MODELING STUDENTS' STRATEGIES WHEN CREATING A GRAPH: A FOCUS ON REFERENCE FRAMES AND COORDINATE SYSTEMS

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We examined students' thinking of graphs around a graphing task from 14 individual interviews, in terms of three layers—frames of reference, coordinate systems, and graphs—and explored their productive and intuitive strategies. As a result, we present a framework that offers a characterization of students' graphing activities. We then discuss implications of the framework.

Keywords: Mathematical Representations, Cognition, Middle School Education.

Graph literacy is important for students to progress in STEM coursework and careers (Paoletti et al., 2020; Costa, 2020) and for making sense of, and responding to, information in the real world (Yore et al., 2007). Sherin (2000) argued researchers should move beyond identifying students' difficulties to explore students' natural inclinations when developing graphical representations and how these inclinations can be leveraged to support graph literacy. In line with researchers who have focused on asset-based accounts of students' strategies, the work we report in this paper was guided by the question, 'What cognitive strategies and intuitive insights do middle school students invent or draw upon when representing quantities in a graphical representation?' To address this question, we present a framework we developed and refined through analyzing interviews with 14 middle school students on the Family Frenzy graphing task. We close by discussing the broader implications of the presented framework.

Some Relevant Literature and Brief Theoretical Underpinnings

Researchers have identified many difficulties students encounter with graphs. Of relevance to this report, researchers identified that students often treat graphs as literal representations of a situation (Bell & Janvier, 1981; Clement, 1989; Lai et al., 2016; Oehrtman et al., 2008). For example, Clement (1989) described students interpreting a speed-height graph of a bike rider as representing a hill the bike rider traveled over. To explore ways students may reason as they construct graphs, we modified Swan's (1985) "Bus Stop Queue" task (Figure 1a), which requested students to interpret a scatterplot by matching each person in the picture to their appropriate point. Note that height and age were labeled along the horizontal and vertical axis, respectively; from this we inferred one goal of the task was to perturb students who interpreted graphs as literal pictures, i.e., interpreted the height of a point as the height of a person. We modified the task by switching the axes labels (Figure 1b) and asking students to create their own graph, as our goal was to examine students' generative activities and intuitions they can build on.

Our work builds on previous work that examined students' generative activities (diSessa et al., 1991; Sarama et al., 2003; Sherin, 2000). Sherin (2000) described students' intuitive representations when tasked to create a picture to describe a motorist's motion over time. Students' depictions often contained pictorial features (i.e., using symbols such as lines to represent more or less of a quantity) that could lead to ideas akin to conventional graphs. However, as Sherin stated, he did not "attempt to be more specific about how this collection is constituted in detail (for example, in terms of knowledge structures)" (p. 413). In this paper, we account for cognitive strategies students draw upon to identify knowledge structures (i.e., thinking patterns that might be involved in students' graph literacy).

