Oja's Algorithm for Streaming Sparse PCA

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Abstract

Oja's algorithm for Streaming Principal Component Analysis (PCA) for n datapoints in a d dimensional space achieves the same sin-squared error $O(r_{\mathsf{eff}}/n)$ as the offline algorithm in $O(d)$ space and $O(nd)$ time and a single pass through the datapoints. Here r_{eff} is the effective rank (ratio of the trace and the principal eigenvalue of the population covariance matrix Σ). Under this computational budget, we consider the problem of sparse PCA, where the principal eigenvector of Σ is s-sparse, and r_{eff} can be large. In this setting, to our knowledge, *there are no known single-pass algorithms* that achieve the minimax error bound in $O(d)$ space and $O(nd)$ time without either requiring strong initialization conditions or assuming further structure (e.g., spiked) of the covariance matrix. We show that a simple single-pass procedure that thresholds the output of Oja's algorithm (the Oja vector) can achieve the minimax error bound under some regularity conditions in $O(d)$ space and $O(nd)$ time. We present a nontrivial and novel analysis of the entries of the unnormalized Oja vector, which involves the projection of a product of independent random matrices on a random initial vector. This is completely different from previous analyses of Oja's algorithm and matrix products, which have been done when the r_{eff} is bounded.

1 Introduction

Principal Component Analysis (PCA) [Pea01, Jol03] is a classical statistical method for data analysis and visualization. Given a dataset $\{X_i\}_{i=1,\ldots,n}$ where $X_i \in \mathbb{R}^d$, sampled independently from a distribution with mean zero and covariance matrix Σ, the goal in PCA is to find the directions that explain most of the variance in the data. It is well known [Wed72, JJK$^+$16, Ver10] that the leading eigenvector, $\hat{v}$, of the empirical covariance matrix, $\hat{\Sigma}$, provides an optimal error rate under suitable tail conditions on the datapoints.

Computing $\hat{v}$ can be inefficient for large sample sizes, n, and dimensions d. Oja's algorithm [Oja82a] offers a comparable error rate in $O(nd)$ time and $O(d)$ space. Going back to the Canadian psychologist Donald Hebb's research [Heb49], it has attracted a lot of attention in theoretical Statistics and Computer Science communities [JJK$^+$16, AZL17, CYWZ18, YHW18, HW19a, HNW21, MP22, LSW21, Mon22, HNWW21]. In these works, the error metric is the $\sin^2$ error between the estimated vector and the principal eigenvector of Σ (true population eigenvector v_1). Notably, [JJK$^+$16], [AZL17], and [HNW21] establish that Oja's algorithm achieves the same $O(r_{\mathsf{eff}}/n)$ sin-squared error as the *offline algorithm* that estimates the top eigenvector of the empirical covariance matrix.

However, when the effective rank, r_{eff}, of Σ (defined as $\mathrm{Tr}(\Sigma)/\|\Sigma\|$) is large, PCA has been shown to be inconsistent [Pau07, JM09, JL09]. This setting comes up in sparse PCA problems, when v_1 is s-sparse (i.e. has only s nonzero entries). Let $\|.\|_0$ denote the l_0 norm, i.e, the count of non-zero vector entries. Then, sparse PCA can be formally framed as the optimization problem:

$$\widehat{v}_{\text{sparse}} := \arg\max_{w \in \mathbb{R}^d} \sum_i \left(X_i^T w\right)^2, \text{ under constraints } \|w\|_2 = 1, \ \|w\|_0 = s \tag{1}$$

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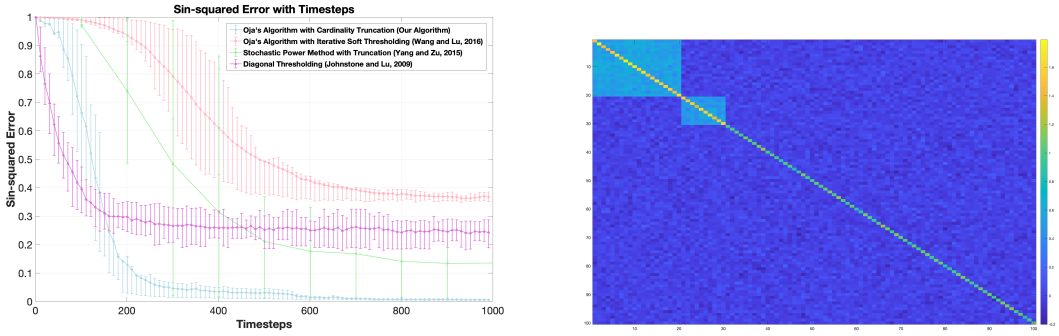


(a) (b)

Figure 1: Comparison of Sparse PCA algorithms for identifying leading eigenvector, v_1, operating in $O(d)$ space and $O(nd)$ time with population covariance matrix specified in [QLR19], Section 5.1. Figure (a) plots [JL09] (Purple), [YX15] (Black), [WL16] (Orange) and our proposed Algorithm 2 (Blue) for $n = d = 1000$, with error bars over 100 random runs. Figure (b) shows an image of the covariance matrix with $n = d = 100$.

In general, without further assumptions, Problem (1) is non-convex and NP-hard [MWA06], as it reduces to subset selection in ordinary least squares regression.

[VL12, CMW13] showed a $O\left(\sigma_*^2 s \log(d)/n\right)$ minimax lower bound for the $\sin^2$ error $1 - (v_1^T \widehat{v})^2$, where $\sigma_*^2 := \frac{\lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)^2}$. Here $\lambda_1 > \lambda_2 \geq \ldots \lambda_d$ are the eigenvalues of Σ. Extensive research has been conducted on optimal offline algorithms for sparse PCA, some of which are convex relaxation-based [BR13, dBEG08, VCLR13, STL07, ZX18, DMMW17, AW08, Ma13, CMW13]. Others involve iterated thresholding [JNRS10, Ma13, YZ13], where a truncated power-method is analyzed along to achieve sparsity. For brevity, we only describe algorithms that fit within the computation budget in consideration, i.e., $O(nd)$ time, $O(d)$ space. For a detailed comparison, see Table 1 and Appendix Section A.1.

Support recovery algorithms in $O(nd)$ time, $O(d)$ space: Consider the spiked covariance model

$$\Sigma = \sum_{i \in [r]} \nu_i v_i v_i^T + I_d \tag{2}$$

where I_d is the identity matrix, $\nu_i > 0$, and v_i are sparse. For the general case, we only assume v_1 is s-sparse. When $r = 1$, Σ_{ii} are the largest for $i \in S$. Diagonal thresholding essentially estimates Σ_{ii} within our computational budget and uses thresholding to recover the support [JL09, AW08]. However, as we will show, without knowing the support sizes in each eigenvector and the number of spikes, this algorithm can fail, even in a spiked setting with $r > 1$. Also, for $r = 1$, [BPP18] show how to adapt a black-box algorithm for sparse linear regression for support recovery.

Sparse PCA algorithms in $O(nd)$ time, $O(d)$ space: The streaming sparse PCA algorithms proposed by [YX15] and [WL16] require an initialization u_0 with a sufficiently large $|u_0^T v_1| = \Omega(1)$ (local convergence), which can be hard to find for large d and a general Σ. See Table 1 for details.

In light of this lack of $O(nd)$ time, $O(d)$ space globally convergent algorithms for sparse PCA, we ask the following question in this work:

Goal: *Is there a single-pass algorithm that, under a general Σ with s-sparse v_1, outputs $\hat{v}$ achieving the minimax $\sin^2$ error $(1 - \langle \widehat{v}, v_1 \rangle^2)$ with $O(d)$ space, $O(nd)$ time, without a strong initialization?*

We provide a surprisingly simple answer to the above question:

Theorem 1.1 (Informal)**.** *For a suitable range of the effective rank r_{eff} and the ratio λ_1/λ_2, there exists a single pass algorithm $\mathcal{A}$ that recovers the support of v_1 using Oja's algorithm, operates under $O(d)$ space, $O(nd)$ time and returns $\hat{v}$ with the minimax optimal $\sin^2$ error, $O\left(\sigma_*^2 s \log(d)/n\right)$, for a general covariance matrix.*

Our contributions:

1. **Support recovery:** We show, for a *general* Σ with the only constraint of a s-sparse v_0 that the top k entries of the Oja vector in magnitude include the true support with high probability. The Oja vector is initialized by a random unit vector.

2. **Sparse PCA:** We use the recovered support to achieve a minimax optimal sparse PCA algorithm.

3. **Entrywise analysis:** Our analysis is nontrivial and novel because it deviates from all existing analyses of matrix products and streaming PCA [HNWTW20, HW19b, LSW21, Lia23] which require $\|X_i\|^2/\lambda_1$ or r_{eff} to be bounded to obtain the $O(1/n)\sin^2$ error rate.

Paper(s)	λ_1/λ_2	Σ	Global conv.?	Space	Time	$\sin^2$ error
Johnstone and Lu [JL09]	$1+o(1)$	Spiked	Y	$O(d)$	$O(nd)$	$o(1)$
SDP-based [VCLR13] [dBEG08]	$1+o(1)$	General	Y	$O(d^2)$	$O(n^\omega + d^\omega)$	$O\left(\frac{s^2\log(d)}{n}\right)$
Shen et al. [SSM13]	$\Omega(d^\epsilon), \epsilon>0$	General	Y	$O(d^2)$	$O(nd^2)$	$o(1)$
Ma, Cai et al. [Ma13] [CMW13]	$1+\Omega(1)$	Spiked	Y	$O(d^2)$	$O(nd^2)$	$O\left(\frac{s\log(d)}{n}\right)$
Yuan and Zhang [YZ13]	$1+\Omega(1)$	General	N	$O(d^2)$	$O(nd^2)$	$O\left(\frac{s\log(d)}{n}\right)$
Yang and Xu [YX15]	$1+\Omega(1)$	Spiked	N	$O(d)$	$O(nd)$	$O\left(\frac{s\log(d)}{n}\right)$
Wang and Lu [WL16]	$1+\Omega(1)$	Spiked	N	$O(d)$	$O(nd)$	$o(1)$
Oja's Algorithm [JJK+16]	$1+o(1)$	General	Y	$O(d)$	$O(nd)$	$O\left(\frac{r_{\mathsf{eff}}}{n}\right)$
Deshp et al. [DM+16]	$1+o(1)$	Spiked	Y	$O(d^2)$	$O(nd^2)$	$O\left(\frac{s^2\log(d)}{n}\right)$
Qiu et al.(Cor. 2) [QLR19]	$1+o(1)$	General	Y	$O(d^2)$	$\Omega(nd^2)$ [1]	$O\left(\frac{s^2\log(d)}{n}\right)$
Qiu et al. (Th. 4) [QLR19]	$1+o(1)$	General	Y	$O(d^2)$	$O(nd^2)$	$O\left(\frac{d^2\log(d)}{\sqrt{n}}\right)$
Gataric et al. (Th. 2) [GWS20]	$1+o(1)$ [2]	Spiked	Y	$O(d^2)$	$O(nd^2)$ [3]	$O\left(\frac{s\log(d)}{n}\right)$
Our work	$1+\Omega(1)$	General	Y	$O(d)$	$O(nd)$	$O\left(\frac{s\log(d)}{n}\right)$

Table 1: Comparison of sparse PCA algorithms for estimating v_1, based on various parameters. We require Assumptions 1 and 2. The other algorithms may be valid under weaker assumptions. For ease of comparison, we fix $\frac{\lambda_1}{\lambda_2} = 1+\Omega(1)$ and $\frac{r_{\mathsf{eff}}\log(n)}{n} = O(1)$ for our results in this table.

In Figure 1b, for a simple spiked model with $r=2$, we show the relative performances of all $O(nd)$ time and $O(d)$ algorithms in Table 1. Our thresholded and renormalized Oja algorithm outperforms all other algorithms operating under the same computational budget. The diagonal thresholding algorithm ([JL09]), which is successful for the special case of $r=1$, has a large error in the general case.

We now present an outline of our paper. We start by describing the problem setup and assumptions in Section 2. Then we present our main results in Section 3, which includes our results for Support

[1] The authors do not state the runtime explicitly. The algorithm, as stated, requires at least $\Omega(nd^2)$ computation.

[2] The authors require $\lambda_1/\lambda_2 \geq 1 + O(\sqrt{s^3\log d/n})$

[3] When there are m spikes, Thm 2 of [GWS20] requires $A = \Omega(\frac{\nu_m^2}{\nu_1^2} d^2\log(d))$. When ν_1 and ν_m are the same order, storing the empirical covariance matrix is computationally more efficient.

Recovery (Section 3.1), Sparse PCA (Section 3.3) and Entrywise Deviation bounds (Section 3.5) for the Oja vector. Finally, we provide a sketch of the proof along with the techniques used in Section 4.

2 Problem setup and preliminaries

Notation. We use $\mathbb{E}[.]$ to denote expectation and $[n]$ for $\{1, \ldots, n\}$. The matrix multiplication constant is denoted as $\omega \approx 2.372$. $X \perp\!\!\!\perp Y$ represents statistical independence between random variables X and Y. The ℓ^2 norm for vectors and operator norm for matrices is $\|.\|_2$, the count of nonzero vector elements (ℓ^0 norm) is $\|.\|_0$, and the Frobenius norm for matrices is $\|.\|_F$. For $v \in \mathbb{R}^d, R \subseteq [d]$, $\lfloor v \rfloor_R \in \mathbb{R}^d$ is the truncated vector with entries outside R set to 0. $I_d \in \mathbb{R}^{d\times d}$ is the identity matrix, with i^{th} column $e_i \in \mathbb{R}^{d\times 1}$. For any set $T \subseteq [d]$, $I_T \in \mathbb{R}^{d\times d}$ is defined as $I_T(i,j) = \mathbb{1}(i,j \in T)\mathbb{1}(i=j)$, where $\mathbb{1}(.)$ is the indicator random variable. $\langle A, B\rangle := \mathrm{Tr}(A^T B)$ represents the matrix inner product. $\widetilde{O}$ and $\widetilde{\Omega}$ represent order notations with logarithmic factors. We start by defining subgaussianity for multivariate distributions.

Definition 2.1. *A random mean-zero vector $X \in \mathbb{R}^d$ with covariance matrix Σ is a $\sigma-$subgaussian random vector ($\sigma > 0$) if for all vectors $v \in \mathbb{R}^d$, we have $\mathbb{E}\left[\exp\left(v^T X\right)\right] \leq \exp\left(\sigma^2 v^T \Sigma v/2\right)$. Equivalently, $\exists\, L > 0$, such that $\forall p \geq 2$, $\left(\mathbb{E}\left[|v^T X|^p\right]\right)^{\frac{1}{p}} \leq L\sigma\sqrt{p}\sqrt{v^T\Sigma v}$.* [4]

This definition of subgaussianity has been used in contemporary works on PCA and covariance estimation (See for example [MZ20, JLT20, DKPP23] and Theorem 4.7.1 in [Ver18]). We operate under the following two assumptions, unless otherwise specified,

Assumption 1 (Subgaussianity)**.** *$\{X_i\}_{i\in[n]}$ are of independent and identically distributed σ-subgaussian vectors in $\mathbb{R}^d$ with covariance matrix $\Sigma := \mathbb{E}\left[X_i X_i^T\right]$.*

We denote the eigenvectors of Σ as $v_1, v_2, \cdots v_d$ and the corresponding eigenvalues as $\lambda_1 > \lambda_2 \geq \cdots \lambda_d$. Define $V_\perp := [v_2, v_3, \cdots v_d] \in \mathbb{R}^{d\times(d-1)}$ and $\Lambda_2 \in \mathbb{R}^{(d-1)\times(d-1)} = \mathrm{diag}\left(\lambda_2, \lambda_3, \cdots \lambda_d\right)$.

Assumption 2 (Sparsity and Spectral gap)**.** *We assume that $\max\left\{1, \frac{\lambda_2}{\lambda_1-\lambda_2}\right\}\frac{\mathrm{Tr}(\Lambda_2)}{\lambda_1-\lambda_2} \leq \frac{cn}{\log(n)}$ and $\frac{\lambda_1}{\lambda_1-\lambda_2} \leq c\sqrt{\frac{n}{\log^2(n)}}$ for an absolute constant $c > 0$. The leading eigenvector, v_1, satisfies $\|v\|_0 \leq s$ with support set $S := \{i : v_1(i) \neq 0\}$.*

Remark 2.2. *We note that Assumption 2 allows for r_{eff} to be as large as d, given a sufficient eigengap. This can be observed by setting $\lambda_1 = \lambda_2(1+g_n)$ for some $g_n > 0$. Note that $\frac{\mathrm{Tr}(\Lambda_2)}{\lambda_1-\lambda_2} \leq \min\left(\frac{1+g_n}{g_n} r_{\mathsf{eff}}, \frac{1}{g_n} d\right)$. If $g_n \leq 1$, then,*

$$\max\left\{1, \frac{1}{g_n}\right\}\frac{\mathrm{Tr}(\Lambda_2)}{\lambda_1-\lambda_2} \leq \frac{1}{g_n}\min\left(\frac{1+g_n}{g_n} r_{\mathsf{eff}}, \frac{1}{g_n} d\right) = 2\frac{1}{g_n}\min\left(\frac{1+g_n}{2g_n} r_{\mathsf{eff}}, \frac{1}{2g_n} d\right) \leq 2r_{\mathsf{eff}}/g_n^2$$

If $g_n \gg 1$, then,

$$\max\left\{1, \frac{1}{g_n}\right\}\frac{\mathrm{Tr}(\Lambda_2)}{\lambda_1-\lambda_2} \leq \frac{\mathrm{Tr}(\Lambda_2)}{\lambda_1-\lambda_2} \leq \frac{d}{g_n}$$

therefore, in both cases, as long as $d \leq \frac{ng_n}{\log n}$, r_{eff} can be as large as d, while allowing for Assumption 2 to hold.

