsilon);$$

i.e. $\beta_n = O(\epsilon)$.

Assume that the variance $\sigma_n^2 = \Theta(n^{-1})$, which is the canonical variance decay rate of Monte Carlo estimation. Then, to guarantee $\sigma_n^2 = O(\epsilon)$, we need $n = \Omega(1/\epsilon)$. Yet, if

$\beta_n = \Theta(n^{-1/2})$, we also need $n = \Omega(1/\epsilon^2)$ so as to have $\beta_n = O(\epsilon)$. In this case, the total number of samples used until iteration t is $nt = \tilde{\Omega}(\epsilon^{-3})$.

On the other hand, if we managed to show that $\beta_n = \tilde{\Theta}(n^{-1})$, we see that we only need $n = \tilde{\Omega}(1/\epsilon)$, and hence $nt = \tilde{\Omega}(\epsilon^{-2})$, the correct rate.

Having explained the necessity of $\tilde{O}(n^{-1})$ bias and equipped with the key bias and concentration bounds, we are ready to carry out the proofs of the worst-case sample complexity bounds for Algorithm 1 and 2. We will follow the proof outlined in Section 3.3.

B.2 The Distributionally Robust Q-learning Algorithm 1

B.2.1 PROOF OF PROPOSITION 7

Proof

Recall that the update rule for Algorithm 1 can be written as

$$\begin{aligned} q_{k+1} - q^* &= (1 - \lambda_k)(q_k - q^*) + \lambda_k [\mathbf{T}_{k+1}(q_k) - \mathbf{T}_{k+1}(q^*) + \mathbf{T}_{k+1}(q^*) - \mathcal{T}(q^*)] \\ &= (1 - \lambda_k)(q_k - q^*) + \lambda_k [\mathbf{H}_{k+1}(q_k) + W_{k+1}] \end{aligned} \quad (\text{B.1})$$

where we define $W_{k+1} := \mathbf{T}_{k+1}(q^*) - \mathcal{T}(q^*)$. Since $\mathbf{T}_{k+1}(q^*)$ is a i.i.d. sequence of estimators to $\mathcal{T}(q^*)$,

$$\beta := \text{Bias}(\mathbf{T}_k(q^*)) = E[\mathbf{T}_k(q^*)] - \mathcal{T}(q^*)$$

is independent of k . We can write $W_{k+1} = \beta + U_{k+1}$ where $U_{k+1} := \mathbf{T}_{k+1}(q^*) - \mathcal{T}(q^*) - \beta$ has zero mean.

Next, we would like to apply Proposition 15. Define the auxiliary sequences

$$P_{k+1} = (1 - \lambda_k)P_k + \lambda_k W_{k+1} \quad (\text{B.2})$$

$$Q_{k+1} = (1 - \lambda_k)Q_k + \lambda_k U_{k+1} \quad (\text{B.3})$$

with $Q_0 = P_0 = 0$. Notice that since $\{U_k, k \geq 1\}$ has mean zero, $E[Q_k] = 0$ for all $k \geq 0$. It is easier to analyze the process $\{Q_k, k \geq 0\}$ than $\{P_k, k \geq 0\}$ which correspond to $\{p_k\}$ in Proposition 15.

To use $\{Q_k, k \geq 0\}$, we first show that

$$P_k = Q_k + \beta.$$

We prove this by induction. The base case $P_1 = \lambda_0 W_1 = Q_1 + \beta$ as $\lambda_0 = 1$. Next we check the induction step. By the iterative updates (B.2) and (B.3) and the induction hypothesis, we have that

$$\begin{aligned} P_{k+1} &= (1 - \lambda_k)P_k + \lambda_k W_{k+1} \\ &= (1 - \lambda_k)(Q_k + \beta) + \lambda_k(U_{k+1} + \beta) \\ &= Q_{k+1} + \beta. \end{aligned} \quad (\text{B.4})$$

By Algorithm 1, $q_1 = 0$. We have that by Lemma 20, $\|q_1 - q^*\|_\infty \leq (1 - \gamma)^{-1}$. Therefore, by Proposition 15

$$\begin{aligned}
 \|q_{k+1} - q^*\|_\infty &\leq \lambda_k \left[\frac{\|q_1 - q^*\|_\infty}{\lambda_1} + \gamma \sum_{j=1}^k \|P_j\|_\infty \right] + \|P_{k+1}\|_\infty \\
 &\leq \lambda_k \left[\frac{1}{\lambda_1(1 - \gamma)} + \left(\gamma \sum_{j=1}^k \|Q_j\|_\infty \right) + \gamma k \|\beta\|_\infty \right] + \|Q_{l,k+1}\|_\infty + \|\beta\|_\infty \\
 &\leq \lambda_k \left[\frac{2}{1 - \gamma} + \gamma \sum_{j=1}^k \|Q_j\|_\infty \right] + \|Q_{k+1}\|_\infty + \frac{2\|\beta\|_\infty}{1 - \gamma}.
 \end{aligned} \tag{B.5}$$

w.p.1, where we used $k\lambda_k = 1/(1/k + (1 - \gamma)) \leq 1/(1 - \gamma)$.

Next, we bound the sequence $\{Q_k, k \geq 1\}$.

Lemma 26 *The $\{Q_k, k \geq 1\}$ sequence satisfies*

$$P(\|Q_{k+1}\|_\infty > t) \leq 2|\mathbf{S}||\mathbf{A}| \exp\left(-\frac{t^2}{\lambda_k(8\gamma(1 - \gamma)^{-1}t + 4\|\sigma^2(q^*)\|_\infty)}\right).$$

where $\sigma^2(q^*)(s, a) = \text{Var}(\mathbf{T}_k(q^*)(s, a))$.

The proof of Lemma 26 is in Appendix B.4. By applying Lemma 26, we have that

$$\begin{aligned}
 \|Q_j\|_\infty &\leq \frac{8\lambda_j}{1 - \gamma} \log(2|\mathbf{S}||\mathbf{A}|/\eta) + 2\sqrt{\lambda_j} \|\sigma(q^*)\|_\infty \sqrt{\log(2|\mathbf{S}||\mathbf{A}|/\eta)} \\
 &\leq \left(\frac{8\lambda_j}{1 - \gamma} + 2\sqrt{\lambda_j} \|\sigma(q^*)\|_\infty \right) \log(2|\mathbf{S}||\mathbf{A}|/\eta)
 \end{aligned}$$

w.p. at least $1 - \eta$.

To establish high probability bound using (B.5), we also need the following properties of the stepsize:

Lemma 27 (Proof of Corollary 3, Wainwright (2019b)) *The following inequalities hold:*

$$\sum_{j=1}^k \sqrt{\lambda_j} \leq \frac{2}{(1 - \gamma)\sqrt{\lambda_k}}; \quad \sum_{j=1}^k \lambda_j \leq \frac{\log(1 + (1 - \gamma)k)}{1 - \gamma}.$$

We have that by Lemma 27 and the union bound,

$$\begin{aligned}
 & \gamma \lambda_{k_0} \sum_{j=1}^{k_0} \|Q_j\|_\infty + \|Q_{k_0+1}\|_\infty \\
 & \leq 8\gamma \left(\frac{\lambda_{k_0} \log(1 + (1-\gamma)k_0)}{(1-\gamma)^2} + \frac{\|\sigma(q^*)\|_\infty \sqrt{\lambda_{k_0}}}{1-\gamma} \right) \log(4|\mathbf{S}||\mathbf{A}|k_0/\eta) \\
 & \quad + \left(\frac{8\lambda_{k_0}}{1-\gamma} + 2\|\sigma(q^*)\|_\infty \sqrt{\lambda_{k_0}} \right) \log(4|\mathbf{S}||\mathbf{A}|k_0/\eta) \\
 & \leq 16 \left(\frac{\lambda_{k_0} \log(1 + (1-\gamma)k_0)}{(1-\gamma)^2} + \frac{\|\sigma(q^*)\|_\infty \sqrt{\lambda_{k_0}}}{1-\gamma} \right) \log(4|\mathbf{S}||\mathbf{A}|k_0/\eta) \\
 & \leq 16 \left(\frac{1}{(1-\gamma)^3 k_0} + \frac{20}{\mathfrak{p}_\wedge (1-\gamma)^{5/2} \sqrt{n_0 k_0}} \right) \log(4dk_0/\eta)^2
 \end{aligned}$$

w.p. at least $1 - \eta$, where we utilize Proposition 21 to bound $\|\sigma(q^*)\|_\infty$.

We use Proposition 22 to bound β . Then, from (B.5) we conclude that there exists constant c s.t.

$$\begin{aligned}
 \|q_{k_0+1} - q^*\|_\infty & \leq c \left(\frac{1}{(1-\gamma)^3 k_0} + \frac{1}{\mathfrak{p}_\wedge (1-\gamma)^{5/2} \sqrt{n_0 k_0}} \right) \log(4dk_0/\eta)^2 + c \frac{r_{\max} + \gamma |q|_{\text{span}}}{(1-\gamma) \mathfrak{p}_\wedge^3 n_0} \log(e|\mathbf{S}| \vee |\mathbf{R}|) \\
 & \leq c \left(\frac{1}{(1-\gamma)^3 k_0} + \frac{1}{\mathfrak{p}_\wedge (1-\gamma)^{5/2} \sqrt{n_0 k_0}} + \frac{1}{\mathfrak{p}_\wedge^3 (1-\gamma)^2 n_0} \right) \log(4dk_0/\eta)^2
 \end{aligned}$$

where c can change from line to line.

Finally, note that for $C_1 \geq 1, C_2 \geq e$, $\log(C_1 C_2) = \log(C_1) + \log(C_2) \leq C_1 \log(C_2)$. So, $\log(4dk_0/\eta)^2 \leq \frac{16}{9} \log(3dk_0/\eta)^2$. This completes the proof. \blacksquare

B.3 The Variance-Reduced Distributionally Robust Q-learning Algorithm 2

B.3.1 PROOF OF PROPOSITION 11

Proof Recall that \mathcal{F}_l be the σ -field generated by the random samples used until the end of epoch l and

$$E_l[\cdot] = E[\cdot | \mathcal{F}_l], P_l[\cdot] = P[\cdot | \mathcal{F}_l], \text{ and } \text{Var}_l(\cdot) = \text{Var}(\cdot | \mathcal{F}_l).$$

In the following proof, the probabilities are w.r.t. $P_{l-1}(\cdot)$. Recall that

$$\mathbf{H}_{l,k} = \mathbf{T}_{l,k}(q) - \mathbf{T}_{l,k}(q^*) \text{ and } \tilde{\mathbf{H}}_l = \tilde{\mathbf{T}}_{l,k}(q) - \tilde{\mathbf{T}}_{l,k}(q^*).$$

From the variance-reduced DR-RL (Algorithm 2) update rule, we have at epoch l ,

$$\begin{aligned}
 q_{l,k+1} - q^* & = (1 - \lambda_k)(q_{l,k} - q^*) + \lambda_k \left[\mathbf{T}_{l,k+1}(q_{l,k}) - \mathbf{T}_{l,k+1}(\hat{q}_{l-1}) + \tilde{\mathbf{T}}_l(\hat{q}_{l-1}) - \mathcal{T}(q^*) \right] \\
 & = (1 - \lambda_k)(q_{l,k} - q^*) + \lambda_k \left[\mathbf{H}_{l,k+1}(q_{l,k}) + \mathbf{T}_{l,k+1}(q^*) - \mathbf{T}_{l,k+1}(\hat{q}_{l-1}) + \tilde{\mathbf{T}}_l(\hat{q}_{l-1}) - \mathcal{T}(q^*) \right] \\
 & = (1 - \lambda_k)(q_{l,k} - q^*) + \lambda_k [\mathbf{H}_{l,k+1}(q_{l,k}) + W_{l,k+1}]
 \end{aligned} \tag{B.6}$$

where we define $W_{l,k+1} = \mathbf{T}_{l,k+1}(q^*) - \mathbf{T}_{l,k+1}(\hat{q}_{l-1}) + \tilde{\mathbf{T}}_l(\hat{q}_{l-1}) - \mathcal{T}(q^*)$. Notice that only the first two terms is dependent on k . We can write

$$\begin{aligned} W_{l,k+1} &= \mathbf{T}_{l,k+1}(q^*) - \mathbf{T}_{l,k+1}(\hat{q}_{l-1}) + \tilde{\mathbf{T}}_l(\hat{q}_{l-1}) - \mathcal{T}(q^*) \\ &= -\mathbf{H}_{l,k+1}(\hat{q}_{l-1}) + \tilde{\mathbf{H}}_l(\hat{q}_{l-1}) + \tilde{\mathbf{T}}_l(q^*) - \mathcal{T}(q^*) \\ &= -[\mathbf{H}_{l,k+1}(\hat{q}_{l-1}) - E_{l-1}[\mathbf{H}_{l,k+1}(\hat{q}_{l-1})]] + \tilde{\mathbf{H}}_l(\hat{q}_{l-1}) + \tilde{\mathbf{T}}_l(q^*) - \mathcal{T}(q^*) - E_{l-1}[\mathbf{H}_{l,k+1}(\hat{q}_{l-1})] \\ &= -U_{l,k+1} + D_l \end{aligned} \tag{B.7}$$

where

$$U_{l,k+1} := \mathbf{H}_{l,k+1}(\hat{q}_{l-1}) - E_{l-1}[\mathbf{H}_{l,k+1}(\hat{q}_{l-1})] \tag{B.8}$$

$$D_l := \tilde{\mathbf{H}}_l(\hat{q}_{l-1}) + \tilde{\mathbf{T}}_l(q^*) - \mathcal{T}(q^*) - E_{l-1}[\mathbf{H}_{l,k+1}(\hat{q}_{l-1})]. \tag{B.9}$$

Note that $E_{l-1}[\mathbf{H}_{l,k+1}(\hat{q}_{l-1})]$ is constant in k .

We will apply Proposition 15. Define the auxiliary sequences

$$P_{l,k+1} = (1 - \lambda_k)P_{l,k} + \lambda_k W_{l,k+1} \tag{B.10}$$

$$Q_{l,k+1} = (1 - \lambda_k)Q_{l,k} + \lambda_k(-U_{l,k+1}) \tag{B.11}$$

with $Q_{l,0} = P_{l,0} = 0$. Note that $U_{l,k+1}$ under E_{l-1} are i.i.d. and has mean 0. So $E_{l-1}[Q_{l,k}] = 0$ for any $k \geq 0$. It is easier to analyze the process $\{Q_{l,k}, k \geq 0\}$ than $\{P_{l,k}, k \geq 0\}$ which correspond to $\{p_k\}$ in Proposition 15.

As in the DR Q-learning case (Equation (B.4)), the same induction argument implies that $P_{l,k} = Q_{l,k} + D_l$.

By the algorithm, $q_{l,1} = \hat{q}_{l-1}$, we have that $\|q_{l,1} - q^*\|_\infty \leq b$. Therefore, by Proposition 15

$$\begin{aligned} \|q_{l,k+1} - q^*\|_\infty &\leq \lambda_k \left[\frac{\|q_{l,1} - q^*\|_\infty}{\lambda_1} + \gamma \sum_{j=1}^k \|P_{l,j}\|_\infty \right] + \|P_{l,k+1}\|_\infty \\ &\leq \lambda_k \left[\frac{b}{\lambda_1} + \left(\gamma \sum_{j=1}^k \|Q_{l,j}\|_\infty + \gamma k \|D_l\|_\infty \right) \right] + \|Q_{l,k+1}\|_\infty + \|D_l\|_\infty. \tag{B.12} \\ &\leq \lambda_k \left[2b + \gamma \sum_{j=1}^k \|Q_{l,j}\|_\infty \right] + \|Q_{l,k+1}\|_\infty + \frac{2\|D_l\|_\infty}{1 - \gamma}. \end{aligned}$$

w.p.1, where we used $k\lambda_k = 1/(1/k + (1 - \gamma)) \leq 1/(1 - \gamma)$.

Next, we prove bounds for $\{\|Q_{l,k}\|_\infty, k \geq 0\}$ and $\|D_l\|_\infty$.

Lemma 28 *Under measure $P_{l-1}(\cdot)$,*

$$P_{l-1}(\|Q_{l,j}\|_\infty > t) \leq 2|\mathbf{S}||\mathbf{A}| \exp\left(-\frac{t^2}{4\lambda_j(\gamma\|\zeta_{l-1}\|_\infty t + \|\sigma_{l-1}^2\|_\infty)}\right)$$

where $\zeta_{l-1} = \hat{q}_{l-1} - q^*$ and $\sigma_{l-1}^2(s, a) = \text{Var}_{l-1}(\mathbf{H}_{l,k}(\hat{q}_{l-1})(s, a))$.

The proof of this Lemma is deferred to Appendix B.4. Apply Lemma 28, we have that

$$\|Q_{l,j}\|_\infty \leq 4\lambda_j \|\zeta_{l-1}\|_\infty \log(2|\mathbf{S}||\mathbf{A}|/\eta) + 2\sqrt{\lambda_j} \|\sigma_{l-1}\|_\infty \sqrt{\log(2|\mathbf{S}||\mathbf{A}|/\eta)}.$$

w.p. at least $1 - \eta$.

Recall the definition of σ_{l-1}^2 and Proposition 24. We have that by Lemma 27 and the union bound,

$$\begin{aligned} & \gamma \lambda_{k_{\text{vr}}} \sum_{j=1}^{k_{\text{vr}}} \|Q_{l,j}\|_\infty + \|Q_{l,k_{\text{vr}}+1}\|_\infty \\ & \leq 4\gamma \left(\frac{\lambda_{k_{\text{vr}}} \log(1 + (1-\gamma)k_{\text{vr}}) \|\zeta_{l-1}\|_\infty}{1-\gamma} + \frac{\|\sigma_{l-1}\|_\infty \sqrt{\lambda_{k_{\text{vr}}}}}{1-\gamma} \right) \log(4|\mathbf{S}||\mathbf{A}|k_{\text{vr}}/\eta) \\ & \quad + \left(4\lambda_{k_{\text{vr}}} \|\zeta_{l-1}\|_\infty + 2\|\sigma_{l-1}\|_\infty \sqrt{\lambda_{k_{\text{vr}}}} \right) \log(4|\mathbf{S}||\mathbf{A}|k_{\text{vr}}/\eta). \quad (\text{B.13}) \\ & \leq 8 \left(\frac{\lambda_{k_{\text{vr}}} \log(e + (1-\gamma)k_{\text{vr}}) \|\zeta_{l-1}\|_\infty}{1-\gamma} + \frac{\|\sigma_{l-1}\|_\infty \sqrt{\lambda_{k_{\text{vr}}}}}{1-\gamma} \right) \log(4|\mathbf{S}||\mathbf{A}|k_{\text{vr}}/\eta) \\ & \leq 8 \left(\frac{b}{(1-\gamma)^2 k_{\text{vr}}} + \frac{2^6 b}{\mathfrak{p}_\wedge^{3/2} (1-\gamma)^{3/2} \sqrt{n_{\text{vr}} k_{\text{vr}}}} \right) \log(4|\mathbf{S}||\mathbf{A}|k_{\text{vr}}/\eta)^2 \end{aligned}$$

w.p. at least $1 - \eta$.

For D_l , recall the definition in (B.9). We add and subtract $\mathcal{H}(\hat{q}_{l-1})$ and write:

$$\begin{aligned} D_l &= \left(\tilde{\mathbf{H}}_l(\hat{q}_{l-1}) - \mathcal{H}(\hat{q}_{l-1}) \right) + \left(\tilde{\mathbf{T}}_l(q^*) - \mathcal{T}(q^*) \right) + \left(\mathcal{H}(\hat{q}_{l-1}) - E_{l-1}[\mathbf{H}_{l,k+1}(\hat{q}_{l-1})] \right) \\ &= \left(\tilde{\mathbf{H}}_l(\hat{q}_{l-1}) - \mathcal{H}(\hat{q}_{l-1}) \right) + \left(\tilde{\mathbf{T}}_l(q^*) - \mathcal{T}(q^*) \right) + E_{l-1}[\mathcal{H}(\hat{q}_{l-1}) - \mathbf{H}_{l,k+1}(\hat{q}_{l-1})]. \quad (\text{B.14}) \end{aligned}$$

Recall Propositions 23, 24, and 25, we have that by union bound,

$$\begin{aligned} \|D_l\|_\infty &\leq c \frac{r_{\max} + |q^*|_{\text{span}} + \|\hat{q}_{l-1} - q^*\|_\infty}{\mathfrak{p}_\wedge^{3/2} \sqrt{m_l}} \sqrt{\log(12d/\eta)} + E_{l-1}(\mathcal{H}(\hat{q}_{l-1}) - \mathbf{H}_{l,k+1}(\hat{q}_{l-1})) \\ &\leq c \frac{r_{\max} + |q^*|_{\text{span}} + \|\hat{q}_{l-1} - q^*\|_\infty}{\mathfrak{p}_\wedge^{3/2} \sqrt{m_l}} \sqrt{\log(12d/\eta)} + c \frac{\|\hat{q}_{l-1} - q^*\|_\infty}{\mathfrak{p}_\wedge^{3/2} \sqrt{n_{\text{vr}}}} \sqrt{\log(e|\mathbf{S}|)} \quad (\text{B.15}) \end{aligned}$$

w.p. at least $1 - \eta$, provided that c is a large enough constant $m_l \geq 8\mathfrak{p}_\wedge^{-2} \log(24d/\eta)$, and $n_{\text{vr}} \geq \mathfrak{p}_\wedge^{-1}$.

Finally, recall that $\hat{q}_l = \hat{q}_{l, k_{\text{vr}}+1}$ and $r_{\max} = 1$, combine (B.12), (B.13), and (B.15) we conclude that there exists absolute constant c s.t.

$$\begin{aligned} \|\hat{q}_l - q^*\|_\infty &\leq c \left(\frac{b}{(1-\gamma)^2 k_{\text{vr}}} + \frac{b}{\mathfrak{p}_\wedge^{3/2} (1-\gamma)^{3/2} \sqrt{n_{\text{vr}} k_{\text{vr}}}} \right) \log(8dk_{\text{vr}}/\eta)^2 \\ &\quad + c \frac{r_{\max} + |q^*|_{\text{span}} + b}{\mathfrak{p}_\wedge^{3/2} (1-\gamma) \sqrt{m_l}} \log(24d/\eta) + c \frac{b}{\mathfrak{p}_\wedge^{3/2} (1-\gamma) \sqrt{n_{\text{vr}}}} \sqrt{\log(e|\mathbf{S}|)} \\ &\leq c \left(\frac{b}{(1-\gamma)^2 k_{\text{vr}}} + \frac{b}{\mathfrak{p}_\wedge^{3/2} (1-\gamma)^{3/2} \sqrt{n_{\text{vr}} k_{\text{vr}}}} + \frac{b}{\mathfrak{p}_\wedge^{3/2} (1-\gamma) \sqrt{n_{\text{vr}}}} \right) \log(8dk_{\text{vr}}/\eta)^2 \\ &\quad + c \frac{1}{\mathfrak{p}_\wedge^{3/2} (1-\gamma)^2 \sqrt{m_l}} \sqrt{\log(24d/\eta)} \end{aligned}$$

w.p. at least $1 - \eta$, where we used $|q^*|_{\text{span}} \leq 2\|q^*\|_\infty \leq 2/(1-\gamma)$ and $b \leq 1/(1-\gamma)$, c can change from line to line.