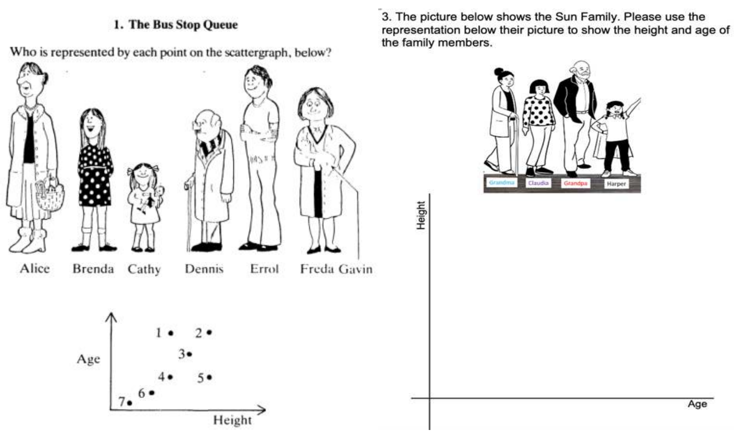

Figure 1: (a) Bus Queue task from Swan (1985); (b) The Family Frenzy task

Frames of Reference, Coordinate System, and Graph

Graphical representations involve spatial depictions of quantities (Thompson, 2011) and are a way to mathematize phenomena. A graphical representation consists of three layers: frames of reference, a coordinate system, and a graph (a collection of points). Frames of reference refer to mental structures used to gauge the relative extents of various attributes in the phenomenon (Levinson, 2003; Lee, 2017; Joshua et al., 2015). Thinking within frames of reference entails attending to and establishing reference points, directionality, and having an idea of what attributes to consider and how to measure them (Joshua et al., 2015; Lee et al., 2020). The nature of graphs and hence, ways of thinking about a graph fundamentally depends on the frames of reference and coordinate systems upon which they are created.

Methods

The data presented here comes from 14 clinical interviews (Ginsburg, 1997) across two projects, both aimed to examine middle school students' (5th to 8th grades) graphing meanings. We collected video recordings, screen recordings, and digital copies of students' written work. The projects recruited students from various mathematical and socio-economic backgrounds. In this paper, we present data from the Family Frenzy task (Figure 1b) which was used in these clinical interviews. We initially examined students' thinking in Family Frenzy and sorted them related to frames of reference, coordinate systems, and graphs (three layers) using the Analytical Framework for Making Sense of Students' Graphical Representations (Lee, 2024). Next, using open and axial techniques (Corbin & Strauss, 1996), we created descriptions of themes within each layer; from these descriptions, we further abstracted and classified the students' strategies, and we present those results in Table 1. We note that the resulting codes are meant to be a holistic characterization of the students' strategies for each attempt they made at the task. Each graphing attempt received a set three of codes where one code was from each category (graphing activity, reference frame activity, coordinate system activity). Results

Students demonstrated a variety of intuitive approaches, which is organized in Table 1. In the table, *representational objects* refers to the (often geometric) objects students physically inscribed on the paper, which included stacked dots, stick people, and bubbles (regions). To distinguish students' inscriptions from the pre-made, two-line segments labeled as Age and Height (what the researchers intended as axes), we call the totality of the two-line segments and the space they span as the *graph space*. We take both the graph space and students' representational objects to constitute their representation of the Sun Family's height and age. We next present one student's strategies to exemplify a subset of these strategies.

Table 1: Summary of Students' Representation Strategies

	Graphing Activity	Reference Frame Activity
Height	• *Spatial Transfer*: Uses fingers or other physical materials to transfer the height of members in the picture to the graph space and marks the height using representational objects. • *Non-physical Transfer*: Estimates relative heights of each member, without using any observable physical action or object to transfer length and indicates such heights in the graph space using representational objects.	• *Pictorial Ordering*: Represents height in the order of the members standing in the picture (e.g., Grandma, Claudia, Grandpa, Harper) in the graph space. • *Quantitative Ordering*: Represents height in ascending or descending order of heights of the members (can be different order than in picture; e.g., Harper, Claudia, Grandma, Grandpa).
Age	• *Indexing*: Estimates relative ages of members based on picture and writes the age of members near the representational object used for height in the graph space. Ages' representations are add-ons to those used for height. • *Non-indexing:* Estimates relative ages of members based on picture and indicates such ages using representational objects in the graph space. Ages' representations are independent of (though could be related to) those used for height.	• *Pictorial Ordering*: Represents age in the order of the members standing in the picture (e.g., Grandma, Claudia, Grandpa, Harper) in the graph space. • *Indexed Ordering*: Represents age in the same order of height in the graph space because age is indexed onto height's representational objects. • *Quantitative Ordering*: Represents age in ascending or descending order of ages of the members (can be in different order than in the picture).
Height and Age Together (Coordinate System Activity)	• *One, implied axis as an ordered number line*: One of the axes in the graph space is acting as an ordered number line while the other is not; 1-D coordination. • *Two, separate, implied axes as number lines*: Both axes in the graph space are acting as an ordered number line for each quantity but the two number lines are used individually; two 1-D coordinations. • *Two, overlapping, implied axes as number lines*: One axis in the graph space acts as an ordered number line for both quantities; both quantities are represented on a single axis: stacked 1-D coordination. • *Two, coordinated, implied axes as number lines*: Each axis in the graph space is acting as an ordered number line for a quantity; both quantities are represented in the two-dimensional space produced by the product of the two axes: 2-D Cartesian coordination	

