"IT DOES SHOW IT BOTH WAYS, THOUGH": EMMA'S REASONING THROUGH GRAPHING CONVENTIONS

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As graph literacy continues to be necessary to communicate in STEM fields, conventions around such graphs have developed for students to work and reason with. We describe a fifth grader's, Emma's, thinking through non-conventional graphical representations of a linear relationship. We argue that Emma relied on mathematical reasoning when faced with conflict in conventions and was able to make sense of unconventional graphs by using quantitative strategies. Although Emma acknowledged her known conventions of graphing, she was not bound by these conventions but rather leaned on her reasoning about quantities and flexible use of reference frames. We use Emma's activity to argue possible implications for research and teaching regarding graphing conventions.

Keywords: Mathematical Representations, Cognition, Middle School Education

Graphical representations are commonly used in STEM fields, and relatedly, the ability to read and write graphical representations is important for students to progress in STEM coursework and careers (Costa, 2020). These graphical representations commonly draw on conventions. For example, many graphical representations are constructed upon the Cartesian plane, with two perpendicular axes (i.e., x and y axes) with the intersection of the axes at (0, 0), named "the origin". Because such conventions are used widely and often, it is important that students know these conventions and use them to communicate ideas with others. However, despite their effectiveness for communication, too much emphasis on conventions can become a hurdle for students. Researchers have shown that students' meanings for graphs are often constrained to a 'a set of rituals' (e.g., Mamolo & Zazkis, 2012; Thompson, 1992). For example, researchers have noted an over-reliance on the vertical line test to determine if a graph represents a function even in cases where this procedure does not apply (Breidenbach et al., 1992; Even, 1993; Montiel et al., 2008; Moore, Silverman, et al. 2019; Oehrtman et al., 2008). Student adherence to conventions used for the Cartesian plane has similarly provoked struggles while creating/interpreting a polar coordinate system (Sayre & Wittman, 2008; Moore et al., 2014). Further, some researchers have shown that some conventions commonly used in math classes are not consistent with how STEM fields use graphical representations in practice. For example, Paoletti et al. (2022) showed that the origin is typically not (0, 0) in graphs used in several STEM fields. Collectively, these studies show that too much attention to conventions might take away students' focus from more important reasoning that could support their graph literacy.

Although the aforementioned studies provide insight into the complexities students can experience when it comes to graphing conventions coming in conflict with their graph reasoning, we note that these studies involved older students, who have had many years of experience with graphing conventions. In our work, we have been working with Grade 5–8 students who are yet to or are in the early stages of learning about graphs in school. We aim to document how students at this earlier stage are capable of reconciling conflict between learned graphing conventions to view them as conventions rather than as required rules, in conjunction with their budding quantitative strategies and thinking within frames of reference (hereafter referred to as "reference frames" (RFs)). In this paper, we describe a fifth grader's, Emma's, thinking about graphical representations of what we deemed to be a linear relationship. We describe how Emma's attention to graphing conventions, quantitative strategies, and thinking within RFs interplayed throughout her engagement with a graphing task. We argue that Emma relied on her quantitative reasoning when faced with conflicts with learned graphing conventions to make sense of unconventional graphs. We close with a discussion on the implications of Emma's work for future research and teaching regarding students' developing meanings for graph conventions.

Theoretical Underpinnings and Relevant Literature

In this section, we discuss the theoretical underpinnings that guided our task design and data analysis. We also review literature relevant to our specific focus on students' interpretation of $y = 2x$ graphs in both conventional and unconventional forms.

Conventions

Thompson (1992) differentiated students' understanding of conventions as conventions (*conventions qua conventions*) versus students' understanding of conventions (to teachers and researchers) as rules that must be followed (*ritual use of conventions*). We used Thompson's distinction between conventions qua conventions and ritual use of conventions to characterize Emma's attention to graphing conventions in our analysis. That is, we attended to whether Emma viewed certain features of graphs presented to her as mere conventions that could be changed or as rules that need to be followed when constructing or interpreting graphs.