Oja's algorithm with constant learning rate. With a constant learning rate, η, and initial vector, u_0, Oja's algorithm [Oja82b], denoted as $\mathsf{Oja}\left(\{X_t\}_{t\in[n]}, \eta, u_0\right)$, performs the updates, $u_t \leftarrow (I + \eta X_t X_t^T)u_{t-1},\ u_t \leftarrow \frac{u_t}{\|u_t\|_2}$. For convenience of analysis, we also define $\forall t \in [n]$,

$$B_t := \left(I + \eta X_t X_t^T\right)\left(I + \eta X_{t-1} X_{t-1}^T\right)\cdots\left(I + \eta X_1 X_1^T\right),\ B_0 = I \tag{3}$$

[4] The results developed in this work follow if instead of subgaussianity, the moment bound holds $\forall\ p \leq 8$.

3 Main results

We present our main contributions in two stages. Firstly, in Section 3.1, we demonstrate that with an upper bound on the support size, the top elements of the Oja vector include the support with constant probability, which can be enhanced using a boosting procedure (SuccessBoost) described in Section 3.4. Secondly, in Section 3.3, we use the support to extract the eigenvector and provide a high-probability $\sin^2$ error guarantee. Section 3.5 details our results on bounding the entrywise deviation of the Oja vector, which are crucial to our proofs and of independent interest. Detailed proofs are in the Appendix, Sections A.4 and A.5, with the learning rate, η, specified in Lemma A.2.4.

3.1 Support recovery

Algorithm 1 OjaSupportRecovery $\left(\{X_i\}_{i\in[n]}, k, \eta\right)$

1: **Input** : Dataset $\{X_i\}_{i\in[n]}$, Cardinality parameter $k \geq s$, learning rate $\eta > 0$
2: $u_0 \sim \mathcal{N}(0, I)$
3: $\hat{v} \leftarrow \mathsf{Oja}\left(\{X_i\}_{i\in[n]}, \eta, y_0\right)$
4: $\widehat{S} \leftarrow$ Indices of k largest values of $|\hat{v}|$
5: **return** $\widehat{S}$

Algorithm 1 provides an estimate, $\widehat{S}$, of the true support set, S. It computes the Oja vector and returns the set of indices corresponding to its k largest entries in absolute value. Our key result in Lemma 3.1 discusses the recovery of the support set, S, for any $k \geq s$, without requiring exact knowledge of the sparsity parameter s. Using Algorithm 1, it provides a set $\widehat{S} \supseteq S$ with probability at least 0.9.

Lemma 3.1 (s-Agnostic Recovery)**.** *Under Assumptions 1,2, for* $\min_i |v_1(i)| = \widetilde{\Omega}\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\left(\frac{d}{n^2}\right)^{\frac{1}{4}}\right)$, $\widehat{S} \leftarrow \mathsf{OjaSupportRecovery}\left(\{X_i\}_{i\in[n]}, k, \eta := \frac{3\log(n)}{n(\lambda_1-\lambda_2)}\right)$ *with* $k \geq s$ *satisfies,* $\mathbb{P}\left(S \subseteq \widehat{S}\right) \geq 0.9$.

If $k = s$, i.e, the size of the support is exactly known, then we can improve the result of Lemma 3.1 to obtain an estimator, $\widehat{S}$, of the support set with high probability. Theorem 3.2 provides the corresponding guarantees. The SuccessBoost algorithm uses geometric aggregation on subsets returned from Algorithm 1 run on $\log(1/\delta)$ disjoint subsets of the data and is described in Section 3.4.

Theorem 3.2 (High probability support recovery)**.** *Let Assumptions 1, 2 hold. For dataset* $\mathcal{D} := \{X_i\}_{i\in[n]}$, *let* $\mathcal{A}$ *be the randomized algorithm which computes* $\widehat{S} \leftarrow \mathsf{OjaSupportRecovery}\left(\{X_i\}_{i\in[n]}, k, \eta\right)$, *where* $\eta := \frac{3\log(n)}{n(\lambda_1-\lambda_2)}$ *and* $k = s$. *Then, for* $\delta \in (0,1)$, $\min_i |v_1(i)| = \widetilde{\Omega}\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\left(\frac{d}{n^2}\right)^{\frac{1}{4}}\right)$, $\tilde{S} \leftarrow \mathsf{SuccessBoost}\left(\{X_i\}_{i\in[n]}, \mathcal{A}, \delta\right)$ *satisfies,*

$$\mathbb{P}\left(\tilde{S} = S\right) \geq 1 - \delta$$

For comparison, existing support recovery algorithms for general Σ are known for convex-relaxation-based algorithms like the SDP-based algorithm of [LV15]. These require a much larger computational budget than ours. In Section 3.3, we show how to use the s-agnostic support recovery in Lemma 3.1 to perform Sparse PCA and obtain a $\sin^2$ error guarantee, where the final high probability error bound is obtained using a similar probability-boosting argument. For the learning rate, we follow the convention in related work ([BDF13, XHS$^+$18, JJK$^+$16, AZL17, HNWW21]) and choose the optimal value of the learning rate, which requires the knowledge of $\lambda_1 - \lambda_2$. We believe an educated guess of η would lead to consistency at the cost of a suboptimal error bound.

3.2 Comparison with other support recovery algorithms

We note that (see Table 1), for the spiked model with $r = 1$ (Eq 2) [JL09] and [AW08] provide a diagonal thresholding algorithm for support recovery using $O(d)$ space and $O(nd)$ time.[5] To achieve a high-probability guarantee for the estimated support set, $\widehat{S}$, of the form $\mathbb{P}\left(\widehat{S} = S\right) \geq 1 - \delta$, they

[5]The algorithm proposed in [AW08] allows for a slight generalization of the spiked model in Eq 2.

require (Proposition 1, [AW08]) $n = \Omega\left(s^2 \log(d) + \log\left(\frac{1}{\delta}\right)\right)$. In comparison, Theorem 3.2 requires a larger sample size.

Remark 3.3. *In practice, we will not know whether Σ is spiked or general. So, we can always augment our support recovery algorithm by taking a union of the support from the Oja vector and diagonal thresholding, still maintaining $O(nd)$ time and $O(d)$ space.*

However, diagonal thresholding only works for the spiked model with a single spike, which our results do not require. It is easy to construct a Σ where the elements in the support of an eigenvector with a small eigenvalue have a larger magnitude than those of v_1 (Eq A.13). Here the diagonal thresholding method fails (see Figure 1a and Proposition 3.4). The explicit construction and the proof of Proposition 3.4 are available in the Appendix Section A.2. Figure 1 a) plots the $\sin^2$ error due to different Sparse PCA algorithms operating in $O(d)$ space and $O(nd)$ time on such a covariance matrix, Σ, which is visualized in Figure 1 a).

Proposition 3.4 (Lower bound for diagonal thresholding)**.** *Let Assumption 1 hold. For any diagonal-thresholding algorithm, $\mathcal{A}$, performing support recovery with sparsity parameter s such that $n = \Omega\left(\sigma^4 s^2 \log(d)\right)$, there exists a covariance matrix Σ with principal eigenvector, v_1, $\|v_1\|_0 = s$, such that, $\mathbb{P}\left(\left|\widehat{S} \bigcap S\right| = 0\right) \geq 1 - d^{-10}$.*

It may seem that if r in Eq 2 is small, and the sparsity parameters of each v_i, $i \leq r$ are known, then diagonal thresholding would work. However, in general, r can be as large as d and the union of supports of v_i can be $[d]$.

3.3 Sparse PCA

In this section, we describe our results for Sparse PCA, which use the support recovery guarantees developed in Section 3.1. For the results in this section, we split the dataset $D := \{X_i\}_{i\in[n]}$ into two halves and estimate the support using the first half as $\widehat{S} \leftarrow \mathsf{OjaSupportRecovery}\left(\{X_i\}_{i\in[\frac{n}{2}]}, k, \eta := \frac{3\log(n)}{n(\lambda_1-\lambda_2)}\right)$ and input, $k \geq s$. The second half of the samples are then used to compute the estimated sparse eigenvector. Algorithm 2 describes a general procedure for Sparse PCA given access to an estimated support set, $\widehat{S}$. We start with an intuitive procedure in Theorem 3.5, which runs Oja's algorithm on the data and then uses the support to truncate the estimated eigenvector.

Algorithm 2 $\mathsf{TruncateOja}\left(\{X_i\}_{i\in[n]}, \widehat{S}, \mathcal{A}, \Theta\right)$

1: **Input** : Dataset $\{X_i\}_{i\in[n]}$, , estimated support set $\widehat{S} \subseteq [d]$, Algorithm $\mathcal{A}$, Parameters Θ
2: $\hat{v} \leftarrow \mathcal{A}\left(\{X_i\}_{i\in[n]}, \Theta\right)$
3: $\hat{v}_{\mathsf{truncvec}} \leftarrow \frac{\lfloor \hat{v} \rfloor_{\hat{S}}}{\left\| \lfloor \hat{v} \rfloor_{\hat{S}} \right\|_2}$
4: **return** $\hat{v}_{\mathsf{truncvec}}$

Theorem 3.5 (Vector Truncation)**.** *Let Assumptions 1 and 2 hold and $k \geq s$. For dataset $D := \{X_i\}_{i\in[n]}$ and $w_0 \sim \mathcal{N}(0, I)$, let $\mathcal{A}$ be the randomized algorithm which computes $\hat{v}_{\mathsf{truncvec}} \leftarrow \mathsf{TruncateOja}\left(\{X_i\}_{i\in\left(\frac{n}{2}, n\right]}, \widehat{S}, \mathsf{Oja}, \{\eta, w_0\}\right)$, where $\eta := \frac{3\log(n)}{n(\lambda_1-\lambda_2)}$. Then, for $\min_i |v_1(i)| = \widetilde{\Omega}\left(\left(\frac{d}{n^2}\right)^{\frac{1}{8}}\right)$, $\tilde{v} \leftarrow \mathsf{SuccessBoost}\left(\{X_i\}_{i\in[n]}, \mathcal{A}, d^{-10}\right)$ satisfies,*

$$\sin^2(\tilde{v}, v_1) \leq C'' \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right)^2 \frac{k \log^2(d)}{n}$$

with probability at least $1 - d^{-10}$, where $C'' \geq 0$ is an absolute constant.

Remark 3.6 (Limitation)**.** *Existing inconsistency results on PCA [JL09] provide a threshold for signal strength $(\lambda_1 - \lambda_2)/\lambda_1$, below which, the principal eigenvector of $\hat{\Sigma}$ is asymptotically orthogonal to v_1. We believe a similar result may hold for the Oja vector, which leads to the signal strength condition in Assumption 2.*

Note that the rate obtained in Theorem 3.5 nearly matches the minimax lower bound proved in [VL12, CMW13], up to a factor of $\frac{\lambda_2}{\lambda_1}$ and $\log(d)$ and has optimal dependence on s, and n. A limitation of Algorithm 2 is that it uses the estimated support, $\widehat{S}$, at the very end after computing the estimated eigenvector to enhance the signal by truncation. Instead, one may run Oja's algorithm on datapoints restricted to the recovered support in the beginning.

To this end, we use the algorithm in [Lia23] (denoted by OptimalOja, see Proposition A.5.3) for subgaussian data, which uses an iteration-dependent sequence of step-sizes $\{\eta_i\}_{i\in[n]}$. We run Algorithm 2 with OptimalOja as the procedure to do sparse PCA. This leads to the minimax error rate, shown in Theorem 3.7. The high probability bounds in Theorem 3.5 and 3.7 both use the support recovery guarantees derived in Section 3.1 and the boosting procedure described in Section 3.4. Detailed proofs for both results can be found in Appendix Section A.5.

Theorem 3.7 (Data Truncation)**.** *Let Assumptions 1 and 2 hold and $k \geq s$. For dataset $\mathcal{D} := \{X_i\}_{i\in[n]}$ and $w_0 \sim \mathcal{N}(0, I)$, let $\mathcal{A}$ be the randomized algorithm which computes $\hat{v}_{\mathsf{truncvec}} \leftarrow \mathsf{TruncateOja}\left(\left\{\lfloor X_i \rfloor_{\widehat{S}}\right\}_{i\in\left(\frac{n}{2},n\right]}, \widehat{S}, \mathsf{OptimalOja}, \{\{\eta_t\}_{t\in[\frac{n}{2}]}, w_0\}\right)$. Then for $\min_i |v_1(i)| = \widetilde{\Omega}\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\left(\frac{d}{n^2}\right)^{\frac{1}{4}}\right)$, $\tilde{v} \leftarrow \mathsf{SuccessBoost}\left(\{X_i\}_{i\in[n]}, \mathcal{A}, d^{-10}\right)$ satisfies,*

$$\sin^2(\tilde{v}, v_1) \leq C'' \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)^2}\frac{k\log(d)}{n}$$

with probability at least $1 - d^{-10}$, where $C'' \geq 0$ is an absolute constant.

Remark 3.8. *Algorithm 2, with both* Oja *and* OptimalOja *as input procedures require a simple initialization vector $w_0 \sim \mathcal{N}(0, I)$. In contrast, the block stochastic power method-based algorithm presented in [YX15] provides local convergence guarantees (see Theorem 1) requiring a block size of $O(s\log(d))$. They provide an initialization procedure, but the theoretical guarantees to achieve such an initialization require block size $\Omega(d)$. [YZ13] also require a close enough initialization. In the particular setting of a single-spiked covariance model, they require $|w_0^T v_1| = \Omega(1)$. In comparison for Algorithm 2, 3.7, it suffices to have $|w_0^T v_1| \geq \frac{\delta}{\sqrt{e}}$ with probability at least $1-\delta$ (see Lemma A.2.1).*

3.4 Probabilistic boosting

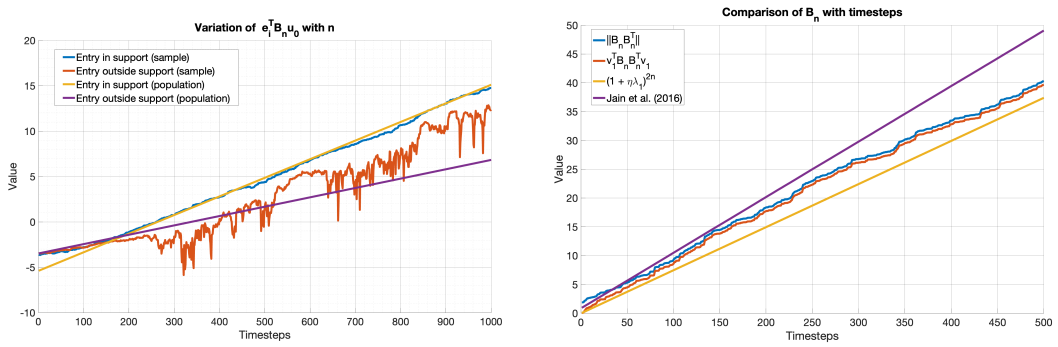
In this section, we describe a generic procedure for boosting the success probability of a given randomized algorithm, $\mathcal{A}$ (also see [KLL+23]). If $\mathcal{A}$ satisfies Definition 3.9, then its probability can be boosted using this procedure. The formal guarantees of the boosting procedure are provided in Lemma 3.10 (proof in Appendix Section A.2). It divides the data evenly into $\log\left(\frac{1}{\delta}\right)$ buckets[6], runs $\mathcal{A}$ on each bucket and aggregates the results via pairwise comparisons.