Finally, note that for $C_1 \geq 1, C_2 \geq e$, $\log(C_1 C_2) = \log(C_1) + \log(C_2) \leq C_1 \log(C_2)$. This completes the proof of Proposition 10. \blacksquare

B.3.2 PROOF OF PROPOSITION 11

Proof By the definition of conditional probability

$$\begin{aligned} &P \left(\bigcap_{l=0}^{l_{\text{vr}}} \left\{ \|\hat{q}_l - q^*\|_\infty \leq 2^{-l} (1-\gamma)^{-1} \right\} \right) \\ &= \prod_{l=0}^{l_{\text{vr}}} P \left(\left. \|\hat{q}_l - q^*\|_\infty \leq 2^{-l} (1-\gamma)^{-1} \right| \bigcap_{n=0}^{l-1} \left\{ \|\hat{q}_n - q^*\|_\infty \leq 2^{-n} (1-\gamma)^{-1} \right\} \right) \\ &= \prod_{l=1}^{l_{\text{vr}}} P \left(\left. \|\hat{q}_l - q^*\|_\infty \leq 2^{-l} (1-\gamma)^{-1} \right| \bigcap_{n=1}^{l-1} \left\{ \|\hat{q}_n - q^*\|_\infty \leq 2^{-n} (1-\gamma)^{-1} \right\} \right) \end{aligned}$$

where we note that $\hat{q}_0 = 0$ and Lemma 20 implies that $\|\hat{q}_0 - q^*\| \leq (1-\gamma)^{-1}$ w.p.1, so the conditioned intersection and product can start from $s = 1$. Let

$$A_{l-1} = \bigcap_{s=1}^{l-1} \left\{ \|\hat{q}_s - q^*\|_\infty \leq 2^{-s} (1-\gamma)^{-1} \right\}.$$

We analyze the probability for $l \geq 1$

$$\begin{aligned} &P \left(\left. \|\hat{q}_l - q^*\|_\infty \leq 2^{-l} (1-\gamma)^{-1} \right| A_{l-1} \right) \\ &= \frac{1}{P(A_{l-1})} E \left[\mathbb{1} \left\{ \|\hat{q}_l - q^*\|_\infty \leq 2^{-l} (1-\gamma)^{-1} \right\} \mathbb{1}_{A_{l-1}} \right] \\ &= \frac{1}{P(A_{l-1})} E \left[\mathbb{1} \left\{ \|\hat{q}_{l-1} - q^*\|_\infty \leq 2^{-(l-1)} (1-\gamma)^{-1} \right\} E \left[\mathbb{1} \left\{ \|\hat{q}_l - q^*\|_\infty \leq 2^{-l} (1-\gamma)^{-1} \right\} \middle| \mathcal{F}_{l-1} \right] \mathbb{1}_{A_{l-1}} \right] \\ &= \frac{1}{P(A_{l-1})} E \left[\mathbb{1} \left\{ \|\hat{q}_{l-1} - q^*\|_\infty \leq 2^{-(l-1)} (1-\gamma)^{-1} \right\} P_{l-1} \left(\mathbb{1} \left\{ \|\hat{q}_l - q^*\|_\infty \leq 2^{-l} (1-\gamma)^{-1} \right\} \right) \mathbb{1}_{A_{l-1}} \right], \end{aligned}$$

By Proposition 10, we recall conditioned on $\|\hat{q}_{l-1} - q^*\|_\infty \leq 2^{-(l-1)}(1-\gamma)^{-1} =: b$

$$\begin{aligned} \|\hat{q}_l - q^*\|_\infty &\leq c \left(\frac{b}{(1-\gamma)^2 k_{\text{vr}}} + \frac{b}{\mathfrak{p}_\wedge^{3/2} (1-\gamma)^{3/2} \sqrt{n_{\text{vr}} k_{\text{vr}}}} + \frac{b}{\mathfrak{p}_\wedge^{3/2} (1-\gamma) \sqrt{n_{\text{vr}}}} \right) \log(3dk_{\text{vr}}/\eta)^2 \\ &\quad + c \frac{1}{\mathfrak{p}_\wedge^{3/2} (1-\gamma)^2 \sqrt{m_l}} \sqrt{\log(3d/\eta)} \end{aligned}$$

w.p. at least $1 - \eta$.

Therefore, by the parameter choice (3.6), we have that for sufficiently large c_{vr} and for events $\omega \in \{\|\hat{q}_{l-1} - q^*\|_\infty \leq 2^{-(l-1)}(1-\gamma)^{-1}\}$,

$$P_{l-1} \left(\mathbb{1} \left\{ \|\hat{q}_l - q^*\|_\infty \leq 2^{-l}(1-\gamma)^{-1} \right\} \right) (\omega) \geq 1 - \frac{\eta}{l_{\text{vr}}}; \quad (\text{B.16})$$

i.e.

$$\mathbb{1} \left\{ \|\hat{q}_{l-1} - q^*\|_\infty \leq 2^{-(l-1)}(1-\gamma)^{-1} \right\} P_{l-1} \left(\mathbb{1} \left\{ \|\hat{q}_l - q^*\|_\infty \leq 2^{-l}(1-\gamma)^{-1} \right\} \right) \geq 1 - \frac{\eta}{l_{\text{vr}}}.$$

Therefore, we have

$$P \left(\|\hat{q}_l - q^*\|_\infty \leq 2^{-l}(1-\gamma)^{-1} \mid A_{l-1} \right) \geq 1 - \frac{\eta}{l_{\text{vr}}},$$

which further gives us

$$P \left(\bigcap_{l=0}^{l_{\text{vr}}} \left\{ \|\hat{q}_l - q^*\|_\infty \leq 2^{-l}(1-\gamma)^{-1} \right\} \right) \geq \left(1 - \frac{\eta}{l_{\text{vr}}} \right)^{l_{\text{vr}}}. \quad (\text{B.17})$$

To finish the proof, we consider the function

$$e(\eta) := \left(1 - \frac{\eta}{l} \right)^l.$$

Clearly, $e(\eta)$ is C^2 with derivatives

$$e'(\eta) = - \left(1 - \frac{\eta}{l} \right)^{l-1}, \quad e''(\eta) = \frac{l-1}{l} \left(1 - \frac{\eta}{l} \right)^{l-2}.$$

Note that $e'' \geq 0$ if $l \geq 1$. So $e'(\eta)$ is non-decreasing. Hence for all $\eta \geq 0$, $e'(\eta) \geq e'(0)$. This implies that

$$\begin{aligned} e(\eta) &= e(0) + \int_0^\eta e'(t) dt \\ &\geq 1 - \eta. \end{aligned}$$

Assumption $\epsilon < (1-\gamma)^{-1}$ implies that $l_{\text{vr}} \geq 1$. Therefore, we plug in this to (B.17) and conclude that

$$P \left(\|\hat{q}_{l_{\text{vr}}} - q^*\|_\infty \leq 2^{-l_{\text{vr}}}(1-\gamma)^{-1} \right) \geq P \left(\bigcap_{l=0}^{l_{\text{vr}}} \left\{ \|\hat{q}_l - q^*\|_\infty \leq 2^{-l}(1-\gamma)^{-1} \right\} \right) \geq 1 - \eta.$$

■

B.4 Proof of Lemma 26 and 28

To prove these two lemma, we introduce the following result:

Lemma 29 (Wainwright (2019b), Lemma 2) *Let $\{Y_k \in \mathbb{R}, k \geq 1\}$ be a sequence of i.i.d. zero mean ζ -bounded r.v.s with variance σ^2 . Define $\{X_k, k \geq 0\}$ by the recursion $X_0 = 0$*

$$X_{k+1} = (1 - \lambda_k)X_k + \lambda_k Y_{k+1},$$

where $\lambda_k = 1/(1 + (1 - \gamma)k)$. Then

$$E[\exp(tX_{k+1})] \leq \exp\left(\frac{t^2\sigma^2\lambda_k}{1 - \zeta\lambda_k|t|}\right)$$

for all $|t| < 1/(\zeta\lambda_k)$.

We first prove Lemma 28.

Proof We use the same steps. Recall (B.11), where $U_{l,k}$ is an i.i.d. sequence under E_{l-1} given by (B.8). By Lemma 20

$$\begin{aligned} \|U_{l,k}\|_\infty &\leq \|\mathbf{T}_{l,k}(\hat{q}_{l-1}) - \mathbf{T}_{l,k}(q^*)\|_\infty + \|E_{l-1}[\mathbf{T}_{l,k}(\hat{q}_{l-1}) - \mathbf{T}_{l,k}(q^*)]\|_\infty \\ &\leq 2\gamma\|\hat{q}_{l-1} - q^*\|_\infty \\ &= 2\gamma\|\zeta_{l-1}\|_\infty. \end{aligned}$$

Notice that by construction, $\mathbf{T}_{l,k}(q^*)(s, a)$ are independent across $s \in \mathbf{S}$, $a \in \mathbf{A}$. Therefore, by Lemma 29,

$$\begin{aligned} E_{l-1} \exp(\lambda\|Q_{l,k+1}\|_\infty) &= E_{l-1} \sup_{(s,a) \in \mathbf{S} \times \mathbf{A}} \max \{\exp(\lambda Q_{l,k+1}(s, a)), \exp(-\lambda Q_{l,k+1}(s, a))\} \\ &\leq \sum_{(s,a) \in \mathbf{S} \times \mathbf{A}} E_{l-1} \exp(\lambda Q_{l,k+1}(s, a)) + E \exp(-\lambda Q_{l,k+1}(s, a)) \\ &\leq 2|\mathbf{S}||\mathbf{A}| \exp\left(\frac{\lambda^2\|\sigma_{l-1}^2\|_\infty\lambda_k}{1 - 2\gamma\|\zeta_{l-1}\|_\infty\lambda_k|\lambda|}\right), \end{aligned}$$

for any $\lambda < 1/(2\gamma\|\zeta_{l-1}\|_\infty\lambda_k)$. Therefore, by the Chernoff bound

$$P_{l-1}(\|Q_{l,k+1}\|_\infty > t) \leq 2|\mathbf{S}||\mathbf{A}| \exp\left(\frac{\lambda^2\|\sigma_{l-1}^2\|_\infty\lambda_k}{1 - 2\gamma\|\zeta_{l-1}\|_\infty\lambda_k|\lambda|}\right) e^{-\lambda t},$$

for any $\lambda \in (0, 1/(2\gamma\|\zeta_{l-1}\|_\infty\lambda_k))$. Choose

$$\lambda = \frac{t}{2\gamma\|\zeta_{l-1}\|_\infty\lambda_k t + 2\|\sigma_{l-1}^2\|_\infty\lambda_k},$$

we conclude that

$$P_{l-1}(\|Q_{l,k+1}\|_\infty > t) \leq 2|\mathbf{S}||\mathbf{A}| \exp\left(-\frac{t^2}{4\lambda_k(\gamma\|\zeta_{l-1}\|_\infty t + \|\sigma_{l-1}^2\|_\infty)}\right).$$

■

Next, we prove Lemma 26. Notice that we only need to modify the bounds on ζ and σ^2 .

Proof Recall that $\{Q_{l,k}, k \geq 0\}$ is given by recursive relation (B.3), where U_k has mean 0. By Lemma 20

$$\begin{aligned}\|U_k\|_\infty &\leq 2\|\mathbf{T}_{k+1}(q^*)\|_\infty \\ &\leq 2r_{\max} + 2\gamma\|q^*\|_\infty \\ &\leq \frac{4\gamma}{1-\gamma}.\end{aligned}$$

and $\text{Var}(\mathbf{T}_{k+1}(q^*)(s, a)) = \sigma^2(q^*)(s, a)$. Therefore, using the same arguments, we conclude that

$$P(\|Q_{k+1}\|_\infty > t) \leq 2|\mathbf{S}||\mathbf{A}| \exp\left(-\frac{t^2}{\lambda_k(8\gamma(1-\gamma)^{-1}t + 4\|\sigma^2(q^*)\|_\infty)}\right).$$

■

Appendix C. Proofs of Properties of the Empirical Bellman Operator: KL Case

C.1 Glossary of Notations and Basic Properties

Before we present our proofs, we first define some technical notations. For finite discrete measurable space $(Y, 2^Y)$, fixed $u \in m2^Y$, and signed measure $\nu \in \mathcal{M}_\pm(Y, 2^Y)$, let

$$\nu[u] = \sum_{y \in Y} \nu(y)u(y)$$

denotes the integral.

For generic probability measure μ on $(Y, 2^Y)$ and random variable $u : Y \rightarrow \mathbb{R}$, let $w = w(\alpha) = e^{-u/\alpha}$; define the KL *dual functional* under the reference measure μ

$$f(\mu, u, \alpha) := -\alpha \log \mu[e^{-u/\alpha}] - \alpha\delta. \quad (\text{C.1})$$

We clarify that $f(\mu, u, 0) = \lim_{\alpha \downarrow 0} f(\mu, u, \alpha) = \text{ess inf}_\mu u$. We present two basic properties of the dual functional f for which the proofs are deferred to Appendix E.

Lemma 30 *For any $\nu \ll \mu$, the dual functional is bounded*

$$-\|u\|_{L^\infty(\mu)} \leq \sup_{\alpha \geq 0} f(\nu, u, \alpha) \leq \|u\|_{L^\infty(\mu)}$$

Lemma 31 *The following bound holds w.p.1.:*

$$\left| \sup_{\alpha \geq 0} f(\mu, u, \alpha) - \sup_{\alpha \geq 0} f(\mu_n, u, \alpha) \right| \leq 2|u|_{\text{span}},$$

where $|u|_{\text{span}} = \max_{s \in S} u(s) - \min_{s \in S} u(s)$.

Let μ_n be the empirical measure form by n i.i.d. samples drawn from μ . In the following development, we need to consider the perturbation analysis on the line of center measures $\{t\mu + (1-t)\mu_n : t \in [0, 1]\}$. So, it is convenient to define for $\mu_{s,a} = p_{s,a}, \nu_{s,a}$

$$\begin{aligned}\mu_{s,a,n}(t) &= t\mu_{s,a} + (1-t)\mu_{s,a,n} \\ m_{s,a,n} &= \mu_{s,a} - \mu_{s,a,n} \\ g_{s,a,n}(t, \alpha) &= f(\mu_{s,a,n}(t), u, \alpha).\end{aligned}\tag{C.2}$$

Note that we will not explicitly indicate the dependence of u for the function g , because it will always be the identity function when $\mu = \nu$ and the value function when $\mu = p$. We will also drop the dependence on (s, a) when clear.

Our analysis involves many derivative computations. We use three type of derivative notations, two of which is explained here and the Radon-Nikodym derivative is introduced in the following paragraph. For a smooth function of multiple arguments $g(t, \alpha_{s,t})$ where $\alpha_{s,t}$ could be dependent on parameters s, t , denote the partial derivatives by $\partial_t, \partial_\alpha$; i.e.

$$\partial_t g(t, \alpha_{s,t}) := \lim_{\epsilon \rightarrow 0} \frac{g(t + \epsilon, \alpha_{s,t})}{\epsilon}, \quad \partial_\alpha g(t, \alpha_{s,t}) := \lim_{\epsilon \rightarrow 0} \frac{g(t, \alpha_{s,t} + \epsilon)}{\epsilon}.$$

On the other hand, when $\alpha_{s,t}$ is also smooth in t , denote the total derivative w.r.t. t by d_t ; i.e.

$$d_t g(t, \alpha_{s,t}) := \lim_{\epsilon \rightarrow 0} \frac{g(t + \epsilon, \alpha_{s,t+\epsilon})}{\epsilon} = \partial_t g(t, \alpha_{s,t}) + \partial_\alpha g(t, \alpha_{s,t}) \partial_t \alpha_{s,t}$$

The intuition behind our ability to remove the $1/\delta$ dependence stems from the mutual absolute continuity (also known as equivalence) between the empirical worst-case transition kernel and reward distribution and the true ones. This holds if δ is sufficiently small and the empirical centers of the uncertainty sets are equivalent to the true centers.

As a result, our techniques rely on the absolute continuity between the empirical measure μ_n and μ . We say that μ is absolute continuous w.r.t. another measure ν , denoted by $\nu \gg \mu$, if for $A \in 2^Y$, $\nu(A) = 0$ implies that $\mu(A) = 0$. We say that μ is equivalent to ν , denoted by $\mu \sim \nu$, if $\nu \gg \mu$ and $\nu \ll \mu$. Note that the empirical measure μ_n always satisfies $\mu_n \ll \mu$ w.p.1. For absolutely continuous measures $\nu \ll \mu$, the Radon-Nikodym derivative is well defined:

$$\frac{d\nu}{d\mu}(y) := \frac{\nu(s)}{\mu(s)} \mathbb{1} \{ \mu(s) \neq 0 \}.$$

The proof strategy we will implement is to consider separately the “good events” on which μ_n and μ are close (so that we have $\mu_n \sim \mu$) and the “bad events” where the empirical measure is not close to the reference model. This motivates us to define for $p > 0$

$$\Omega_{n,p}(\mu) = \left\{ \omega : \sup_y |\mu_n(\omega)(y) - \mu(y)| \leq p \right\}.\tag{C.3}$$

Then, in the DR-RL setting, define

$$\begin{aligned}\Omega_{n,p} &= \bigcap_{s,a} \Omega_{n,p}(p_{s,a}) \cap \bigcap_{s,a} \Omega_{n,p}(\nu_{s,a}) \\ &= \left\{ \omega : \sup_{s,a} \sup_{s'} |p_{s,a,n}(\omega)(s') - p_{s,a}(s')| \leq p, \sup_{s,a} \sup_r |\nu_{s,a,n}(\omega)(r) - \nu_{s,a}(r)| \leq p \right\}.\end{aligned}$$

We frequently make use of the minimum support probability of certain measures such as $\mu, \mu_{s,a}$. This is denoted by $\mu_\wedge := \min \{\mu(s) : \mu(s) > 0\}$, $\mu_{s,a,\wedge} := \min \{\mu(s) : \mu(s) > 0\}$.

It is easy to see that the following lemma holds:

Lemma 32 *Suppose $p < \mu_\wedge$, then on $\Omega_{n,p}(\mu)$, $\mu \sim \mu_n$ and $\inf_{y:\mu(y)>0} \mu_n(y) > \mu_\wedge - p$.*

Moreover, the empirical measures are satisfies the following concentrations:

Lemma 33 *Let μ be any probability measure on finite measure space $(Y, 2^Y)$. Then, for any $k = 1, 2, 3, \dots$*

$$P(\Omega_{n,p}(\mu)^c) \leq \frac{1}{p^{2k} n^k} \log(e^{2k-1} |Y|)^k.$$

In particular, if we choose $k = 1$,

$$P(\Omega_{n,p}(\mu)^c) \leq \frac{1}{p^2 n} \log(e|Y|).$$

This lemma follows from the subgaussian property of empirical measures on finite measure space; i.e. Lemma 39 holds.

For absolutely continuous empirical measures, we also have the following lemma, again as a consequence of subgaussianity and hence Lemma 39.

Lemma 34 *Let ξ_n be another random measure on $(Y, 2^Y)$. Let (Ω, \mathcal{F}, P) be the probability space that supports ξ_n, μ_n . Suppose that $\mu_n \ll \xi_n$, $\mu \ll \xi_n$, and $\xi_n(y) > p$ for all y s.t. $\xi_n(y) \neq 0$. Then, for all $A \in \mathcal{F}$, the following bounds hold:*

$$E \left\| \frac{dm_n}{d\xi_n} \right\|_{L^\infty(\xi_n)} \mathbb{1}_A \leq \frac{1}{p\sqrt{n}} \sqrt{\log(e|Y|)}$$

and

$$E \left\| \frac{dm_n}{d\xi_n} \right\|_{L^\infty(\xi_n)}^2 \mathbb{1}_A \leq \frac{1}{p^2 n} \log(e|Y|).$$

The proofs of these results are deferred to Appendix D.

C.2 Proof of Proposition 21

Proof By definition, we have

$$\begin{aligned} \left| \widehat{\mathbf{T}}(q)(s, a) - \mathcal{T}(q)(s, a) \right| &\leq \sup_{\beta \geq 0} |f(\nu_{s,a,n}, id, \beta) - f(\nu_{s,a}, id, \beta)| \\ &\quad + \gamma \sup_{\alpha \geq 0} |f(p_{s,a,n}, v(q), \alpha) - f(p_{s,a}, v(q), \alpha)|. \end{aligned} \tag{C.4}$$

We will drop the s, a dependence for simplicity. This motivates us to look at the dual functional applied to generic measureable $u : Y \rightarrow \mathbb{R}$. Let's Define $w = e^{-u/\alpha}$.

$$\begin{aligned} |f(\mu_n, u, \alpha) - f(\mu, u, \alpha)| &= |g_n(0, \alpha) - g_n(1, \alpha)| \\ &= \left| \partial_t g_n(t, \alpha) \right|_{t=\tau} \\ &= \alpha \left| \frac{m_n[w]}{\mu_n(\tau)[w]} \right| \end{aligned}$$

for some random variable $\tau \in (0, 1)$. To bound this, we introduce the following lemma for which the proof is deferred to E.

Lemma 35 *Let $m = \mu_1 - \mu_2$ with $\mu_1, \mu_2 \ll \mu$ and $w = e^{-u/\alpha}$, we have that*

$$\sup_{\alpha \geq 0} \frac{\alpha^j m[w]^2}{\mu[w]^2} \leq 3^j \inf_{\kappa \in \mathbb{R}} \|u - \kappa\|_{L^\infty(\mu)}^j \left\| \frac{dm}{d\mu} \right\|_{L^\infty(\mu)}^2.$$

To apply Lemma 35, we consider $p \leq \frac{1}{2}\mu_\wedge$. Then, By Lemma 32, on $\Omega_{n,p}(\mu)$, $\mu_n(t) \sim \mu$ for all $t \in [0, 1]$. So, on $\Omega_{n,p}(\mu)$, we have by Lemma 35

$$\begin{aligned} \sup_{\alpha \geq 0} |f(\mu_n, u, \alpha) - f(\mu, u, \alpha)| &\leq \sup_{\alpha \geq 0} \alpha \left| \frac{m_n[w]}{\mu_n(\tau)[w]} \right| \\ &\leq \inf_{\kappa \in \mathbb{R}} 3 \|u - \kappa\|_{L^\infty(\mu)} \left\| \frac{dm_n}{d\mu_n(\tau)} \right\|_{L^\infty(\mu)} \\ &= 3 |u|_{\text{span}} \left\| \frac{dm_n}{d\mu_n(\tau)} \right\|_{L^\infty(\mu)}. \end{aligned}$$

Therefore, by partitioning Ω into $\Omega_{n,p}(\mu)^c$ and $\Omega_{n,p}(\mu)$, we bound

$$\begin{aligned} E \sup_{\alpha \geq 0} |f(\mu_n, u, \alpha) - f(\mu, u, \alpha)|^2 \\ \leq 9 |u|_{\text{span}}^2 E \left\| \frac{dm_n}{d\mu_n(\tau)} \right\|_{L^\infty(\mu)}^2 \mathbb{1}_{\Omega_{n,p}(\mu)} + 4 |u|_{\text{span}}^2 P(\Omega_{n,p}(\mu)^c) \end{aligned} \tag{C.5}$$

where on $\Omega_{n,p}(\mu)^c$, we use the bound in Lemma 31.