Moore and colleagues examined students' interpretations of simple graphs, like $y = 3x$, constructed in nonconventional variations of the Cartesian plane (Moore & Thompson, 2015; Moore, 2016; Moore, Stevens et al., 2019; Moore, Silverman et al., 2019). Graphing tasks like this were used to develop models of students' graphing activity, with specific attention to what aspects of the graphs were prioritized in students' focus. In doing so, the researchers were also able to examine students' meanings for conventions interplaying with their reasoning about quantitative relationships. Moore, Stevens et al. (2019) provided numerous examples of pre-service teachers (PSTs) whose graphing activity was constrained to maintaining conventions as rules. In many cases, the PSTs' reliance on conventions took precedence over their quantitative meanings for the situation, leading them to claim that mathematically accurate graphs (from the researchers' perspective) were wrong due to the graphs differing from their expected conventions in some way. For example, only 31% of PSTs from the study deemed an accurate graph of $y = 3x$ with x and y represented on the vertical and horizontal axis, respectively, to be an accurate representation of the relationship defined by $y = 3x$. Inspired by this line of work, we designed the "Variations of $y = 2x$" task to vary conventional features of the canonical $y = 2x$ graph and asked students to check whether the graph accurately depicted the relationship between x and y. Variations included changing the axes and/or the orientation of axes like in Moore and colleagues' work. Other features, such as the location of the origin and the scale of each axis, were also varied (see the Methods section for more details).

Quantitative Reasoning and Reference Frames

We conjectured students could rely on their quantitative reasoning to develop meanings for graphing conventions as conventions. We adopt Steffe, Thompson, and colleagues' (e.g., Smith & Thompson, 2008; Steffe, 1991) description of quantitative reasoning, which characterizes *quantities* as conceptual entities individuals construct to interpret their experiential worlds (von Glasersfeld, 1995). Quantitative reasoning, then, entails an individual conceiving of and reasoning about the relationships between quantities (Smith & Thompson, 2008; Thompson, 2011). Engaging with algebraic situations should entail quantitative reasoning (Smith & Thompson, 2008; Steffe & Izsák, 2002). With respect to "$y = 2x$" in our work, a student reasoning quantitatively may quantify a relationship between y and x as multiplicative (i.e., the y-value is always twice the x-value).

In the context of quantitative reasoning, Joshua et al. (2015) defined a RF as "a set of mental actions through which an individual might organize processes and products of quantitative reasoning" (p. 2). Joshua et al. identified three related mental actions—committing to a unit of measure, committing to a reference point, and committing to a directionality of measure comparison (p. 32). Further, Joshua et al. defined a coordinate system as the product of the mental activity involved in conceptualizing and coordinating multiple RFs, which allows individuals "to represent the measures of different quantities simultaneously when those measures stem from potentially different frames of reference" (ibid., p. 35).

Similarly, but more broadly, we use RFs to refer to mental structures used to gauge the relative extent of various attributes in the phenomenon (Levinson, 2003; Lee, 2017; Joshua et al., 2015). Thinking within RFs entails attending to and establishing reference objects, directionality, and having an idea of what and how to measure the quantities being depicted (Joshua et al., 2015; Lee et al., 2019). For example, to create or interpret the graphical representations like those in Figure 1, an individual will need to establish x and y in terms of where they start, in which direction they move/change, and how each quantity is measured (e.g., unit of measure). Relatedly, coordinate systems refer to the geometric coordination of the RFs (e.g., axes). A coordinate system allows an individual to systematically express and coordinate RFs; a graph refers to a collection of points depicted upon the underlying coordinate system. Considering such a collection of points, an individual can hold in mind both quantities' (potentially varying) magnitudes simultaneously (Thompson et al., 2017). The nature of graphs and hence, ways of thinking about a graph, fundamentally depends on the RFs and coordinate systems upon which the graphs are created and how individuals make sense of the quantities depicted.

Lee et al. (2019) documented shifts in how a PST constructed and reasoned within RFs when engaging in graphing activities with non-canonical coordinate systems. Specifically, Lee et al. attended to the PST's reference points and directionality of measure comparison, which shifted from relying on perceptual features of graphs to focusing on coordinated actions such as quantitative relationships. The researchers hypothesized that the PST's shift was supported by perturbations from the unconventional graphs. Building on this work, in our work with Emma, we attended to her RFs, specifically, her attention to some reference point(s) and directionality of measure comparison.

Guided by these ideas, our research question is, “When faced with unconventional graphical representations of $y = 2x$, what reasoning does one fifth grader employ between her conventions, quantitative meanings, and reference frames?”

Methods

In this paper, we present data from a larger project that uses clinical interviews (Ginsburg, 1997; Clement, 2000; Goldin, 2000) to examine students’ current ways of graph thinking. The project goal is to examine middle school students’ graphing activities that could inform theory and practice.