Definition 3.9. *Let $\mathcal{T}$ be a set with metric ρ and $\mathcal{A}$ be a randomized algorithm which takes as input n i.i.d datapoints $D := \{X_i\}_{i\in[d]}$ and possibly additional statistically independent parameters θ, and returns an estimate $q \in \mathcal{T}$, which satisfies $\mathbb{P}(\rho(q, q_*) \geq \epsilon) \leq \frac{1}{3}$ for a fixed $q_* \in \mathcal{T}$. Then, $\mathcal{A}$ is said to be a constant success oracle with parameters $(D, \theta, \mathcal{T}, \rho, q_*, \epsilon)$, denoted as $\mathcal{A} := \mathsf{ConstantSuccessOracle}(D, \theta, \mathcal{T}, \rho, q_*, \epsilon)$.*

Algorithm 3 $\mathsf{SuccessBoost}\left(\{X_i\}_{i\in[n]}, \mathcal{A}, \delta\right)$

1: **Input** : Dataset $D := \{X_i\}_{i\in[n]}$, $\mathcal{A} := \mathsf{ConstantSuccessOracle}(D, \theta, \mathcal{T}, \rho, q_*, \epsilon)$, Required failure probability δ
2: **Return** : An estimate $\tilde{q} \in \mathcal{T}$ such that $\mathbb{P}(\rho(\tilde{q}, q_*) \leq 3\epsilon) \geq 1-\delta$
3: $S \leftarrow 30\log\left(\frac{1}{\delta}\right)$, $B \leftarrow n/S$
4: $\forall t \in [S], q_t \leftarrow \mathcal{A}\left(\left\{X_{B(t-1)+i}\right\}_{i\in[B]}, \theta, \mathcal{T}, \rho, q_*, \epsilon\right)$, $\mathcal{C}_t \leftarrow \{t' \in [S] : \rho(q_t, q_{t'}) \leq 2\epsilon\}$
5: **If** $\exists q_t$ such that $|\mathcal{C}_t|/S \geq 0.4$ **Return** q_t **Else Return** $\perp$

[6]For simplicity, we assume S, B in Algorithm 3 are integers.



(a) (b)

Figure 2: We use Σ used in [QLR19], Section 5.1. (a) Variation of $\log\left(|e_i^\top B_n u_0|\right)$ for $i \in S$ and $i \notin S$ (y-axis) with n (x-axis) for a fixed unit vector u_0. η is set as Theorem 3.5 and n grows from 1 to 1000. The lines labelled "sample" plot $\log(|e_i^\top B_n u_0|)$, whereas the "population" curves plot $\log(|\mathbb{E}\left[e_i^\top B_n u_0\right]|)$. (b) Variation of $\log\left(\left\|B_n B_n^T\right\|\right)$ and $\log\left(v_1^T B_n B_n^T v_1\right)$ (y-axis) with $n \in [300]$ (x-axis). We also plot log of the bound of $\left\|B_n B_n^T\right\|$ as in [JJK+16] and $2n\log(1+\eta\lambda_1)$ for comparison.

Lemma 3.10 (Geometric Aggregation for Boosting)**.** *Let* $\mathcal{A} := \mathsf{ConstantSuccessOracle}(D, \theta, \mathcal{T}, \rho, q_*, \epsilon)$ *(Definition 3.9) for dataset* $D := \{X_i\}_{i\in[n]}$*. Then for* $\delta \in (0,1)$, $\tilde{q} \leftarrow \mathsf{SuccessBoost}\left(\{X_i\}_{i\in[n]}, \mathcal{A}, \delta\right)$ *satisfies* $\mathbb{P}(\rho(\tilde{q}, q_*) \le 3\epsilon) \ge 1-\delta$.

3.5 Entrywise deviation of the Oja vector

To analyze the success probability of recovering the indices in S, we will define the following event, $\mathcal{E} := \{S \subseteq \widehat{S}\}$. We now upper bound $\mathbb{P}(\mathcal{E}^c)$. Define an element of the unnormalized Oja vector as $r_i := e_i^T B_n u_0,\ \ i \in [d]$. Here $u_0 \sim \mathcal{N}(0, I)$ is the initialization used in Algorithm 1. Observe that

$$\mathcal{E} \iff \exists \tau_n > 0 \text{ such that } \{\forall i \in S, |r_i| \ge \tau_n\} \bigcap \{|\{i : i \notin S, |r_i| \ge \tau_n\}| \le k - s\}$$

or equivalently,

$$\mathcal{E}^c \iff \forall \tau_n > 0, \{\exists i \in S, |r_i| \le \tau_n\} \bigcup \{|\{i : i \notin S, |r_i| \ge \tau_n\}| > k - s\}$$

Therefore, for any fixed $\tau_n > 0$, $\mathcal{E}^c \implies \{\exists i \in S, |r_i| \le \tau_n\} \bigcup \{|\{i : i \notin S, |r_i| \ge \tau_n\}| > k - s\}$ We will, therefore, be interested in the tail behavior of r_i for $i \in S$ and $i \notin S$. Before presenting our theorems, we will use Figure 2 to emphasize the daunting nature of what we aim to prove. Consider the quantity $\mathbb{E}[r_i|u_0] = \mathbb{E}\left[e_i^T B_n u_0 | u_0\right]$. We use $X = C \pm \Delta$ to denote $|X - C| \le \Delta$.

$$\begin{aligned} \mathbb{E}[r_i|u_0] &= e_i^T \mathbb{E}[B_n] v_1 v_1^T u_0 + e_i^T \mathbb{E}[B_n] V_\perp V_\perp^T u_0 \\ &= \begin{cases} e_i^T v_1 v_1^T u_0 (1+\eta\lambda_1)^n \pm \left|e_i^T V_\perp V_\perp^T u_0\right| (1+\eta\lambda_2)^n & \text{For } i \in S \\ \pm \left|e_i^T V_\perp V_\perp^T u_0\right| (1+\eta\lambda_2)^n & \text{For } i \notin S \end{cases} \end{aligned} \tag{4}$$

Thus, traditional wisdom would make us hope that the elements, r_i, will concentrate around their respective expectations, whose absolute values are off by a ratio $|v_1(i)||u_0^T v_1| \exp(n\eta(\lambda_1 - \lambda_2))$.

However, Figure 2(a) shows that while the elements in the support seem *close* to their expectation, those not in support are, on average, *much larger* than their expectation. First, note that elementwise analysis of the Oja vector has not been done even in the low dimensional regime where $r_{\text{eff}}/n \to 0$. In this regime, there is very recent related work for eigenvectors of the empirical covariance matrix $\hat{\Sigma}$ [AFW22] which are not applicable here. In the high-dimensional case, an analog can be drawn with elements $\hat{\Sigma}$, which concentrate around their mean individually. Yet, $\|\hat{\Sigma} - \Sigma\|$ is not small. Thus, thresholding $\hat{\Sigma}$ obtains consistent estimates of Σ under sparsity assumptions [BL09, DM+16, Nov23].

A similar principle is applied by [SSM11] where the eigenvector of $\hat{\Sigma}$ is truncated. They assume that n is fixed, and $\lambda_1/\lambda_2 = d^\alpha \to \infty$ as $d \to \infty$. In comparison, our analysis is about products of random matrices, not sums, and hence, completely different. We will show that $\hat{v}_1(i)$, even when $i \in S$, do not concentrate. But for a suitably chosen threshold, They are large with high probability,

whereas those outside S are much lower with high probability. Proving this is also difficult because the analysis involves the concentration of the projection of a product of independent high dimensional matrices on some initial random vector. Lemma 3.11 establishes exactly that for elements in the support.

Lemma 3.11 (Tail bound in support)**.** *Fix a $\delta \in (0.1, 1)$. Define the event $\mathcal{G} := \left\{|v_1^T u_0| \geq \frac{\delta}{\sqrt{e}}\right\}$ and threshold $\tau_n := \frac{\delta}{\sqrt{2e}} \min_{i \in S} |v_1(i)| (1 + \eta\lambda_1)^n$. Let the learning rate be set as in Lemma 3.1. Then, for an absolute constant $C_H > 0$,*

$$\forall i \in S, \quad \mathbb{P}\left(|r_i| \leq \tau_n \middle| \mathcal{G}\right) \leq C_H \left[\eta\lambda_1 \log(n) + \eta\lambda_1 \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right) \frac{1}{v_1(i)^2}\right]$$

Our next result provides a bound for $i \notin S$.

Lemma 3.12 (Tail bound outside support)**.** *Fix a $\delta \in (0.1, 1)$. Let the learning rate be set as in Lemma 3.1 and define the threshold $\tau_n := \frac{\delta}{\sqrt{2e}} \min_{i \in S} |v_1(i)| (1 + \eta\lambda_1)^n$. Then, for $\min_i |v_1(i)| = \widetilde{\Omega}\left(\frac{\lambda_1}{\lambda_1 - \lambda_2} \left(\frac{d}{n^2}\right)^{\frac{1}{4}}\right)$ and an absolute constant $C_T > 0$ we have,*

$$\forall i \notin S, \quad \mathbb{P}\left(|r_i| > \tau_n\right) \leq C_T \left[\eta^2\lambda_1^2 \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right)^2 \left(\frac{1}{\delta^2 \min_{i \in S_{hi}} v_1(i)^2}\right)^2\right]$$

The proofs of Lemmas 3.11 and 3.12 are based on tail-bounds involving the second and fourth moments of $r_i := e_i^T B_n u_0$. The details of obtaining the tail bounds are deferred to the Appendix Section A.3. The results developed in this section are used to analyze the support recovery and $\sin^2$ error guarantees provided in Section 3.1 and 3.3. We provide a brief proof sketch in Section 4.

4 Proof technique

In this section, we outline the proof techniques for the entrywise deviation bounds in Lemmas 3.11 and 3.12. These bounds are crucial for analyzing both the support recovery results (Lemma 3.1 and Theorem 3.2) and the sparse PCA results (Theorems 3.5 and 3.7). The proof involves deriving bounds on the expectation and second moment of $u_0^T B_n U U^T B_n u_0$, where $U \in \mathbb{R}^{d \times k}$ is a fixed matrix and $u_0 \sim \mathcal{N}(0, I)$. It then applies Chebyshev's inequality to obtain the tail bound. For the proof sketch, we use $U = e_i$, but we maintain general notation for broader applicability in Theorem 3.5. For our results, we also need to bound this quantity with $U = I_S$ (see Lemma A.5.1 for details). Our techniques to bound $\mathbb{E}[u_0^T B_n U U^T B_n u_0]$ are detailed in Section 4.1.

4.1 Solving a linear system of recursions

One can show that (see Lemma A.2.11 in Appendix),

$$\mathbb{E}\left[u_0^T B_n^T U U^T B_n u_0\right] = \underbrace{\mathbb{E}\left[v_1^T B_n^T U U^T B_n v_1\right]}_{=:\alpha_n} + \underbrace{\mathbb{E}\left[\mathrm{Tr}\left(V_\perp^T B_n^T U U^T B_n V_\perp\right)\right]}_{=:\beta_n} \tag{5}$$

We start by showing how to bound α_n and β_n. Before we dive into our techniques, we note that the analysis of Oja's algorithm [JJK$^+$16] in the non-sparse setting provides some tools that we could potentially use here. Using the recursion from Lemma 9 in [JJK$^+$16], we get

$$\left\|\mathbb{E}\left[B_n U U^T B_n^T\right]\right\| \leq \exp(2n\eta\lambda_1 + n\eta^2\mathcal{V}) \left\|U U^T\right\|, \tag{6}$$

where $\mathcal{V}$ is a variance parameter defined as $\left\|\mathbb{E}\left[(A_1 - \Sigma)(A_1 - \Sigma)^T\right]\right\|$. Lemma A.2.3 shows that for σ-subgaussian X (definition 2.1),

$$\mathcal{V} := \left\|\mathbb{E}\left[(A_1 - \Sigma)(A_1 - \Sigma)^T\right]\right\| = \left\|\mathbb{E}\left[A_1 A_1^T\right] - \Sigma^2\right\| \leq 2L^4\sigma^4\lambda_1 \,\mathrm{Tr}(\Sigma) + \lambda_1^2$$

This provides an upper bound on $\alpha_n \leq \left\|\mathbb{E}\left[B_n U U^T B_n^T\right]\right\|$. While this bound is tight when r_{eff} is bounded by a constant, in the high dimensional setting (Assumption 2) considered in this work, this bound is too loose. This is evident from Figure 2(B), which plots $\left\|\mathbb{E}\left[B_n U U^T B_n^T\right]\right\|$ for $U = I$, along with the bound achieved using Eq 6, labeled as *Jain et al. (2016)*. Note that the plots are in

the log-scale so a difference in the slopes translates to a significant multiplicative difference. This warrants a more fine-grained analysis of α_n.

Let us examine α_n more closely to obtain a finer bound. Using the structure of the matrix product, B_n, from Eq 3, we have:

$$\alpha_n = \alpha_{n-1}\left(1 + 2\eta\lambda_1\right) + \eta^2 \mathbb{E}\left[\left(v_1^T X_n X_n^T v_1\right)\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)\right]$$

Now, as a consequence of subgaussianity (see Lemma A.2.2), for $K := (2L^2\sigma^2)^2$, with the Cauchy-Schwartz inequality, the second term in the RHS can be bounded further using:

$$\mathbb{E}\left[(v_1^T X_n X_n^T v_1)^2\right] \leq K\lambda_1^2, \;\; \mathbb{E}\left[\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)^2 \middle| \mathcal{F}_{n-1}\right] \leq K \operatorname{Tr}(U^T B_{n-1}^T \Sigma B_{n-1} U)^2$$

Therefore, using the above bound along with the eigen-decomposition $\Sigma := \lambda_1 v_1 v_1^T + V_\perp \Lambda_2 V_\perp^T$,

$$\alpha_n \leq \left(1 + 2\eta\lambda_1 + 4L^2\eta^2\sigma^4\lambda_1^2\right)\alpha_{n-1} + 4L^2\eta^2\sigma^4\lambda_1\lambda_2\beta_{n-1} \tag{7}$$

Similarly, β_n can also be upper bounded as follows:

$$\beta_n \leq \left(1 + 2\eta\lambda_2 + 4\eta^2 L^4\sigma^4\lambda_2 \operatorname{Tr}(\Sigma)\right)\beta_{n-1} + 4\eta^2 L^4\sigma^4\lambda_1 \operatorname{Tr}(\Sigma)\,\alpha_{n-1} \tag{8}$$

Note that upper bounding and eliminating α_{n-1} or β_{n-1} from Eq 8, 7 respectively, would simplify the recursion but lead to a weaker bound as in Eq 6. Therefore, we solve Eq 7 and 8 as a system of linear recursions in α_n and β_n.

$$\begin{pmatrix}\alpha_n \\ \beta_n\end{pmatrix} = \underbrace{\begin{pmatrix}1 + 2\eta\lambda_1 + O\left(\eta^2\lambda_1^2\right) & O\left(\lambda_1\lambda_2\right) \\ O\left(\operatorname{Tr}(\Sigma)\right) & 1 + 2\eta\lambda_2 + O\left(\eta^2 \operatorname{Tr}(\Sigma)\right)\end{pmatrix}}_{:=P}\begin{pmatrix}\alpha_{n-1} \\ \beta_{n-1}\end{pmatrix} \tag{9}$$

Estimating elements of P^n, where P is the defined 2×2 matrix, is crucial. [Wil92] gives a compact expression for these elements using $\lambda_1(P)$ and $\lambda_2(P)$. Under our assumptions, we have $P_{11} > P_{22}$. A naive upper bound on $\lambda_1(P)$ using Weyl's inequality [Die15] is $1 + 2\eta\lambda_1 + c_3 \operatorname{Tr}(\Sigma)$, similar to Eq 6. Since recursions like Eq 9 are common in our analysis, we provide a general solution in Lemma A.2.5 (detailed in the Appendix Section A.2).

An important consequence of this is that we now have the following bounds on α_n for $U = I$:

$$\alpha_n \leq \left(1 + 2\eta\lambda_1 + c_1\eta^2\lambda_1^2\right)^n \left(1 + O(\eta\lambda_1)\right) \tag{10}$$

which is much tighter than Eq 6 in our high-dimensional regime. Furthermore, observing Figure 2(b), we see that Eq 10 presents a much tighter upper bound, matching $\left(1 + \eta\lambda_1\right)^{2n}$ up to constant factors.

Recall that the bounds obtained in this section deal with α_n, β_n defined in Eq 5. A similar system of recursions can be obtained to get tight bounds on $\mathbb{E}\left[\left(v_1^T B_n^T U U^T B_n v_1\right)^2\right]$ and $\mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T B_n^T U U^T B_n V_\perp\right)^2\right]$, details of which we defer to the Appendix in Lemmas A.2.9, A.2.10.