By Lemma 32, on $\Omega_{n,p}(\mu)$ for y s.t. $\mu(y) > 0$, $\mu_n(y) \geq \mu_\wedge - p \geq \frac{1}{2}\mu_\wedge \geq p$. Since $\mu(y) > 0$ implies that $\mu(y) \geq \mu_\wedge$, we have that $\mu_n(t)(y) \geq p$ for any $t \in [0, 1]$. Therefore, Lemma 34 applies. On the other hand, Lemma 33 also applies and is used to bound $P(\Omega_{n,p}(\mu)^c)$.

Therefore, continue from (C.5), we have

$$E \sup_{\alpha \geq 0} |f(\mu_n, u, \alpha) - f(\mu, u, \alpha)|^2 \leq 13 \frac{|u|_{\text{span}}^2}{p^2 n} \log(e|Y|).$$

We conclude that choosing $p = \frac{1}{2}\mathfrak{p}_\wedge \leq \frac{1}{2} \min \{\nu_{s,a,\wedge}, p_{s,a,\wedge}\}$,

$$\begin{aligned} \text{Var}(\widehat{\mathbf{T}}(q)(s, a)) &\leq 2E \sup_{\beta \geq 0} |f(\nu_{s,a,n}, id, \beta) - f(\nu_{s,a}, id, \beta)|^2 \\ &\quad + 2\gamma^2 E \sup_{\alpha \geq 0} |f(p_{s,a,n}, v(q), \alpha) - f(p_{s,a}, v(q), \alpha)|^2 \\ &\leq 26 \frac{\|id\|_{\nu_{s,a}, \text{span}}^2}{p^2 n} \log(e|\mathbf{R}|) + 26\gamma^2 \frac{\|v(q)\|_{p_{s,a}, \text{span}}^2}{p^2 n} \log(e|\mathbf{S}|) \\ &\leq 26 \frac{r_{\max}^2 + \gamma^2 |q|_{\text{span}}^2}{p^2 n} \log(e(|\mathbf{R}| \vee |\mathbf{S}|)). \end{aligned}$$

Plugging in $p = \frac{1}{2}\mathfrak{p}_\wedge$, we obtain the claimed inequality in Proposition 21. ■

C.3 Proof of Proposition 22

Proof We consider for generic u and measure μ on $(Y, 2^Y)$. We assume $\delta < \frac{1}{24}\mu_\wedge$, which will be guaranteed by Assumption 1.

Since $\alpha \rightarrow f(\mu, u, \alpha)$ is continuous, and from Si et al. (2020) it is sufficient to optimize the Lagrange multiplier on compact set $[0, \delta^{-1}\|u\|_{L^\infty(\mu)}]$, there is an optimal Lagrange multiplier $\alpha_n^*(t)$ that achieves $\sup_{\alpha \geq 0} f(\mu_n(t), u, \alpha)$.

The bias of the dual functional

$$\begin{aligned} & \text{Bias}(f(\mu_n, u, \alpha_n^*)) \\ &= E(g_n(0, \alpha_n^*(0)) - g_n(1, \alpha^*)) \mathbb{1}_{\Omega_{n,p}(\mu)} + E(g_n(0, \alpha_n^*(0)) - g_n(1, \alpha^*)) \mathbb{1}_{\Omega_{n,p}(\mu)^c} \\ &=: E_1 + E_2. \end{aligned} \quad (\text{C.6})$$

We fix $p \leq \frac{1}{4}\mu_\wedge$. Notice that by assumption,

$$\delta < \frac{1}{24}\mu_\wedge < \frac{1}{2}\mu_\wedge \leq -\log\left(1 - \frac{1}{2}\mu_\wedge\right). \quad (\text{C.7})$$

Then, the following Lemma 36 holds.

Lemma 36 (Differentiability of the Dual Functional) *Suppose $\delta < -\log(1 - \frac{1}{2}\mu_\wedge)$ and $p \leq \frac{1}{4}\mu_\wedge$, then*

- On $\Omega_{n,p}(\mu)$, $t \rightarrow \sup_{\alpha \geq 0} g_n(t, \alpha)$ is $C^2((0, 1)) \cap C[0, 1]$.
- $\alpha^* = 0$ iff u is μ essentially constant. So, $\alpha_n^*(t) \equiv 0$ and $\sup_{\alpha \geq 0} g_n(t, \alpha) \equiv \mu[u]$
- If $\alpha^* > 0$, then $\alpha_n^*(t) > 0$ for all $t \in [0, 1]$ with

$$d_t \sup_{\alpha \geq 0} g_n(t, \alpha) = -\alpha_n^*(t) \frac{m_n[w]}{\mu_n(t)[w]}$$

and

$$\begin{aligned} & d_t d_t \sup_{\alpha \geq 0} g_n(t, \alpha) \\ &= -\alpha_n^*(t) \frac{m_n[w]^2}{\mu_n(t)[w]^2} \\ & \quad - \left(\frac{\alpha_n^*(t)^3}{\text{Var}_{\mu_n^*(t)}(u)} \right) \left(\frac{m_n[w]}{\mu_n(t)[w]} + \frac{m_n[uw]}{\alpha_n^*(t)\mu_n(t)[w]} - \frac{m_n[w]\mu_n(t)[uw]}{\alpha_n^*(t)\mu_n(t)[w]^2} \right)^2. \end{aligned} \quad (\text{C.8})$$

The proof of this result is deferred to Appendix E.

So, on $\Omega_{n,p}(\mu)$, $t \rightarrow g_n(t, \alpha_n^*(t))$ is $C^2(0, 1) \cap C[0, 1]$. By the (second order) mean value theorem, there exists random variable $\tau \in [0, 1]$ s.t.

$$\begin{aligned} E_1 &= E \left(-d_t g_n(t, \alpha_n^*(t)) \Big|_{t=1} + \frac{1}{2} d_t d_t g_n(t, \alpha_n^*(t)) \Big|_{t=\tau} \right) \mathbb{1}_{\Omega_{n,p}(\mu)} \\ &= E \left(\alpha^* \frac{m_n[w]}{\mu[w]} + \frac{1}{2} d_t d_t g_n(t, \alpha_n^*(t)) \Big|_{t=\tau} \right) \mathbb{1}_{\Omega_{n,p}(\mu)} \\ &= \alpha^* \frac{E m_n[w]}{\mu[w]} - E \alpha^* \frac{m_n[w]}{\mu[w]} \mathbb{1}_{\Omega_{n,p}(\mu)^c} + E \left[\frac{1}{2} d_t d_t g_n(t, \alpha_n^*(t)) \Big|_{t=\tau} \mathbb{1}_{\Omega_{n,p}(\mu)} \right] \\ &= 0 - E_{1,1} + E_{1,2} \end{aligned}$$

where $Em_n[u] = 0$ for any function u . Recall Lemma 35. Since naturally $\mu \gg \mu, \mu_n$,

$$\begin{aligned} |E_{1,1}| &\leq 3 |u|_{\text{span}} E \left\| \frac{dm_n}{d\mu} \right\|_{L^\infty(\mu)} \mathbb{1}_{\Omega_{n,p}(\mu)^c} \\ &\leq 3 \frac{|u|_{\text{span}}}{\mu_\wedge} P(\Omega_{n,p}(\mu)^c) \\ &\leq 3 \frac{|u|_{\text{span}}}{\mu_\wedge p^2 n} \log(e|Y|), \end{aligned}$$

where we use Lemma 33 for the last inequality.

On $\Omega_{n,p}(\mu)$, by Lemma 36, for all $t \in [0, 1]$ either $\alpha_n^*(t) = 0$ or $\alpha_n^*(t) > 0$. In the first case, we have trivially $E_{1,2} = 0$. In the second case,

$$\begin{aligned} -d_t d_t g_n(t, \alpha_n^*(t)) &= -d_t \partial_t g_n(t, \alpha_n^*(t)) \\ &= \alpha_n^*(t) \frac{m_n[w]^2}{\mu_n(t)[w]^2} \\ &\quad + \left(\frac{\alpha_n^*(t)^3}{\text{Var}_{\mu_n^*(t)}(u)} \right) \left(\frac{m_n[w]}{\mu_n(t)[w]} + \frac{m_n[uw]}{\alpha_n^*(t)\mu_n(t)[w]} - \frac{\mu_n(t)[uw]m_n[w]}{\alpha_n^*(t)\mu_n(t)[w]^2} \right)^2. \end{aligned}$$

Next, we prove a finer characteristic when δ goes to 0. We need the following Lemma:

Lemma 37 *On $\Omega_{n,p}(\mu)$ with $p < \mu_\wedge$*

$$\sup_{\alpha \geq 0} \alpha^3 \left(\frac{m_n[w]}{\mu_n(t)[w]} + \frac{m_n[uw]}{\alpha\mu_n(t)[w]} - \frac{m_n[w]\mu_n(t)[uw]}{\alpha\mu_n(t)[w]^2} \right)^2 \leq 136 \inf_{\kappa \in \mathbb{R}} \|u - \kappa\|_{L^\infty(\mu)}^3 \left\| \frac{dm_n}{d\mu_n(t)} \right\|_{L^\infty(\mu)}^2.$$

Applying Lemma 35 and 37, we have that on $\Omega_{n,p}(\mu)$

$$\begin{aligned} &|d_t d_t g_n(t, \alpha_n^*(t))| \mathbb{1}_{\Omega_{n,p}(\mu)} \\ &\leq 3 \inf_{\kappa \in \mathbb{R}} \|u - \kappa\|_{L^\infty(\mu)} \left\| \frac{dm_n}{d\mu_n(t)} \right\|_{L^\infty(\mu)}^2 + 136 \frac{\inf_{\kappa \in \mathbb{R}} \|u - \kappa\|_{L^\infty(\mu)}^3}{\text{Var}_{\mu_n^*(t)}(u)} \left\| \frac{dm_n}{d\mu_n(t)} \right\|_{L^\infty(\mu)}^2 \\ &\leq 3 |u|_{\text{span}} \left\| \frac{dm_n}{d\mu_n(t)} \right\|_{L^\infty(\mu)}^2 + 136 |u|_{\text{span}} \frac{\|u - \mu_n^*[u]\|_{L^\infty(\mu_n^*)}^2}{\|u - \mu_n^*(t)[u]\|_{L^2(\mu_n^*)}^2} \left\| \frac{dm_n}{d\mu_n(t)} \right\|_{L^\infty(\mu)}^2 \end{aligned}$$

To bound the second ratio in the last inequality, we introduce the following lemma, whose proof is deferred to Appendix E as well.

Lemma 38 *Suppose $\delta \leq \frac{1}{24}\mu_\wedge$ and $p \leq \frac{1}{4}\mu_\wedge$. When the optimal Lagrange multiplier $\alpha^* > 0$, worst-case measures $\mu_n^*(t) = \mu_n(t)[w]/\mu_n(t)[w]$ satisfies $\mu_n^*(t)(y) \geq \frac{1}{2}\mu_\wedge$ on $\Omega_{n,p}(\mu)$.*

For $\delta \leq \frac{1}{24}\mu_\wedge$, by Lemma 38, for some y' s.t. $\mu_n^*(t)(y') > 0$,

$$\begin{aligned} \frac{\|u - \mu_n^*[u]\|_{L^\infty(\mu_n^*)}^2}{\|u - \mu_n^*(t)[u]\|_{L^2(\mu_n^*)}^2} &= \frac{|u(y') - \mu_n^*[u]|^2}{\mu_n^*(t)(y')|u(y') - \mu_n^*[u]|^2 + \sum_{y \neq y'} \mu_n^*(t)(y)|u(y) - \mu_n^*[u]|^2} \\ &\leq \frac{|u(y') - \mu_n^*[u]|^2}{\mu_n^*(t)(y')|u(y') - \mu_n^*[u]|^2} \\ &\leq \frac{2}{\mu_\wedge} \end{aligned}$$

As in the proof of Proposition 21, under the choice $p \leq \frac{1}{4}\mu_\wedge$, Lemma 34 applies. Therefore,

$$|E_{1,2}| \leq 275 \frac{|u|_{\text{span}}}{\mu_\wedge p^2 n} \log(e|Y|)$$

For E_2 in (C.6), we use Lemma 31 and previous bound on $P(\Omega_{n,p}(\mu)^c)$

$$\begin{aligned} |E_2| &\leq E |f(\mu_n, u, \alpha_n^*(0)) - f(\mu, u, \alpha^*)| \mathbb{1}_{\Omega_{n,p}(\mu)^c} \\ &\leq 2 |u|_{\text{span}} P(\Omega_{n,p}(\mu)^c) \\ &\leq 2 \frac{|u|_{\text{span}}}{p^2 n} \log(e|Y|) \\ &\leq \frac{|u|_{\text{span}}}{\mu_\wedge p^2 n} \log(e|Y|) \end{aligned}$$

Therefore, going back to (C.6), we have

$$\left| \text{Bias} \left(\sup_{\alpha \geq 0} f(\mu_n, u, \alpha) \right) \right| \leq 280 \frac{|u|_{\text{span}}}{\mu_\wedge p^2 n} \log(e|Y|).$$

Apply this to the empirical Bellman operator with $p = \frac{1}{4}\mathfrak{p}_\wedge \leq \frac{1}{4} \min \{p_{s,a,\wedge}, \mu_{s,a,\wedge}\}$ and Assumption 1 holds. So, $\delta < \frac{1}{24}\mathfrak{p}_\wedge$ implies that $\delta < \frac{1}{24} \min \{p_{s,a,\wedge}, \mu_{s,a,\wedge}\}$. Therefore, we have

$$\begin{aligned} |\text{Bias}(\widehat{\mathbf{T}}(q)(s, a))| &= \left| \text{Bias} \left(\sup_{\beta \geq 0} f(\nu_{s,a,n}, id, \beta) \right) + \gamma \text{Bias} \left(\sup_{\alpha \geq 0} f(p_{s,a,n}, v(q), \alpha) \right) \right| \\ &\leq 4480 \frac{\|id\|_{\nu_{s,a}, \text{span}} + \gamma |v(q)|_{\text{span}}}{\mathfrak{p}_\wedge^3 n} \log(e|\mathbf{S}| \vee |\mathbf{R}|) \\ &\leq 4480 \frac{r_{\max} + \gamma |q|_{\text{span}}}{\mathfrak{p}_\wedge^3 n} \log(e|\mathbf{S}| \vee |\mathbf{R}|). \end{aligned}$$

■

C.4 Proof of Proposition 23

Proof We recall the bound (C.4) and the subsequent result

$$\sup_{\alpha \geq 0} |f(\mu_n, u, \alpha) - f(\mu, u, \alpha)| \leq 3 |u|_{\text{span}} \left\| \frac{dm_n}{d\mu_n(\tau)} \right\|_{L^\infty(\mu)}.$$

Again, we consider $p \leq \frac{1}{2}\mu_\wedge$. Also recall the definition (C.3) of $\Omega_{n,p}(\mu)$. By Lemma 32, on $\Omega_{n,p}(\mu)$ for y s.t. $\mu(y) > 0$, $\mu_n(y) \geq \mu_\wedge - p \geq \frac{1}{2}\mu_\wedge \geq p$. Since $\mu(y) > 0$ implies that

$\mu(y) \geq \mu_\wedge$, we have that $\mu_n(t)(y) \geq p$ for any $t \in [0, 1]$. Therefore, we have

$$\begin{aligned}
 & P \left(\sup_{\alpha \geq 0} |f(\mu_n, u, \alpha) - f(\mu, u, \alpha)| > t \right) \\
 & \leq P(\Omega_{n,p}(\mu)^c) + P \left(3|u|_{\text{span}} \left\| \frac{dm_n}{d\mu_n(\tau)} \right\|_{L^\infty(\mu)} > t, \Omega_{n,p}(\mu) \right) \\
 & \leq P \left(\sup_y |\mu_n(y) - \mu(y)| > p \right) + P \left(\frac{3|u|_{\text{span}}}{p} \sup_y |m_n(y)| > t \right) \\
 & \leq 2 \sum_y \left(\exp(-2p^2n) + \exp \left(-\frac{2p^2t^2n}{9|u|_{\text{span}}^2} \right) \right) \\
 & \leq 2|Y| \left(\exp(-2p^2n) + \exp \left(-\frac{2p^2t^2n}{9|u|_{\text{span}}^2} \right) \right)
 \end{aligned}$$

where we used Hoeffding's inequality and union bound.

Therefore, going back to the DR Bellman operator setting, we choose $p = \frac{1}{4}\mathfrak{p}_\wedge$ and by union bound

$$\begin{aligned}
 & P(\|\widehat{\mathbf{T}}(q) - \mathcal{T}(q)\|_\infty > t) \\
 & \leq P \left(\sup_{s,a} \sup_{\beta \geq 0} |f(\nu_{s,a,n}, id, \beta) - f(\nu_{s,a}, id, \beta)| > \frac{t}{2} \right) \\
 & \quad + P \left(\sup_{s,a} \sup_{\alpha \geq 0} |f(p_{s,a,n}, v(q), \beta) - f(p_{s,a}, v(q), \beta)| > \frac{t}{2} \right) \\
 & \leq 2(|\mathbf{S}|^2|\mathbf{A}| + |\mathbf{S}||\mathbf{A}||\mathbf{R}|) \exp \left(-\frac{\mathfrak{p}_\wedge^2 n}{8} \right) + 2|\mathbf{S}||\mathbf{A}||\mathbf{R}| \exp \left(-\frac{\mathfrak{p}_\wedge^2 t^2 n}{288r_{\max}^2} \right) \\
 & \quad + 2|\mathbf{S}|^2|\mathbf{A}| \exp \left(-\frac{\mathfrak{p}_\wedge^2 t^2 n}{288\gamma^2|q|_{\text{span}}^2} \right).
 \end{aligned}$$

We set each of the three terms to be less than $\eta/3$ and find that it suffices to have

$$n \geq \frac{8}{\mathfrak{p}_\wedge^2} \log(12|\mathbf{S}||\mathbf{A}|(|\mathbf{S}| \vee |\mathbf{R}|)/\eta)$$

and

$$t \geq \frac{17(r_{\max} + \gamma|q|_{\text{span}})}{\mathfrak{p}_\wedge \sqrt{n}} \sqrt{\log(6|\mathbf{S}||\mathbf{A}|(|\mathbf{S}| \vee |\mathbf{R}|)/\eta)}.$$

This implies the statement of the proposition. ■

C.5 Proof of Proposition 24

Proof We define

$$V := \mathcal{H}(\hat{q}) - \widehat{\mathbf{H}}(\hat{q}) = (\mathcal{T}(\hat{q}) - \mathcal{T}(q_*)) - (\widehat{\mathbf{T}}(\hat{q}) - \widehat{\mathbf{T}}(q_*)). \quad (\text{C.9})$$

Recall the dual formulation

$$\mathcal{T}(q)(s, a) = \sup_{\beta \geq 0} f(\nu_{s,a}, id, \beta) + \gamma \sup_{\alpha \geq 0} f(p_{s,a}, v(q), \alpha).$$

The first term is not dependent on q , hence canceled in V . We have that

$$|V(s, a)| = \gamma \left| \sup_{\alpha \geq 0} f(p_{s,a}, v(\hat{q}), \alpha) - \sup_{\alpha \geq 0} f(p_{s,a}, v(q^*), \alpha) - \sup_{\alpha \geq 0} f(p_{s,a,n}, v(\hat{q}), \alpha) + \sup_{\alpha \geq 0} f(p_{s,a,n}, v(q^*), \alpha) \right|$$

Note that if $v(\hat{q})$ and $v(q^*)$ are both μ essentially constant, then $V = 0$, and the claim of Proposition 24 holds trivially. Therefore, moving forward, we consider the case at least one of $v(\hat{q})$ and $v(q^*)$ is not μ essentially constant.

To analyze V while keeping the consistency of our notations, we define $v_t = tv(\hat{q}) + (1-t)v(q^*)$, $\mu = p_{s,a}$, $\mu_n = p_{s,a,n}$, $m = \mu - \mu_n$, and $\mu(t) = t\mu - (1-t)\mu_n$. Because Assumption 1 is imposed, we have that $\delta < \frac{1}{24}\mu_\wedge$.

We consider the parametric function for $s, t \in [0, 1]$

$$h(s, t) := \sup_{\alpha \geq 0} f(\mu(t), v_s, \alpha) = f(\mu(t), v_s, \alpha_{s,t}^*). \quad (\text{C.10})$$

To motivates our analysis, we assume that $h(s, \cdot)$ is $C^1(0, 1) \cap C[0, 1]$ and $\partial_t h(\cdot, t)$ is $C^1(0, 1) \cap C[0, 1]$ as well. Then the fundamental theorem of calculus implies that

$$\begin{aligned} |V(s, a)| &= \gamma |h(1, 0) - h(0, 0) - h(1, 1) + h(0, 1)| \\ &= \gamma \left| - \int_0^1 \partial_t h(1, t) dt + \int_0^1 \partial_t h(0, t) dt \right| \\ &= \gamma \left| \int_0^1 \int_0^1 \partial_s \partial_t h(s, t) ds dt \right| \\ &\leq \gamma \int_0^1 \int_0^1 |\partial_s \partial_t h(s, t)| ds dt \end{aligned} \quad (\text{C.11})$$

where $\partial_s \partial_t h(s, t)$ is easier to analyze. We proceed to show that (C.11) is valid (with some minor modification) on $\Omega_{n,p}(\mu)$.

As in the proof of Proposition 22, Lemma 36 applies when we consider $p \leq \frac{1}{4}\mu_\wedge$. So, for $p \leq \frac{1}{4}\mu_\wedge$, on $\Omega_{n,p}(\mu)$, $h(s, \cdot)$ is $C^2(0, 1) \cap C[0, 1]$ with derivative

$$\partial_t h(s, t) = d_t \sup_{\alpha \geq 0} f(\mu(t), v_s, \alpha) = -\alpha_{s,t}^* \frac{m[w_s]}{\mu(t)[w_s]}.$$

Here, by Lemma 36, $\alpha_{s,t}^*$ is the unique optimal Lagrange multiplier, and $w_s = e^{-v_s/\alpha_{s,t}^*}$.