Participant and Data Collection

The participants were recruited locally via social media and ranged from fifth to eighth grade. Four students met with the researchers on a university campus in the southern United States to participate in a sequence of four hour-long individual clinical interviews. Interviews had an interviewer (IR) and witness-researcher (WR) present; they were video-recorded with a focus on student work and any interactions and gestures between the student and IR. We digitized student work through scanning and screen-recordings. The participant we focus on in this paper, Emma, was a fifth grader. Specific to the task, Emma self-reported that in school, she had not seen graphs like the ones from the task. However, Emma did describe exposure in school to using coordinate grids to plot points, where the origin would be placed at (0, 0). Although she had experience with “conventional” coordinate systems in school, these conventions had not necessarily been emphasized yet in relation to linear graphs such as $y = 2x$. We note that Emma reported studying additional mathematics outside of school, and she demonstrated familiarity with linear graphs throughout her interviews.

This paper focuses on data from one task in Emma’s third interview, “Variations of $y = 2x$”, implemented through the online, interactive teaching and learning platform, Desmos. We designed the task while considering the work discussed above with unconventional coordinate systems and graphs. Our task contained four slides, where each slide contained a graph of the line $y = 2x$ with differing orientations of axes, scaling, or origin changes (Figure 1). Specifically, Graphs A and B (Figure 1a and b) showed the x- and y-axis with differing scales, Graph C (Figure 1c) had positive x-values oriented to the left and positive y-axis values oriented downwards, and Graph D (Figure 1d) showed the axes intersecting at (-2, 0). When opening each slide, we asked Emma if the graph represented the relationship between x and y in the equation $y = 2x$ by selecting “Yes,” “No,” or “I don’t know”. Because our goal was to investigate how the student might make sense of the quantitative relationship and not their ability to read an equation, if the student had a difficult time interpreting the equation $y = 2x$, we explained to the student that the equation meant the y value is always twice the value of x.

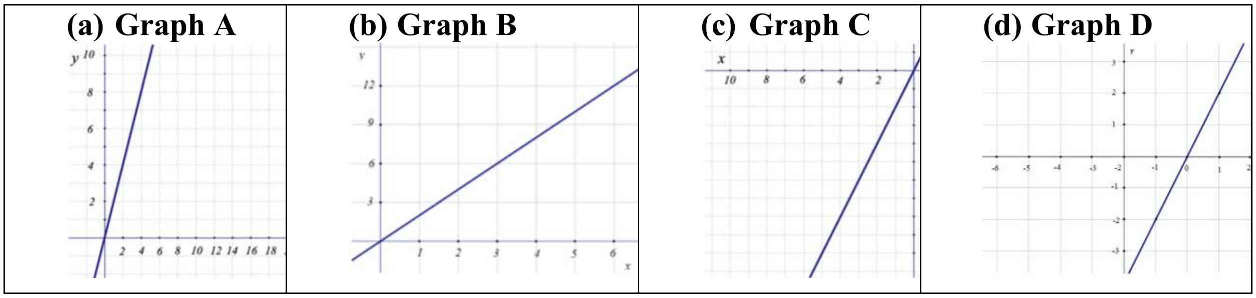

Figure 1: "Variations of $y = 2x$" Graphs

Data Analysis

In our analysis, we created a thick description of Emma's activity with the task (Geertz, 1973). We used this description to build a model of Emma's current meanings through conceptual analysis (von Glasersfeld & Steffe, 1991; Thompson, 2008). As we attempted to build this model, we characterized Emma's quantitative reasoning, attention to conventions, and thinking within RFs. Specifically, we examined Emma's activity for her quantitative reasoning, potential habitual use of conventions, relevant RFs Emma used, and shifts between habitual use of conventions and using conventions qua conventions. During this process, we re-examined previous parts of the description to support our working model, identify possible shifts in Emma's reasoning over the episodes, or negate our original interpretations.

Results

Although Emma expressed her known conventions around graphs, she was able to rely on her quantitative meanings for the relationship $y = 2x$, in conjunction with the use of flexible RFs, to determine if a(n unconventional) graph accurately depicted the relationship. Notably, her flexible use of RFs included interpreting shifts in directionality (i.e., representing positive x-values to the left), unconventional units (i.e., tick marks not representing 1 unit), and different reference points (i.e., unconventional intersection of axes). In all four graphs, Emma consistently used quantitative reasoning and RFs to resolve conflicts that arose when aspects of a graph did not match the conventions she assumed needed to be maintained.