5 Conclusion

Oja's algorithm for streaming PCA has been extensively studied in the recent theoretical literature, typically assuming that $\|X_i\|^2/\lambda_1$ is bounded or a slowly growing covariance matrix effective rank r_{eff}. This paper addresses the high-dimensional sparse PCA setting where the effective rank r_{eff} can be as large as $n/\log n$ while v_1 is s-sparse. In this context, while there has been a vast body of work that achieves minimax error bounds, we are unaware of any single-pass algorithm that works in $O(nd)$ time, $O(d)$ space, on a general Σ, without any strong initialization. Surprisingly, our thresholded estimator achieves the minimax error bound of $O(s \log d/n)$, whereas the error rate of Oja's algorithm is $O(r_{\mathsf{eff}}/n)$. Empirically, the elements of the unnormalized Oja vector do not concentrate in this regime. Through an analysis that uncouples the projection of a product of independent random matrices on v_1 and its orthogonal subspace, we show that the entries of the Oja vector within the support of v_1 are large, while those outside are much smaller.

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References

[AFW22] Emmanuel Abbe, Jianqing Fan, and Kaizheng Wang. An lp theory of pca and spectral clustering. *The Annals of Statistics*, 50(4):2359–2385, 2022.

[AW08] Arash A Amini and Martin J Wainwright. High-dimensional analysis of semidefinite relaxations for sparse principal components. In *2008 IEEE international symposium on information theory*, pages 2454–2458. IEEE, 2008.

[AZL17] Zeyuan Allen-Zhu and Yuanzhi Li. First efficient convergence for streaming k-pca: a global, gap-free, and near-optimal rate. In *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 487–492. IEEE, 2017.

[BB19] Matthew Brennan and Guy Bresler. Optimal average-case reductions to sparse pca: From weak assumptions to strong hardness. In *Conference on Learning Theory*, pages 469–470. PMLR, 2019.

[BDF13] Akshay Balsubramani, Sanjoy Dasgupta, and Yoav Freund. The fast convergence of incremental pca. In C. J. C. Burges, L. Bottou, M. Welling, Z. Ghahramani, and K. Q. Weinberger, editors, *Advances in Neural Information Processing Systems 26*, pages 3174–3182. Curran Associates, Inc., 2013.

[BKW20] Afonso S Bandeira, Dmitriy Kunisky, and Alexander S Wein. Computational hardness of certifying bounds on constrained pca problems. In *11th Innovations in Theoretical Computer Science Conference (ITCS 2020)*, volume 151, page 78. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2020.

[BL09] Peter J. Bickel and Elizaveta Levina. Covariance regularization by thresholding. *arXiv: Statistics Theory*, 2009.

[BPP18] Guy Bresler, Sung Min Park, and Madalina Persu. Sparse pca from sparse linear regression. *Advances in Neural Information Processing Systems*, 31, 2018.

[BR13] Quentin Berthet and Philippe Rigollet. Optimal detection of sparse principal components in high dimension. 2013.

[CMW13] T Tony Cai, Zongming Ma, and Yihong Wu. Sparse pca: Optimal rates and adaptive estimation. 2013.

[CYWZ18] Minshuo Chen, Lin Yang, Mengdi Wang, and Tuo Zhao. Dimensionality reduction for stationary time series via stochastic nonconvex optimization. *Advances in Neural Information Processing Systems*, 31, 2018.

[dBEG08] Alexandre d'Aspremont, Francis Bach, and Laurent El Ghaoui. Optimal solutions for sparse principal component analysis. *Journal of Machine Learning Research*, 9(7), 2008.

[Die15] Rainer Dietmann. Weyl's inequality and systems of forms. *Quarterly Journal of Mathematics*, 66(1):97–110, 2015.

[DKPP23] Ilias Diakonikolas, Daniel Kane, Ankit Pensia, and Thanasis Pittas. Nearly-linear time and streaming algorithms for outlier-robust pca. In *International Conference on Machine Learning*, pages 7886–7921. PMLR, 2023.

[DKWB23] Yunzi Ding, Dmitriy Kunisky, Alexander S Wein, and Afonso S Bandeira. Subexponential-time algorithms for sparse pca. *Foundations of Computational Mathematics*, pages 1–50, 2023.

[DM+16] Yash Deshp, Andrea Montanari, et al. Sparse pca via covariance thresholding. *Journal of Machine Learning Research*, 17(141):1–41, 2016.

[DMMW17] Santanu S. Dey, Rahul Mazumder, Marco Molinaro, and Guanyi Wang. Sparse principal component analysis and its l_1-relaxation, 2017.

[GMZ17] Chao Gao, Zongming Ma, and Harrison H Zhou. Sparse cca: Adaptive estimation and computational barriers. 2017.

[GWS20] Milana Gataric, Tengyao Wang, and Richard J Samworth. Sparse principal component analysis via axis-aligned random projections. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 82(2):329–359, 2020.

[Heb49] Donald O. Hebb. *The organization of behavior: A neuropsychological theory*. Wiley, New York, June 1949.

[HJS+22] Baihe Huang, Shunhua Jiang, Zhao Song, Runzhou Tao, and Ruizhe Zhang. Solving sdp faster: A robust ipm framework and efficient implementation. In *2022 IEEE 63rd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 233–244. IEEE, 2022.

[HNW21] De Huang, Jonathan Niles-Weed, and Rachel Ward. Streaming k-pca: Efficient guarantees for oja's algorithm, beyond rank-one updates. *CoRR*, abs/2102.03646, 2021.

[HNWTW20] De Huang, Jonathan Niles-Weed, Joel A. Tropp, and Rachel Ward. Matrix concentration for products, 2020.

[HNWW21] De Huang, Jonathan Niles-Weed, and Rachel Ward. Streaming k-pca: Efficient guarantees for oja's algorithm, beyond rank-one updates, 2021.

[HW19a] Amelia Henriksen and Rachel Ward. AdaOja: Adaptive Learning Rates for Streaming PCA. *arXiv e-prints*, page arXiv:1905.12115, May 2019.

[HW19b] Amelia Henriksen and Rachel Ward. Concentration inequalities for random matrix products. *arXiv e-prints*, page arXiv:1907.05833, July 2019.

[JJK+16] Prateek Jain, Chi Jin, Sham Kakade, Praneeth Netrapalli, and Aaron Sidford. Streaming pca: Matching matrix bernstein and near-optimal finite sample guarantees for oja's algorithm. In *Proceedings of The 29th Conference on Learning Theory (COLT)*, June 2016.

[JKL+20] Haotian Jiang, Tarun Kathuria, Yin Tat Lee, Swati Padmanabhan, and Zhao Song. A faster interior point method for semidefinite programming. In *2020 IEEE 61st annual symposium on foundations of computer science (FOCS)*, pages 910–918. IEEE, 2020.

[JKL+24] Arun Jambulapati, Syamantak Kumar, Jerry Li, Shourya Pandey, Ankit Pensia, and Kevin Tian. Black-box k-to-1-pca reductions: Theory and applications. In Shipra Agrawal and Aaron Roth, editors, *Proceedings of Thirty Seventh Conference on Learning Theory*, volume 247 of *Proceedings of Machine Learning Research*, pages 2564–2607. PMLR, 30 Jun–03 Jul 2024.

[JL09] Iain M. Johnstone and Arthur Yu Lu. On consistency and sparsity for principal components analysis in high dimensions. *Journal of the American Statistical Association*, 104(486):682–693, 2009. PMID: 20617121.

[JLT20] Arun Jambulapati, Jerry Li, and Kevin Tian. Robust sub-gaussian principal component analysis and width-independent schatten packing. *Advances in Neural Information Processing Systems*, 33:15689–15701, 2020.

[JM09] Sungkyu Jung and J. S. Marron. PCA consistency in high dimension, low sample size context. *The Annals of Statistics*, 37(6B):4104 – 4130, 2009.

[JNRS10] Michel Journée, Yurii Nesterov, Peter Richtárik, and Rodolphe Sepulchre. Generalized power method for sparse principal component analysis. *J. Mach. Learn. Res.*, 11:517–553, mar 2010.

[Jol03] Ian T Jolliffe. Principal component analysis. *Technometrics*, 45(3):276, 2003.

[KLL+23] Jonathan Kelner, Jerry Li, Allen X Liu, Aaron Sidford, and Kevin Tian. Semi-random sparse recovery in nearly-linear time. In *The Thirty Sixth Annual Conference on Learning Theory*, pages 2352–2398. PMLR, 2023.

[Lia23] Xin Liang. On the optimality of the oja's algorithm for online pca. *Statistics and Computing*, 33(3):62, 2023.

[LSH22] Hanbyul Lee, Qifan Song, and Jean Honorio. Support recovery in sparse pca with incomplete data. *Advances in Neural Information Processing Systems*, 35:27321–27332, 2022.

[LSW21] Robert Lunde, Purnamrita Sarkar, and Rachel Ward. Bootstrapping the error of oja's algorithm. *Advances in Neural Information Processing Systems*, 34:6240–6252, 2021.

[LV15] Jing Lei and Vincent Q Vu. Sparsistency and agnostic inference in sparse pca. 2015.

[Ma13] Zongming Ma. Sparse principal component analysis and iterative thresholding. *The Annals of Statistics*, 41(2):772 – 801, 2013.

[Mac08] Lester Mackey. Deflation methods for sparse pca. *Advances in neural information processing systems*, 21, 2008.

[Mon22] Jean-Marie Monnez. Stochastic approximation of eigenvectors and eigenvalues of the q-symmetric expectation of a random matrix. *Communications in Statistics-Theory and Methods*, pages 1–15, 2022.

[MP22] Nikos Mouzakis and Eric Price. Spectral guarantees for adversarial streaming pca, 2022.

[MWA06] Baback Moghaddam, Yair Weiss, and Shai Avidan. Generalized spectral bounds for sparse lda. In *Proceedings of the 23rd international conference on Machine learning*, pages 641–648, 2006.

[MZ20] Shahar Mendelson and Nikita Zhivotovskiy. Robust covariance estimation under l_4-l_2 norm equivalence. 2020.

[Nov23] Gleb Novikov. Sparse pca beyond covariance thresholding. *arXiv preprint arXiv:2302.10158*, 2023.

[Oja82a] Erkki Oja. Simplified neuron model as a principal component analyzer. *Journal of Mathematical Biology*, 15(3):267–273, November 1982.

[Oja82b] Erkki Oja. Simplified neuron model as a principal component analyzer. *Journal of mathematical biology*, 15:267–273, 1982.

[Pau07] Debashis Paul. Asymptotics of sample eigenstructure for a large dimensional spiked covariance model. *Statistica Sinica*, pages 1617–1642, 2007.

[Pea01] Karl Pearson. Liii. on lines and planes of closest fit to systems of points in space. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 2(11):559–572, 1901.

[QLR19] Yixuan Qiu, Jing Lei, and Kathryn Roeder. Gradient-based sparse principal component analysis with extensions to online learning, 2019.

[SSM11] Dan Shen, Haipeng Shen, and J. S. Marron. Consistency of sparse pca in high dimension, low sample size contexts. *J. Multivar. Anal.*, 115:317–333, 2011.

[SSM13] Dan Shen, Haipeng Shen, and James Stephen Marron. Consistency of sparse pca in high dimension, low sample size contexts. *Journal of Multivariate Analysis*, 115:317–333, 2013.

[STL07] Bharath K. Sriperumbudur, David A. Torres, and Gert R. G. Lanckriet. Sparse eigen methods by d.c. programming. In *Proceedings of the 24th International Conference on Machine Learning*, ICML '07, page 831–838, New York, NY, USA, 2007. Association for Computing Machinery.

[VCLR13] Vincent Q Vu, Juhee Cho, Jing Lei, and Karl Rohe. Fantope projection and selection: A near-optimal convex relaxation of sparse pca. In C.J. Burges, L. Bottou, M. Welling, Z. Ghahramani, and K.Q. Weinberger, editors, *Advances in Neural Information Processing Systems*, volume 26. Curran Associates, Inc., 2013.

[Ver10] Roman Vershynin. Introduction to the non-asymptotic analysis of random matrices. *arXiv preprint arXiv:1011.3027*, 2010.

[Ver18] Roman Vershynin. *High-Dimensional Probability*. Cambridge University Press, Cambridge, UK, 2018.

[VL12] Vincent Vu and Jing Lei. Minimax rates of estimation for sparse pca in high dimensions. In Neil D. Lawrence and Mark Girolami, editors, *Proceedings of the Fifteenth International Conference on Artificial Intelligence and Statistics*, volume 22 of *Proceedings of Machine Learning Research*, pages 1278–1286, La Palma, Canary Islands, 21–23 Apr 2012. PMLR.

[Wai19] Martin J Wainwright. *High-dimensional statistics: A non-asymptotic viewpoint*, volume 48. Cambridge university press, 2019.

[Wed72] Per-Åke Wedin. Perturbation bounds in connection with singular value decomposition. *BIT Numerical Mathematics*, 12:99–111, 1972.

[Wil92] Kenneth S. Williams. The nth power of a 2×2 matrix. *Mathematics Magazine*, 65(5):336–336, 1992.

[WL16] Chuang Wang and Yue M. Lu. Online learning for sparse pca in high dimensions: Exact dynamics and phase transitions, 2016.

[XHS+18] Peng Xu, Bryan He, Christopher De Sa, Ioannis Mitliagkas, and Chris Re. Accelerated stochastic iteration. volume 84 of *Proceedings of Machine Learning Research*, pages 58–67, Playa Blanca, Lanzarote, Canary Islands, 09–11 Apr 2018. PMLR.

[YHW18] Puyudi Yang, Cho-Jui Hsieh, and Jane-Ling Wang. History pca: A new algorithm for streaming pca. *arXiv preprint arXiv:1802.05447*, 2018.

[YX15] Wenzhuo Yang and Huan Xu. Streaming sparse principal component analysis. In Francis Bach and David Blei, editors, *Proceedings of the 32nd International Conference on Machine Learning*, volume 37 of *Proceedings of Machine Learning Research*, pages 494–503, Lille, France, 07–09 Jul 2015. PMLR.

[YZ13] Xiao-Tong Yuan and Tong Zhang. Truncated power method for sparse eigenvalue problems. *Journal of Machine Learning Research*, 14(4), 2013.

[ZX18] Hui Zou and Lingzhou Xue. A selective overview of sparse principal component analysis. *Proceedings of the IEEE*, 106(8):1311–1320, 2018.

A Appendix

The Appendix is organized as follows:

1. Section A.1 provides further details about related work
2. Section A.2 provides some useful results used in subsequent analyses
3. Section A.3 provides Entrywise deviation bounds for the Oja vector (Lemmas 3.11, 3.12)
4. Section A.4 proves convergence of Support Recovery results (Lemma 3.1, Theorem 3.2)
5. Section A.5 proves convergence of Sparse PCA results (Theorems 3.5,3.7)
6. Section A.6 provides another alternative way of truncation using a value-based thresholding (Theorem A.6.1)

A.1 Further details on related work

There has been a lot of work on computational computational hardness of sparse PCA [GMZ17, BB19, DKWB23, BKW20].

Minimax optimal Sparse PCA algorithms with global convergence: These consist of SDP-based algorithms such as [AW08, VCLR13, dBEG08], which do not scale well in high-dimensions (see [BR13, Wai19]). The state-of-the-art SDP solvers [JKL$^+$20, HJS$^+$22] currently have a runtime $\Omega\left(n^\omega + d^\omega\right)$, where $\omega \approx 2.732$ is the matrix multiplication exponent. Algorithms proposed in [Ma13, CMW13, JNRS10, DM$^+$16] involve forming the entire $(d \times d)$ sample covariance matrix, which can itself be challenging from the perspective of space and time complexity. Furthermore, [Ma13, CMW13, DM$^+$16] have been analyzed under the *spiked covariance model* in Eq 2. [QLR19] propose a computationally efficient modification of the Fantope projection-based algorithm of [VCLR13], which requires $O(d^2)$ space, and $\Omega(nd^2)$ time.

Single-pass online sparse PCA algorithms with $O(d^2)$ storage and $O(nd^2)$ time [QLR19] also provide a single-pass online algorithm and state that this algorithm (Theorem 4) *is the first to provably obtain the global optima in a streaming setting without any initialization, under a general* Σ. However, this method requires $O(d^2)$ storage, $O(nd^2)$ time, and the estimation error is $O\left(\frac{d^2}{\sqrt{n}}\right)$ (Theorem 4, [QLR19]). The algorithm does d sparse linear regression problems to achieve this.

Support recovery algorithms with $O(d^2)$ storage and $O(nd^2)$ time : [LV15, LSH22] use an SDP-based approach and [BPP18] use sparse linear regression for support recovery.

More details on streaming PCA algorithms [YX15] provides an online block version of the truncated power method in [YZ13] under the spiked model (Eq 2). They require an initialization u_0 with a sufficiently large $|u_0^T v_1| = \Omega(1)$ (local convergence). Their proposed initialization with streaming PCA algorithm until reaching a specific accuracy threshold, for which there is no known theoretical guarantee under the spiked high-dimensional setting. [WL16] provides an analysis of streaming sparse PCA under Eq 2 via partial differential equations (PDE), but they only prove asymptotic convergence. Similar to [YZ13], they also require $|u_0^T v_1| = \Omega(1)$ which can be hard to find in high dimensions for a general Σ. Recent results provide a black-box way to obtain the top-k principal components (k-PCA) given an algorithm to extract the top eigenvector (see [a]) which could be employed treating our algorithm as a 1-PCA oracle (see [JKL$^+$24, Mac08]). We believe that our analysis can be extended to obtain top-k principal components simultaneously via QR decomposition and thresholding.

