Supporting Learning Through Interpreting Others' Solutions From a Radical Constructivist Perspective: A Theoretical Report

Teo Paoletti University of Delaware	Claudine Margolis University of Michigan	Allison L. Gantt University of Delaware
Hamilton L. Hardison Texas State University	Hwa Young Lee Texas State University	Allison Olshefke-Clark University of Delaware

Although much research has made the case for the value of students' making sense of others' solutions, explanatory mechanisms for how such learning occurs are lacking. In this theoretical report, we consider how students' making sense of others' mathematical solutions may support learning from a radical constructivist perspective. We elaborate on radical constructivist constructs–social goals, cognitive perturbations, and reflective abstraction–and use these constructs to model how engagement with others' mathematical solutions may engender learning. We illustrate our model with a task we designed to promote students' meanings for spatial coordinate systems. We conclude with implications for research and teaching.

Keywords: Learning Theory; Cognition; Problem-based Learning

Students' making sense of classmates' solutions to mathematical problems (e.g., Webb et al., 2014) and with worked examples (e.g., Barbieri et al., 2023) has been positively associated with mathematics achievement. Although researchers have suggested reasons why such student activity may translate to achievement gains (Brown et al., 1992; Webb et al., 2023), they have not provided explanatory mechanisms for how such learning occurs for an individual. As a theory of learning focused on ways individuals develop knowledge, radical constructivism can provide such explanations. Broadly, radical constructivism explains how individuals build knowledge through persistent cognitive adaptation in concert with their lived experiences, and it defines learning in terms of shifts in an individual's mental structures (von Glasersfeld, 1995). In this theoretical report, we consider how students' working to make sense of others' mathematical solutions may support learning from a radical constructivist perspective (Table 1 provides colloquial descriptions of key terms).

Table 1. Colloquial descriptions of terms to orient the reader

Term	Brief colloquial description of constructs to orient the reader
Perturbation	Anything that creates a disturbance in a student's equilibrated (or settled) state. A perturbation is neutralized when the disturbance no longer exists.
Experiential Provocation	A perturbation that a student conceives as external in origin (i.e., from their interaction with their environment). Examples include other people, tasks, etc.
Social Goal	A goal a student establishes and assumes is shared with others.
Scheme	Goal-directed regularities in a student's functioning consisting of: (1) a perceived situation, (2) activity triggered by the situation, and (3) an anticipated result.
Cognitive Perturbation	A perturbation in which a student experiences discrepancies in their use of a scheme.
Affective experience	Experience in which a person's feelings, attitudes, or moods are activated.
Reflective Abstraction	A mechanism to explain an individual's modification of their schemes toward greater cohesion and generality.

Learning in Radical Constructivism and Connections to Social Interactions

In this section, we present and coordinate constructs to yield explanatory mechanisms through which a student may learn from others' solutions. First, we conceptualize that a student prompted to interpret a solution can experience a disturbance to their settled cognitive state (i.e., a perturbation). Second, as the student works to understand the solution, they can enact schemes relevant to their understanding of, and goals for, interpreting the solution. If the student experiences a cognitive perturbation, they may modify or reorganize their schemes to neutralize the disturbance. Such reorganizations can result in learning at a higher cognitive level (i.e., reflective abstraction). Below, we offer more elaboration on this process.

Schemes, Goals, and Learning

As students engage with others' solutions, we posit they would draw upon *schemes*. Drawing on von Glasersfeld (1995), Hackenberg (2014) defined schemes as "goal-directed regularities in a person's functioning that consist of three parts: a situation, an activity triggered by how the person perceives the situation, and a result of the activity that a person assimilates to her or his expectations" (p. 87). A student's schemes are activated through interactive experiences and involve their anticipation of the outcome of the interaction; thus, a student's goals are central to the schemes they enact. Any interaction prompting the student to draw on one or more schemes is a disturbance to the student's settled (equilibrated) state.