Conventions, Quantitative Reasoning, and RFs Aligned: Graph A

In Graph A (Figure 1a), Emma's meanings for conventions, RFs, and quantitative reasoning aligned. After some conversation about how $y = 2x$ may be represented in a graph, the IR asked Emma what she thought about the relationship as meaning y is always twice x. Emma first implicitly considered if the graph represented a rule in which x was two more then y by checking if the point (0, 2) was on the graph before realizing she should consider if y-values were double x-values. She then moved her cursor to (0, 0) and over horizontally to $x = 2$, claiming, "If x is that [two], y is that [*moving her cursor up vertically to intersect the graph and then horizontally over to $y = 4$ on the y axis*]." With the cursor on (2, 4) on the graphed line, Emma argued that this point was correct based on four being "two times x". Emma decided to answer "yes" to the prompt and provided more explanation to back up her claim, such as (4, 8) being another point on the graph reflecting her quantitative meaning for $y = 2x$ of y being "two times x".

Kosko, K. W., Caniglia, J., Courtney, S., Zolfaghari, M., & Morris, G. A., (2024). *Proceedings of the forty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Kent State University.

Across her activity, we infer Emma used the *x*- and *y*-axis each as a RF. She identified 0-points for each axis, worked with an implicit direction, and understood each tick to represent the appropriate number of units. For example, for Emma, $x = 2$ meant starting at 0 and moving two units right via 1 tick mark jump. Finally, we note Emma relied on a quantitative meaning for the relationship (*y* is "two times *x*") to determine if the graph reflected the relationship. Emma continued to use this *quantitative meaning* in the rest of the graphs of the task. In some cases when Emma became perturbed as she addressed a novel graph, the IR referred back to her quantitative meaning to help remind Emma of the connection of the equation to the relationship.

Conventions Superseded by Quantitative Reasoning and RFs: Graph B and C

The unconventional nature of Graphs B and C (Figure 1b, 1c) created perturbations for Emma as she attempted to interpret novel coordinate systems. However, Emma leveraged her quantitative meanings along with flexible reasoning about RFs to interpret both graphs as accurate representations of the relationship $y = 2x$.

In each case, as Emma tried to apply her quantitative meaning, the unconventional nature of the graph created a complexity. When initially addressing each graph, Emma decided that the graphs did not reflect the relationship. In Graph B, this happened as Emma was looking for $x = 2$ and $y = 4$ to touch on the graph; as she moved up from $x = 2$ to the graph, she said, "It doesn't [represent the relationship]. Four would be right there [*motioning over the graphed line above x = 2 between the y-values tick values of 3 and 6*]." We conjecture the point not being at the intersection of two gridlines created a complexity for Emma. As the *y*-axis was scaled by increments of 3, 4 was not represented on the scale or by a gridline; we infer this broke from the convention Emma (implicitly) used in the prior graph that each tick mark along the *y*-axis represented a change of 1. Emma rejected Graph C even faster, calling it "wrong" due to its unconventional nature, saying, "From what I see, those [*referring to the x and y values on the left and down of the intersection of the axes*] have to be negative numbers because that is the… I think that's the third quadrant…" In each case, we inferred that conventions around coordinate systems, implicitly in Graph B and explicitly in Graph C, influenced Emma's initial decisions for if the graphs represented the relationship.

However, Emma reconsidered each graph as she returned to her quantitative meanings and adapted her RFs when asked to explain her original decision. In Graph B, Emma reorganized her RFs such that the unit of measure of each tick mark represented matched those depicted. After her last comment above regarding Graph B and her conflict with the point (2, 4), she tilted her head and wondered aloud, "actually, it does [represent the relationship]." She then decided to check that $x = 3$ corresponded to $y = 6$ in the given graph, confirming that the graph represented the quantitative relationship. Emma then returned to checking $x = 2$. She placed the cursor directly above the *x*-axis and defined the distance between the *x*-axis and her cursor as a unit length, "the top of the circle [cursor] would be one". She then iterated that length by moving the cursor up three more times, intersecting the graph at the *y*-value of 4. We infer Emma had re-established her RFs constituting each axis to attend to the non-normative scaling of the *y*-axis as compared to Graph A. Using this reorganization in conjunction with her quantitative reasoning of checking the pairs of points, she determined that Graph B accurately depicted the relationship.