Next, we show that for every fixed t , there is a function $D_s \partial_t h$ s.t.

$$\int_0^1 D_s \partial_t h(s, t) ds = \partial_t h(1, t) - \partial_t h(0, t). \quad (\text{C.12})$$

We note that by Lemma 36, $\alpha_{s,t}^* = 0$ if and only if v_s is essentially constant. This can only happen at one particular $s = s^*$. Otherwise, if there are some $0 \leq s_1 < s_2 \leq 1$,

$s_1 v(\hat{q}) + (1 - s_1)v(q^*) = c_1 e$ and $s_2 v(\hat{q}) + (1 - s_2)v(q^*) = c_2 e$ w.p.1 under μ , where e is the vector of all ones, then for all $a, b \geq 0$,

$$\frac{as_1 + bs_2}{a + b} v(\hat{q}) + \left(1 - \frac{as_1 + bs_2}{a + b}\right) v(q^*) = (ac_1 + bc_2)e.$$

This would imply that $v(\hat{q})$ and $v(q^*)$ are both essentially constant.

We consider two cases:

Case 1: v_s is never essentially constant for all $s \in [0, 1]$.

In this case, $\alpha_{s,t}^* > 0$ for all $s \in [0, 1]$. Note that $s \rightarrow e^{-v_s/\alpha}$ is clearly C^∞ for $\alpha > 0$. So, on $\Omega_{n,p}(\mu)$ if $\alpha_{s,t}^*$ is $C^1(0, 1)$ in s , then $s \rightarrow \partial_t h(s, t)$ is $C^1(0, 1) \cap C[0, 1]$.

We show differentiability of $s \rightarrow \alpha_{s,t}^*$ by invoking the implicit function theorem as in the proof of Lemma 36. When $\alpha_{s,t}^* > 0$, as shown in Lemma 36, it is the unique solution to the optimality condition

$$\alpha_{s,t}^*(-\log \mu(t)[w_s] - \delta) - \frac{\mu(t)[v_s w_s]}{\mu(t)[w_s]} =: F(s, \alpha_{s,t}^*) = 0. \quad (\text{C.13})$$

Define the optimal measure

$$\mu^*(s, t)[\cdot] = \frac{\mu(t)[w_s]}{\mu(t)[w_s]}.$$

Since for all fixed t , $\alpha_{s,t}^* > 0$ on $(0, 1)$ and F is infinite smooth. The implicit function theorem then implies that $\alpha_{s,t}^*$ is $C^1(0, 1) \cap C[0, 1]$ and $s \rightarrow \partial_t h(s, t)$ is $C^1(0, 1) \cap C[0, 1]$.

We compute the derivative $\partial_s \partial_t h$ in this case. Let $\Delta_v = v(\hat{q}) - v(q^*)$. Rewrite the optimality equation as

$$\alpha_{s,t}^*(-\log \mu(t)[w_s] - \delta) = \frac{\mu(t)[v_s w_s]}{\mu(t)[w_s]}.$$

Differentiate w.r.t. s on both side

$$\begin{aligned} \text{LHS} &= \partial_s \alpha_{s,t}^*(-\log \mu(t)[w_s] - \delta) + \frac{\mu(t)[\Delta_v w_s]}{\mu(t)[w_s]} - \partial_s \alpha_{s,t}^* \frac{\mu(t)[v_s w_s]}{\alpha_{s,t}^* \mu(t)[w_s]} \\ &= \partial_s \alpha_{s,t}^*(-\log \mu(t)[w_s] - \delta) + \mu^*(s, t)[\Delta_v] - \partial_s \alpha_{s,t}^* \mu^*(s, t)[v_s / \alpha_{s,t}^*] \\ \text{RHS} &= \frac{\mu(t)[\Delta_v w_s] \mu(t)[v_s w_s]}{\alpha_{s,t}^* \mu(t)[w_s]^2} + \frac{\mu(t)[\Delta_v w_s]}{\mu(t)[w_s]} - \frac{\mu(t)[\Delta_v v_s w_s]}{\alpha_{s,t}^* \mu(t)[w_s]} \\ &\quad + \partial_s \alpha_{s,t}^* \left(-\frac{\mu(t)[v_s w_s]^2}{(\alpha_{s,t}^*)^2 \mu(t)[w_s]^2} + \frac{\mu(t)[v_s^2 w_s]}{(\alpha_{s,t}^*)^2 \mu(t)[w_s]} \right) \\ &= -\text{Cov}_{\mu^*(s, t)}(\Delta_v, v_s / \alpha_{s,t}^*) + \mu^*(s, t)[\Delta_v] + \partial_s \alpha_{s,t}^* \text{Var}_{\mu^*(s, t)}(v_s / \alpha_{s,t}^*) \end{aligned}$$

From the optimality equation and the LHS and RHS derivatives, we have

$$\begin{aligned} \partial_s \alpha_{s,t}^* (\log \mu(t)[w_s] + \delta + \mu^*(s, t)[v_s / \alpha_{s,t}^*] + \text{Var}_{\mu^*(s, t)}(v_s / \alpha_{s,t}^*)) &= \text{Cov}_{\mu^*(s, t)}(\Delta_v, v_s / \alpha_{s,t}^*) \\ \partial_s \alpha_{s,t}^* \text{Var}_{\mu^*(s, t)}(v_s / \alpha_{s,t}^*) &= \text{Cov}_{\mu^*(s, t)}(\Delta_v, v_s / \alpha_{s,t}^*) \\ \partial_s \alpha_{s,t}^* &= \frac{\text{Cov}_{\mu^*(s, t)}(\Delta_v, v_s / \alpha_{s,t}^*)}{\text{Var}_{\mu^*(s, t)}(v_s / \alpha_{s,t}^*)}. \end{aligned} \quad (\text{C.14})$$

Therefore, when $\alpha_{s,t}^* > 0$,

$$\begin{aligned}
 \partial_s \partial_t h(s, t) &= \partial_s \frac{-\alpha_{s,t}^* m[w_s]}{\mu(t)[w_s]} \\
 &= -\frac{m[w_s]\mu(t)[\Delta_v w_s]}{\mu(t)[w_s]^2} + \frac{m[\Delta_v w_s]}{\mu(t)[w_s]} - \partial_s \alpha_{s,t}^* \frac{m[w_s]}{\mu(t)[w_s]} \\
 &\quad + \partial_s \alpha_{s,t}^* \left(-\frac{m[v_s w_s]}{\alpha_{s,t}^* \mu(t)[w_s]} + \frac{m[w_s]\mu(t)[v_s w_s]}{\alpha_{s,t}^* \mu(t)[w_s]^2} \right) \\
 &=: D_1 + D_2 + D_3 + D_4.
 \end{aligned} \tag{C.15}$$

Case 2: There is a unique $s^* \in [0, 1]$ s.t. v_s is essentially constant.

In this case, the previous argument implies that $s \rightarrow \partial_t h(s, t)$ is $C^1(0, s^*)$, $C^1(s^*, 1)$, and continuous at 0, 1. The derivative is also given by (C.15).

We need to show the existence of $D_s \partial_t h$ that satisfy (C.12). Observe that if $s \rightarrow \partial_t h(s, t)$ is continuous at s^* , then applying the fundamental theorem of calculus on the interval $[0, s^*]$ and $[s^*, 1]$ separately, we will have that

$$\partial_t h(1, t) - \partial_t h(0, t) = \int_0^{s^*} \partial_s \partial_t h(s, t) ds + \int_{s^*}^1 \partial_s \partial_t h(s, t) ds.$$

Hence, taking $D_s \partial_t h(s, t) = \partial_s \partial_t h(s, t)$ for every $s \neq s^*$ and $D_s \partial_t h(s^*, t) = 0$ will suffice to produce (C.12).

It is left to check the continuity at s^* . As analyzed in (E.1),

$$\lim_{\alpha \downarrow 0} \alpha_{s,t}^* \frac{m[w_s]}{\mu(t)[w_s]} = 0.$$

So we can conclude the continuity of $s \rightarrow \partial_t h(s, t)$ at s^* , if we can show that when $v_s \rightarrow ce$ for some constant c , then $\alpha_{s,t}^* \downarrow 0$.

To prove this, we assume to the contrary that there is a subsequential limit $\alpha_{s_n,t}^* \rightarrow \beta > 0$ for some sequence $s_n \rightarrow s^*$. But since F defined (C.13) in s and α when $\alpha > 0$, we must have that

$$0 = \lim_{n \rightarrow \infty} F(s_n, \alpha_{s_n,t}^*) = \beta(-\log \mu(t)[e^{-ce/\beta}] - \delta) - c = -\delta\beta$$

raising a contradiction. This implies that $s \rightarrow \partial_t h(s, t)$ is continuous at s^* , and hence (C.12) holds with $D_s \partial_t h(s, t) = \partial_s \partial_t h(s, t)$ for every $s \neq s^*$ and $D_s \partial_t h(s^*, t) = 0$.

Therefore, we have shown that the bound (C.11) is valid on $\Omega_{n,p}(\mu)$ with $p \leq \frac{1}{4}\mu_\wedge$.

Now we bound $\partial_s \partial_t h(s, t)$ using the decomposition (C.15). $|D_1|$ and $|D_2|$ can be bounded using the change of measure techniques: on $\Omega_{n,p}(\mu)$

$$\begin{aligned}
 |D_1| &\leq \frac{\mu(t)[\frac{dm}{d\mu(t)} w_s] \mu(t)[|\Delta_v| w_s]}{\mu(t)[w_s]^2} \\
 &\leq \|\Delta_v\|_\infty \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)}
 \end{aligned}$$

and

$$\begin{aligned} |D_2| &\leq \frac{\mu(t)[\frac{dm}{d\mu(t)}|\Delta_v|w_s]}{\mu(t)[w_s]} \\ &\leq \|\Delta_v\|_\infty \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)} \end{aligned}$$

To bound $|D_3|$ and $|D_4|$, recall $\partial_s \alpha_{s,t}^*$ from (C.14).

$$\begin{aligned} |D_3| &= \left| \partial_s \alpha_{s,t}^* \frac{m[w_s]}{\mu(t)[w_s]} \right| \\ &\leq \frac{|\text{Cov}_{\mu^*(s,t)}(\Delta_v, v_s/\alpha_{s,t}^*)|}{\text{Var}_{\mu^*(s,t)}(v_s/\alpha_{s,t}^*)} \frac{m[w_s]}{\mu(t)[w_s]} \\ &\leq \frac{|\text{Cov}_{\mu^*(s,t)}(\Delta_v, v_s)|}{\text{Var}_{\mu^*(s,t)}(v_s)} \alpha_{s,t}^* \frac{m[w_s]}{\mu(t)[w_s]} \\ &\stackrel{(i)}{\leq} 3 \frac{|\text{Cov}_{\mu^*(s,t)}(\Delta_v, v_s)|}{\text{Var}_{\mu^*(s,t)}(v_s)} \inf_{\kappa \in \mathbb{R}} \|v_s - \kappa\|_{L^\infty(\mu)} \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)} \\ &\leq 3 \sqrt{\text{Var}_{\mu^*(s,t)}(\Delta_v)} \frac{\|v_s - \mu^*(s,t)[v_s]\|_{L^\infty(\mu)}}{\sqrt{\text{Var}_{\mu^*(s,t)}(v_s)}} \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)} \\ &\leq 3 \|\Delta_v\|_\infty \frac{\|v_s - \mu^*(s,t)[v_s]\|_{L^\infty(\mu^*(s,t))}}{\|v_s - \mu^*(s,t)[v_s]\|_{L^2(\mu^*(s,t))}} \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)} \end{aligned}$$

where (i) used Lemma 35 with $j = 1$. Since $\alpha_{s,t}^* > 0$ for $s \in (0, 1)$ and v_s is not essentially constant, by Lemma 38, for some $s' \in \mathbf{S}$ s.t. $v_s(s') - \mu^*(s,t)[v_s] \neq 0$

$$\begin{aligned} &\frac{\|v_s - \mu^*(s,t)[v_s]\|_{L^\infty(\mu^*(s,t))}^2}{\|v_s - \mu^*(s,t)[v_s]\|_{L^2(\mu^*(s,t))}^2} \\ &= \frac{|v_s(s') - \mu^*(s,t)[v_s]|^2}{\mu^*(s,t)(s')|v_s(s') - \mu^*(s,t)[v_s]|^2 + \sum_{s'' \neq s'} \mu^*(s,t)(s'')|v_s(s'') - \mu^*(s,t)[v_s]|^2} \\ &\leq \frac{|v_s(s') - \mu^*(s,t)[v_s]|^2}{\mu^*(s,t)(s')|v_s(s') - \mu^*(s,t)[v_s]|^2} \\ &\leq \frac{2}{\mu_\wedge} \end{aligned}$$

So,

$$|D_3| \leq \frac{5\|\Delta_v\|_\infty}{\sqrt{\mu_\wedge}} \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)}.$$

From (C.14), by the property of variance,

$$|\partial_s \alpha_{s,t}^*| \leq \sqrt{\frac{\text{Var}_{\mu^*(s,t)}(\Delta_v)}{\text{Var}_{\mu^*(s,t)}(v_s/\alpha_{s,t}^*)}} \leq \frac{\|\Delta_v\|_\infty}{\sqrt{\text{Var}_{\mu^*(s,t)}(v_s/\alpha_{s,t}^*)}}.$$

Hence applying similar analysis,

$$\begin{aligned}
 |D_4| &= \left| \partial_s \alpha_{s,t}^* \left(-\frac{m[v_s w_s]}{\alpha_{s,t}^* \mu(t)[w_s]} + \frac{m[w_s] \mu(t)[v_s w_s]}{\alpha_{s,t}^* \mu(t)[w_s]^2} \right) \right| \\
 &= |\partial_s \alpha_{s,t}^*| \left| -\mu^*(s, t) \left[\frac{dm}{d\mu(t)} v_s / \alpha_{s,t}^* \right] + \mu^*(s, t) \left[\frac{dm}{d\mu(t)} \right] \mu^*(s, t) [v_s / \alpha_{s,t}^*] \right| \\
 &\leq \frac{\|\Delta_v\|_\infty}{\sqrt{\text{Var}_{\mu^*(s,t)}(v_s / \alpha_{s,t}^*)}} \text{Cov}_{\mu^*(s,t)} \left(\frac{dm}{d\mu(t)}, v_s / \alpha_{s,t}^* \right) \\
 &\leq \|\Delta_v\|_\infty \sqrt{\text{Var}_{\mu^*(s,t)} \left(\frac{dm}{d\mu(t)} \right)} \\
 &\leq \|\Delta_v\|_\infty \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)}.
 \end{aligned}$$

By (C.11), we have

$$\begin{aligned}
 E|V| &\leq E|V| \mathbb{1}_{\Omega_{n,p}(\mu)^c} + \gamma \int_0^1 \int_0^1 E |\partial_s \partial_t h(s, t)| \mathbb{1}_{\Omega_{n,p}(\mu)} ds dt \\
 &\leq E|V| \mathbb{1}_{\Omega_{n,p}(\mu)^c} + \gamma \sup_{s, t \in (0, 1)} E(|D_1| + |D_2| + |D_3| + |D_4|) \mathbb{1}_{\Omega_{n,p}(\mu)}.
 \end{aligned}$$

Recall the definition (C.9) of V ,

$$\begin{aligned}
 \|V\|_\infty &= \|(\mathcal{T}(\hat{q}) - \mathcal{T}(q_*)) - (\widehat{\mathbf{T}}(\hat{q}) - \widehat{\mathbf{T}}(q_*))\|_\infty \\
 &\leq \|\mathcal{T}(\hat{q}) - \mathcal{T}(q_*)\|_\infty + \|\widehat{\mathbf{T}}(\hat{q}) - \widehat{\mathbf{T}}(q_*)\|_\infty \\
 &\leq 2\gamma \|\hat{q} - q_*\|_\infty.
 \end{aligned}$$

So, by Lemma 33,

$$\begin{aligned}
 E|V| \mathbb{1}_{\Omega_{n,p}(\mu)^c} &\leq 2\gamma \|\hat{q} - q_*\|_\infty P(\Omega_{n,p}(\mu)^c) \\
 &\leq \frac{2\gamma \|\hat{q} - q_*\|_\infty}{p^2 n} \log(e|\mathbf{S}|) \tag{C.16}
 \end{aligned}$$

By the previous bounds on $|D_i|$, $i = 1, 2, 3, 4$,

$$\begin{aligned}
 E|V| &\leq \frac{2\gamma \|\hat{q} - q_*\|_\infty}{p^2 n} \log(e|\mathbf{S}|) + \gamma \sup_{s, t \in (0, 1)} \frac{8}{\sqrt{\mu_\wedge}} E\|\Delta_v\|_\infty \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)} \mathbb{1}_{\Omega_{n,p}(\mu)} \\
 &\leq \frac{2^5 \gamma \|\hat{q} - q_*\|_\infty}{\mu_\wedge^2 n} \log(e|\mathbf{S}|) + \frac{8\|\Delta_v\|_\infty}{p\sqrt{\mu_\wedge}\sqrt{n}} \sqrt{\log(e|\mathbf{S}|)} \\
 &\leq \frac{2^5 \gamma \|\hat{q} - q_*\|_\infty}{\mu_\wedge^2 n} \log(e|\mathbf{S}|) + \frac{2^5 \|\hat{q} - q_*\|_\infty}{\mu_\wedge^{3/2} \sqrt{n}} \sqrt{\log(e|\mathbf{S}|)} \\
 &\leq \frac{2^6 \|\hat{q} - q_*\|_\infty}{\mathfrak{p}_\wedge^{3/2} \sqrt{n}} \log(e|\mathbf{S}|) \tag{C.17}
 \end{aligned}$$

where we choose $p = \frac{1}{4}\mu_\wedge \leq \frac{1}{4}\mathfrak{p}_\wedge$ and the last inequality follows from the assumption in Proposition 24 that $n \geq \mathfrak{p}_\wedge^{-1}$.

To bound the variance, note that $\text{Var}(\widehat{\mathbf{T}}(\bar{x}) - \widehat{\mathbf{T}}(x_*)) \leq EV^2$ and

$$EV^2 \mathbb{1}_{\Omega_{n,p}(\mu)} \leq \gamma^2 \int_0^1 \int_0^1 (\partial_s \partial_t h(s, t))^2 ds dt$$

which follows from applying Jensen's inequality to the $[0, 1]^2$ integral. Therefore,

$$\begin{aligned} & \text{Var}(\widehat{\mathbf{T}}(\hat{q}) - \widehat{\mathbf{T}}(q^*)) \\ & \leq 8\gamma^2 \|\hat{q} - q_*\|_\infty^2 P(\Omega_{n,p}(\mu)^c) + \gamma^2 E \int_0^1 \int_0^1 4(D_1^2 + D_2^2 + D_3^2 + D_4^2) ds dt \mathbb{1}_{\Omega_{n,p}(\mu)} \\ & \leq \frac{2^7 \gamma^2 \|\hat{q} - q_*\|_\infty^2}{\mu_\wedge^2 n} \log(e|\mathbf{S}|) + \frac{112}{\mu_\wedge} \|\Delta_v\|_\infty^2 \sup_{s,t \in (0,1)} E \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)}^2 \mathbb{1}_{\Omega_{n,p}(\mu)} \\ & \leq \frac{2^7 \gamma^2 \|\hat{q} - q_*\|_\infty^2}{\mu_\wedge^2 n} \log(e|\mathbf{S}|) + \frac{2^{11} \|\hat{q} - q_*\|_\infty^2}{\mu_\wedge^3 n} \log(e|\mathbf{S}|). \\ & \leq \frac{2^{12} \|\hat{q} - q_*\|_\infty^2}{\mathfrak{p}_\wedge^3 n} \log(e|\mathbf{S}|). \end{aligned} \tag{C.18}$$

■

C.6 Proof of Proposition 25

Proof Recall the notations and definitions in as the proof of Proposition 24 in Appendix C.5 and, in particular, the definition (C.9) and bound (C.11) for V . We again choose $p \leq \frac{1}{4}\mu_\wedge = \frac{1}{4}p_{s,a,\wedge}$. As Appendix C.5, we have that

$$|V(s, a)| \leq |V| \mathbb{1}_{\Omega_{n,p}(\mu)^c} + \gamma \sup_{s,t \in (0,1)} (|D_1| + |D_2| + |D_3| + |D_4|) \mathbb{1}_{\Omega_{n,p}(\mu)}$$

where $\mu = p_{s,a}$.

Since Assumption 1 is assumed, the bounds on D_1, D_2, D_3, D_4 are still applicable. Therefore, by Hoeffding's inequality and union bound

$$\begin{aligned} & P(|V(s, a)| > t) \\ & \leq P(\Omega_{n,p}(p_{s,a})^c) + P \left(\gamma \sup_{s,t \in (0,1)} (|D_1| + |D_2| + |D_3| + |D_4|) > t, \Omega_{n,p}(p_{s,a}) \right) \\ & \leq P \left(\sup_{s' \in \mathbf{S}} |p_{s,a,n}(s') - p_{s,a}(s')| > p \right) + P \left(\frac{8\|\hat{q} - q_*\|_\infty}{(p_{s,a,\wedge} - p)\sqrt{p_{s,a,\wedge}}} \sup_{s' \in \mathbf{S}} |m_n(s')| > t \right) \\ & \leq \sum_{s' \in \mathbf{S}} \left(P(|m_n(s')| > p) + P \left(\frac{11\|\hat{q} - q_*\|_\infty}{p_{s,a,\wedge}^{3/2}} |m_n(s')| > t \right) \right) \\ & \leq 2|\mathbf{S}| \left(\exp(-2p^2 n) + \exp \left(-\frac{p_{s,a,\wedge}^{3/2} t^2 n}{56\|\hat{q} - q_*\|_\infty^2} \right) \right) \end{aligned}$$

where $m_n = p_{s,a,n} - p_{s,a}$. Then, as $\mathfrak{p}_\wedge \leq p_{s,a,\wedge}$ for all $(s, a) \in \mathbf{S} \times \mathbf{A}$, by union bound

$$P(\|V\|_\infty > t) \leq 2|\mathbf{S}|^2|\mathbf{A}| \left(\exp\left(-\frac{\mathfrak{p}_\wedge^2 n}{8}\right) + \exp\left(-\frac{\mathfrak{p}_\wedge^3 t^2 n}{56\|\hat{q} - q_*\|_\infty^2}\right) \right).$$

We first control the first term to be less than $\eta/2$, which is implied by

$$n \geq \frac{8}{\mathfrak{p}_\wedge^2} \log(4|\mathbf{S}|^2|\mathbf{A}|/\eta).$$

Finally, the second term less than $\eta/2$ is implied by choosing

$$t^2 = \frac{56\|\hat{q} - q^*\|}{\mathfrak{p}_\wedge^3 n} \log(4|\mathbf{S}|^2|\mathbf{A}|/\eta).$$

This proves the claimed result. \blacksquare

Appendix D. Proof of Technical Lemmas: Empirical Measures and Concentrations

The proofs in the rest of this section is based on the following concentration property of maximum subgaussian random variables.