A.2 Useful results

Lemma A.2.1. *(Fact 2.9 [DKPP23]) For any symmetric $d \times d$ matrix A, we have $\mathrm{Var}_{z\sim\mathcal{N}(0,I)}\left[z^T A z\right] = 2\|A\|_F^2$. If A is a PSD matrix, then for any $\beta > 0$, it holds that*

$$\mathbb{P}_{z\sim\mathcal{N}(0,I)}\left[z^T A z \geq \beta \operatorname{Tr}(A)\right] \geq 1 - \sqrt{e\beta}$$

Proof. We give a short proof here. Since A is a symmetric matrix, let $A = P\Lambda P^T$ where P is an orthonormal matrix and Λ is a diagonal matrix. Then, denoting $y := P^T z$ we note that $y \sim \mathcal{N}(0, I)$.

Therefore,

$$z^T Az = z^T P\Lambda P^T z = y^T \Lambda y = \sum_{i=1}^{d} \lambda_i y_i^2$$

Therefore,

$$\mathbb{E}_{z\sim\mathcal{N}(0,I)}\left[z^T Az\right] = \mathbb{E}_{y\sim\mathcal{N}(0,I)}\left[\sum_{i=1}^{d}\lambda_i y_i^2\right] = \sum_{i=1}^{d}\lambda_i = \operatorname{Tr}(A)$$

and

$$\begin{aligned}\mathbb{E}_{z\sim\mathcal{N}(0,I)}\left[\left(z^T Az\right)^2\right] = \mathbb{E}_{y\sim\mathcal{N}(0,I)}\left[\left(\sum_{i=1}^{d}\lambda_i y_i^2\right)^2\right] &= \mathbb{E}_{y\sim\mathcal{N}(0,I)}\left[\sum_{i=1}^{d}\lambda_i^2 y_i^4 + \sum_{i,j,i\neq j}^{d}\lambda_i\lambda_j y_i^2 y_j^2\right] \\ &= 3\sum_{i=1}^{d}\lambda_i^2 + \sum_{i,j,i\neq j}^{d}\lambda_i\lambda_j = 2\operatorname{Tr}\left(A^2\right) + \operatorname{Tr}(A)^2\end{aligned}$$

To get the tail lower bound, note that it trivially follows if $\beta > 1$. Therefore we proceed with $\beta \in (0,1)$. We have

$$\begin{aligned}\mathbb{P}_{z\sim\mathcal{N}(0,I)}\left[z^T Az \leq \beta\operatorname{Tr}(A)\right] &= \mathbb{P}_{y\sim\mathcal{N}(0,I)}\left[\sum_{i=1}^{d}\lambda_i y_i^2 \leq \beta\sum_{i=1}^{d}\lambda_i\right] \\ &\leq \mathbb{E}\left[\exp\left(t\left(\beta\sum_{i=1}^{d}\lambda_i - \sum_{i=1}^{d}\lambda_i y_i^2\right)\right)\right], t>0 \\ &= \exp\left(t\beta\sum_{i=1}^{d}\lambda_i\right)\mathbb{E}\left[\exp\left(-t\sum_{i=1}^{d}\lambda_i y_i^2\right)\right] \\ &= \exp\left(t\beta\sum_{i=1}^{d}\lambda_i\right)\prod_{i=1}^{d}\left(1+2\lambda_i t\right)^{-\frac{1}{2}}\end{aligned}$$

Let $t = \frac{1}{2\sum_{i=1}^{d}\lambda_i}\left(\frac{1}{\beta}-1\right)$. Then,

$$\begin{aligned}\mathbb{P}_{z\sim\mathcal{N}(0,I)}\left[z^T Az \leq \beta\operatorname{Tr}(A)\right] &\leq \exp\left(\frac{1-\beta}{2}\right)\prod_{i=1}^{d}\left(1+\frac{\lambda_i}{\sum_{i=1}^{d}\lambda_i}\left(\frac{1}{\beta}-1\right)\right)^{-\frac{1}{2}} \\ &\leq \exp\left(\frac{1-\beta}{2}\right)\left(1+\left(\frac{1}{\beta}-1\right)\right)^{-\frac{1}{2}} \\ &= \exp\left(\frac{1-\beta}{2}\right)\sqrt{\beta} \\ &\leq \sqrt{e\beta}\end{aligned}$$

Hence proved. □

Lemma A.2.2. *Let $X \in \mathbb{R}^d$ be a σ-subgaussian random vector with covariance matrix Σ. Then, for any matrix $M \in \mathbb{R}^{d\times m}$ and any positive integer $p \geq 2$,*

$$\mathbb{E}\left[\left(X^T MM^T X\right)^p\right] \leq \left(L^2\sigma^2 p\right)^p \operatorname{Tr}\left(M^T\Sigma M\right)^p$$

Proof. Let the eigendecomposition of MM^T be $P\Lambda P^T$. Define $Y := P^T X$. Then,

$$\begin{aligned}\mathbb{E}\left[\left(X^T MM^T X\right)^p\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{d}\lambda_i y_i^2\right)^p\right] \\ &= \sum_{k_1+k_2+\cdots k_d=p;\ k_1,k_2,\cdots k_d\geq 0}\binom{n}{k_1,k_2,\cdots k_d}\mathbb{E}\left[\prod_{i=1}^{d}\lambda_i^{k_i} y_i^{2k_i}\right] \qquad \text{(A.11)}\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left[\prod_{i=1}^{d}\lambda_i^{k_i}y_i^{2k_i}\right] &= \left(\prod_{i=1}^{d}\lambda_i^{k_i}\right)\mathbb{E}\left[\prod_{i=1}^{d}y_i^{2k_i}\right] \\
&\leq \left(\prod_{i=1}^{d}\lambda_i^{k_i}\right)\prod_{i=1}^{d}\left(\mathbb{E}\left[\left(y_i^{2k_i}\right)^{\frac{p}{k_i}}\right]\right)^{\frac{k_i}{p}}, \text{ using Holder's inequality since } \sum_{i=1}^{d}k_i = p, \\
&= \left(\prod_{i=1}^{d}\lambda_i^{k_i}\right)\prod_{i=1}^{d}\left(\mathbb{E}\left[y_i^{2p}\right]\right)^{\frac{k_i}{p}} \qquad \text{(A.12)}
\end{aligned}$$

Using the definition of sub-gaussianity (Definition 2.1) we have,

$$\begin{aligned}
\mathbb{E}\left[y_i^{2p}\right] &= \mathbb{E}\left[\left(e_i^TP^TX\right)^{2p}\right] \\
&= \mathbb{E}\left[\left(\left(Pe_i\right)^TX\right)^{2p}\right] \\
&\leq L^{2p}\sigma^{2p}\left(\sqrt{p}\right)^{2p}\left(\|Pe_i\|_\Sigma\right)^{2p} \\
&= L^{2p}\sigma^{2p}p^p\left(e_i^TP^T\Sigma Pe_i\right)^p
\end{aligned}$$

Susbstituting in Eq A.12 we have,

$$\begin{aligned}
\mathbb{E}\left[\prod_{i=1}^{d}\lambda_i^{k_i}y_i^{2k_i}\right] &\leq \left(\prod_{i=1}^{d}\lambda_i^{k_i}\right)\left(\prod_{i=1}^{d}L^{2k_i}\sigma^{2k_i}p^{k_i}\left(e_i^TP^T\Sigma Pe_i\right)^{k_i}\right) \\
&= \left(L^2\sigma^2p\right)^p\prod_{i=1}^{d}\left(\lambda_ie_i^TP^T\Sigma Pe_i\right)^{k_i}
\end{aligned}$$

Substituting in Eq A.11 we have,

$$\begin{aligned}
\mathbb{E}\left[\left(X^TMM^TX\right)^p\right] &\leq \left(L^2\sigma^2p\right)^p\sum_{k_1+k_2+\cdots k_d=p;\ k_1,k_2,\cdots k_d\geq 0}\binom{n}{k_1,k_2,\cdots k_d}\prod_{i=1}^{d}\left(\lambda_ie_i^TP^T\Sigma Pe_i\right)^{k_i} \\
&= \left(L^2\sigma^2p\right)^p\left(\sum_{i=1}^{d}\lambda_ie_i^TP^T\Sigma Pe_i\right)^p \\
&= \left(L^2\sigma^2p\right)^p\left(\mathrm{Tr}\left(\left(\sum_{i=1}^{d}\lambda_iPe_ie_i^TP^T\right)\Sigma\right)\right)^p = \left(L^2\sigma^2p\right)^p\mathrm{Tr}\left(M^T\Sigma M\right)^p
\end{aligned}$$

Hence proved. □

Lemma A.2.3. *Let $X \in \mathbb{R}^d$ be a σ-subgaussian random vector with covariance matrix Σ. Then,*

$$\left\|\mathbb{E}\left[\left(XX^T\right)^2\right]\right\| \leq 4L^4\sigma^4\lambda_1\,\mathrm{Tr}\left(\Sigma\right)$$

Proof. For any fixed unit vector $u \in \mathbb{R}^d$, we have

$$\begin{aligned}
u^T\mathbb{E}\left[\left(XX^T\right)^2\right]u &= \mathbb{E}\left[\left(X^TX\right)\left(X^Tu\right)^2\right] \\
&\leq \sqrt{\mathbb{E}\left[\left(X^TX\right)^2\right]\mathbb{E}\left[\left(X^Tu\right)^4\right]} \\
&= \sqrt{\mathbb{E}\left[\left(X^TX\right)^2\right]\mathbb{E}\left[\left(X^Tuu^TX\right)^2\right]} \\
&\leq \left(2L^2\sigma^2\right)^2\mathrm{Tr}\left(\Sigma\right)\mathrm{Tr}\left(u^T\Sigma u\right) \\
&\leq \left(2L^2\sigma^2\right)^2\lambda_1\,\mathrm{Tr}\left(\Sigma\right)
\end{aligned}$$

where we used Lemma A.2.2 with $p = 2$ and $M = I$. □

Proposition 3.4 (Lower bound for diagonal thresholding)**.** *Let Assumption 1 hold. For any diagonal-thresholding algorithm, $\mathcal{A}$, performing support recovery with sparsity parameter s such that $n = \Omega\left(\sigma^4 s^2 \log(d)\right)$, there exists a covariance matrix Σ with principal eigenvector, v_1, $\|v_1\|_0 = s$, such that, $\mathbb{P}\left(\left|\widehat{S} \bigcap S\right| = 0\right) \geq 1 - d^{-10}$.*

Proof. Let s be a multiple of 3 for ease of analysis. Consider a dataset with a covariance matrix,

$$\begin{aligned}
&\Sigma := \beta_1 v_1 v_1^\top + \beta_2 v_2 v_2^\top + \beta_3 v_2 v_2^\top + \beta_4 v_2 v_2^\top + \frac{1}{2} I, \beta_1 = 2\beta_2 = 2.1\beta_3 = 2.2\beta_4 \\
&\forall i \in [s], |v_1(i)| = \frac{1}{\sqrt{s}}, \quad \forall i \in \left(s, \frac{4s}{3}\right], |v_2(i)| = \sqrt{\frac{3}{s}} \\
&\forall i \in \left(\frac{4s}{3}, \frac{5s}{3}\right], |v_3(i)| = \sqrt{\frac{3}{s}} \quad \forall i \in \left(\frac{5s}{3}, 2s\right], |v_4(i)| = \sqrt{\frac{3}{s}}
\end{aligned} \tag{A.13}$$

where $\beta_1 = \frac{1}{2}$. Based on Eq A.13, we have for,

$$\begin{aligned}
&i \in (1, s], \Sigma_{i,i} = \frac{1}{2} + \frac{\beta_1}{s} \\
&i \in \left(s, \frac{4s}{3}\right], \Sigma_{i,i} = \frac{1}{2} + \frac{3\beta_2}{s} = \frac{1}{2} + \frac{3\beta_1}{2s} \\
&i \in \left(\frac{4s}{3}, \frac{5s}{3}\right], \Sigma_{i,i} = \frac{1}{2} + \frac{3\beta_3}{s} = \frac{1}{2} + \frac{3\beta_1}{2.1s} \\
&i \in \left(\frac{5s}{3}, 2s\right], \Sigma_{i,i} = \frac{1}{2} + \frac{3\beta_4}{s} = \frac{1}{2} + \frac{3\beta_1}{2.2s} \\
&i \in (2s, d], \Sigma_{i,i} = \frac{1}{2}
\end{aligned}$$

Note that the largest eigenvalue of Σ, $\lambda_1 = \beta_1 + \frac{1}{2}$. Let $t_n := 10\sigma^2\lambda_1\sqrt{\frac{\log(d)}{n}}$. Using Lemma 6.26 from [Wai19], we have, for the empirical covariance matrix, $\hat{\Sigma}$,

$$\mathbb{P}\left(\max_{i,j \in [d]} \left|\hat{\Sigma}_{i,j} - \Sigma(i,j)\right| \geq t_n\right) \leq \frac{1}{d^{10}}$$

Define the event, $\mathcal{E} := \max_{i,j\in[d]} \left|\hat{\Sigma}(i,j) - \Sigma(i,j)\right| \leq t_n$ and note that due to the sample complexity bound on n, under event $\mathcal{E}$,

$$\min_{i \in (s,2s]} \hat{\Sigma}_{i,i} > \max_{i \in [1,s]} \hat{\Sigma}_{i,i} \geq \min_{i \in [1,s]} \hat{\Sigma}_{i,i} \geq \max_{i > 2s} \hat{\Sigma}_{i,i}$$

Therefore, under event $\mathcal{E}$, the s largest diagonal entries of $\hat{\Sigma}$ are $i \in (s, 2s]$, and therefore, $|\hat{S} \bigcap S| = 0$, which completes our proof. □

Lemma 3.10 (Geometric Aggregation for Boosting)**.** *Let $\mathcal{A} :=$ $\mathsf{ConstantSuccessOracle}(D, \theta, \mathcal{T}, \rho, q_*, \epsilon)$ (Definition 3.9) for dataset $D := \{X_i\}_{i\in[n]}$. Then for $\delta \in (0,1)$, $\tilde{q} \leftarrow \mathsf{SuccessBoost}\left(\{X_i\}_{i\in[n]}, \mathcal{A}, \delta\right)$ satisfies $\mathbb{P}(\rho(\tilde{q}, q_*) \leq 3\epsilon) \geq 1 - \delta$.*

Proof. Consider the indicator random variables $\chi_i := \mathbb{1}(\rho(q_i, q_*) \leq \epsilon)$. Let $p := \frac{1}{3}$ and $r = 300\log\left(\frac{1}{\delta}\right)$ for convenience of notation. Then, $\forall i \in [r]$, $\mathbb{P}(\chi_i = 1) \geq 1 - p$. Define the set $\mathcal{S} := \{i : i \in [r], \chi_i = 1\}$. We note that using standard Chernoff bounds for sums of independent Bernoulli random variables, for $\theta \in (0,1)$,

$$\mathbb{P}(|\mathcal{S}| \leq (1-\theta)\mathbb{E}[|\mathcal{S}|]) \leq \exp\left(-\frac{\theta^2 \mathbb{E}[|\mathcal{S}|]}{2}\right)$$

We have, $\mathbb{E}\left[|\mathcal{S}|\right] \geq r\left(1-p\right)$ using linearity of expectation. Therefore,

$$\begin{aligned} &\mathbb{P}\left(|\mathcal{S}| \leq \left(1-\theta\right)\left(1-p\right)r\right) \leq \exp\left(-\frac{\theta^2\left(1-p\right)r}{2}\right) \\ \implies &\mathbb{P}\left(|\mathcal{S}| \leq 0.9\left(1-p\right)r\right) \leq \exp\left(-\frac{\left(1-p\right)r}{200}\right), \text{ for } \theta := \frac{1}{10} \end{aligned} \tag{A.14}$$

Recall that Algorithm 3 defines $\tilde{q}$ as:

$$\tilde{q} := q_i, \text{ such that } \frac{\left|\{j \in [r] : \rho\left(q_i, q_j\right) \leq 2\epsilon\}\right|}{r} \geq 0.9\left(1-p\right) \tag{A.15}$$

Note that the definition of $\tilde{q}$ does not require knowledge of q_* and it can be computed by calculating $\rho(.)$ error between all distinct $\binom{r}{2}$ pairs $\left(q_i, q_j\right)_{i,j\in[r],i\neq j}$.