A student uses their schemes as part of their goal-directed activity. Moreover, in classrooms where sharing solutions is prioritized, the student may conceive they are working toward a *social goal* with others. Steffe and Thompson (2000) describe that an observer can identify a student working toward a social goal when the observer can infer the student (1) understands others in the group have intentions and (2) pursues a goal they assume is shared with the others. For example, a student may assume their classmates are working toward articulating solutions that are understandable to others and adopt this to inform their own goal-directed activity. We note even if other students are working toward a different goal (e.g., only intending to obtain a correct solution), a student's perception of a social goal can drive their goal-directed activity.

Experiential Provocations and Perturbations

If a student works to interpret classmates' solutions, they have experienced a disturbance to their equilibrium; von Glasersfeld (1980) broadly defines a *perturbation* as any input that creates a disturbance in a student's equilibrium. In this report, we use Steffe and Thompson's (2000) distinction between two kinds of disturbances (experiential provocation versus perturbation) based on the student's perception of the source of the disturbance (external versus internal). We refer to *experiential provocations* as disturbances that the student conceives to originate from their interaction with their environment (Steffe & Olive, 2002; Steffe & Thompson, 2000; Steffe & Tzur, 1994). We note the environment may include tasks, tools, teachers, other students, and, relevant to this report, others' solutions. Prompts that require a student to interpret a classmate's solutions can serve as experiential provocations from the student's perspective; in some cases, such an experiential provocation may lead to the student establishing a social goal with their classmates such as solving a particular problem or understanding a novel solution.

Experiential provocations are perturbations, but not all perturbations are *cognitive perturbations*. Students might be able to neutralize some perturbations using their current schemes and without experiencing any discrepancies as they activate and anticipate the results; such perturbations are *not* cognitive perturbations. Neutralizing other perturbations may involve a student experiencing discrepancies in their use of a scheme (Steffe & Olive, 2009; von

Glasersfeld, 1995). Such perturbations *are* cognitive perturbations. Cognitive perturbations can lead to a student reorganizing or modifying their existing schemes to achieve an equilibrated state (Steffe, 1991a, 1991b; Tillema & Gatza, 2024; von Glasersfeld, 1995). We conceptualize learning occurs through adjusting schemes in response to cognitive perturbations.

To illustrate, a student engaging in the goal-directed activity of interpreting a classmate's solution encounters a perturbation in the form of an experiential provocation. If they can interpret the classmate's solution without modification to their schemes (such as when the classmate's solution is nearly identical to their own), then the experiential provocation did not create a cognitive perturbation. However, if the student has difficulty making sense of a solution, then the experiential provocation could result in a cognitive perturbation. Such a cognitive perturbation could require the student to modify or reorganize their schemes to achieve an equilibrated state. We elaborate on this example and provide more details in the next section.

Types of Cognitive Perturbations and Affective Experiences

When a student experiences a cognitive perturbation, they may experience a minor or major cognitive perturbation. Many researchers have equated perturbations with major cognitive conflict or experiencing a 'problem' (e.g., Booker, 1996; Lerman, 1996; Simon et al., 2010). Although a major conflict is one type of cognitive perturbation, students can also experience minor cognitive perturbations without (consciously) experiencing a problem (Steffe, 2011; Steffe & Olive, 2002, 2009). If an individual neutralizes a perturbation, an observer's hypotheses about the steps taken can serve to differentiate major and minor cognitive perturbations. Major cognitive perturbations require more significant accommodations to a student's schemes to resolve, whereas minor cognitive perturbations may be easily resolvable with small adjustments to a student's schemes. Students may also experience non-neutralizable cognitive perturbations when they do not yet have schemes that allow them to neutralize the perturbation.

To exemplify this distinction, we again turn to a student work. A student might experience a minor cognitive perturbation when a solution has some feature or way of reasoning that is novel, and the student is able to neutralize the perturbation with minor modifications to their current schemes. If an observer infers the student undertakes a major modification or reorganization of their schemes, then the observer could characterize the perturbation as major. Finally, the student may experience a non-neutralizable cognitive perturbation if the student's current schemes do not support them in satisfactorily interpreting (from the student's perspective) the solution.