D.1 Subgaussian Maximum Inequality

Lemma 39 *Let $\{Y_i, i = 1 \dots n\}$ be σ^2 -sub-Gaussian with zero means, not necessarily independent, then*

$$EZ := E \max_{i=1 \dots n} |Y_i|^k \leq 2^k \sigma^k (k - 1 + \log n)^{k/2}.$$

Proof For any $\lambda > 0$, consider an increasing function $\phi_\lambda(z) = \exp(\lambda z^{1/k})$ for $z \geq 0$. Since $Z \geq 0$,

$$\begin{aligned} \phi_\lambda(EZ) &= \phi_\lambda(EZ \mathbf{1}\{Z > u\} + EZ \mathbf{1}\{Z \leq u\}) \\ &\leq \phi_\lambda(EZ \mathbf{1}\{Z > u\} + uP(Z \leq u)) \\ &\leq \phi_\lambda(EZ + u) \end{aligned}$$

Take second derivatives,

$$\phi_\lambda''(z) = k^{-2} \lambda z^{1/k-2} e^{\lambda z^{1/k}} (\lambda z^{1/k} - k + 1);$$

one can see that $\phi_\lambda(z)$ is convex for $z \geq (k-1)^k \lambda^{-k}$. Let $u = (k-1)^k \lambda^{-k}$. By Jensen's inequality

$$\begin{aligned} \phi_\lambda(EZ) &\leq E\phi_\lambda(Z + (k-1)^k \lambda^{-k}) \\ &= e^{k-1} E \exp(\lambda \max_{i=1 \dots n} |Y_i|) \\ &\leq e^{k-1} \sum_{i=1}^n E e^{\lambda |Y_i|} \end{aligned}$$

Since $\{Y_i\}$ are Sub-Gaussian,

$$P(|Y_i| > t) \leq 2 \exp\left(-\frac{t^2}{2\sigma^2}\right)$$

By Rigollet (2015, Lemmas 1.4 and 1.5), one can show that

$$\log E e^{\lambda|Y_i|} \leq \log(E[e^{\lambda Y_i} + e^{-\lambda Y_i}]) \leq \log(2 \exp(\sigma^2 \lambda^2/2)) \leq 4\sigma^2 \lambda^2.$$

Therefore,

$$\begin{aligned} \lambda \left(E \max_{i=1 \dots n} |Y_i|^k \right)^{1/k} &= \log \phi_\lambda(EZ) \\ &\leq k - 1 + \log n + 4\sigma^2 \lambda^2. \end{aligned}$$

Rearrange and take infimum over $\lambda > 0$, we conclude

$$\begin{aligned} E \max_{i=1 \dots n} |Y_i|^k &\leq \left(\inf_{\lambda > 0} \frac{k - 1 + \log n}{\lambda} + 4\sigma^2 \lambda \right)^k \\ &\leq 2^k \sigma^k (k - 1 + \log n)^{k/2} \end{aligned}$$

■

D.2 Proof of Lemma 33

Proof By definition and Markov's inequality

$$\begin{aligned} P(\Omega_{n,p}(\mu)^c) &= P\left(\sup_y |\mu_n(y) - \mu(y)| > p\right) \\ &\leq \frac{1}{p^{2k}} E\left[\sup_y |\mu_n(y) - \mu(y)|^{2k}\right] \\ &= \frac{1}{p^{2k} n^{2k}} E\left[\sup_y \left(\sum_{i=1}^n \mathbb{1}\{Y_i = y\} - \mu(y)\right)^{2k}\right] \end{aligned}$$

Since $\sum_{i=1}^n \mathbb{1}\{Y_i = y\} - \mu(y)$ is $n/4$ sub-Gaussian, by Lemma 39

$$P(\Omega_{n,p}(\mu)^c) \leq \frac{1}{p^{2k} n^k} (2k - 1 + \log(|Y|))^k = \frac{1}{p^{2k} n^k} \log(e^{2k-1} |Y|)^k$$

as claimed. ■

D.3 Proof of Lemma 34

Proof Note that by Jensen's inequality,

$$E \left\| \frac{dm_n}{d\xi_n} \right\|_{L^\infty(\xi_n)}^2 \mathbb{1}_A \geq \left(E \left\| \frac{dm_n}{d\xi_n} \right\|_{L^\infty(\xi_n)} \mathbb{1}_A \right)^2.$$

So it suffices to show the second claim. By assumption,

$$E \left\| \frac{dm_n}{d\xi_n} \right\|_{L^\infty(\xi_n)}^2 \mathbb{1}_A \leq \frac{1}{p^2} E \sup_y |m_n(y)|^2 \mathbb{1}_A.$$

Same as the proof of Lemma 33, we use Lemma 39 to conclude that

$$E \left\| \frac{dm_n}{d\xi_n} \right\|_{L^\infty(\xi_n)}^2 \mathbb{1}_A \leq \frac{1}{p^{2n}} \log(e|Y|).$$

■

Appendix E. Proof of Technical Lemmas: KL Case

E.1 Proof of Lemma 30

Proof

$$\sup_{\alpha \geq 0} f(\nu, u, \alpha) \geq \lim_{\alpha \downarrow 0} f(\nu, u, \alpha) = \operatorname{ess\,inf}_{\nu} u \geq \operatorname{ess\,inf}_{\mu} u \geq -\|u\|_{L^\infty(\mu)}$$

On the other hand, since the sup is achieved on compact K . For optimal $\alpha_\nu^* > 0$,

$$\begin{aligned} \sup_{\alpha \geq 0} f(\nu, \alpha) &\leq \|u\|_{L^\infty(\nu)} - \alpha_\nu^* \log \nu [e^{-(u - \|u\|_{L^\infty(\nu)})/\alpha_\nu^*}] \\ &\leq \|u\|_{L^\infty(\mu)} \end{aligned}$$

where the last line follows from that $\nu[e^{-(u - \|u\|_{L^\infty(\nu)})/\alpha_\nu^*}] > 0$ and $\nu \ll \mu$. Also, if $\alpha_\nu^* = 0$, the above holds trivially. ■

E.2 Proof of Lemma 31

Proof Let α^* and α_n^* Use Lemma 30,

$$\begin{aligned} &\left| \sup_{\alpha \geq 0} f(\mu, u, \alpha) - \sup_{\alpha \geq 0} f(\mu_n, u, \alpha) \right| \\ &= \left| \alpha_n^* \log \mu_n [e^{-u/\alpha_n^*}] + \alpha_n^* \delta - \alpha^* \log \mu [e^{-u/\alpha^*}] - \alpha^* \delta \right| \\ &= \inf_{\kappa \in \mathbb{R}} \left| \alpha_n^*(0) \log \mu_n [e^{-(u-\kappa)/\alpha_n^*(0)}] + \alpha_n^* \delta - \alpha^* \log \mu [e^{-(u-\kappa)/\alpha^*}] - \alpha^* \delta \right| \\ &\leq \inf_{\kappa \in \mathbb{R}} |f(\mu_n, u - \kappa, \alpha_n^*(0))| + |f(\mu, u - \kappa, \alpha^*)| \\ &\leq 2 \inf_{\kappa \in \mathbb{R}} \|u - \kappa\|_{L^\infty(\mu)} \\ &= 2 |u|_{\text{span}} \end{aligned}$$

■

E.3 Proof of Lemma 32

By definition, on $\Omega_{n,p}(\mu)$, $|\mu_n(y) - \mu(y)| \leq p$. So for all y s.t. $\mu(y) > 0$

$$0 < \mu_\wedge - p \leq \mu(y) - p \leq \mu_n(y).$$

Moreover, if $\mu_n(y) = 0$, then $0 \geq \mu(y) - p$, we must have that $\mu(y) = 0$. So, $\mu_n \gg \mu$ and hence $\mu_n \sim \mu$.

E.4 Proof of Lemma 35

Proof First we note that for any $\kappa \in \mathbb{R}$

$$\frac{m[e^{-u/\alpha}]}{\mu[e^{-u/\alpha}]} = \frac{m[e^{-(u-\kappa)/\alpha}]}{\mu[e^{-(u-\kappa)/\alpha}]}.$$

Therefore, it suffices to show that for $m = \mu_1 - \mu_2$ s.t. $\mu \gg \mu_1, \mu_2$

$$\sup_{\alpha \geq 0} \frac{\alpha^j m[w]^2}{\mu[w]^2} \leq 9 \|u\|_{L^\infty(\mu_{s,a})}^j \left\| \frac{dm}{d\mu} \right\|_{L^\infty(\mu)}^2.$$

Fix any $c > 0$, write

$$\begin{aligned} \sup_{\alpha \geq 0} \frac{\alpha^j m[w]^2}{\mu[w]^2} &= \max \left\{ \sup_{\alpha \in [0, c\|u\|_\infty]} \frac{\alpha^j m[w]^2}{\mu[w]^2}, \sup_{\alpha \geq c\|u\|_\infty} \frac{\alpha^j m[w]^2}{\mu[w]^2} \right\} \\ &=: \max \{J_1(c), J_2(c)\}. \end{aligned}$$

We first bound $J_2(c)$

$$J_2(c) = \sup_{\alpha \geq c\|u\|_{L^\infty(\mu)}} \frac{\alpha^j m[e^{-(u+\|u\|_{L^\infty(\mu)})/\alpha}]^2}{\mu_n[e^{-(u+\|u\|_{L^\infty(\mu)})/\alpha}]^2}$$

For simplicity, let $w' := e^{-(u+\|u\|_{L^\infty(\mu)})/\alpha}$. Recall that $m = \mu_n - \mu$, so $m[1] = 0$ and

$$\alpha^j m[e^{-(u+\|u\|_{L^\infty(\mu)})/\alpha}]^2 = (m[\alpha^{j/2}(e^{-(u+\|u\|_{L^\infty(\mu)})/\alpha} - 1)])^2.$$

Define and note that $v := \alpha^{j/2}(e^{-(u+\|u\|_{L^\infty(\mu)})/\alpha} - 1) < 0$. Then

$$\begin{aligned} \frac{\alpha m[w']^2}{\mu[w']^2} &= \frac{m[v]^2}{\mu[w']^2} \\ &= \frac{1}{\mu[w']^2} \mu \left[\frac{dm}{d\mu} v \right]^2 \\ &\leq \frac{\mu[-v]^2}{\mu[w']^2} \left\| \frac{dm}{d\mu} \right\|_{L^\infty(\mu)}^2 \\ &\leq \left\| \frac{v}{w'} \right\|_{L^\infty(\mu)}^2 \left\| \frac{dm}{d\mu} \right\|_{L^\infty(\mu)}^2 \end{aligned}$$

We defer the proof of the following claim:

Lemma 40 For any $j \in [0, 2]$

$$\sup_{\alpha \geq c\|u\|_{L^\infty(\mu)}} \left\| \frac{v}{w'} \right\|_{L^\infty(\mu)} \leq (c\|u\|_{L^\infty(\mu)})^{j/2} (e^{2/c} - 1).$$

Therefore,

$$J_2(c) \leq c^j \|u\|_{L^\infty(\mu)}^j (e^{2/c} - 1)^2 \left\| \frac{dm}{d\mu} \right\|_{L^\infty(\mu)}^2.$$

Choose $c = 2/\log 2$

$$\begin{aligned} \sup_{\alpha \geq 0} \frac{\alpha m[w]^2}{\mu[w]^2} &= \max \{J_1(c), J_2(c)\} \\ &\leq \max \left\{ c^j \|u\|_{L^\infty(\mu)}^j, c^j \|u\|_{L^\infty(\mu)}^j (e^{2/c} - 1)^2 \right\} \left\| \frac{dm}{d\mu} \right\|_{L^\infty(\mu)}^2 \\ &\leq 9 \|u\|_{L^\infty(\mu)}^j \left\| \frac{dm}{d\mu} \right\|_{L^\infty(\mu)}^2 \end{aligned}$$

which completes the proof. \blacksquare

E.4.1 PROOF OF LEMMA 40

Proof We bound

$$\begin{aligned} \left\| \frac{v}{w'} \right\|_{L^\infty(\mu)} &= \operatorname{ess\,sup}_{\mu} \alpha^{j/2} (e^{(u(s) + \|u\|_{L^\infty(\mu)})/\alpha} - 1) \\ &\leq \alpha^{j/2} (e^{2\|u\|_{L^\infty(\mu)}/\alpha} - 1) \end{aligned}$$

Compute derivative: let $\beta = 2\|u\|_{L^\infty(\mu)}/\alpha$

$$\frac{d}{d\alpha} \alpha^{j/2} (e^{2\|u\|_{L^\infty(\mu)}/\alpha} - 1) = \alpha^{j/2-1} ((e^\beta - 1)j/2 - \beta e^\beta)$$

Notice that when $\beta = 0$, the above expression is 0. Moreover, for $j \in [0, 2]$

$$\frac{d}{d\beta} ((e^\beta - 1)j/2 - \beta e^\beta) = (j/2 - 1 - \beta) e^\beta < 0;$$

i.e. $(e^\beta - 1)j/2 - \beta e^\beta$ is decreasing. Therefore, for $\alpha > 0$

$$\frac{d}{d\alpha} \alpha^{j/2} (e^{2\|u\|_{L^\infty(\mu)}/\alpha} - 1) < 0;$$

i.e. $\alpha^{j/2} (e^{2\|u\|_{L^\infty(\mu)}/\alpha} - 1)$ is decreasing in α . Hence

$$\begin{aligned} \sup_{\alpha \geq c\|u\|_{L^\infty(\mu)}} \left\| \frac{v}{w'} \right\|_{L^\infty(\mu)} &\leq \sup_{\alpha \geq c\|u\|_{L^\infty(\mu)}} \alpha^{j/2} (e^{2\|u\|_{L^\infty(\mu)}/\alpha} - 1) \\ &= (c\|u\|_{L^\infty(\mu)})^{j/2} (e^{2/c} - 1) \end{aligned}$$

establishing the claim. \blacksquare

E.5 Proof of Lemma 37

Proof Let $u' = u + \|u\|_{L^\infty(\mu)}$ and $w' = e^{-u'/\alpha}$.

$$\begin{aligned}
 & \sup_{\alpha \geq 0} \alpha^3 \left(\frac{m_n[w]}{\mu_n(t)[w]} + \frac{m_n[uw]}{\alpha \mu_n(t)[w]} - \frac{m_n[w]\mu_n(t)[uw]}{\alpha \mu_n(t)[w]^2} \right)^2 \\
 &= \sup_{\alpha \geq 0} \alpha^3 \left(\frac{m_n[w']}{\mu_n(t)[w']} + \frac{m_n[u'w']}{\alpha \mu_n(t)[w']} - \frac{m_n[w']\mu_n(t)[u'w']}{\alpha \mu_n(t)[w']^2} \right)^2 \\
 &\leq 2 \sup_{\alpha \geq 0} \alpha^3 \left(\frac{m_n[w']}{\mu_n(t)[w']} + \frac{m_n[u'w']}{\alpha \mu_n(t)[w']} \right)^2 + 2 \sup_{\alpha \geq 0} \alpha \frac{m_n[w']^2 \mu_n(t)[u'w']^2}{\mu_n(t)[w']^4} \\
 &=: 2OPT_1 + 2OPT_2
 \end{aligned}$$

We first analyze OPT_1 . Fix $c \geq 0$, we separately consider $\alpha \geq c\|u\|_{L^\infty(\mu)}$ and $\alpha \in [0, c\|u\|_{L^\infty(\mu)}]$. The first two terms

$$\begin{aligned}
 \sup_{\alpha \geq c\|u\|_{L^\infty(\mu)}} \alpha^3 \left(\frac{m_n[w']}{\mu_n(t)[w']} + \frac{m_n[u'w']}{\alpha \mu_n(t)[w']} \right)^2 &= \sup_{\alpha \geq c\|u\|_{L^\infty(\mu)}} \alpha^3 \frac{m_n[(1 + u'/\alpha)w']^2}{\mu_n(t)[w']^2} \\
 &= \sup_{\alpha \geq c\|u\|_{L^\infty(\mu)}} \frac{m_n[\alpha^{3/2}((1 + u'/\alpha)w' - 1)]^2}{\mu_n(t)[w']^2}.
 \end{aligned}$$

Recall that $1 + x \leq e^x$; i.e. $(1 + u'/\alpha)w' - 1 \leq 0$. Also, by Lemma 32, on $\Omega_{n,p}(\mu)$, $p < \mu_\wedge$, $\mu_n(t) \sim \mu_n \sim \mu$. So,

$$\begin{aligned}
 \frac{m_n[\alpha^{3/2}((1 + u'/\alpha)w' - 1)]^2}{\mu_n(t)[w']^2} &\leq \left\| \frac{\alpha^{3/2}(1 - (1 + u'/\alpha)w')}{w'} \right\|_{L^\infty(\mu)}^2 \left\| \frac{dm_n}{d\mu_n(t)} \right\|_{L^\infty(\mu)}^2 \\
 &\leq \left\| \alpha^{3/2}(e^{u'/\alpha} - (1 + u'/\alpha)) \right\|_{L^\infty(\mu)}^2 \left\| \frac{dm_n}{d\mu_n(t)} \right\|_{L^\infty(\mu)}^2
 \end{aligned}$$

Recall the Taylor series of e^x . For all $s \in \mathbf{S}$, we have that

$$\alpha^{3/2}(e^{u'(s)/\alpha} - (1 + u'(s)/\alpha)) = \sum_{k=2}^{\infty} \frac{u'(s)^k}{\alpha^{k-3/2} k!}.$$

Notice that $k - 3/2 > 0$ for $k \geq 2$ and the terms in the sum are non-negative. So, the above expression suggests that $\alpha \rightarrow \alpha^{3/2}(e^{u'(s)/\alpha} - (1 + u'(s)/\alpha))$ is decreasing. Therefore, on $\Omega_{n,p}(\mu)$

$$\sup_{\alpha \geq c\|u\|_{L^\infty(\mu)}} \alpha^3 \left(\frac{m_n[w']}{\mu_n(t)[w']} + \frac{m_n[u'w']}{\alpha \mu_n(t)[w']} \right)^2 \leq c^3 \|u\|_{L^\infty(\mu)}^3 (e^{2/c} - 1)^2 \left\| \frac{dm_n}{d\mu_n(t)} \right\|_{L^\infty(\mu)}^2.$$

Also,

$$\begin{aligned}
 & \sup_{\alpha \in [0, c\|u\|_{L^\infty(\mu)}]} \alpha^3 \left(\frac{m_n[w']}{\mu_n(t)[w']} + \frac{m_n[u'w']}{\alpha\mu_n(t)[w']} \right)^2 \\
 & \leq 2 \sup_{\alpha \in [0, c\|u\|_{L^\infty(\mu)}]} \left(\frac{\alpha^3 m_n[w']^2}{\mu_n(t)[w']^2} + \frac{\alpha m_n[u'w']^2}{\mu_n(t)[w']^2} \right) \\
 & \leq 2 \sup_{\alpha \in [0, c\|u\|_{L^\infty(\mu)}]} \left(\alpha^3 \left\| \frac{dm_n}{d\mu_n(t)} \right\|_{L^\infty(\mu)}^2 + \alpha \|u'\|_{L^\infty(\mu)}^2 \left\| \frac{dm_n}{d\mu_n(t)} \right\|_{L^\infty(\mu)}^2 \right) \\
 & \leq 2(c^3 + 4c) \|u\|_{L^\infty(\mu)}^3 \left\| \frac{dm_n}{d\mu_n(t)} \right\|_{L^\infty(\mu)}^2
 \end{aligned}$$

Choose $c = 2$, we conclude that

$$OPT_1 \leq 32 \|u\|_{L^\infty(\mu)}^3 \left\| \frac{dm_n}{d\mu_n(t)} \right\|_{L^\infty(\mu)}^2$$

For OPT_2 , we use Lemma 35.

$$\begin{aligned}
 OPT_2 & \leq 9 \|u\|_{L^\infty(\mu)} \left\| \frac{dm_n}{d\mu_n(t)} \right\|_{L^\infty(\mu)}^2 \frac{\mu_n(t)[u'w']^2}{\mu_n(t)[w']^2} \\
 & \leq 9 \|u\|_{L^\infty(\mu)} \left\| \frac{dm_n}{d\mu_n(t)} \right\|_{L^\infty(\mu)}^2 \|u'\|_{L^\infty(\mu)}^2 \\
 & \leq 36 \|u\|_{L^\infty(\mu)}^3 \left\| \frac{dm_n}{d\mu_n(t)} \right\|_{L^\infty(\mu)}^2
 \end{aligned}$$

Therefore, we conclude that on $\Omega_{n,p}(\mu)$

$$\sup_{\alpha \geq 0} \alpha^3 \left(\frac{m_n[w]}{\mu_n(t)[w]} + \frac{m_n[uw]}{\alpha\mu_n(t)[w]} - \frac{m_n[w]\mu_n(t)[uw]}{\alpha\mu_n(t)[w]^2} \right)^2 \leq 136 \|u\|_{L^\infty(\mu)}^3 \left\| \frac{dm_n}{d\mu_n(t)} \right\|_{L^\infty(\mu)}^2.$$

The lemma follows from considering $u - \kappa$, which won't change the left hand side. \blacksquare

E.6 Proof of Lemma 36

Proof From Si et al. (2020), it is sufficient to consider $\alpha \in [0, \delta^{-1}\|u\|_{L^\infty(\mu)}] =: K$. For $\alpha > 0$ fixed,

$$\partial_t g_n(t, \alpha) = -\alpha \frac{m_n[w]}{\mu_n(t)[w]}.$$

Also, for $\alpha = 0$, by Lemma 32 and $p \leq \frac{1}{4}\mu_\wedge$, $g_n(t, 0) \equiv \text{ess inf}_\mu u$; hence $\partial_t g_n(t, 0) \equiv 0$. Again by Lemma 32 $\mu_n(t) \sim \mu$ on $\Omega_{n,p}(\mu)$. So, the Radon-Nikodym theorem applies: For

fixed $t \in [0, 1]$,

$$\begin{aligned}
 \lim_{\alpha \downarrow 0} \sup_{s \in (t \pm \epsilon) \cap [0, 1]} |\partial_t g_n(t, \alpha)| &\leq \lim_{\alpha \downarrow 0} \sup_{t \in [0, 1]} \alpha \left| \frac{m_n[w]}{\mu_n(t)[w]} \right| \\
 &= \lim_{\alpha \downarrow 0} \sup_{t \in [0, 1]} \alpha \left| \frac{1}{\mu_n(t)[w]} \mu_n(t) \left[\frac{dm_n}{d\mu_n(t)} w \right] \right| \\
 &\leq \lim_{\alpha \downarrow 0} \sup_{t \in [0, 1]} \alpha \left\| \frac{dm_n}{d\mu_n(t)} \right\|_{L^\infty(\mu)} \\
 &\leq \lim_{\alpha \downarrow 0} \frac{\alpha}{\mu_\wedge - p} \\
 &= 0.
 \end{aligned} \tag{E.1}$$

where we used Hölder's inequality to get the second last line. Therefore, $\partial_t g(\cdot, \cdot)$ is continuous on $[0, 1] \times K$.

Next define

$$\Theta(t) := \arg \max_{\alpha \in K} g(t, \alpha).$$

To simplify notation, we use the w to denote $w = w_n^*(t) = e^{-u/\alpha_n^*(t)}$. We discuss two cases:

1. If u is μ -essentially constant with $\|u\|_{L^\infty(\mu)} = \bar{u}$, then

$$\sup_{\alpha \in K} -\alpha \log e^{-\bar{u}/\alpha} - \alpha \delta = \sup_{\alpha \in K} \bar{u} - \alpha \delta;$$

i.e. $\Theta(t) = \{0\}$.