Let $\mathcal{E}$ be the event $\{|\mathcal{S}| > \ 0.9\left(1-p\right)r\}$ and denote $f \ := \ 0.9\left(1-p\right)$ for convenience of notation. Let us now operate conditioned on $\mathcal{E}$. Note that conditioned on $\mathcal{E}$, such a $\tilde{q}$ always exists since any point in $\mathcal{S}$ is a valid selection of $\tilde{q}$. This is true since

$$\rho\left(q_i, q_j\right) \leq \rho\left(q_i, q_*\right) + \rho\left(q_j, q_*\right) \leq 2\epsilon$$

Here we used the property of the event $\mathcal{E}$ and the triangle inequality for ρ. We further have, conditioned on $\mathcal{E}$ using triangle inequality for some $i \in \mathcal{S}$,

$$\rho\left(\tilde{q}, q_*\right) \leq \rho\left(\tilde{q}, q_i\right) + \rho\left(q_i, q_*\right) \leq 3\epsilon \tag{A.16}$$

Therefore, we have

$$\begin{aligned} \mathbb{P}\left(\rho\left(\tilde{q}, q_*\right) \geq 3\epsilon\right) &= \mathbb{P}\left(\mathcal{E}\right)\mathbb{P}\left(\rho\left(\tilde{q}, q_*\right) \geq 3\epsilon | \mathcal{E}\right) + \mathbb{P}\left(\mathcal{E}^c\right)\mathbb{P}\left(\rho\left(\tilde{q}, q_*\right) \geq 3\epsilon | \mathcal{E}^c\right) \\ &= 0 + \mathbb{P}\left(\mathcal{E}^c\right)\mathbb{P}\left(\rho\left(\tilde{q}, q_*\right) \geq 3\epsilon | \mathcal{E}^c\right) \text{ using Eq A.16} \\ &\leq \mathbb{P}\left(\mathcal{E}^c\right) \\ &\leq \exp\left(-\frac{\left(1-p\right)r}{200}\right) \text{ using Eq A.14} \end{aligned}$$

which completes our proof. □

Lemma A.2.4 (Learning rate schedule)**.** *Let the learning rate be set as* $\eta := \dfrac{\kappa \log\left(n\right)}{n\left(\lambda_1 - \lambda_2\right)}$ *for a positive constant* $\kappa > 0$*. For constant* $c \leq \frac{1}{8\kappa}\min\left\{\frac{1}{\sqrt{C}}\frac{1}{C}\right\}$*, let*

$$\max\left\{1, \frac{\lambda_2}{\lambda_1 - \lambda_2}\right\}\frac{\mathrm{Tr}\left(\Lambda_2\right)}{\lambda_1 - \lambda_2} \leq \frac{cn}{\log\left(n\right)}, \quad \frac{\lambda_1}{\lambda_1 - \lambda_2} \leq c\sqrt{\frac{n}{\log^2\left(n\right)}}$$

If $\kappa \geq 2 + o\left(1\right),\ n = \Omega\left(s^2\log\left(d\right)\right)$*, the following hold:*

1. $\eta \leq \frac{1}{C}\frac{\left(\lambda_1 - \lambda_2\right)}{\lambda_2\,\mathrm{Tr}\left(\Lambda_2\right)}$
2. $C\eta \leq \frac{1}{4}\min\left\{\frac{1}{\lambda_1}, \frac{1}{\mathrm{Tr}\left(\Lambda_2\right)}, \frac{1}{\sqrt{\lambda_1\,\mathrm{Tr}\left(\Lambda_2\right)}}\right\}$
3. $C\eta^2 n\lambda_1^2 \leq \frac{1}{4}$
4. $\exp\left(-rn\eta\left(\lambda_1 - \lambda_2\right)\right) \leq \eta\lambda_1$ *for* $r \geq \frac{1}{2}$

where $C := 100\left(L^4\sigma^4 + L^2\sigma^2\right) + 16$*. We state another useful restatement of Claim (1) used in subsequent analysis,*

$$\exists\theta \in \left(0.5, 1\right),\ \ \left(1-\theta\right)\left(\lambda_1 - \lambda_2\right) + 50L^4\sigma^4\eta\lambda_1^2 = 50L^4\sigma^4\log\left(n\right)\eta\lambda_2\,\mathrm{Tr}\left(\Sigma\right)$$

Proof.

$$\eta\frac{\lambda_2\,\mathrm{Tr}\left(\Lambda_2\right)}{\lambda_1 - \lambda_2} = \kappa\frac{\log\left(n\right)}{n}\frac{\mathrm{Tr}\left(\Lambda_2\right)}{\lambda_1 - \lambda_2}\frac{\lambda_2}{\lambda_1 - \lambda_2} \leq \kappa c$$

Therefore, the first claim follows for $c \leq \frac{1}{\kappa C}$. For the second claim,

$$C\eta\lambda_1 = \frac{\kappa C \log(n)\,\lambda_1}{n(\lambda_1 - \lambda_2)} \leq \kappa C c \frac{1}{\sqrt{n}} \leq \frac{1}{4}$$

where the last inequality holds for $c \leq \frac{1}{4\kappa C}$ and $n \geq 1$. Furthermore we have

$$C\eta \operatorname{Tr}(\Lambda_2) = C\frac{\kappa \log(n)}{n(\lambda_1 - \lambda_2)} \operatorname{Tr}(\Lambda_2) \leq \kappa C c \leq \frac{1}{4}$$

where the last inequality holds for $\kappa C c \leq \frac{1}{4}$. Note that $\eta C \leq \min\left\{\frac{1}{4\lambda_1}, \frac{1}{4}\frac{1}{\operatorname{Tr}(\Lambda_2)}\right\}$ imply $4\eta C \leq \frac{1}{\sqrt{\lambda_1 \operatorname{Tr}(\Lambda_2)}}$.

For the third claim, we have

$$C\eta^2 n\lambda_1^2 = \eta\lambda_1 \frac{\kappa C \log(n)\,\lambda_1}{(\lambda_1 - \lambda_2)} = \frac{\kappa^2 C \log^2(n)}{n}\left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right)^2 \leq \kappa^2 C c^2 \leq \frac{1}{4}$$

where the last inequality holds when $c \leq \frac{1}{\sqrt{4\kappa^2 C}}$.

Next, we have the last claim,

$$\exp(-rn\eta(\lambda_1 - \lambda_2)) = \exp(-r\kappa\log(n)) = \frac{1}{n^{r\kappa}}$$

Therefore, it suffices to ensure

$$\frac{1}{n^{r\kappa}} \leq \frac{1}{n^{\frac{\kappa}{2}}} \leq \frac{\kappa\log(n)}{n} \overset{(iv)}{\leq} \frac{\kappa\lambda_1\log(n)}{n(\lambda_1 - \lambda_2)}$$

where (iv) follows since $\frac{\lambda_1}{(\lambda_1 - \lambda_2)} \geq 1$ as $\lambda_1 > \lambda_2$. Therefore, we require

$$\frac{1}{n^{\frac{\kappa}{2}}} \leq \frac{\kappa\log(n)}{n}$$

which holds for $\kappa = 2 + o(1)$ and sufficiently large n. □

Lemma A.2.5. *For constants $c_1, c_2, c_3, c_4, c_5 > 0$, consider the following system of recursions -*

$$\alpha_n \leq \left(1 + c_1\eta\lambda_1 + c_2\eta^2\lambda_1^2\right)\alpha_{n-1} + c_3\eta^2\lambda_1\lambda_2\beta_{n-1},$$
$$\beta_n \leq \left(1 + c_1\eta\lambda_2 + c_4\eta^2\lambda_2\operatorname{Tr}(\Sigma)\right)\beta_{n-1} + c_5\eta^2\lambda_1\operatorname{Tr}(\Sigma)\,\alpha_{n-1}$$

Let $\exists\theta \in (0.5, 1)$, which satisfies $c_1(1-\theta)(\lambda_1 - \lambda_2) + c_2\eta\lambda_1^2 = c_4\eta\lambda_2\operatorname{Tr}(\Sigma)$ and

$$\frac{4c_3c_5}{c_1^2}\eta^2\lambda_2\operatorname{Tr}(\Sigma)\left(\frac{\lambda_1}{\theta(\lambda_1 - \lambda_2)}\right)^2 \leq 1, \;\; 4\eta\lambda_1\left(\frac{c_2\lambda_1}{c_1(\lambda_1 - \lambda_2)}\right) \leq 1 - \theta$$

Then we have,

$$\alpha_n \leq \lambda_1(P)^n\left[\alpha_0 + \eta\lambda_1\left(\frac{2c_3\lambda_1}{c_1\theta(\lambda_1 - \lambda_2)}\right)\left(\beta_0 + \alpha_0\frac{c_5}{c_4}\left(\frac{1-\theta}{\theta}\right)\right)\right],$$
$$\beta_n \leq \beta_0\lambda_2(P)^n + \left[\eta\lambda_1\left(\frac{2c_5\lambda_1}{c_1\theta(\lambda_1 - \lambda_2)}\right)\left(\alpha_0\frac{\operatorname{Tr}(\Sigma)}{\lambda_1} + \beta_0\frac{c_3}{c_4}\left(\frac{1-\theta}{\theta}\right)\right)\right]\lambda_1(P)^n$$

where

$$\left|\lambda_1(P) - 1 - c_1\eta\lambda_1 - c_2\eta^2\lambda_1^2\right| \leq \frac{c_3c_5}{c_4}\eta^2\lambda_1^2\left(\frac{1-\theta}{\theta}\right)$$
$$\left|\lambda_2(P) - 1 - c_1\eta\lambda_2 - c_4\eta^2\lambda_2\operatorname{Tr}(\Sigma)\right| \leq \frac{c_3c_5}{c_4}\eta^2\lambda_1^2\left(\frac{1-\theta}{\theta}\right)$$

Proof. Writing the recursions in a matrix form, we have

$$\begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix} = \begin{pmatrix}1+c_1\eta\lambda_1+c_2\eta^2\lambda_1^2 & c_3\eta^2\lambda_1\lambda_2\\ c_5\eta^2\lambda_1\operatorname{Tr}(\Sigma) & 1+c_1\eta\lambda_2+c_4\eta^2\lambda_2\operatorname{Tr}(\Sigma)\end{pmatrix}\begin{pmatrix}\alpha_{n-1}\\ \beta_{n-1}\end{pmatrix} \tag{A.17}$$

Define

$$P := \begin{pmatrix}1+c_1\eta\lambda_1+c_2\eta^2\lambda_1^2 & c_3\eta^2\lambda_1\lambda_2\\ c_5\eta^2\lambda_1\operatorname{Tr}(\Sigma) & 1+c_1\eta\lambda_2+c_4\eta^2\lambda_2\operatorname{Tr}(\Sigma)\end{pmatrix}$$

Then $P := I + c_1\eta M$, where

$$M := \begin{pmatrix}\lambda_1+u\eta\lambda_1^2 & v\eta\lambda_1\lambda_2\\ w\eta\lambda_1\operatorname{Tr}(\Sigma) & \lambda_2+x\eta\lambda_2\operatorname{Tr}(\Sigma)\end{pmatrix}$$

and $u := \frac{c_2}{c_1}$, $v = \frac{c_3}{c_1}$, $x = \frac{c_4}{c_1}$, $w = \frac{c_5}{c_1}$. We now compute eigenvalues of M. The trace and determinants are given as -

$$\begin{aligned}T &:= \lambda_1+\lambda_2+u\eta\lambda_1^2+x\eta\lambda_2\operatorname{Tr}(\Sigma)\\ D &:= \lambda_1\lambda_2+\eta\lambda_1\lambda_2\left(x\operatorname{Tr}(\Sigma)+u\lambda_1\right)+ux\eta^2\lambda_1^2\lambda_2\operatorname{Tr}(\Sigma)-vw\eta^2\lambda_1^2\lambda_2\operatorname{Tr}(\Sigma)\end{aligned}$$

Next we compute $\frac{T^2}{4}-D$,

$$\begin{aligned}\frac{T^2}{4}-D &= \frac{(\lambda_1-\lambda_2)^2}{4}+\eta\left(\frac{(\lambda_1+\lambda_2)\left(u\lambda_1^2+x\lambda_2\operatorname{Tr}(\Sigma)\right)-2\lambda_1\lambda_2\left(x\operatorname{Tr}(\Sigma)+u\lambda_1\right)}{2}\right)\\ &\quad+\frac{\eta^2\left(u\lambda_1^2+x\lambda_2\operatorname{Tr}(\Sigma)\right)^2}{4}-(ux-vw)\eta^2\lambda_1^2\lambda_2\operatorname{Tr}(\Sigma)\\ &= \left[\frac{(\lambda_1-\lambda_2)^2}{4}-2\left(\frac{x\eta\lambda_2\operatorname{Tr}(\Sigma)}{2}\right)\left(\frac{\lambda_1-\lambda_2}{2}\right)+\left(\frac{x\eta\lambda_2\operatorname{Tr}(\Sigma)}{2}\right)^2\right]\\ &\quad+\eta u\lambda_1^2\left(\frac{\lambda_1-\lambda_2}{2}+\frac{\eta u\lambda_1^2}{4}\right)+\left(vw-\frac{ux}{2}\right)\eta^2\lambda_1^2\lambda_2\operatorname{Tr}(\Sigma)\\ &= \frac{1}{4}\left((\lambda_1-\lambda_2)-x\eta\lambda_2\operatorname{Tr}(\Sigma)\right)^2+\frac{\eta u\lambda_1^2}{2}\left((\lambda_1-\lambda_2)-x\eta\lambda_2\operatorname{Tr}(\Sigma)\right)+\frac{\eta^2\lambda_1^2}{4}\left(u^2\lambda_1^2+4vw\lambda_2\operatorname{Tr}(\Sigma)\right),\\ &= \frac{1}{4}\left[(\lambda_1-\lambda_2)-x\eta\lambda_2\operatorname{Tr}(\Sigma)+\eta u\lambda_1^2\right]^2+vw\eta^2\lambda_1^2\lambda_2\operatorname{Tr}(\Sigma)\end{aligned}$$

Let $(\lambda_1-\lambda_2)-x\eta\lambda_2\operatorname{Tr}(\Sigma)+\eta u\lambda_1^2=\theta(\lambda_1-\lambda_2)$ for $\theta\in(0,1)$,

$$\begin{aligned}\frac{T^2}{4}-D &= \frac{\theta^2(\lambda_1-\lambda_2)^2}{4}+vw\eta^2\lambda_1^2\lambda_2\operatorname{Tr}(\Sigma)\\ &= \frac{\theta^2(\lambda_1-\lambda_2)^2}{4}\left(1+4\eta^2\lambda_2\operatorname{Tr}(\Sigma)\frac{vw\lambda_1^2}{\theta^2(\lambda_1-\lambda_2)^2}\right)\end{aligned}$$

Let $\frac{\eta^2vw\lambda_1^2\lambda_2\operatorname{Tr}(\Sigma)}{\theta^2(\lambda_1-\lambda_2)^2}\le\frac{1}{4}$. Then, using the identity $1-\frac{x}{2}\le\sqrt{1+x}\le1+\frac{x}{2}$ for $x\in(0,1)$ we have,

$$\frac{\theta}{2}(\lambda_1-\lambda_2)\left(1-\frac{2\eta^2vw\lambda_1^2\lambda_2\operatorname{Tr}(\Sigma)}{\theta^2(\lambda_1-\lambda_2)^2}\right)\le\sqrt{\frac{T^2}{4}-D}\le\frac{\theta}{2}(\lambda_1-\lambda_2)\left(1+\frac{2\eta^2vw\lambda_1^2\lambda_2\operatorname{Tr}(\Sigma)}{\theta^2(\lambda_1-\lambda_2)^2}\right) \tag{A.18}$$

Let us simplify $\frac{\eta^2vw\lambda_1^2\lambda_2\operatorname{Tr}(\Sigma)}{\theta^2(\lambda_1-\lambda_2)^2}$ using the definition of θ. We have

$$\frac{\eta^2vw\lambda_1^2\lambda_2\operatorname{Tr}(\Sigma)}{\theta^2(\lambda_1-\lambda_2)^2}=\frac{\eta vw\lambda_1^2}{x}\left(\frac{(1-\theta)}{\theta^2(\lambda_1-\lambda_2)}+\frac{\eta u\lambda_1^2}{\theta^2(\lambda_1-\lambda_2)^2}\right)$$

Let $(1-\theta) \geq \frac{4\eta u\lambda_1^2}{(\lambda_1-\lambda_2)}$. Then,

$$\begin{aligned}\frac{\theta}{2}\left(\lambda_1-\lambda_2\right) \times \frac{\eta^2 vw\lambda_1^2\lambda_2 \operatorname{Tr}(\Sigma)}{\theta^2\left(\lambda_1-\lambda_2\right)^2} &= \frac{\eta vw\lambda_1^2}{2x}\left(\frac{1-\theta}{\theta}+\frac{4\eta u\lambda_1^2}{\theta\left(\lambda_1-\lambda_2\right)}\right) \\ &= \frac{\eta vw\lambda_1^2}{2\theta x}\left(1-\theta+\frac{4\eta u\lambda_1^2}{\left(\lambda_1-\lambda_2\right)}\right) \\ &\leq \eta\lambda_1^2\frac{vw}{x}\left(\frac{1-\theta}{\theta}\right) \end{aligned} \tag{A.19}$$