Emma similarly switched decisions with Graph C by reasoning flexibly with RFs and maintaining a focus on the quantitative relationship. In particular, after verbally identifying the unconventional axes different from Emma's expected signs for "Quadrant 3," the IR asked Emma what her answer to the prompt would be. After a 6-second pause, Emma responded "it does [represent the relationship]." Emma then gestured to the 2 on the *x*-axis down to the graphed line and horizontally over to the 4 on the *y*-axis explaining, "It shows it because the two and the four touch right there, on the line." Emma smiled and decided confidently to answer "Yes." She further demonstrated how when *x* equaled 4, *y* equaled 8 on the line as "another way I can prove it." Although Emma's initial reaction was to reject Graph C, she re-considered her decision after reorganizing her RFs, attending to the changed direction of the values on the *x*- and *y*-axis. This reorganization of her RFs allowed her to use her quantitative meaning for the relationship to confirm Graph C did, in fact, represent the relationship. Emma's work with Graphs B and C evidence her understanding of convention qua conventions, where she leaned on her re-organization and use of RFs and quantitative reasoning to overcome an initial hesitation towards the representation that was depicted differently than she seemed to expect.

Conventions in Conflict with Quantitative Reasoning and RFs: Graph D

The unconventional location of the intersection of the axes in Graph D created a greater complexity for Emma as she considered if the graph correctly represented the relationship. However, as before, she eventually was able to reorganize her RFs and leverage her quantitative meanings to interpret the graph as correct. When first viewing Graph D, Emma expressed concern with the intersection of the axes:

> Why do they have…Um. I think that this line [*traces y-axis on the screen*] has to be moved over more… the *y* line, has to be moved over more [*gestures to the zero on the x-axis*] to the zero because, um, I, well, maybe it doesn't… uh, it does. Um, it has to be*,* the origin is always (0, 0).

Emma's reaction seems indicative of a ritual use of conventions regarding the intersection of the axes ("always (0,0)"). In fact, her tone changed as she emphasized the origin "*has to be*" (0, 0). However, there was also a note of suspicion that "maybe it doesn't" have to be at zero. Immediately after making this comment, Emma critically investigated between the current origin (Figure 2a) and her desired origin, (0, 0) (Figure 2b). She then discussed (0,0)'s placement, "Hmm. That does… That shows zero, too. That's showing… Hmm, actually… Actually, that shows (0, 0). But I don't think, was it… I don't understand this. How are the, why is the *y* line like that?" We infer that Emma realized the point (0, 0) was on the given graph, which she understood was consistent with the given relationship $y = 2x$. However, the unconventional placement of the *y*-axis persisted in creating confusion as Emma again declared that the graph would not represent the relationship.

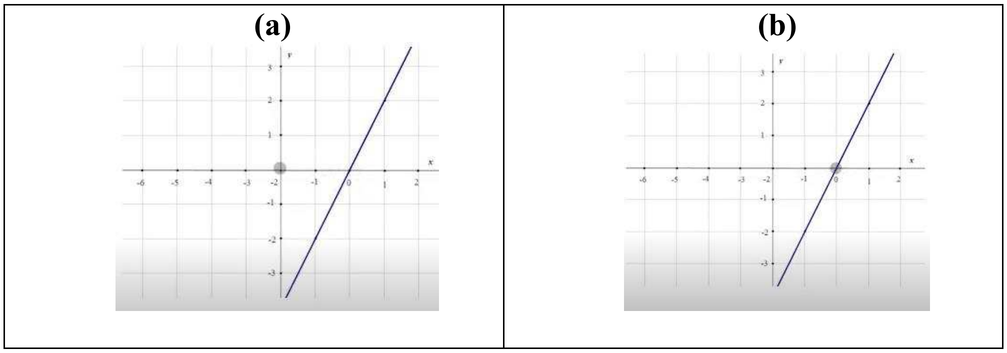

Figure 2: Graph D's (a) Depicted Origin, and (b) Emma's Desired Origin at (0, 0)