2. u is not μ -essentially constant. Note that when $\alpha > 0$, $w > 0$; we can define a new measure

$$\mu_n^*(t)[\cdot] = \frac{\mu_n(t)[w \cdot]}{\mu_n(t)[w]}.$$

We have that

$$\begin{aligned}
 \partial_\alpha \partial_\alpha g_n(t, \alpha) &= -\frac{\mu_n(t)[u^2 w]}{\alpha^3 \mu_n(t)[w]} + \frac{\mu_n(t)[uw]^2}{\alpha^3 \mu_n(t)[w]^2} \\
 &= -\frac{\mu_n^*(t)[u^2]}{\alpha^3} + \frac{\mu_n^*(t)[u]^2}{\alpha^3} \\
 &= -\frac{\text{Var}_{\mu_n^*(t)}(u)}{\alpha^3} \\
 &< 0;
 \end{aligned}$$

i.e. $g_n(t, \cdot)$ is strictly concave for $\alpha > 0$. Also, recall that $g_n(t, \cdot)$ is continuous at 0. So, in this case either $\Theta(t) = \{0\}$ or $\Theta(t) = \{\alpha_n^*(t)\}$ where $\delta^{-1} \|u\|_{L^\infty(\mu)} \geq \alpha_n^*(t) > 0$.

In particular, $\Theta(t)$ is a singleton which we will denote by $\alpha_n^*(t)$ in both cases. We conclude that by Shapiro et al. (2014) Theorem 7.21, the following derivative exists

$$d_t \sup_{\alpha \in K} g_n(t, \alpha) = \sup_{\alpha \in \Theta(t)} \partial_t g_n(t, \alpha) = \partial_t g_n(t, \alpha_n^*(t)).$$

Next, we analyze the second derivative. We prove that under Assumption 1, we have that on $\Omega_{n,p}(\mu)$ $\alpha^* = 0$ or $\alpha^* > 0$ will imply that $\alpha_n^*(t) = 0$ or $\alpha_n^*(t) > 0$ respectively.

Let $\rho = \mu(\{y : u(y) = \text{ess inf}_\mu u\})$ and $\rho_n(t)$ the mixed version. Since $\mu_n \ll \mu$, if $\rho = 1$ (thence $\alpha^* = 0$), then we automatically have that $\rho_n(t) \equiv 1$ and $\alpha_n^*(t) \equiv 0$.

Now we consider the case $\rho \neq 1$. Notice that by definition of $\Omega_{n,p}(\mu)$, $\rho - p \leq \rho_n \leq \rho + p$. There are two cases:

1. $\alpha^* = 0$. From Hu and Hong (2013), $\alpha^* = 0$ iff $\rho \geq e^{-\delta}$. If we want $\alpha_n^*(t) = 0$ for all $t \in [0, 1]$, a sufficient condition is that $\rho_n(t) \geq \rho - p \geq e^{-\delta}$.
2. $\alpha^* > 0$ iff $\rho < e^{-\delta}$. If we want $\alpha_n^*(t) > 0$ for all $t \in [0, 1]$, a sufficient condition is that $\rho_n(t) \leq \rho + p < e^{-\delta}$.

Therefore, for any $e^{-\delta} \neq \rho \subset \{\mu(\{y : u(y) \leq t\}) : t \in \mathbb{R}\}$, we can always choose p small enough s.t. for $\omega \in \Omega_{n,p}(\mu)$, $\rho_n(t)$ is close to ρ for all t and the above sufficient conditions hold.

Remark 41 While this generalizes to all but finitely many δ , for simplicity of presentation, we assume Assumption 1 that $\mu_\wedge/2 \geq 1 - e^{-\delta}$.

So, if $\rho \neq 1$, then $1 - \rho \geq \mu_\wedge > 1 - e^{-\delta}$; i.e. $\rho < e^{-\delta}$ and case 1 cannot happen. Therefore, $\alpha^* = 0$ iff u is μ essentially constant. Moreover, by our choice $p \leq \frac{1}{4}\mu_\wedge$,

$$\rho + p \leq 1 - \frac{3}{4}\mu_\wedge < 1 - \frac{1}{2}\mu_\wedge \leq e^{-\delta}$$

satisfying the sufficient condition in case 2. Hence our assumption on p implies that if $\alpha^* = 0$ or $\alpha^* > 0$, then on $\omega \in \Omega_{n,p}(\mu)$, $\alpha_n^*(t) = 0$ or $\alpha_n^*(t) > 0$ for all $t \in [0, 1]$ respectively.

1. $\alpha^* = 0$, then $g_n(t, \alpha_n^*(t)) = g_n(t, 0)$ is constant. Hence $d_t d_t g_n(t, \alpha_n^*(t)) = 0$.
2. $\alpha^* > 0$, then $\alpha_n^*(t_1), \alpha_n^*(t_2) > 0$. Since $g_n(t, \cdot)$ is strictly convex, $\alpha_n^*(t)$ is the unique solution to the first order optimality condition

$$0 = \partial_\alpha g_n(t, \alpha_n^*(t)) = -\log \mu_n(t)[w] - \delta - \frac{\mu_n(t)[uw]}{\alpha_n^*(t)\mu_n(t)[w]}. \quad (\text{E.2})$$

Note that $\partial_\alpha g_n \in C^\infty([0, 1] \times \mathbb{R}_{++})$ and that $\partial_\alpha \partial_\alpha g_n(t, \alpha_n^*(t)) < 0$. The implicit function theorem implies that $\alpha_n^*(t) \in C^1((0, 1))$ with derivative

$$\begin{aligned} d_t \alpha_n^*(t) &= -\frac{\partial_t \partial_\alpha g_n(t, \alpha_n^*(t))}{\partial_\alpha \partial_\alpha g_n(t, \alpha_n^*(t))} \\ &= \left(\frac{\alpha_n^*(t)^3}{\text{Var}_{\mu_n^*(t)}(u)} \right) \left(-\frac{m_n[w]}{\mu_n(t)[w]} + \frac{\mu_n(t)[uw]m_n[w]}{\alpha_n^*(t)\mu_n(t)[w]^2} - \frac{m_n[uw]}{\alpha_n^*(t)\mu_n(t)[w]} \right) \end{aligned}$$

We conclude that

$$\partial_t g_n(t, \alpha_n^*(t)) = -\alpha_n^*(t) \frac{m_n[w]}{\mu_n(t)[w]}$$

is $C^1((0, 1))$ as a function of t . Therefore, $g_n(t, \alpha_n^*(t))$ is $C^2((0, 1))$ with derivative

$$\begin{aligned} & d_t d_t g_n(t, \alpha_n^*(t)) \\ &= d_t \partial_t g_n(t, \alpha_n^*(t)) \\ &= -\alpha_n^*(t) \frac{m_n[w]^2}{\mu_n(t)[w]^2} + d_t \alpha_n^*(t) \left(\frac{m_n[w]}{\mu_n(t)[w]} + \frac{m_n[uw]}{\alpha_n^*(t)\mu_n(t)[w]} - \frac{m_n[w]\mu_n(t)[uw]}{\alpha_n^*(t)\mu_n(t)[w]^2} \right) \\ &= -\alpha_n^*(t) \frac{m_n[w]^2}{\mu_n(t)[w]^2} - \left(\frac{\alpha_n^*(t)^3}{\text{Var}_{\mu_n^*(t)}(u)} \right) \left(\frac{m_n[w]}{\mu_n(t)[w]} + \frac{m_n[uw]}{\alpha_n^*(t)\mu_n(t)[w]} - \frac{m_n[w]\mu_n(t)[uw]}{\alpha_n^*(t)\mu_n(t)[w]^2} \right)^2. \end{aligned}$$

Therefore, Lemma 36 summarizes these two cases. \blacksquare

E.7 Proof of Lemma 38

Proof

First, we note that if $\mu_n^*(t)(y) \geq \mu_n(t)(y) > 0$, then by Lemma 32, $\mu_n^*(t)(y) \geq \mu_n(t)(y) \geq \frac{3}{4}\mu_\wedge$. So, we will only consider cases where $\mu_n^*(t)(y) < \mu_n(t)(y)$. We now fix any such y .

By Lemma 36, under the given assumptions $\alpha^* > 0$ implies that $\alpha_n^*(t) > 0$. So, the KL constraint is binding; i.e. $\delta = D_{\text{KL}}(\mu_n^*(t) \parallel \mu_n(t))$. By the log-sum inequality,

$$\delta = D_{\text{KL}}(\mu_n^*(t) \parallel \mu_n(t)) \geq \mu_n^*(t)(y) \log \left(\frac{\mu_n^*(t)(y)}{\mu_n(t)(y)} \right) + (1 - \mu_n^*(t))(y) \log \left(\frac{1 - \mu_n^*(t)(y)}{1 - \mu_n(t)(y)} \right)$$

Define

$$kl(q, b) = q \log \left(\frac{q}{b} \right) + (1 - q) \log \left(\frac{1 - q}{1 - b} \right).$$

where we think of $b = \mu_n(t)(y)$. Observe that for $q \in (0, b)$

$$\partial_q \partial_q kl(q, b) = \frac{1}{q} + \frac{1}{1 - q} > 0;$$

i.e. $kl(\cdot, b)$ is strictly convex and the maximum is achieved at $q = 0$, $kl(0, b) = \log(1/(1-b))$. Since $b \in [\frac{3}{4}\mu_\wedge, 1 - \frac{3}{4}\mu_\wedge]$, we have that $\log(1/(1-b)) \geq \log(1/(1 - \frac{3}{4}\mu_\wedge)) > \frac{3}{4}\mu_\wedge > \delta$. So, by the convexity, continuity of $kl(\cdot, b)$ and $kl(b, b) = 0$, there is unique $q^* \in (0, b)$ s.t. $kl(q^*, b) = \delta$. Now we bound such q^* .

Since $d_q kl(q, b) < 0$ for $q < b$, by the fundamental theorem of calculus and convexity

$$\begin{aligned} kl(q, b) &= - \int_q^b \partial_x kl(x, b) dx \\ &= \int_q^b \log \left(\frac{1 - x}{1 - b} \right) - \log \left(\frac{x}{b} \right) dx \\ &\geq \int_q^b \left(\frac{1}{b} + \frac{1}{1 - b} \right) (x - b) dx \\ &= \frac{(b - q)^2}{2b(1 - b)} \\ &=: \zeta(q, b) \end{aligned}$$

Note that for $q < b$

$$d_b\zeta(q, b) = \frac{(b - q)(q + b - 2qb)}{2(1 - b)^2 b^2} > 0$$

i.e. $\zeta(q, \cdot)$ is increasing. Suppose to the contrary $q^* < \frac{1}{2}\mu_\wedge$, then

$$\begin{aligned} kl(q^*, b) &\geq \zeta(q^*, b) \\ &\geq \inf_{b \in [\frac{3}{4}\mu_\wedge, 1 - \frac{3}{4}\mu_\wedge]} \zeta(q^*, b) \\ &= \zeta\left(q^*, \frac{3}{4}\mu_\wedge\right) \\ &> \frac{1}{24}\mu_\wedge. \end{aligned}$$

However, by assumption, $\frac{1}{24}\mu_\wedge \geq \delta \geq kl(q^*, \mu_n(t)(y)) > \mu_\wedge$. Hence $q^* \geq \frac{1}{2}\mu_\wedge$. We conclude that $\mu_n^*(t) \geq \frac{1}{2}\mu_\wedge$. \blacksquare

Appendix F. The Empirical Robust Bellman Operator: χ_2 Case

To analyze the variance-reduced Q-learning for the χ_2 case, we establish important statistical properties of the empirical DR Bellman operator $\widehat{\mathbf{T}}$ and its recentered version $\widehat{\mathbf{H}}$. We defer the proofs to Appendix H. The proof techniques are similar to that in Appendix C.

We let $\widehat{\mathbf{T}}$ be the empirical DR Bellman operator formed by n samples defined in (5.5). Define the recentered operators $\widehat{\mathbf{H}}, \mathcal{H}$ as in (A.1). We fix $\hat{q} \in \mathbb{R}^{\mathbf{S} \times \mathbf{A}}$.

Proposition 42 *Suppose Assumption 3 is enforced. Then*

$$|E[\widehat{\mathbf{H}}(\hat{q})(s, a) - \mathcal{H}(\hat{q})(s, a)]| \leq \frac{2^6 \|\hat{q} - q_*\|_\infty}{\mathfrak{p}_\wedge \sqrt{n}} \log(e|\mathbf{S}|),$$

provided $n \geq \mathfrak{p}_\wedge^{-2}$, and

$$\text{Var}(\widehat{\mathbf{H}}(q^*))(s, a) \leq \frac{2^{11} \|\hat{q} - q_*\|_\infty^2}{\mathfrak{p}_\wedge^2 n} \log(e|\mathbf{S}|)$$

for all $n \geq 1$.

Proposition 43 *Assume Assumption 3. Then w.p. at least $1 - \eta$*

$$\|\mathcal{H}(\hat{q}) - \widehat{\mathbf{H}}(\hat{q})\|_\infty \leq \frac{6 \|\hat{q} - q^*\|_\infty}{\mathfrak{p}_\wedge \sqrt{n}} \sqrt{\log(4|\mathbf{S}|^2|\mathbf{A}|/\eta)}$$

provided that $n \geq 8\mathfrak{p}_\wedge^{-2} \log(4|\mathbf{S}|^2|\mathbf{A}|/\eta)$

Proposition 44 *The empirical DR Bellman operator*

$$\|\widehat{\mathbf{T}}(q) - \mathcal{T}(q)\|_\infty \leq \frac{8(r_{\max} + \gamma \|q\|_\infty)}{\mathfrak{p}_\wedge \sqrt{n}} \sqrt{\log(6|\mathbf{S}||\mathbf{A}|(|\mathbf{S}| \vee |\mathbf{R}|)/\eta)}$$

w.p. at least $1 - \eta$, provided that $n \geq 8\mathfrak{p}_\wedge^{-2} \log(12|\mathbf{S}||\mathbf{A}|(|\mathbf{S}| \vee |\mathbf{R}|)/\eta)$.

Appendix G. Analysis of the Variance-Reduced Q-Learning: χ_2 Case

We proceed with the analysis of the variance-reduced DR Q-learning Algorithm 2 in the χ_2 divergence case, similar to the KL case. Specifically, we aim to show that if the q -function from the last variance-reduced algorithm epoch, \hat{q}_{l-1} , is within a certain error b of the optimal q^* , then \hat{q}_l will have a better concentration bound by a geometric factor. This is summarized in Proposition 45, which is analogous to Proposition 10 in the KL case.

Recall that \mathcal{F}_l denotes the σ -field generated by the random samples used until the end of epoch l . We define the conditional expectation $E_{l-1}[\cdot] = E[\cdot | \mathcal{F}_{l-1}]$.

Proposition 45 *Assuming that Assumptions 2 and 3 are satisfied. On $\{\omega : \|\hat{q}_{l-1} - q^*\|_\infty \leq b\}$ for some $b \leq 1/(1-\gamma)$, under measure $P_{l-1}(\cdot) := E_{l-1}[\mathbf{1}\{\cdot\}]$, we have that there exists numerical constant c s.t.*

$$\begin{aligned} \|\hat{q}_l - q^*\|_\infty &\leq c \left(\frac{b}{(1-\gamma)^2 k_{\text{vr}}} + \frac{b}{\mathfrak{p}_\wedge(1-\gamma)^{3/2} \sqrt{n_{\text{vr}} k_{\text{vr}}}} + \frac{b}{\mathfrak{p}_\wedge(1-\gamma) \sqrt{n_{\text{vr}}}} \right) \log(3dk_{\text{vr}}/\eta)^2 \\ &\quad + c \frac{1}{\mathfrak{p}_\wedge(1-\gamma)^2 \sqrt{m_l}} \sqrt{\log(3d/\eta)} \end{aligned}$$

w.p. at least $1 - \eta$, provided that $m_l \geq 8\mathfrak{p}_\wedge^{-2} \log(24d/\eta)$ and $n_{\text{vr}} \geq \mathfrak{p}_\wedge^{-1}$.

Proof [Proof of Proposition 45] We recall the proof of Proposition 10 in Appendix B.3.1. We have that by (B.12), under P_{l-1} , on $\{\omega : \|\hat{q}_{l-1} - q^*\|_\infty \leq b\}$

$$\|q_{l,k+1} - q^*\|_\infty \leq \lambda_k \left[2b + \gamma \sum_{j=1}^k \|Q_{l,j}\|_\infty \right] + \|Q_{l,k+1}\|_\infty + \frac{2\|D_l\|_\infty}{1-\gamma} \quad (\text{G.1})$$

w.p.1. The sequence $\{Q_{l,j} : j = 1, \dots, k+1\}$, by (B.13), satisfies

$$\begin{aligned} &\gamma \lambda_{k_{\text{vr}}} \sum_{j=1}^{k_{\text{vr}}} \|Q_{l,j}\|_\infty + \|Q_{l,k_{\text{vr}}+1}\|_\infty \\ &\leq 8 \left(\frac{\lambda_{k_{\text{vr}}} \log(e + (1-\gamma)k_{\text{vr}}) \|\zeta_{l-1}\|_\infty}{1-\gamma} + \frac{\|\sigma_{l-1}\|_\infty \sqrt{\lambda_{k_{\text{vr}}}}}{1-\gamma} \right) \log(4|\mathbf{S}||\mathbf{A}|k_{\text{vr}}/\eta) \end{aligned}$$

w.p. at least $1 - \eta$, where we recall that

$$\begin{aligned} \|\zeta_{l-1}\|_\infty &= \|\hat{q}_{l-1} - q^*\|, \\ \|\sigma_{l-1}^2\|_\infty &= \max_{(s,a) \in \mathbf{S} \times \mathbf{A}} \text{Var}_{l-1}(\mathbf{H}_{l,k}(\hat{q}_{l-1})(s, a)). \end{aligned}$$

Therefore, by Proposition 42, we have that

$$\begin{aligned} &\gamma \lambda_{k_{\text{vr}}} \sum_{j=1}^{k_{\text{vr}}} \|Q_{l,j}\|_\infty + \|Q_{l,k_{\text{vr}}+1}\|_\infty \\ &\leq c \left(\frac{b}{(1-\gamma)^2 k_{\text{vr}}} + \frac{b}{\mathfrak{p}_\wedge(1-\gamma)^{3/2} \sqrt{n_{\text{vr}} k_{\text{vr}}}} \right) \log(4|\mathbf{S}||\mathbf{A}|k_{\text{vr}}/\eta)^2 \end{aligned}$$

for some constant c .

Moreover, recall the definition of D_l in (B.14). By Propositions 42, 43, and 44, we have that

$$\begin{aligned}\|D_l\|_\infty &\leq c \frac{r_{\max} + |q^*|_{\text{span}} + \|\hat{q}_{l-1} - q^*\|_\infty}{\mathfrak{p}_\wedge \sqrt{m_l}} \sqrt{\log(12d/\eta)} + c \frac{\|\hat{q}_{l-1} - q^*\|_\infty}{\mathfrak{p}_\wedge \sqrt{n_{\text{vr}}}} \sqrt{\log(e|\mathbf{S}|)} \\ &\leq c \frac{1}{\mathfrak{p}_\wedge (1-\gamma) \sqrt{m_l}} \sqrt{\log(12d/\eta)} + c \frac{b}{\mathfrak{p}_\wedge \sqrt{n_{\text{vr}}}} \sqrt{\log(e|\mathbf{S}|)}\end{aligned}$$

for some constant c that can change from line to line.

Combining these bound with (G.1) and apply union bound, we conclude that

$$\begin{aligned}\|q_{l,k_{\text{vr}}+1} - q^*\|_\infty &\leq c \left(\frac{b}{(1-\gamma)^2 k_{\text{vr}}} + \frac{b}{\mathfrak{p}_\wedge (1-\gamma)^{3/2} \sqrt{n_{\text{vr}} k_{\text{vr}}}} + \frac{b}{\mathfrak{p}_\wedge (1-\gamma) \sqrt{n_{\text{vr}}}} \right) \log(8dk_{\text{vr}}/\eta)^2 \\ &\quad + c \frac{1}{\mathfrak{p}_\wedge (1-\gamma)^2 \sqrt{m_l}} \sqrt{\log(24d/\eta)}\end{aligned}$$

w.p. at least $1 - \eta$. Recall the definition in Algorithm 2 that $q_{l,k_{\text{vr}}+1} = \hat{q}_l$.

Finally, we adjust the constant in the log factor using the inequality for $C_1 \geq 1, C_2 \geq e$, $\log(C_1 C_2) = \log(C_1) + \log(C_2) \leq C_1 \log(C_2)$. This completes the proof. \blacksquare

Given Proposition 46, we apply the analysis techniques for the variance-reduction iterates in the proof of 46. This yields the following Proposition.

Proposition 46 *Assume Assumptions 2 and 3. For $\epsilon < (1-\gamma)^{-1}$, define parameters according to (5.6). Then, the statement of Proposition 11 hold; i.e. the sequence $\{\hat{q}_l, 0 \leq l \leq l_{\text{vr}}\}$ produced by Algorithm 2 satisfies the pathwise property that $\|\hat{q}_l - q^*\|_\infty \leq 2^{-l}(1-\gamma)^{-1}$ for all $0 \leq l \leq l_{\text{vr}}$ w.p. at least $1 - \eta$. In particular, the final estimator $\hat{q}_{l_{\text{vr}}}$ satisfies $\|\hat{q}_{l_{\text{vr}}} - q^*\|_\infty \leq 2^{-l_{\text{vr}}}(1-\gamma)^{-1}$ w.p. at least $1 - \eta$.*

Proof [Proof of Proposition 46] Follow the proof of Proposition 11, we only to validate (B.16) given the parameter choice in (5.6). By Proposition 45, conditioned on $\|\hat{q}_{l-1} - q^*\|_\infty \leq 2^{-(l-1)}(1-\gamma)^{-1} =: b$

$$\begin{aligned}\|\hat{q}_l - q^*\|_\infty &\leq c \left(\frac{b}{(1-\gamma)^2 k_{\text{vr}}} + \frac{b}{\mathfrak{p}_\wedge (1-\gamma)^{3/2} \sqrt{n_{\text{vr}} k_{\text{vr}}}} + \frac{b}{\mathfrak{p}_\wedge (1-\gamma) \sqrt{n_{\text{vr}}}} \right) \log(3dk_{\text{vr}}/\eta)^2 \\ &\quad + c \frac{1}{\mathfrak{p}_\wedge (1-\gamma)^2 \sqrt{m_l}} \sqrt{\log(3d/\eta)}\end{aligned}$$

w.p. at least $1 - \eta$.