Moreover, as the above descriptions suggest, the perturbations students encounter, cognitive or otherwise, may bring forth different affective responses (Steffe & Wiegel, 1996; von Glasersfeld, 1995). For example, Steffe (2011) described, "A perturbation may be a surprise (e.g., unanticipated success, elation, or a sensation of pain)" (p. 258) and von Glasersfeld (1995) noted a perturbation may include "disappointment" (p. 65). A classmate's solution that a student finds hard to follow may lead them to experience frustration (i.e., a negative affective experience). In contrast, a solution they recognize as different from their own may result in a positive affective experience (e.g., pleasant surprise or delight). Thus, a student's affective experiences are important considerations in the design of learning environments that may engender reorganizations in students' schemes, thereby leading to learning.

Reorganization of Schemes and Reflective Abstraction

To further describe the reorganization of schemes that may occur after a perturbation, we use Piaget's (2001) notion of abstraction. Abstraction is a mechanism explaining an individual's modification of their schemes toward greater cohesion and generality. In this report, we use the concept of *reflective abstraction*. In broad strokes, reflective abstraction entails two processes: a projection of actions or schemes to a higher level of thought and a reorganization that occurs at this higher level (Ellis et al., 2024; Piaget, 2001; Steffe, 2024; Tallman & O'Bryan, 2024; Tallman & Uscanga, 2020; von Glasersfeld, 1995). The reorganization can involve the creation of a coherent relationship or network of relationships between existing schemes as well as with new schemes (Piaget, 2001; Tallman & Uscanga, 2020). Such a reorganization involves taking prior meanings as input for further operating and thus can be considered a "higher" level. We note the cognizing subject need not be consciously aware of any reorganization.

Connecting to a students' affective experiences, a student might experience what an observer would call reflective abstraction as the student considers aspects of a solution they do and do not appreciate. This consideration could result in the student reorganizing their schemes to incorporate aspects of solutions they appreciated and avoid aspects they did not perceive as valuable (e.g., adopting steps or language from a particular solution they find useful and avoiding confusing lines of reasoning). In this way, affective experiences can work in concert with cognitive perturbations, leading to students' reorganizing their meanings at a higher level.

Other researchers have argued for the importance of supporting reflective abstraction and have provided suggestions for doing so. First, providing students repeated opportunities to develop schemes relevant to particular meanings can support their connecting their schemes and reasoning across similar (and different) contexts (Tallman & O'Bryan, 2024; Thompson, 2013). Second, providing occasions for students to compare activity across tasks can support reflective abstraction (Ellis et al., 2024; Piaget, 1976, 2001; Tallman & O'Bryan, 2024). We use these two suggestions to try to create opportunities for reflective abstraction as we describe below.

Exemplifying the Constructs: X-marks the Spot

We designed the *X-Marks the Spot Task* leveraging the above constructs to support students' work with spatial coordinate systems (described below). We included occasions for students to engage with their classmates' mathematical ideas with the intention they would set social goals. We conjectured multiple rounds of describing (to classmates) and interpreting descriptions (written by classmates) of locations in space could occasion experiential provocations for students. We hypothesized these experiential provocations could lead to major or minor cognitive perturbations along with associated affective experiences. Further, we provided deliberate opportunities for students to reflect on location descriptions at a higher level of thought. We intended these experiences to support students in engaging in reflective abstraction as they reorganize their meanings for organizing space.

Before describing the task, we first provide important background information about spatial coordinate systems. We then describe the task with the above constructs in mind. We provide examples from when we used the described tasks with elementary pre-service teachers (EPSTs) in a classroom setting to help contextualize how we view the constructs relating to the task.

Task Background: Spatial Coordinate Systems and Conventions

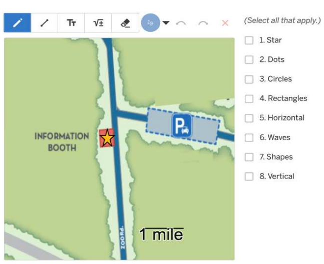
Researchers in mathematics education and spatial cognition have explored ways individuals conceive of and describe two-dimensional space (Lee, 2017; Piaget & Inhelder, 1967; Taylor & Tversky, 1996). In this report, we focus on a task designed to support students' developing meanings for spatial coordinate systems (Lee, 2017; Lee & Hardison, 2016; Lee et al., 2020; Paoletti et al., 2022). A spatial coordinate system is a coordinate system (CS) that entails either mentally overlaying a CS onto some perceived space or overlaying a space onto an already established CS. In either case, objects within the space can be located via coordinates. Radar on a ship and GPS are different examples of spatial CSs (i.e., polar and Cartesian CSs, respectively).