Emma continued to consider whether the intersection of the axes at (0, 0) was a rule that must be followed in relation to her inferences regarding RFs and her quantitative meaning. As Emma considered Graph D, she motioned along each axis to show that the graph represented should depict $x = 2$ corresponded to $y = 4$. However, we infer that in the moment, Emma still considered the graph to be incorrect. She opted to check another point, moving her cursor along the x-axis to 1, then moving up and horizontal to the y-value of 2. As she did this, she paused and looked closer, "Wait… but it does! It shows it… Hmm, it does." She then moved on to the point (0, 0) and reasoned that doubling zero should achieve that point, laughing to herself, seemingly with surprise. The IR then asked her where zero should be on the graph, and Emma repeated her original reasoning, "If I could have the zero anywhere, I would have the zero right here [*places cursor on the intersection that currently had x = -2 in Figure 2a*]." We infer Emma wanted the intersection of the axes to be (0, 0), not (-2, 0). She stared at the screen again for about five seconds and calmly decided, "It does show it both ways, though… because, I can do it with the one and the two [*gestures up to the graph from x = 1*]… Oh, one and a half would be about there [*puts mouse between the 1 and 2 on the x-axis*]… One and a half, three [*motioning from the x-axis to the graph at y=3*]!" She then pulled herself back and smiled, concluding, "I think it's actually yes". We infer that Emma's initial reaction to the graph involved the intersection of the axes at (0, 0) to be a rule rather than a choice. However, as she focused on the RFs represented by each axis (rather than the intersection point), she reconsidered the graph in terms of her quantitative meaning, concluding the graph accurately reflected the relationship. Although she still expressed preference towards the intersection of the axes to be at (0, 0), Emma treated this as a conventional choice (convention qua convention) rather than a rule that must be followed (ritual use of convention).

Although Emma initially rejected each of Graphs B, C, and D due to something unconventional about each, she eventually reorganized her RFs to consider if the graph reflected the underlying quantitative relationship. Reflecting a conscious awareness of the unconventional nature of such graphs, Emma referred to unconventional aspects of the graphs such as the axes as possible "mistakes" or "there to confuse [me]." But, consistent with understanding graphical conventions qua conventions, Emma understood each graph as reflecting the quantitative relationship defined by $y = 2x$.

Discussion

Addressing our research question, we showed interactions between Emma's meanings for conventions, quantitative reasoning, and use of RFs as she explored representations of $y = 2x$. Emma's strategies were powerful in leading her to reconcile unconventional graphs by focusing on the quantitative relationship and re-organizing her RFs. Emma's flexibility was illustrated through her reasoning through unconventional axes directions, scaling, and origin as she continued to rely on y being twice x and thus checking if appropriate points met on the graph. Emma's activity exemplifies the merit in students grappling with conventions on their own before directly being adopted throughout their schooling; we conjecture such discussions that allow students to consider quantitative meanings for algebraic equations and RFs may be fruitful in supporting students understanding graphing conventions as conventions. Further, such unconventional graphs can also be fruitful for supporting students in moving beyond a ritual use of conventions, such as realizing the intersection of the axes did not have to be "always (0, 0)".

Connecting back to the literature, several researchers have conjectured that students' meanings for algebra and graphs as a set of rituals may stem from a lack of opportunities to construct and reason about relationships between quantities (Moore, Silverman et al., 2014; Moore, Silverman et al., 2019; Paoletti, 2020; Paoletti et al., 2018; Thompson & Thompson, 1995). We note how Emma, as a fifth grader, was focused on a quantitative relationship in her activity, which allowed her to exhibit more flexible reasoning than the PSTs reported on addressing similar tasks (Moore, Silverman et al., 2019; Moore, Stevens et al., 2019). We conjecture Emma's flexibility relative to the PSTs may be due to her having significantly less school experiences adhering to conventions. That is, we conjecture conventions become rules for students when they are always used without explicit conversations or opportunities to consider other choices. In reality, students need this flexibility when faced with unconventional representations found to be applied in real-life contexts (e.g., as in STEM fields), especially as fields continue to evolve unpredictably over time along with possible new developments for representing quantities and needs for students reasoning within those developments arising.

Based on Emma's interactions with the given representations moving beyond conventions to determine the quantitative relationship depicted, we conjecture providing students with such unconventional coordinate systems early in their learning about graphs could support them in developing meanings for conventions qua conventions. However, our sample consisting of one student in one session limits our ability to evidence such conjecture. We call for future research to explore this possibly. Such research can support teachers and researchers in understanding and supporting flexible meanings for graphs that support students across STEM fields and real-world contexts.

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