Therefore, it is easy to see that by the parameter choice (5.6), we have that for sufficiently large c_{vr} and for events $\omega \in \{\|\hat{q}_{l-1} - q^*\|_\infty \leq 2^{-(l-1)}(1-\gamma)^{-1}\}$,

$$P_{l-1} \left(\mathbb{1} \left\{ \|\hat{q}_l - q^*\|_\infty \leq 2^{-l}(1-\gamma)^{-1} \right\} \right) (\omega) \geq 1 - \frac{\eta}{l_{\text{vr}}};$$

validating (B.16). Following the same arguments as in proof of Proposition 11 will yield Proposition 46. \blacksquare

Now, we prove Theorem 18.

Proof [Proof of Theorem 18] By Proposition 46, under the parameter choice (5.6), $\|\hat{q}_{l_{\text{vr}}} - q^*\|_\infty \leq \epsilon$ w.p. at least $1 - \eta$. The total number of samples used is

$$|\mathbf{S}||\mathbf{A}| \left(l_{\text{vr}} n_{\text{vr}} k_{\text{vr}} + \sum_{l=1}^{l_{\text{vr}}} m_l \right) = \tilde{O} \left(|\mathbf{S}||\mathbf{A}| \left(\frac{1}{\mathbf{p}_\lambda^2 (1-\gamma)^4} + \frac{4^{l_{\text{vr}}}}{\mathbf{p}_\lambda^2 (1-\gamma)^2} \right) \right).$$

This yields the sample complexity bound in Theorem 18. \blacksquare

Appendix H. Proofs of Properties of the Empirical Bellman Operator: χ_2 Case

We first define some notations that mimic the definitions in Appendix C. Again, we override the notations for the KL case. For generic probability measure μ on $(Y, 2^Y)$ and random variable $u : Y \rightarrow \mathbb{R}$, let $w = (\alpha - u)_+$; define the χ_2 dual functional under the reference measure μ as

$$f(\mu, u, \alpha) := \alpha - c(\delta) \mu[w^2]^{\frac{1}{2}}. \quad (\text{H.1})$$

Recall the dual formulation of the DR Bellman operator (5.4), we have that

$$\mathcal{T}(q)(s, a) = \sup_{\beta \in \mathbb{R}} f(\nu_{s,a}, id, \beta) + \gamma \sup_{\alpha \in \mathbb{R}} f(p_{s,a}, v(q), \alpha). \quad (\text{H.2})$$

Next, we present two important lemmas that underlie our analysis of the DR Bellman operator in the χ_2 case. First, we characterize the optimal Lagrange multiplier in the dual formulation (5.4).

Lemma 47 *For $\delta > 0$, $f(\mu, u, \alpha)$ is second continuously differentiable and concave for $\alpha > \text{ess inf}_\mu u$. The supremum is achieved at $\text{ess inf}_\mu u \leq \alpha^* < \infty$, i.e. $\sup_{\alpha \in \mathbb{R}} f(\mu, u, \alpha) = f(\mu, u, \alpha^*)$, satisfying*

$$\mu[w^2] = c(\delta)^2 \mu[w]^2. \quad (\text{H.3})$$

Moreover, if $\alpha > \text{ess inf}_\mu u$, then

$$\mu^*(\cdot) := \frac{\mu[w \mathbb{1}\{\cdot\}]}{\mu[w]}. \quad (\text{H.4})$$

is a worst-case measure satisfying

$$\mu^*[u] = f(\mu, u, \alpha^*) = \inf_{\mu' : D_{\chi_2}(\mu' \parallel \mu) \leq \delta} \mu'[u] = \inf_{\mu' : D_{\chi_2}(\mu' \parallel \mu) = \delta} \mu'[u];$$

i.e. the χ_2 constraint is active.

Finally, if $\alpha^* = \text{ess inf}_\mu u$, then the measure

$$\mu^*(\cdot) := \frac{\mu[\mathbb{1}\{U \cap \cdot\}]}{\mu(U)} \quad (\text{H.5})$$

where $U := \{s : \mu(s) > 0, u(s) = \text{ess inf}_\mu u\}$ is a worst-case measure.

With this lemma, we can show that under Assumption 3, the optimal Lagrange multiplier α^* is sufficiently large so that $w = (\alpha^* - v)_+ = \alpha^* - v$ a.s. μ .

Lemma 48 *If $\delta < \frac{1}{2}\mu_\wedge := \min_{s: \mu(s) > 0} \mu(s)$, then $\alpha^* \geq \text{ess sup}_\mu u$. Moreover, if u is not μ essentially constant, then $\alpha^* > \text{ess sup}_\mu u$.*

The proofs of these Lemmas are deferred to Appendix I.

H.1 Proof of Proposition 42

As in Appendix C.5, call $V := \mathcal{H}(\hat{q}) - \hat{\mathbf{H}}(\hat{q}) = (\mathcal{T}(\hat{q}) - \mathcal{T}(q_*)) - (\hat{\mathbf{T}}(\hat{q}) - \hat{\mathbf{T}}(q_*))$.

Recall the following notations in Appendix C.5: $v_t = tv(\hat{q}) + (1-t)v(q_*)$, $\mu = p$, $m = p - p_n$, and $\mu(t) = tp - (1-t)p_n$. Let

$$h(s, t) := \sup_{\alpha \in \mathbb{R}} f(\mu(t), v_s, \alpha).$$

We consider $\Omega_{n,p}(\mu)$ with $p \leq \frac{1}{4}\mu_\wedge$. Then, by Lemma 32, we have that $\mu \sim \mu_n \sim \mu(t)$ on $\Omega_{n,p}(\mu)$. Also, recall that $\alpha_{s,t}^*$ is the optimal Lagrange multiplier that satisfies the conclusions of Lemma 47.

First we note that if $v(\hat{q})$ and $v(q^*)$ are both μ essentially constant, then $V = 0$, and the claim of Proposition 42 holds trivially. Moving forward, we consider the case at least one of $v(\hat{q})$ and $v(q^*)$ is not μ essentially constant.

We proceed to show the differentiability of h in this setting. This is summarized by Lemma 49. The proof of this result is deferred to Appendix I.

Note that Assumption 3 implies that $\delta < \frac{1}{2}\mu_\wedge$.

Lemma 49 *Suppose $\delta < \frac{1}{2}\mu_\wedge$ and $p \leq \frac{1}{4}\mu_\wedge$. If at least one of $v(\hat{q})$ and $v(q^*)$ is not μ essentially constant, then on $\Omega_{n,p}(\mu)$ there exists function $s, t \rightarrow D^2h(s, t)$ s.t.*

$$|V(s, a)| \leq \gamma \int_0^1 \int_0^1 |D^2h(s, t)| ds dt \quad (\text{H.6})$$

w.p.1, where

$$\begin{aligned} D^2h(s, t) &= \left[\frac{\mu(t)[\Delta_v w_s]m[w_s^2]}{2\mu(t)[w_s]\mu(t)[w_s^2]} - \frac{m[\Delta_v w_s]}{\mu(t)[w_s]} \right] + \partial_s \alpha_{s,t}^* \left(\frac{m[w_s^2]}{2\mu(t)[w_s^2]} - \frac{m[w_s]}{\mu(t)[w_s]} \right) \\ &=: D_1 + D_2 \end{aligned} \quad (\text{H.7})$$

with

$$\partial_s \alpha_{s,t}^* = \frac{c(\delta)^2 \mu(t)[w_s] \mu(t)[\Delta_v] - \mu(t)[w_s \Delta_v]}{(c(\delta)^2 - 1)\mu(t)[w_s]} \quad (\text{H.8})$$

We analyze the two terms separately. Recall that $w_s \geq 0$. Similar to the techniques in Appendix C.5, we have that on $\Omega_{n,p}(\mu)$ with $\mu \sim \mu_n \sim \mu(t)$,

$$\left| \frac{\mu(t)[\Delta_v w_s]m[w_s^2]}{2\mu(t)[w_s]\mu(t)[w_s^2]} \right| \leq \left| \frac{\|\Delta_v\|_\infty m[w_s^2]}{2\mu(t)[w_s^2]} \right| \leq \frac{1}{2} \|\Delta_v\|_\infty \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)}$$

and

$$\frac{m[\Delta_v w_s]}{\mu(t)[w_s]} \leq \|\Delta_v\|_\infty \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)}.$$

Hence on $\Omega_{n,p}(\mu)$,

$$|D_1| \leq \frac{3}{2} \|\Delta_v\|_\infty \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)}.$$

For D_2 , we note that

$$\begin{aligned} |\partial_s \alpha_{s,t}^*| &= \left| \frac{c(\delta)^2 \mu(t)[w_s] \mu(t)[\Delta_v] - \mu(t)[w_s \Delta_v]}{(c(\delta)^2 - 1) \mu(t)[w_s]} \right| \\ &= \frac{1}{c(\delta)^2 - 1} \left| c(\delta)^2 \mu(t)[\Delta_v] - \frac{\mu(t)[w_s \Delta_v]}{\mu(t)[w_s]} \right| \\ &\leq \frac{c(\delta)^2 + 1}{c(\delta)^2 - 1} \|\Delta_v\| \end{aligned}$$

Next, we consider

$$\begin{aligned} \frac{m[w_s^2]}{2\mu(t)[w_s^2]} - \frac{m[w_s]}{\mu(t)[w_s]} &= \frac{m[(w_s - \mu(t)[w_s])^2] + 2m[w_s] \mu(t)[w_s]}{2\mu(t)[w_s^2]} - \frac{m[w_s]}{\mu(t)[w_s]} \\ &= \frac{m[(w_s - \mu(t)[w_s])^2]}{2c(\delta)^2 \mu(t)[w_s]^2} - \frac{(c(\delta)^2 - 1)m[w_s]}{c(\delta)^2 \mu(t)[w_s]} \end{aligned}$$

where we use the optimality condition (E.2) to replace $\mu(t)[w_s^2]$ with $c(\delta)^2 \mu(t)[w_s]^2$. Then,

$$\begin{aligned} |m[(w_s - \mu(t)[w_s])^2]| &= \left| \mu(t) \left[\frac{dm}{d\mu(t)} (w_s - \mu(t)[w_s])^2 \right] \right| \\ &\leq \mu(t) [(w_s - \mu(t)[w_s])^2] \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)} \\ &= (\mu(t)[w_s^2] - \mu(t)[w_s]^2) \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)} \\ &= (c(\delta)^2 - 1) \mu(t)[w_s]^2 \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)} \end{aligned}$$

where we also apply (E.2) and $\mu(t) \sim \mu$. So,

$$\begin{aligned} \left| \frac{m[w_s^2]}{2\mu(t)[w_s^2]} - \frac{m[w_s]}{\mu(t)[w_s]} \right| &\leq \left| \frac{m[(w_s - \mu(t)[w_s])^2]}{2c(\delta)^2 \mu(t)[w_s]^2} \right| + \left| \frac{(c(\delta)^2 - 1)m[w_s]}{c(\delta)^2 \mu(t)[w_s]} \right| \\ &\leq \frac{3}{2} \frac{c(\delta)^2 - 1}{c(\delta)^2} \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)}. \end{aligned}$$

Therefore, we have that

$$\begin{aligned}
 |D_2| &= |\partial_s \alpha_{s,t}^*| \left| \frac{m[w_s^2]}{2\mu(t)[w_s^2]} - \frac{m[w_s]}{\mu(t)[w_s]} \right| \\
 &\leq \frac{3(c^2 + 1)}{2c^2} \|\Delta_v\|_\infty \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)} \\
 &\leq 3 \|\Delta_v\|_\infty \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)}
 \end{aligned}$$

as $c(\delta)^2 = 1 + 2\delta \geq 1$.

So, on $\Omega_{n,p}(\mu)$,

$$|\partial_s \partial_t h(s, t)| \leq |D_1| + |D_2| \leq \frac{9}{2} \|\Delta_v\|_\infty \left\| \frac{dm}{d\mu} \right\|_{L^\infty(\mu)}.$$

Recall (C.16) and (C.17), we have that

$$\begin{aligned}
 E|V| &\leq \frac{2\gamma\|\hat{q} - q_*\|_\infty}{p^2 n} \log(e|\mathbf{S}|) + 5\gamma\|\Delta_v\|_\infty \sup_{s,t \in (0,1)} E \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)} \mathbb{1}_{\Omega_{n,p}(\mu)} \\
 &\leq \frac{2^5 \|\hat{q} - q_*\|_\infty}{\mu_\wedge^2 n} \log(e|\mathbf{S}|) + \frac{5\|\hat{q} - q_*\|_\infty}{p\sqrt{n}} \sqrt{\log(e|\mathbf{S}|)} \\
 &\leq \frac{2^5 \|\hat{q} - q_*\|_\infty}{\mu_\wedge^2 n} \log(e|\mathbf{S}|) + \frac{20\|\hat{q} - q_*\|_\infty}{\mu_\wedge \sqrt{n}} \sqrt{\log(e|\mathbf{S}|)} \\
 &\leq \frac{2^6 \|\hat{q} - q_*\|_\infty}{\mathfrak{p}_\wedge \sqrt{n}} \log(e|\mathbf{S}|)
 \end{aligned}$$

where we choose $p = \frac{1}{4}\mu_\wedge \leq \frac{1}{4}\mathfrak{p}_\wedge$ and the last inequality follows from the assumption that $n \geq \mathfrak{p}_\wedge^{-2}$.

To bound the variance, we use the same techniques as in (C.18) and conclude that for $n \geq 1$

$$\begin{aligned}
 \text{Var}(\hat{\mathbf{T}}(\hat{q}) - \hat{\mathbf{T}}(q^*)) &\leq 8\gamma^2 \|\hat{q} - q_*\|_\infty^2 P(\Omega_{n,p}(\mu)^c) + \gamma^2 E \int_0^1 \int_0^1 2(D_1^2 + D_2^2) ds dt \mathbb{1}_{\Omega_{n,p}(\mu)} \\
 &\leq \frac{2^7 \|\hat{q} - q_*\|_\infty^2}{\mu_\wedge^2 n} \log(e|\mathbf{S}|) + 24 \|\Delta_v\|_\infty^2 \sup_{s,t \in (0,1)} E \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)}^2 \mathbb{1}_{\Omega_{n,p}(\mu)} \\
 &\leq \frac{2^7 \|\hat{q} - q_*\|_\infty^2}{\mu_\wedge^2 n} \log(e|\mathbf{S}|) + \frac{2^{10} \|\hat{q} - q_*\|_\infty^2}{\mu_\wedge n} \log(e|\mathbf{S}|) \\
 &\leq \frac{2^{11} \|\hat{q} - q_*\|_\infty^2}{\mathfrak{p}_\wedge n} \log(e|\mathbf{S}|).
 \end{aligned}$$

This is the variance of $\mathbf{H}(\hat{q})$ as $\mathcal{H}(\hat{q})$ is deterministic.

H.2 Proof of Proposition 43

Proof Given Lemma 49, we directly apply the arguments in Appendix C.6.

We have that w.p.1,

$$|V(s, a)| \leq |V| \mathbb{1}_{\Omega_{n,p}(\mu)^c} + \gamma \sup_{s, t \in (0,1)} (|D_1| + |D_2|) \mathbb{1}_{\Omega_{n,p}(\mu)}$$

where $\mu = p_{s,a}$. Recall the choice $p \leq \frac{1}{4}\mu_{\wedge} = \frac{1}{4}p_{s,a,\wedge}$. By Hoeffding's inequality and the union bound

$$\begin{aligned} P(|V(s, a)| > t) &\leq P(\Omega_{n,p}(p_{s,a})^c) + P\left(\gamma \sup_{s, t \in (0,1)} (|D_1| + |D_2|) > t, \Omega_{n,p}(p_{s,a})\right) \\ &\leq P\left(\sup_{s' \in S} |p_{s,a,n}(s') - p_{s,a}(s')| > p\right) + P\left(\frac{5\gamma\|\hat{q} - q_*\|_{\infty}}{p_{s,a,\wedge} - p} \sup_{s \in S} |m(s)| > t\right) \\ &\leq \sum_{s \in S} \left(P(|m(s)| > p) + P\left(\frac{8\|\hat{q} - q_*\|_{\infty}}{p_{s,a,\wedge}} |m(s)| > t\right) \right) \\ &\leq 2|\mathbf{S}| \left(\exp(-2p^2n) + \exp\left(-\frac{p_{s,a,\wedge}^2 t^2 n}{32\|\hat{q} - q_*\|_{\infty}^2}\right) \right) \end{aligned}$$

Then, as $\mathfrak{p}_{\wedge} \leq p_{s,a,\wedge}$ for all $(s, a) \in \mathbf{S} \times \mathbf{A}$, by union bound

$$P(\|V\|_{\infty} > t) \leq 2|\mathbf{S}|^2|\mathbf{A}| \left(\exp\left(-\frac{\mathfrak{p}_{\wedge}^2 n}{8}\right) + \exp\left(-\frac{\mathfrak{p}_{\wedge}^2 t^2 n}{32\gamma^2\|\hat{q} - q_*\|_{\infty}^2}\right) \right).$$

We first control the first term to be less than $\eta/2$, which is implied by

$$n \geq \frac{8}{\mathfrak{p}_{\wedge}^2} \log(4|\mathbf{S}|^2|\mathbf{A}|/\eta).$$

Finally, the second term less than $\eta/2$ is implied by choosing

$$t^2 = \frac{32\gamma^2\|\hat{q} - q^*\|}{\mathfrak{p}_{\wedge}^2 n} \log(4|\mathbf{S}|^2|\mathbf{A}|/\eta).$$

This proves the claimed result. ■

H.3 Proof of Proposition 44

Proof We recall the bound (C.4). If $v(q)$ is essentially constant w.r.t. $p_{s,a}$, then $\widehat{\mathbf{T}}(q)(s, a) = \mathcal{T}(q)(s, a)$. Therefore, we then focus on the case that $v(q)$ is not essentially constant.

Again, we fix $p \leq \frac{1}{4}\mathfrak{p}_{\wedge} \leq \frac{1}{4}\mu_{\wedge}$ and thus on $\Omega_{n,p}(\mu)$, $\mu \sim \mu_n$ where $\mu = \nu_{s,a}$ or $p_{s,a}$. So, if u is not essentially constant, by Assumption 3 and Lemma 48, we have that

$$\sup_{\alpha \in \mathbb{R}} f(\mu_n, u, \alpha) = f(\mu_n, u, \alpha_n^*), \quad \sup_{\alpha \in \mathbb{R}} f(\mu, u, \alpha) = f(\mu, u, \alpha^*)$$

for some $\alpha_n^*, \alpha^* > \text{ess sup}_{\mu} u =: u_{\vee}$.

Then, as in (C.4) we analyze

$$\sup_{\alpha > u_{\vee}} |f(\mu_n, u, \alpha) - f(\mu, u, \alpha)|.$$

Since $\alpha > u_\vee$, $\mu[w^2] > 0$ and f is differentiable in μ on $\Omega_{n,p}(\mu)$. By the mean value theorem,

$$\begin{aligned} |f(\mu_n, u, \alpha) - f(\mu, u, \alpha)| &= c(\delta) \frac{1}{2} \left| \mu(\tau)[w^2]^{-\frac{1}{2}} m[w^2] \right| \\ &= \frac{1}{2} \left| \frac{m[(\alpha - u)^2]}{\mu(\tau)[\alpha - u]} \right| \end{aligned}$$

for some $\tau \in [0, 1]$ where we used (H.3) and $\mu(t) = t\mu + (1-t)\mu_n$ and $m = \mu - \mu_n$.

We first consider when $\alpha > 2\|u\|_\infty$,

$$\begin{aligned} &\sup_{\alpha > 2\|u\|_\infty} |f(\mu_n, u, \alpha) - f(\mu, u, \alpha)| \\ &\leq \sup_{\alpha > 2\|u\|_\infty} \frac{1}{2} \left| \frac{m[\alpha^2 - 2\alpha u + u^2]}{\alpha - \mu(\tau)[u]} \right| \\ &\leq \sup_{\alpha > 2\|u\|_\infty} \left| \frac{\alpha m[u]}{\alpha - \mu(\tau)[u]} \right| + \frac{1}{2} \left| \frac{m[u^2]}{\alpha - \mu(\tau)[u]} \right| \\ &\leq \sup_{\alpha > 2\|u\|_\infty} \left| \frac{(\alpha - \mu(\tau)[u])m[u]}{\alpha - \mu(\tau)[u]} \right| + \left| \frac{\mu(\tau)[u]m[u]}{\alpha - \mu(\tau)[u]} \right| + \frac{1}{2} \left| \frac{m[u^2]}{\alpha - \mu(\tau)[u]} \right| \\ &\stackrel{(i)}{\leq} |m[u]| + \left| \frac{\mu(\tau)[u]}{\|u\|_\infty} \right| |m[u]| + \frac{1}{2} \frac{m[u^2]}{\|u\|_\infty} \\ &\leq \|u\|_\infty \sup_{y \in Y} |m(y)| \end{aligned}$$

where (i) uses that $\alpha \geq 2\|u\|_\infty$ and hence $\alpha - \mu(\tau)[u] \geq \|u\|_\infty$.

On the other hand, if $u_\vee < \alpha \leq 2\|u\|_\infty$

$$\begin{aligned} \sup_{u_\vee < \alpha \leq 2\|u\|_\infty} |f(\mu_n, u, \alpha) - f(\mu, u, \alpha)| &\leq \sup_{u_\vee < \alpha \leq 2\|u\|_\infty} \frac{1}{2} \left| \frac{m[(\alpha - u)^2]}{\mu(\tau)[\alpha - u]} \right| \\ &\leq \sup_{u_\vee < \alpha \leq 2\|u\|_\infty} \frac{1}{2} \|\alpha - u\|_\infty \left| \frac{m[\alpha - u]}{\mu(\tau)[\alpha - u]} \right| \\ &\leq \frac{3}{2} \|u\|_\infty \left\| \frac{dm}{d\mu(\tau)} \right\|_{L^\infty(\mu)} \\ &\leq \frac{3}{2} \|u\|_\infty \frac{1}{\mu_\wedge - p} \sup_{y \in Y} |m(y)| \\ &\leq \frac{2\|u\|_\infty}{\mu_\wedge} \sup_{y \in Y} |m(y)| \end{aligned}$$

where the last two inequalities follow from Lemma 32 and $p \leq \frac{1}{4}\mu_\wedge$.

Therefore, we have

$$\begin{aligned}
 & P \left(\sup_{\alpha \in \mathbb{R}} |f(\mu_n, u, \alpha) - f(\mu, u, \alpha)| > t \right) \\
 & \leq P(\Omega_{n,p}(\mu)^c) + P \left(\sup_{\alpha > v_Y} |f(\mu_n, u, \alpha) - f(\mu, u, \alpha)| > t, \Omega_{n,p}(\mu) \right) \\
 & \leq P \left(\sup_y |\mu_n(y) - \mu(y)| > p \right) + P \left(\frac{2 \|u\|_\infty}{\mu_\wedge} \sup_{y \in Y} |m_n(y)| > t \right) \\
 & \leq 2 \sum_y \left(\exp(-2p^2 n) + \exp \left(-\frac{\mu_\wedge^2 t^2 n}{2 \|u\|_\infty^2} \right) \right) \\
 & \leq 2|Y| \left(\exp(-2p^2 n) + \exp \left(-\frac{\mu_\wedge^2 t^2 n}{2 \|u\|_\infty^2} \right) \right)
 \end{aligned}$$

where we used Hoeffding's inequality and union bound.