We note there are countless ways an individual can organize space or construct a spatial CS. Cartesian and polar coordinates are two conventional spatial CSs. Conventions, including conventional coordinate systems, involve choices often developed or adopted for the purposes of efficiency and communication (Moore et al., 2019; Zazkis, 2008). Given the communicative value of such conventions, we conjectured we could support students in developing a social goal by providing them repeated opportunities to describe locations in space, with the anticipation of classmates' interpreting it, and to interpret classmates' descriptions of locations. This goal could lead to activity that supported students in reorganizing their schemes for organizing space towards more clear and efficient strategies. This conjecture is supported by Lee (2017), who found the goal of relaying instructions to others when locating an object in space supported ninth-grade students in developing viable spatial CSs much like conventional ones.

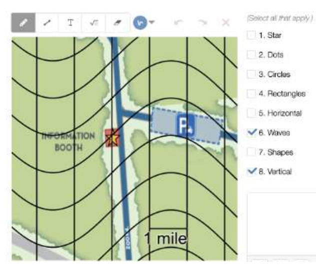
The Design of the *X Marks the Spot Task*

In the *X Marks the Spot Task*, we provide students with the map, buttons, and prompt shown in Figure 1a/b. Students can try each button and observe different overlays. Several overlays can be used to generate conventional coordinate systems: Star and Circles when coordinated can create a polar-like coordinate system; Vertical and Horizontal together can create a Cartesian-like coordinate system. Other overlays can be used to create unconventional yet viable ways to organize space (e.g., Figure 1b shows the map with the Waves and Vertical overlays). We conjecture the introduction of the task itself could serve as an experiential provocation for each student, and a student may experience minor cognitive perturbations as they develop associations between the action of clicking each button and the resulting change to the map.

After exploring the overlays, we ask students to mark an X. To support the creation of a social goal, we prompt students to use the overlays to provide a written description for their X's location that classmates will use. Thus, a student may assume their classmates have a shared goal of marking the same location. To achieve this social goal, a student may try to write a description clear enough for their classmates to mark a precise location.



Play with the different overlays. In the next slides you will use the overlays either (a) to describe the location of an X or (b) interpret classmates' descriptions



(a) (b) (c)

Figure 1. (a) The initial map, (b) initial prompt and (b) the map with Wave and Vertical overlays.

In classroom settings, we have EPSTs write their initial descriptions in groups. We share each group's description and the other EPSTs mark an X based on that description. After this, we engage in a whole class discussion using each group's description, original X, and an overlay of all the classmates' marked Xs (e.g., Figure 2). By comparing the locations of all the marked Xs, the EPSTs can determine whether they have achieved the *social goal* of marking their Xs in the same location. After several rounds of this activity, we provide numerous hypothetical classmate descriptions (Table 2) and ask students to mark an X from the description and provide feedback on the description to the hypothetical classmate. These hypothetical descriptions are intentionally designed to (a) move from vague to more specific while (b) leading to more viable and/or conventional spatial CSs (i.e., polar and Cartesian).

Table 2. X Marks the Spot Task hypothetical descriptions and reflection opportunity

#	Description	Intention
1	Click the Star (1) and Circles (3) options. The X is on the 2nd circle on the line above the line going to the left from the star.	Polar-like CS that is intentionally vague
2	Click the Star (1) and Circles (3) options. Imagine the star is like a clock with the line going straight up being 12 o'clock and the line going straight down being 6 o'clock. The X is on the 4 o'clock line halfway between the 2nd and 3rd circle.	Polar-like CS that should provide enough information to locate an exact location that lies on a reference object in the CS.
3	Click the Star (1) and Circles (3) options. Imagine the star is like a clock with the line going straight up being 12 o'clock and the line going straight down being 6 o'clock. The X is 1.25 miles from the star halfway between 10 and 11 o'clock.	Polar-like CS that should provide enough information to locate an exact location that does not lie on a reference object in the CS.
[Reflection prompt for #1-3] Below are all the descriptions that used the Star (1) and Circles (3). Talk with your neighbor about which part of these descriptions was most helpful		
#4-6 provide similar phrasing with Cartesian-like CSs. There is also an opportunity for reflection on #4-6.		