Then, using Eq A.18 and A.19, the eigenvalues of M are given as $\lambda_1(M) := \frac{T}{2}+\sqrt{\frac{T^2}{4}-D}$ and $\lambda_2(M) := \frac{T}{2}-\sqrt{\frac{T^2}{4}-D}$ such that

$$\left|\lambda_1(M)-\lambda_1-u\eta\lambda_1^2\right|,\ \ \left|\lambda_2(M)-\lambda_2-x\eta\lambda_2\operatorname{Tr}(\Sigma)\right| \ \leq\ \eta\lambda_1^2\frac{vw}{x}\left(\frac{1-\theta}{\theta}\right)$$

The eigenvalues of P are given as $\lambda_1(P) := 1+c_1\eta\lambda_1(M)$ and $\lambda_2(P) := 1+c_1\eta\lambda_2(M)$. Then we have,

$$\left|\lambda_1(P)-1-c_1\eta\lambda_1-c_2\eta^2\lambda_1^2\right| = \left|\lambda_1(P)-P_{1,1}\right| \ \leq\ \frac{c_3c_5}{c_4}\eta^2\lambda_1^2\left(\frac{1-\theta}{\theta}\right) \tag{A.20}$$

$$\left|\lambda_2(P)-1-c_1\eta\lambda_2-c_4\eta^2\lambda_2\operatorname{Tr}(\Sigma)\right| = \left|\lambda_2(P)-P_{2,2}\right| \ \leq\ \frac{c_3c_5}{c_4}\eta^2\lambda_1^2\left(\frac{1-\theta}{\theta}\right) \tag{A.21}$$

We then use the result from [Wil92] to compute P^n and α_n, β_n. To compute P^n, we first compute the matrices X and Y -

$$\begin{aligned} X &= \frac{P-\lambda_2(P)I}{\lambda_1(P)-\lambda_2(P)} = \frac{1}{\lambda_1(P)-\lambda_2(P)}\begin{pmatrix}P_{1,1}-\lambda_2(P) & P_{1,2}\\ P_{2,1} & P_{2,2}-\lambda_2(P)\end{pmatrix}, \\ Y &= \frac{P-\lambda_1(P)I}{\lambda_2(P)-\lambda_1(P)} = \frac{1}{\lambda_1(P)-\lambda_2(P)}\begin{pmatrix}\lambda_1(P)-P_{1,1} & -P_{1,2}\\ -P_{2,1} & \lambda_1(P)-P_{2,2}\end{pmatrix} \end{aligned}$$

Then, $P^n = \lambda_1(P)^n X + \lambda_2(P)^n Y$, which gives

$$P^n = \begin{pmatrix}P_{1,1}a_n-b_n & P_{1,2}a_n\\ P_{2,1}a_n & P_{2,2}a_n-b_n\end{pmatrix}$$

where

$$a_n := \left(\frac{\lambda_1(P)^n-\lambda_2(P)^n}{\lambda_1(P)-\lambda_2(P)}\right),\ \ b_n := \left(\frac{\lambda_1(P)^n\lambda_2(P)-\lambda_1(P)\lambda_2(P)^n}{\lambda_1(P)-\lambda_2(P)}\right)$$

Therefore, for $y_0 = \begin{bmatrix}\alpha_0\\ \beta_0\end{bmatrix}$, we have

$$\alpha_n = e_1^T P^n y_0 = (\alpha_0 P_{1,1}+\beta_0 P_{1,2})\left(\frac{\lambda_1(P)^n-\lambda_2(P)^n}{\lambda_1(P)-\lambda_2(P)}\right) - \alpha_0\lambda_1(P)\lambda_2(P)\left(\frac{\lambda_1(P)^{n-1}-\lambda_2(P)^{n-1}}{\lambda_1(P)-\lambda_2(P)}\right), \tag{A.22}$$

$$\beta_n = e_2^T P^n y_0 = (\alpha_0 P_{2,1}+\beta_0 P_{2,2})\left(\frac{\lambda_1(P)^n-\lambda_2(P)^n}{\lambda_1(P)-\lambda_2(P)}\right) - \beta_0\lambda_1(P)\lambda_2(P)\left(\frac{\lambda_1(P)^{n-1}-\lambda_2(P)^{n-1}}{\lambda_1(P)-\lambda_2(P)}\right) \tag{A.23}$$

Therefore, using Eq A.20 and Eq A.21,

$$\begin{aligned}\alpha_n &\leq \alpha_0\lambda_1\left(P\right)^n + \left(\beta_0 P_{1,2} + \alpha_0\frac{c_3c_5}{c_4}\eta^2\lambda_1^2\left(\frac{1-\theta}{\theta}\right)\right)\left(\frac{\lambda_1\left(P\right)^n - \lambda_2\left(P\right)^n}{\lambda_1\left(P\right) - \lambda_2\left(P\right)}\right) \\ &= \alpha_0\lambda_1\left(P\right)^n + \left(\beta_0 c_3\eta^2\lambda_1\lambda_2 + \alpha_0\frac{c_3c_5}{c_4}\eta^2\lambda_1^2\left(\frac{1-\theta}{\theta}\right)\right)\left(\frac{\lambda_1\left(P\right)^n - \lambda_2\left(P\right)^n}{\lambda_1\left(P\right) - \lambda_2\left(P\right)}\right), \quad \text{(A.24)} \\ \beta_n &\leq \beta_0\lambda_2\left(P\right)^n + \left(\alpha_0 P_{2,1} + \beta_0\frac{c_3c_5}{c_4}\eta^2\lambda_1^2\left(\frac{1-\theta}{\theta}\right)\right)\left(\frac{\lambda_1\left(P\right)^n - \lambda_2\left(P\right)^n}{\lambda_1\left(P\right) - \lambda_2\left(P\right)}\right) \\ &= \beta_0\lambda_2\left(P\right)^n + \left(\alpha_0 c_5\eta^2\lambda_1\operatorname{Tr}\left(\Sigma\right) + \beta_0\frac{c_3c_5}{c_4}\eta^2\lambda_1^2\left(\frac{1-\theta}{\theta}\right)\right)\left(\frac{\lambda_1\left(P\right)^n - \lambda_2\left(P\right)^n}{\lambda_1\left(P\right) - \lambda_2\left(P\right)}\right) \quad \text{(A.25)}\end{aligned}$$

Recall that using Eq A.20 and Eq A.21

$$\begin{aligned}\left|\lambda_1\left(P\right) - \lambda_2\left(P\right)\right| &\geq \left|P_{1,1} - P_{2,2}\right| - \frac{2c_3c_5}{c_4}\eta^2\lambda_1^2\left(\frac{1-\theta}{\theta}\right) \\ &= c_1\eta\left(\left(\lambda_1 - \lambda_2\right) - x\eta\lambda_2\operatorname{Tr}\left(\Sigma\right) + \eta u\lambda_1^2\right) - \frac{2c_3c_5}{c_4}\eta^2\lambda_1^2\left(\frac{1-\theta}{\theta}\right) \\ &= c_1\theta\eta\left(\lambda_1 - \lambda_2\right) - \frac{2c_3c_5}{c_4}\eta^2\lambda_1^2\left(\frac{1-\theta}{\theta}\right) \\ &\geq \frac{1}{2}c_1\theta\eta\left(\lambda_1 - \lambda_2\right)\end{aligned}$$

Substituting in Eq A.24 and Eq A.25 we have,

$$\begin{aligned}\alpha_n &\leq \lambda_1\left(P\right)^n\left[\alpha_0 + \eta\lambda_1\left(\frac{2c_3\lambda_1}{c_1\theta\left(\lambda_1 - \lambda_2\right)}\right)\left(\beta_0 + \alpha_0\frac{c_5}{c_4}\left(\frac{1-\theta}{\theta}\right)\right)\right], \\ \beta_n &\leq \beta_0\lambda_2\left(P\right)^n + \left[\eta\lambda_1\left(\frac{2c_5\lambda_1}{c_1\theta\left(\lambda_1 - \lambda_2\right)}\right)\left(\alpha_0\frac{\operatorname{Tr}\left(\Sigma\right)}{\lambda_1} + \beta_0\frac{c_3}{c_4}\left(\frac{1-\theta}{\theta}\right)\right)\right]\lambda_1\left(P\right)^n\end{aligned}$$

Hence proved. □

Lemma A.2.6. *Let $U \in \mathbb{R}^{d\times m}$ then, for all $t > 0$, under subgaussianity (Definition 2.1) and the step-size η satisfying $(1-\theta)\left(\lambda_1 - \lambda_2\right) + 2L^4\sigma^4\eta\lambda_1^2 = 2L^4\sigma^4\eta\lambda_2\operatorname{Tr}\left(\Sigma\right)$ for $\theta \in \left(\frac{1}{2}, 1\right)$ then we have*

$$\begin{aligned}&\lambda_1\mathbb{E}\left[U^T B_n^T v_1 v_1^T B_n U\right] \leq \gamma_1^n\left[\alpha_0 + \eta\lambda_1\left(\frac{2\lambda_1}{\theta\left(\lambda_1 - \lambda_2\right)}\right)\left(\beta_0 + \alpha_0\left(\frac{1-\theta}{\theta}\right)\right)\right], \\ &\mathbb{E}\left[\operatorname{Tr}\left(U^T B_n^T V_\perp\Lambda_2 V_\perp^T B_n U\right)\right] \leq \beta_0\gamma_2^n + \left[\eta\lambda_1\left(\frac{2\lambda_1}{\theta\left(\lambda_1 - \lambda_2\right)}\right)\left(\alpha_0\frac{\operatorname{Tr}\left(\Sigma\right)}{\lambda_1} + \beta_0\left(\frac{1-\theta}{\theta}\right)\right)\right]\gamma_1^n\end{aligned}$$

where B_n is defined in Eq 3, $\alpha_0 = \lambda_1 v_1^T U U^T v_1$, $\beta_0 = \operatorname{Tr}\left(U^T V_\perp\Lambda_2 V_\perp^T U\right)$ and

$$\begin{aligned}\left|\gamma_1 - 1 - 2\eta\lambda_1 - 4L^4\sigma^4\eta^2\lambda_1^2\right| &\leq 4L^4\sigma^4\eta^2\lambda_1^2\left(\frac{1-\theta}{\theta}\right) \\ \left|\gamma_2 - 1 - 2\eta\lambda_2 - 4L^4\sigma^4\eta^2\lambda_2\operatorname{Tr}\left(\Sigma\right)\right| &\leq 4L^4\sigma^4\eta^2\lambda_1^2\left(\frac{1-\theta}{\theta}\right)\end{aligned}$$

Proof. Let $\alpha_n := \lambda_1\mathbb{E}\left[\operatorname{Tr}\left(v_1^T B_n U U^T B_n^T v_1\right)\right]$, $\beta_n := \mathbb{E}\left[\operatorname{Tr}\left(U^T B_n^T V_\perp\Lambda_2 V_\perp^T B_n U\right)\right]$ such that

$$\alpha_n + \beta_n = \mathbb{E}\left[\operatorname{Tr}\left(U^T B_n^T\Sigma B_n U\right)\right]$$

Define $A_n := X_n X_n^T$ and let $\mathcal{F}_n$ denote the filtration for observations $i \in [n]$. Then,

$$
\begin{aligned}
\alpha_n &= \lambda_1 \mathbb{E}\left[v_1^T B_n U U^T B_n^T v_1\right] \\
&= \lambda_1 \mathbb{E}\left[v_1^T \left(I + \eta A_n\right) B_{n-1} U U^T B_{n-1}^T \left(I + \eta A_n\right) v_1\right] \\
&= \alpha_{n-1} + 2\eta\lambda_1 \mathbb{E}\left[v_1^T A_n B_{n-1} U U^T B_{n-1}^T v_1\right] + \eta^2 \lambda_1 \mathbb{E}\left[v_1^T A_n B_{n-1} U U^T B_{n-1}^T A_n v_1\right] \\
&= \alpha_{n-1} + 2\eta\lambda_1 \mathbb{E}\left[v_1^T \Sigma B_{n-1} U U^T B_{n-1}^T v_1\right] + \eta^2 \lambda_1 \mathbb{E}\left[\left(v_1^T X_n X_n^T v_1\right)\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)\right] \\
&= \alpha_{n-1}\left(1 + 2\eta\lambda_1\right) + \eta^2 \lambda_1 \mathbb{E}\left[\left(v_1^T X_n X_n^T v_1\right)\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)\right] \\
&= \alpha_{n-1}\left(1 + 2\eta\lambda_1\right) + \eta^2 \mathbb{E}\left[\mathbb{E}\left[\left(v_1^T X_n X_n^T v_1\right)\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)\middle| \mathcal{F}_{n-1}\right]\right] \\
&\leq \alpha_{n-1}\left(1 + 2\eta\lambda_1\right) + \eta^2 \lambda_1 \mathbb{E}\left[\sqrt{\mathbb{E}\left[\left(v_1^T X_n X_n^T v_1\right)^2 \middle| \mathcal{F}_{n-1}\right] \mathbb{E}\left[\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)^2 \middle| \mathcal{F}_{n-1}\right]}\right] \\
&= \alpha_{n-1}\left(1 + 2\eta\lambda_1\right) + \eta^2 \lambda_1 \mathbb{E}\left[\sqrt{\mathbb{E}\left[\left(X_n^T v_1 v_1^T X_n\right)^2 \middle| \mathcal{F}_{n-1}\right] \mathbb{E}\left[\left(X_n^T B_{n-1} U U^T B_{n-1} X_n\right)^2 \middle| \mathcal{F}_{n-1}\right]}\right] \\
&\leq \alpha_{n-1}\left(1 + 2\eta\lambda_1\right) + 4\eta^2 L^4 \sigma^4 \lambda_1 \operatorname{Tr}\left(\Sigma, v_1 v_1^T\right) \mathbb{E}\left[\operatorname{Tr}\left(\Sigma, B_{n-1} U U^T B_{n-1}^T\right)\right], \text{ using Lemma A.2.2 with } p = 2 \\
&= \alpha_{n-1}\left(1 + 2\eta\lambda_1\right) + 4\eta^2 L^4 \sigma^4 \lambda_1^2 \left(\mathbb{E}\left[\operatorname{Tr}\left(\lambda_1 v_1 v_1^T + V_\perp \Lambda_2 V_\perp^T, B_{n-1} U U^T B_{n-1}^T\right)\right]\right) \\
&= \left(1 + 2\eta\lambda_1 + 4\eta^2 L^4 \sigma^4 \lambda_1^2\right) \alpha_{n-1} + 4\eta^2 L^4 \sigma^4 \lambda_1^2 \beta_{n-1} \qquad \text{(A.26)}
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
\beta_n &= \mathbb{E}\left[\operatorname{Tr}\left(U^T B_n^T V_\perp \Lambda_2 V_\perp^T B_n U\right)\right] \\
&= \mathbb{E}\left[\operatorname{Tr}\left(\Lambda_2^{\frac{1}{2}} V_\perp^T B_n U U^T B_n^T V_\perp \Lambda_2^{\frac{1}{2}}\right)\right] \\
&= \mathbb{E}\left[\operatorname{Tr}\left(\Lambda_2^{\frac{1}{2}} V_\perp^T B_{n-1} U U^T B_{n-1}^T V_\perp \Lambda_2^{\frac{1}{2}}\right)\right] + 2\eta \mathbb{E}\left[\operatorname{Tr}\left(\Lambda_2^{\frac{1}{2}} V_\perp^T A_n B_{n-1} U U^T B_{n-1}^T V_\perp \Lambda_2^{\frac{1}{2}}\right)\right] + \\
&\quad + \eta^2 \mathbb{E}\left[\operatorname{Tr}\left(\Lambda_2^{\frac{1}{2}} V_\perp^T A_n B_{n-1} U U^T B_{n-1}^T A_n V_\perp \Lambda_2^{\frac{1}{2}}\right)\right] \\
&= \beta_{n-1} + 2\eta \mathbb{E}\left[\operatorname{Tr}\left(\Lambda_2^{\frac{1}{2}} V_\perp^T \Sigma B_{n-1} U U^T B_{n-1}^T V_\perp \Lambda_2^{\frac{1}{2}}\right)\right] + \eta^2 \mathbb{E}\left[\left(X_n^T V_\perp \Lambda_2 V_\perp^T X_n\right)\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)\right] \\
&= \beta_{n-1} + 2\eta \mathbb{E}\left[\operatorname{Tr}\left(\Lambda_2^2 V_\perp^T \Sigma B_{n-1} U U^T B_{n-1}^T V_\perp\right)\right] + \eta^2 \mathbb{E}\left[\mathbb{E}\left[\left(X_n^T V_\perp \Lambda_2 V_\perp^T X_n\right)\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)\middle| \mathcal{F}_{n-1}\right]\right] \\
&\leq \left(1 + 2\eta\lambda_2\right) \beta_{n-1} + \eta^2 \mathbb{E}\left[\sqrt{\mathbb{E}\left[\left(X_n^T V_\perp \Lambda_2 V_\perp^T X_n\right)^2 \middle| \mathcal{F}_{n-1}\right] \mathbb{E}\left[\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)^2 \middle| \mathcal{F}_{n-1}\right]}\right] \\
&\leq \left(1 + 2\eta\lambda_2\right) \beta_{n-1} + 4\eta^2 L^4 \sigma^4 \operatorname{Tr}\left(\Sigma V_\perp \Lambda_2 V_\perp^T\right) \mathbb{E}\left[\operatorname{Tr}\left(\Sigma B_{n-1} U U^T B_{n-1}^T\right)\right], \text{ using Lemma A.2.2 with } p = 2 \\
&= \left(1 + 2\eta\lambda_2 + 4\eta^2 L^4 \sigma^4 \operatorname{Tr}\left(\Lambda_2^2\right)\right) \beta_{n-1} + 4\eta^2 L^4 \sigma^4 \operatorname{Tr}\left(\Lambda_2^2\right) \alpha_{n-1} \\
&\leq \left(1 + 2\eta\lambda_2 + 4\eta^2 L^4 \sigma^4 \lambda_2 \operatorname{Tr}\left(\Lambda_2\right)\right) \beta_{n-1} + 4\eta^2 L^4 \sigma^4 \operatorname{Tr}\left(\Lambda_2^2\right) \alpha_{n-1} \qquad \text{(A.27)}
\end{aligned}
$$