Therefore, going back to the DR Bellman operator setting, we choose $p = \frac{1}{4} \mathfrak{p}_\wedge$. By union bound

$$\begin{aligned}
 & P(\|\widehat{\mathbf{T}}(q) - \mathcal{T}(q)\|_\infty > t) \\
 & \leq P \left(\sup_{s,a} \sup_{\beta \in \mathbb{R}} |f(\nu_{s,a,n}, id, \beta) - f(\nu_{s,a}, id, \beta)| > \frac{t}{2} \right) \\
 & \quad + P \left(\sup_{s,a} \sup_{\alpha \in \mathbb{R}} |f(p_{s,a,n}, v(q), \beta) - f(p_{s,a}, v(q), \beta)| > \frac{t}{2} \right) \\
 & \leq 2(|\mathbf{S}|^2 |\mathbf{A}| + |\mathbf{S}| |\mathbf{A}| |\mathbf{R}|) \exp \left(-\frac{\mathfrak{p}_\wedge^2 n}{8} \right) + 2|\mathbf{S}| |\mathbf{A}| |\mathbf{R}| \exp \left(-\frac{\mathfrak{p}_\wedge^2 t^2 n}{64 r_{\max}^2} \right) \\
 & \quad + 2|\mathbf{S}|^2 |\mathbf{A}| \exp \left(-\frac{\mathfrak{p}_\wedge^2 t^2 n}{64 \gamma^2 \|q\|_\infty^2} \right).
 \end{aligned}$$

We set each of the three terms to be less than $\eta/3$ and find that it suffices to have

$$n \geq \frac{8}{\mathfrak{p}_\wedge^2} \log(12|\mathbf{S}| |\mathbf{A}| (|\mathbf{S}| \vee |\mathbf{R}|) / \eta)$$

and

$$t \geq \frac{8(r_{\max} + \gamma |q|_{\text{span}})}{\mathfrak{p}_\wedge \sqrt{n}} \sqrt{\log(6|\mathbf{S}| |\mathbf{A}| (|\mathbf{S}| \vee |\mathbf{R}|) / \eta)}.$$

This implies the statement of the proposition. ■

Appendix I. Proof of Technical Lemmas: χ_2 Case

I.1 Proof of Lemma 47

Proof First, we note that for every u and μ , f is continuous in α . Differentiate, we see that $f(\mu, u, \cdot)$ is C^1 with derivative

$$\partial_\alpha f(\mu, u, \alpha) = 1 - c(\delta)\mu[w^2]^{-\frac{1}{2}}\mu[w] \quad (\text{I.1})$$

which is again continuous. Differentiate again, we get that

$$\begin{aligned} \partial_\alpha \partial_\alpha f(\mu, u, \alpha) &= c(\delta) \left(\mu[w^2]^{-\frac{3}{2}}\mu[w]^2 - \mu[w^2]^{-\frac{1}{2}}\mu[1\{\alpha > v\}] \right) \\ &= c(\delta)\mu[w^2]^{-\frac{3}{2}} \left(\mu[w1\{\alpha > v\}]^2 - \mu[w^2]\mu[1\{\alpha > v\}]^2 \right) \\ &\stackrel{(i)}{\leq} 0 \end{aligned} \quad (\text{I.2})$$

when $\alpha > \text{ess inf}_\mu u$, where (i) follows from Jensen's inequality. Moreover, this expression is continuous for $\alpha >$. Therefore, f is second differentiable and convex in α when $\alpha > \text{ess inf}_\mu u$.

As we commented after Lemma 17, it suffices to optimize over $\alpha \geq \text{ess inf}_\mu u$. By the continuity of f and $\partial_\alpha f$ in α and convexity, if the optimizer $\text{ess inf}_\mu u < \alpha^* < \infty$, it must satisfies

$$0 = \partial_\alpha f(\mu, u, \alpha^*) = 1 - c(\delta)\mu[w^2]^{-\frac{1}{2}}\mu[w];$$

which is (H.3).

Next, we handle the boundary cases $\alpha^* = \infty$ and $\alpha^* = \text{ess inf}_\mu u$. Notice that rewriting (H.3) as

$$\mu \left[\left(\frac{w}{\mu[w]} \right)^2 \right] = c(\delta)^2$$

we see that for $\delta > 0$, $\alpha^* \neq \infty$, because otherwise $\frac{w}{\mu[w]} = 1$ a.s. μ . and the above equality cannot hold.

On the other hand, if $\alpha^* = \text{ess inf}_\mu u$, then (H.3) holds trivially with $w = 0$.

Then, we show that (H.4) is a worst-case measure. It suffices to check that $\mu^*[u] = f(\mu, u, \alpha^*)$ and $D_{\chi_2}(\mu^* \parallel \mu) = \delta$. We have that

$$\begin{aligned} \mu^*[u] &= \frac{\mu[wu]}{\mu[w]} \\ &= \alpha^* - \frac{\mu[(\alpha^* - u)1\{\alpha > u\}(\alpha^* - u)]}{\mu[w]} \\ &= \alpha^* - \frac{\mu[w^2]}{\mu[w]} \\ &\stackrel{(i)}{=} \alpha^* - c(\delta)^2\mu[w] \\ &\stackrel{(ii)}{=} \alpha^* - c(\delta)\mu[w^2]^{\frac{1}{2}} \\ &= f(\mu, u, \alpha^*) \end{aligned}$$

where (i) and (ii) follows from (H.3). Moreover, by definition (5.1),

$$\begin{aligned}
 D_{\chi_2}(\mu^* \|\mu) &= \frac{1}{2}\mu \left[\left(\frac{d\mu^*}{d\mu} - 1 \right)^2 \right] \\
 &= \frac{1}{2}\mu \left[\left(\frac{w}{\mu[w]} - 1 \right)^2 \right] \\
 &= \frac{1}{2} \left(\frac{\mu[w^2]}{\mu[w]^2} + 1 - 2 \right) \\
 &\stackrel{(i)}{=} \frac{1}{2} (c(\delta)^2 - 1) \\
 &= \delta
 \end{aligned}$$

again (i) follows from (H.3).

Finally, clearly μ^* defined in (H.5) satisfies $\mu^*[u] = \text{ess inf}_\mu u = f(\mu, u, \alpha^*)$. So, to show that μ^* is a worst-case measure, it suffices to check that $D_{\chi_2}(\mu^* \|\mu) \leq \delta$.

To show this, we observe that if $\alpha^* = \text{ess inf}_\mu u$, then by convexity we must have that for all sufficiently small $\epsilon > 0$, $\partial_\alpha f(\mu, u, \alpha^* + \epsilon) \leq 0$. Otherwise, $\alpha^* = \text{ess inf}_\mu u$ cannot be optimal. In particular, let $w(\epsilon) = (\alpha^* + \epsilon - u)_+$, then by (I.1), we have that

$$\mu[w(\epsilon)^2] \leq c(\delta)^2 \mu[w(\epsilon)]^2.$$

Note that if ϵ is sufficiently small, i.e. when $\alpha^* + \epsilon < u(s)$ for all $s \notin U$ and $\mu(s) > 0$, then $w(\epsilon) = \epsilon \mathbf{1}_U$. Therefore, we must have that

$$\mu[\mathbf{1}_U] \leq c(\delta)^2 \mu[\mathbf{1}_U]^2;$$

i.e. $\mu(U)^{-1} \leq c(\delta)^2$. With this bound, we now compute

$$\begin{aligned}
 D_{\chi_2}(\mu^* \|\mu) &= \frac{1}{2}\mu \left[\left(\frac{\mathbf{1}_U}{\mu(U)} - 1 \right)^2 \right] \\
 &= \frac{1}{2} \left(\frac{1}{\mu(U)} - 1 \right) \\
 &\leq \frac{1}{2} (c(\delta)^2 - 1) \\
 &= \delta.
 \end{aligned}$$

Therefore, this proves Lemma 47. ■

I.2 Proof of Lemma 48

Proof

If u is μ essentially constant, then $\text{ess inf}_\mu u = \text{ess sup}_\mu u = \alpha^*$; i.e. the statement of Lemma 48 holds.

Next, we prove that if u is not μ essentially constant, then $\delta < \frac{1}{2}\mu_\wedge$ implies $\alpha^* \geq \text{ess sup}_\mu u$. To achieve this, we first show that $\alpha^* > \text{ess inf}_\mu u$ under these assumptions.

We prove this by assuming $\alpha^* = \text{ess inf}_\mu u$ and raising a contradiction. By Lemma 47, μ^* defined in (H.5) is a worst-case measure. Hence,

$$\begin{aligned}\delta &\geq D_{\chi_2}(\mu^* \parallel \mu) \\ &= \frac{1}{2}\mu \left[\left(\frac{\mathbf{1}_U}{\mu(U)} - 1 \right)^2 \right] \\ &= \frac{1}{2} \left(\frac{1}{\mu(U)} - 1 \right) \\ &\stackrel{(i)}{\geq} \frac{1}{2} \frac{\mu_\wedge}{1 - \mu_\wedge}\end{aligned}$$

where (i) follows from the assumption that u is not μ essentially constant, so

$$U = \left\{ s : \mu(s) > 0, u(s) = \text{ess inf}_\mu u \right\}$$

cannot be of probability 1. In particular, by the definition of μ_\wedge , $\mu(U) \leq 1 - \mu_\wedge$. Therefore, rearrange terms, we have that

$$\frac{\delta}{\mu_\wedge} \geq \frac{1}{2} \frac{1}{1 - \mu_\wedge} \geq \frac{1}{2};$$

i.e. $\delta \geq \frac{1}{2}\mu_\wedge$, contradicting our assumption. Therefore, $\alpha^* > \text{ess inf}_\mu u$.

Using this, we then show that if u is not μ essentially constant, $\delta < \frac{1}{2}\mu_\wedge$, and $\alpha^* > \text{ess inf}_\mu u$, then $\alpha^* \geq \text{ess sup}_\mu u$.

We prove by contradiction, assuming that $\text{ess inf}_\mu u < \alpha^* \leq \text{ess sup}_\mu u$. Since $\alpha^* \leq \text{ess sup}_\mu u$, we must have that for some $s' \in \mathbf{S}$ s.t. $\mu(s') > 0$, $w(s') = (\alpha^* - u(s'))_+ = 0$. By Lemma 47, μ^* defined in (H.4) is a worst-case measure when $\alpha^* > \text{ess inf}_\mu u$. Moreover,

$$\begin{aligned}\delta &= D_{\chi_2}(\mu^* \parallel \mu) \\ &= \frac{1}{2}\mu \left[\left(\frac{w}{\mu[w]} - 1 \right)^2 \right] \\ &\geq \frac{1}{2}\mu(s')\end{aligned}$$

contradicting the assumption. Therefore, $\alpha^* > \text{ess sup}_\mu u$. This completes the proof of Lemma 48. \blacksquare

I.3 Proof of Lemma 49

Proof

By assumption, we are interested in empirical measures that satisfy $\Omega_{n,p}(\mu)$ (c.f. (C.3)) with $p \leq \frac{1}{4}\mu_\wedge$. Then, by Lemma 32, we have that $\mu \sim \mu_n \sim \mu(t)$ on $\Omega_{n,p}(\mu)$.

We first fix $s \in [0, 1]$. Let us denote $v_{s,\vee} := \text{ess sup}_{\mu} v_s$. Recall that by Lemma 48, when $\delta < \frac{1}{2}\mu_{\wedge}$, it suffices to optimize the Lagrange multiplier in $[v_{s,\vee}, \infty)$. We have

$$\partial_t f(\mu(t), v_s, \alpha) = -\frac{1}{2}c(\delta)m[w_s^2]\mu(t)[w_s^2]^{-\frac{1}{2}}$$

where $w_s = (\alpha - v_s)_+ = \alpha - v$. It is not hard to see that $\partial_t f(\mu(t), v_s, \alpha)$ is continuous on $[0, 1] \times [v_{s,\vee}, \infty)$ even if v_s is essentially constant (in this case we note that $\partial_t f(\mu(t), v_s, v_{s,\vee}) = 0$).

Next define

$$\Theta(t) := \arg \max_{\alpha > v_{s,\vee}} f(\mu(t), v_s, \alpha).$$

We discuss two cases:

1. If v_s is μ essentially constant, then for $\alpha > v_{s,\vee}$

$$f(\mu(t), v_s, \alpha) = \alpha - c(\delta)(\alpha - v_{s,\vee}) = (1 - c(\delta))\alpha + c(\delta)v_{s,\vee}.$$

Since $c(\delta) = 1 + 2\delta > 0$, this is maximized at $\Theta(t) = \{v_{s,\vee}\}$.

2. v_s is not μ essentially constant. Note that then by Lemma 48, $\alpha > \text{ess sup}_{\mu} v_s$, $\alpha > v_s$ a.s. μ (hence $\mu(t)$). Recall that the second derivative in (I.2),

$$\begin{aligned} & \partial_{\alpha} \partial_{\alpha} f(\mu(t), v_s, \alpha) \\ &= c(\delta)\mu(t)[w_s^2]^{-\frac{3}{2}} \left(\mu(t)[w_s \mathbb{1}\{\alpha > v_s\}]^2 - \mu(t)[w_s^2]\mu(t)[\mathbb{1}\{\alpha > v_s\}]^2 \right) \\ &= c(\delta)\mu(t)[w_s^2]^{-\frac{3}{2}} (\mu(t)[w_s]^2 - \mu(t)[w_s^2]) \\ &< 0 \end{aligned} \tag{I.3}$$

where the last inequality follows from that w_s is not $\mu(t)$ constant, hence the variance is positive. So, in this case $f(\mu(t), v_s, \cdot)$ is strictly concave. Thus, $\Theta(t)$ is a singleton.

Therefore, in both case, $\Theta(t)$ is a singleton. We conclude that by Shapiro et al. (2014, Theorem 7.21), the following derivative exists

$$\begin{aligned} d_t \sup_{\alpha > v_{s,\vee}} f(\mu(t), v_s, \alpha) &= \sup_{\alpha \in \Theta(t)} \partial_t f(\mu(t), v_s, \alpha) \\ &= \partial_t f(\mu(t), v_s, \alpha_{s,t}^*) \\ &= -\frac{1}{2}c(\delta)m[w_s^2]\mu(t)[w_s^2]^{-\frac{1}{2}}. \end{aligned} \tag{I.4}$$

where it is understood that $w_s = (\alpha_{s,t}^* - v_s)_+ = \alpha_{s,t}^* - v_s$. Therefore, we have shown that $t \rightarrow h(s, t)$ is $C^1(0, 1) \cap C[0, 1]$. Hence,

$$\begin{aligned} |V(s, a)| &= \gamma |h(1, 0) - h(0, 0) - h(1, 1) + h(0, 1)| \\ &= \gamma \left| \int_0^1 \partial_t h(1, t) - \partial_t h(0, t) dt \right| \end{aligned}$$

Next, we show that for any fixed t , there exists a mapping $s \rightarrow D_s \partial_t h(s, t)$ s.t. (C.12) holds.

We note that by Lemma 48, $\alpha_{s,t}^* = v_{s,\vee}$ only when v_s is essentially constant. Again, assuming that at least one of $v(\hat{q})$ and $v(q^*)$ is not μ essentially constant, as in the proof of Proposition 24, this can only happen at one particular $s = s^*$.

We separately consider these two cases:

Case 1: v_s is never essentially constant for all $s \in [0, 1]$.

In this case, $\alpha_{s,t}^* > v_{s,\vee}$ for all $s \in [0, 1]$. Note that $w_s = \alpha_{s,t}^* - v_s > 0$. So, if on $\Omega_{n,p}(\mu)$, $\alpha_{s,t}^*$ is $C^1(0, 1) \cap C[0, 1]$ in s , then by chain rule, $s \rightarrow \partial_t h(s, t)$ in (I.4) is $C^1(0, 1) \cap C[0, 1]$.

As in the proof of Proposition 24, we show differentiability of $s \rightarrow \alpha_{s,t}^*$ by invoking the implicit function theorem. By the strict convexity (I.3), $\alpha_{s,t}^*$ is the unique solution to the optimality condition (H.3)

$$0 = c(\delta)^2 \mu(t) [w_s]^2 - \mu(t) [w_s^2] =: F(s, \alpha_{s,t}^*).$$

Since F is infinite smooth, the implicit function theorem implies that $\alpha_{s,t}^*$ is $C^1(0, 1) \cap C[0, 1]$ and $s \rightarrow \partial_t h(s, t)$ is $C^1(0, 1) \cap C[0, 1]$.

We compute the derivative $\partial_s \partial_t h$ in this case. Recall $\Delta_v = v(\hat{q}) - v(q^*)$. Differentiate w.r.t. s on both side, we have

$$0 = c(\delta)^2 2\mu(t) [w_s] \mu(t) [\partial_s \alpha_{s,t}^* - \Delta_v] - 2\mu(t) [w_s (\partial_s \alpha_{s,t}^* - \Delta_v)].$$

Rearranging terms, we have

$$\partial_s \alpha_{s,t}^* = \frac{c(\delta)^2 \mu(t) [w] \mu(t) [\Delta_v] - \mu(t) [w_s \Delta_v]}{(c(\delta)^2 - 1) \mu(t) [w_s]}$$

This gives (H.8). Moreover, when $\alpha_{s,t}^* > 0$,

$$\begin{aligned} \partial_s \partial_t h(s, t) &= \frac{1}{2} c(\delta) \mu(t) [w_s^2]^{-\frac{3}{2}} \mu(t) [\Delta_v w_s^2] m[w_s^2] - c(\delta) \mu(t) [w_s^2]^{-\frac{1}{2}} m[\Delta_v w_s] \\ &\quad + \partial_s \alpha_{s,t}^* \left(-c(\delta) \mu(t) [w_s^2]^{-\frac{1}{2}} m[w_s] + \frac{1}{2} c(\delta) \mu(t) [w_s^2]^{-\frac{3}{2}} \mu(t) [w_s] m[w_s^2] \right) \\ &= c(\delta) \mu(t) [w_s^2]^{-\frac{1}{2}} \mu(t) [w_s] \left(\frac{\mu(t) [\Delta_v w_s] m[w_s^2]}{2\mu(t) [w_s] \mu(t) [w_s^2]} - \frac{m[\Delta_v w_s]}{\mu(t) [w_s]} \right) \\ &\quad + c(\delta) \mu(t) [w_s^2]^{-\frac{1}{2}} \mu(t) [w_s] \partial_s \alpha_{s,t}^* \left(\frac{m[w_s^2]}{2\mu(t) [w_s^2]} - \frac{m[w_s]}{\mu(t) [w_s]} \right) \\ &\stackrel{(i)}{=} \left[\frac{\mu(t) [\Delta_v w_s] m[w_s^2]}{2\mu(t) [w_s] \mu(t) [w_s^2]} - \frac{m[\Delta_v w_s]}{\mu(t) [w_s]} \right] + \partial_s \alpha_{s,t}^* \left(\frac{m[w_s^2]}{2\mu(t) [w_s^2]} - \frac{m[w_s]}{\mu(t) [w_s]} \right) \end{aligned}$$

where (i) uses the optimality equation (H.3). This is consistent with (H.7).

Case 2: There is a unique $s^* \in [0, 1]$ s.t. v_s is essentially constant.

As in the proof of Proposition 24, in this case, the previous argument implies that $s \rightarrow \partial_t h(s, t)$ is $C^1(0, s^*)$, $C^1(s^*, 1)$, and continuous at 0, 1. The derivative is also given by (H.7).

Again, we show the existence of $D_s \partial_t h$ that satisfy (C.12). Observe that if $s \rightarrow \partial_t h(s, t)$ is continuous at s^* , then applying the fundamental theorem of calculus on the interval $[0, s^*]$ and $[s^*, 1]$ separately, we will have that

$$\partial_t h(1, t) - \partial_t h(0, t) = \int_0^{s^*} \partial_s \partial_t h(s, t) ds + \int_{s^*}^1 \partial_s \partial_t h(s, t) ds.$$

Hence, taking $D_s \partial_t h(s, t) = \partial_s \partial_t h(s, t)$ for every $s \neq s^*$ and $D_s \partial_t h(s^*, t) = 0$ will suffice to produce (C.12).

It is left to check the continuity at s^* of

$$\partial_t h(s, t) = \partial_t f(\mu(t), v_s, \alpha_{s,t}^*) = -\frac{1}{2} c(\delta) m [w_s^2] \mu(t) [w_s^2]^{-\frac{1}{2}}$$

from (I.4). Note that on $\Omega_{n,p}(\mu)$, for all $s \in [0, 1]$, $\alpha \geq v_{s,\vee}$,

$$\begin{aligned} \left| -\frac{1}{2} c(\delta) m [\alpha - v_s] \mu(t) [(\alpha - v_s)^2]^{-\frac{1}{2}} \right| &\leq \left| \frac{1}{2} \mu(t) [(\alpha - v_s)^2]^{\frac{1}{2}} \right| \left\| \frac{dm}{d\mu(t)} \right\|_{L^\infty(\mu)} \\ &\stackrel{(i)}{\leq} \left| \frac{1}{2} \|\alpha - v_s\|_{L^\infty(\mu)} \frac{1}{\mu_\wedge - p} \right| \\ &\leq \left| \frac{1}{2} \|\alpha - v_s\|_{L^\infty(\mu)} \frac{1}{\frac{3}{4} \mu_\wedge} \right| \end{aligned}$$

where (i) follows from Lemma 32. Also, $\partial_t h(s^*, t) = 0$. Therefore, if $\partial_s \alpha_{s,t}^* \rightarrow v_{s^*,\vee}$ as $s \rightarrow s^*$, then $\|\alpha - v_s\|_{L^\infty(\mu)} \rightarrow 0$ as $s \rightarrow s^*$, implying continuity at s^* .

It is left to check that $\partial_s \alpha_{s,t}^* \rightarrow v_{s^*,\vee}$ as $s \rightarrow s^*$. To prove this, we assume to the contrary that there is a subsequential limit $\alpha_{s_n,t}^* \rightarrow \beta + v_{s^*,\vee}$ for some sequence $s_n \rightarrow s^*$ and $\beta > 0$. But by Lemma 47, we must have that

$$0 = \lim_{n \rightarrow \infty} c(\delta)^2 \mu(t) [\alpha_{s_n,t}^* - v_{s_n}]^2 - \mu(t) [(\alpha_{s_n,t}^* - v_{s_n})^2] = \delta \beta$$

raising a contradiction. This implies that $s \rightarrow \partial_t h(s, t)$ is continuous at s^* , and hence (C.12) holds with $D_s \partial_t h(s, t) = \partial_s \partial_t h(s, t)$ for every $s \neq s^*$ and $D_s \partial_t h(s^*, t) = 0$.

Therefore, in both cases (H.6) holds with $|D^2 h(s, t)|$ is given by (H.7). This gives the claim of the lemma. \blacksquare