Thomas' Representation and His Strategies

Six students used a *spatial transfer* strategy when graphing the family's height. Transferring was evidenced by measuring the height in the picture in some manner (e.g., using a ruler, using the span of two fingers) and then marking this measurement directly in the graph space, resulting in a literal copy of the cartoon's height. Figure 2 shows Thomas enacting spatial transfer (and his final representation). Thomas partitioned the Height axis into what he called centimeters. He then used his fingers to measure Grandma's height and then maintained this gap to represent her height on the vertical axis (Figure 2 left and middle). He used this strategy for all the family members, which yielded a set of stacked names on the y-axis (Figure 2 right). Further, this strategy yielded a *quantitative ordering for heights* in that the heights of family members were ordered from shortest to tallest in his representation.

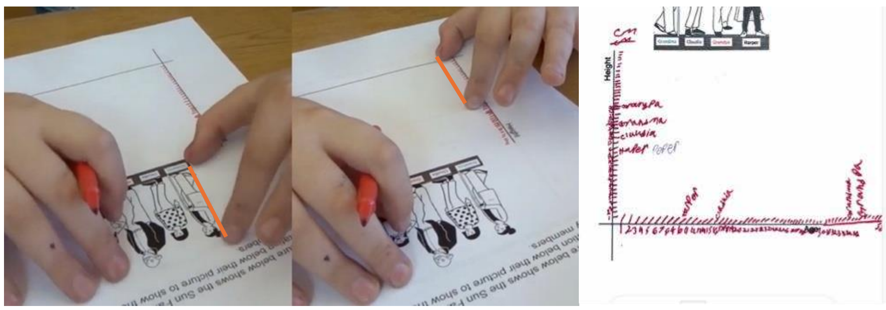

Figure 2: Thomas' Strategy and Final Representation

Thomas used a *non-indexing strategy for age* as he inferred ages based on the picture and represented them along the horizontal axis in the graph space. Specifically, he placed 60 tick marks on the Age axis, and plotted the family members from youngest (Harper) to oldest (Grandpa) along the axis. Thomas ordered the ages in ascending order (see Figure 2 right), and we inferred this order was independent of his representations of height, yielding a *quantitative ordering for age*. Thomas' graphing was indicative of using *two, separate, implied axes as number lines*. Based on how he partitioned each axis into unit-heights and unit-ages and plotted family members' height and age on each axis, we inferred he treated each axis as a number line. Note, Thomas plotted each family member twice, once along each axis. When the interviewer asked if he could find a way to mark each family member only once, Thomas maintained that age and height could not be represented together with a single point. Thus, we inferred his graph space remained as two, separate, implied axes as number lines.

Discussion

We presented a framework characterizing a variety of strategies students used when creating graphical representations given a pictorial scenario. Our framework attends to students' graphing activities of each quantity, height and age before potentially being coordinated together. The framework provides more nuanced "knowledge structures" (Sherin, 2000, p. 413) that students draw on when constructing graphs than previously described, attending to their graphing activities in relation to their reference frame and coordinate system activities. These activities refer to mental actions we inferred from observing students' physical graphing actions. We do not intend our framework to be exhaustive, but instead a starting point for future research that can contribute additional strategies to the framework. We believe the students' strategies in the framework can be leveraged to support students in achieving more conventional graphing meanings. For example, we can build from students' creations of 1-dimensional graphs as conceptual starting points to motivate the potential construction of a 2-dimensional coordinate system from their 1-dimensional graphs. While most research has described students' literal translations as hindering, we view it as a tool that could be productively used and subsequently modified to lead to more productive graphing meanings. We will be further examining these constructions as we continue in our research.

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