The act of interpreting others' responses occasion *experiential provocations* which could result in a student experiencing a *cognitive perturbation*. We conjecture students would be able to interpret many descriptions using their current schemes (i.e., *without cognitive perturbation*). We also conjecture interpreting vague descriptions (e.g., Description 1 in Table 2) or observing discrepancies in the location of marked Xs for the same description (e.g., Figure 2c) could create *cognitive perturbations*. For example, consider one group's description for the marked X (Figure 2a/b). During a class conversation, the teacher presented an overlay of their classmates' marked Xs for their description (Figure 2c). Observing the variance of the marked Xs, the EPSTs had opportunities to consider ways each student interpreted "two boxes up from the bottom" (i.e., does this include the bottommost box?). In this case, an EPST may have experienced a *cognitive perturbation* as they attempted to understand their classmates' descriptions or interpretations.

"With the horizontal and vertical overlays selected, the x is in lower right hand side below the parking garage. It is one box over from the right and two boxes up from the bottom."

(a) (b) (c)

Figure 2. (a) A description EPSTs gave for the X marked in (b), and (c) the Xs marked by the rest of the students

We anticipate the descriptions could lead to a student having a positive affective experience in concert with a cognitive perturbation. A student might have positive affective experiences as they interpret descriptions they feel are interesting, clear, or lead to precise locations (e.g., #2-3 in Table 2). For example, when tasked to provide feedback on Description 3, one EPST stated, "No advice, very precise. I like that clock analogy." We conjecture the EPST might have experienced a minor cognitive perturbation along with a positive affective experience as they made sense of the clock analogy for the first time ("very precise. I like…").

Further, we conjecture tasking EPSTs to provide feedback to the author of each description can engender reflective abstraction as students reorganize their meanings for organizing space. For instance, we observed some EPSTs adopted phrases (and associated reasoning) that included elements from the classmate description involving the clock analogy when engaging the final prompt. This prompt asked them to describe the location of a novel X (Figure 3a). Using the visual in Figure 3b, one EPST described: “Using the idea of a clock with the top line being 12, go in between 4 and 5 o clock in the third outermost circle. The x is in the middle of this box.” We note how the clock language this student incorporated aligns closely with a polar-like CS used in Descriptions #2-3 (Table 2). Thus, we conjecture reflecting across descriptions supported the student in reorganizing their meanings for organizing space to incorporate elements they found useful, lending themselves toward constructing a viable and conventional spatial CS.

(a) (b)

Figure 3. (a) the last X described by EPSTs and (b) an EPST’s use of polar-like coordinates

Contribution, Implications, and Areas for Future Research

In this report, we elaborated on social goals, experiential provocations, cognitive perturbations, and reflective abstraction. Importantly, we characterize how affective experiences are intertwined with cognitive perturbations in ways that can engender learning. We are unaware of examples in which radical constructivist researchers detail ways positive or negative affective experiences interact with students’ learning. Given the emphasis on collaborative group work, such learning is likely to occur in classrooms where students interpret others’ responses positively and reorganize their own meanings as a result of these interpretations.

We conjecture the use of others’ descriptions to potentially provoke cognitive perturbations was productive in this task due to the communicative nature of the mathematics at hand. We conjecture providing students with repeated opportunities to first generate their own descriptions and then interpret descriptions that communicate more or less effectively and efficiently increases the chances students would experience positive affective experiences and cognitive perturbations that could result in their reorganizing their meanings for organizing space. There are likely other concepts that rely heavily on communicative goals such as conventions, in which students could be supported in learning via the use of others’ solutions. We call for additional research exploring this possibility. This and other research could build on prior work showing how examining others’ solutions to mathematical problems can support learning (Barbieri et al., 2023; Webb et al., 2014). Further, such research could leverage the constructs outlined here to provide explanations for *how* examining others’ solutions can lead to learning.

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