Writing the recursions in a matrix form, we have

$$
\begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = \begin{pmatrix} 1 + 2\eta\lambda_1 + 4\eta^2 L^4 \sigma^4 \lambda_1^2 & 4\eta^2 L^4 \sigma^4 \lambda_1^2 \\ 4\eta^2 L^4 \sigma^4 \operatorname{Tr}\left(\Lambda_2^2\right) & 1 + 2\eta\lambda_2 + 4\eta^2 L^4 \sigma^4 \lambda_2 \operatorname{Tr}\left(\Lambda_2\right) \end{pmatrix} \begin{pmatrix} \alpha_{n-1} \\ \beta_{n-1} \end{pmatrix} \qquad \text{(A.28)}
$$

The result then follows by using Lemma A.2.5. □

Lemma A.2.7. *Let $U \in \mathbb{R}^{d \times m}$ then, for all $t > 0$, under subgaussianity (Definition 2.1) and the step-size η satisfying $(1-\theta)(\lambda_1 - \lambda_2) + 2L^4\sigma^4\eta\lambda_1^2 = 2L^4\sigma^4\eta\lambda_2 \operatorname{Tr}(\Sigma)$ for $\theta \in \left(\frac{1}{2}, 1\right)$ then we have*

$$
\begin{aligned}
&\mathbb{E}\left[v_1^T B_n U U^T B_n^T v_1\right] \leq \gamma_1^n \left[\alpha_0 + \eta\lambda_1 \left(\frac{2\lambda_1}{\theta\left(\lambda_1 - \lambda_2\right)}\right)\left(\beta_0 + \alpha_0 \left(\frac{1-\theta}{\theta}\right)\right)\right], \\
&\mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T B_n U U^T B_n^T V_\perp\right)\right] \leq \beta_0 \gamma_2^n + \left[\eta\lambda_1 \left(\frac{2\lambda_1}{\theta\left(\lambda_1 - \lambda_2\right)}\right)\left(\alpha_0 \frac{\operatorname{Tr}(\Sigma)}{\lambda_1} + \beta_0 \left(\frac{1-\theta}{\theta}\right)\right)\right] \gamma_1^n
\end{aligned}
$$

where B_n is defined in Eq 3, $\alpha_0 = v_1^T U U^T v_1$, $\beta_0 = \operatorname{Tr}\left(V_\perp^T U U^T V_\perp\right)$ and

$$\left|\gamma_1 - 1 - 2\eta\lambda_1 - 4L^4\sigma^4\eta^2\lambda_1^2\right| \le 4L^4\sigma^4\eta^2\lambda_1^2\left(\frac{1-\theta}{\theta}\right)$$
$$\left|\gamma_2 - 1 - 2\eta\lambda_2 - 4L^4\sigma^4\eta^2\lambda_2\operatorname{Tr}(\Sigma)\right| \le 4L^4\sigma^4\eta^2\lambda_1^2\left(\frac{1-\theta}{\theta}\right)$$

Proof. Let $\alpha_n := \mathbb{E}\left[\operatorname{Tr}\left(v_1^T B_n U U^T B_n^T v_1\right)\right]$, $\beta_n := \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T B_n U U^T B_n^T V_\perp\right)\right]$. Define $A_n := X_n X_n^T$ and let $\mathcal{F}_n$ denote the filtration for observations $i \in [n]$. Then,

$$\begin{aligned}
\alpha_n &= \mathbb{E}\left[v_1^T B_n U U^T B_n^T v_1\right] \\
&= \mathbb{E}\left[v_1^T\left(I+\eta A_n\right)B_{n-1}UU^TB_{n-1}^T\left(I+\eta A_n\right)v_1\right] \\
&= \alpha_{n-1} + 2\eta\mathbb{E}\left[v_1^T A_n B_{n-1}UU^TB_{n-1}^T v_1\right] + \eta^2\mathbb{E}\left[v_1^T A_n B_{n-1}UU^TB_{n-1}^T A_n v_1\right] \\
&= \alpha_{n-1} + 2\eta\mathbb{E}\left[v_1^T\Sigma B_{n-1}UU^TB_{n-1}^T v_1\right] + \eta^2\mathbb{E}\left[\left(v_1^TX_nX_n^Tv_1\right)\left(X_n^TB_{n-1}UU^TB_{n-1}^TX_n\right)\right] \\
&= \alpha_{n-1}\left(1+2\eta\lambda_1\right) + \eta^2\mathbb{E}\left[\left(v_1^TX_nX_n^Tv_1\right)\left(X_n^TB_{n-1}UU^TB_{n-1}^TX_n\right)\right] \\
&= \alpha_{n-1}\left(1+2\eta\lambda_1\right) + \eta^2\mathbb{E}\left[\mathbb{E}\left[\left(v_1^TX_nX_n^Tv_1\right)\left(X_n^TB_{n-1}UU^TB_{n-1}^TX_n\right)\middle|\mathcal{F}_{n-1}\right]\right] \\
&\le \alpha_{n-1}\left(1+2\eta\lambda_1\right) + \eta^2\mathbb{E}\left[\sqrt{\mathbb{E}\left[\left(v_1^TX_nX_n^Tv_1\right)^2\middle|\mathcal{F}_{n-1}\right]\mathbb{E}\left[\left(X_n^TB_{n-1}UU^TB_{n-1}^TX_n\right)^2\middle|\mathcal{F}_{n-1}\right]}\right] \\
&= \alpha_{n-1}\left(1+2\eta\lambda_1\right) + \eta^2\mathbb{E}\left[\sqrt{\mathbb{E}\left[\left(X_n^Tv_1v_1^TX_n\right)^2\middle|\mathcal{F}_{n-1}\right]\mathbb{E}\left[\left(X_n^TB_{n-1}UU^TB_{n-1}X_n\right)^2\middle|\mathcal{F}_{n-1}\right]}\right] \\
&\le \alpha_{n-1}\left(1+2\eta\lambda_1\right) + 4\eta^2L^4\sigma^4\operatorname{Tr}\left(\Sigma, v_1v_1^T\right)\mathbb{E}\left[\operatorname{Tr}\left(\Sigma, B_{n-1}UU^TB_{n-1}^T\right)\right], \text{ using Lemma A.2.2 with } p=2 \\
&= \alpha_{n-1}\left(1+2\eta\lambda_1\right) + 4\eta^2L^4\sigma^4\lambda_1\left(\mathbb{E}\left[\operatorname{Tr}\left(\lambda_1v_1v_1^T + V_\perp\Lambda_2V_\perp^T, B_{n-1}UU^TB_{n-1}^T\right)\right]\right) \\
&\le \left(1+2\eta\lambda_1+4\eta^2L^4\sigma^4\lambda_1^2\right)\alpha_{n-1} + 4\eta^2L^4\sigma^4\lambda_1\lambda_2\beta_{n-1} \qquad \text{(A.29)}
\end{aligned}$$

and similarly,

$$\begin{aligned}
\beta_n &= \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^TB_nUU^TB_n^TV_\perp\right)\right] \\
&= \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T\left(I+\eta A_n\right)B_{n-1}UU^TB_{n-1}^T\left(I+\eta A_n\right)V_\perp\right)\right] \\
&= \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^TB_{n-1}UU^TB_{n-1}^TV_\perp\right)\right] + 2\eta\mathbb{E}\left[\operatorname{Tr}\left(V_\perp^TA_nB_{n-1}UU^TB_{n-1}^TV_\perp\right)\right] + \\
&\quad + \eta^2\mathbb{E}\left[\operatorname{Tr}\left(V_\perp^TA_nB_{n-1}UU^TB_{n-1}^TA_nV_\perp\right)\right] \\
&= \beta_{n-1} + 2\eta\mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T\Sigma B_{n-1}UU^TB_{n-1}^TV_\perp\right)\right] + \eta^2\mathbb{E}\left[\left(X_n^TV_\perp V_\perp^TX_n\right)\left(X_n^TB_{n-1}UU^TB_{n-1}^TX_n\right)\right] \\
&= \beta_{n-1} + 2\eta\mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T\Sigma B_{n-1}UU^TB_{n-1}^TV_\perp\right)\right] + \eta^2\mathbb{E}\left[\mathbb{E}\left[\left(X_n^TV_\perp V_\perp^TX_n\right)\left(X_n^TB_{n-1}UU^TB_{n-1}^TX_n\right)\middle|\mathcal{F}_{n-1}\right]\right] \\
&\le \left(1+2\eta\lambda_2\right)\beta_{n-1} + \eta^2\mathbb{E}\left[\sqrt{\mathbb{E}\left[\left(X_n^TV_\perp V_\perp^TX_n\right)^2\middle|\mathcal{F}_{n-1}\right]\mathbb{E}\left[\left(X_n^TB_{n-1}UU^TB_{n-1}^TX_n\right)^2\middle|\mathcal{F}_{n-1}\right]}\right] \\
&\le \left(1+2\eta\lambda_2\right)\beta_{n-1} + 4\eta^2L^4\sigma^4\operatorname{Tr}\left(\Sigma V_\perp V_\perp^T\right)\mathbb{E}\left[\operatorname{Tr}\left(\Sigma B_{n-1}UU^TB_{n-1}^T\right)\right], \text{ using Lemma A.2.2 with } p=2 \\
&= \left(1+2\eta\lambda_2+4\eta^2L^4\sigma^4\lambda_2\operatorname{Tr}\left(\Lambda_2\right)\right)\beta_{n-1} + 4\eta^2L^4\sigma^4\lambda_1\operatorname{Tr}\left(\Lambda_2\right)\alpha_{n-1} \qquad \text{(A.30)}
\end{aligned}$$

The result then follows by using Lemma A.2.5. □

Lemma A.2.8. *For all $t > 0$, under subgaussianity (Definition 2.1), let $U \in \mathbb{R}^{d\times m}$. Let the step-size η be set according to Lemma A.2.4 then we have,*

$$\mathbb{E}\left[\left(v_1^TB_nUU^TB_n^Tv_1\right)^2\right] \le \mu_1^{2n}\left[\alpha_0 + \eta\lambda_1\left(\frac{2\lambda_1}{\theta\left(\lambda_1-\lambda_2\right)}\right)\left(\beta_0 + \alpha_0\left(\frac{1-\theta}{\theta}\right)\right)\right]^2,$$
$$\mathbb{E}\left[\operatorname{Tr}\left(V_\perp^TB_nUU^TB_n^TV_\perp\right)^2\right] \le \left(\beta_0\mu_2^n + \left[\eta\lambda_1\left(\frac{2\lambda_1}{\theta\left(\lambda_1-\lambda_2\right)}\right)\left(\alpha_0\frac{\operatorname{Tr}\left(\Sigma\right)}{\lambda_1} + \beta_0\left(\frac{1-\theta}{\theta}\right)\right)\right]\mu_1^n\right)^2$$

where B_n is defined in Eq 3, $\alpha_0 = v_1^T U U^T v_1$, $\beta_0 = \operatorname{Tr}\left(V_\perp^T U U^T V_\perp\right)$ and

$$\left|\mu_1 - 1 - 2\eta\lambda_1 - 50\eta^2 L^4\sigma^4\lambda_1^2\right| \le 50L^4\sigma^4\eta^2\lambda_1^2\left(\frac{1-\theta}{\theta}\right)$$
$$\left|\mu_2 - 1 - 2\eta\lambda_2 - 50\eta^2 L^4\sigma^4\lambda_2\operatorname{Tr}(\Sigma)\right| \le 50L^4\sigma^4\eta^2\lambda_1^2\left(\frac{1-\theta}{\theta}\right)$$

Proof. Let $\alpha_n := \mathbb{E}\left[\left(v_1^T B_n U U^T B_n^T v_1\right)^2\right]$, $\beta_n := \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T B_n U U^T B_n^T V_\perp\right)^2\right]$. Then, using Lemma A.2.9 we have,

$$\alpha_n \le \left(1 + 4\eta\lambda_1 + 100\eta^2 L^4\sigma^4\lambda_1^2\right)\alpha_{n-1} + 100\eta^2 L^4\sigma^4\lambda_1^2\sqrt{\alpha_{n-1}\beta_{n-1}} + 600\eta^4 L^8\sigma^8\lambda_1^4\beta_{n-1}$$

and using Lemma A.2.10 we have,

$$\begin{aligned}\beta_n \le{}& \left(1 + 4\eta\lambda_2 + 100\eta^2 L^4\sigma^4\lambda_2\operatorname{Tr}(\Sigma)\right)\beta_{n-1} + 100\eta^2 L^2\sigma^2\lambda_1\operatorname{Tr}(\Sigma)\sqrt{\alpha_{n-1}\beta_{n-1}} \\ &+ 600\eta^2 L^4\sigma^4\lambda_1^2\alpha_{n-1}\end{aligned}$$

Define $a_n := \sqrt{\alpha_n}$ and $b_n := \sqrt{\beta_n}$. Then using $\sqrt{1+x} \le 1 + \frac{x}{2}$, we have

$$\begin{aligned}a_n &\le \sqrt{1 + 4\eta\lambda_1 + 100\eta^2 L^4\sigma^4\lambda_1^2}a_{n-1} + 25\eta^2 L^2\sigma^2\lambda_1^2 b_{n-1}, \\ &\le \left(1 + 2\eta\lambda_1 + 50\eta^2 L^4\sigma^4\lambda_1^2\right)a_{n-1} + 25\eta^2 L^2\sigma^2\lambda_1^2 b_{n-1} \\ b_n &\le \sqrt{1 + 4\eta\lambda_2 + 100\eta^2 L^4\sigma^4\lambda_2\operatorname{Tr}(\Sigma)}b_{n-1} + 25\eta^2 L^2\sigma^2\lambda_1^2 a_{n-1}, \\ &\le \left(1 + 2\eta\lambda_2 + 50\eta^2 L^4\sigma^4\lambda_2\operatorname{Tr}(\Sigma)\right)b_{n-1} + 25\eta^2 L^2\sigma^2\lambda_1^2 a_{n-1}\end{aligned}$$

The result then follows from Lemma A.2.5. □

Lemma A.2.9. *For all $t > 0$, under subgaussianity (Definition 2.1), let $U \in \mathbb{R}^{d\times m}$, $\alpha_n := \mathbb{E}\left[\left(v_1^T B_n U U^T B_n^T v_1\right)^2\right]$, $\beta_n := \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T B_n U U^T B_n^T V_\perp\right)^2\right]$ and $\eta L^2\sigma^2\lambda_1 \le \frac{1}{4}$ then*

$$\alpha_n \le \left(1 + 4\eta\lambda_1 + 100\eta^2 L^4\sigma^4\lambda_1^2\right)\alpha_{n-1} + 100\eta^2 L^4\sigma^4\lambda_1^2\sqrt{\alpha_{n-1}\beta_{n-1}} + 600\eta^4 L^8\sigma^8\lambda_1^4\beta_{n-1}$$

where B_n is defined in Eq 3.

Proof. Let $A_n := X_n X_n^T$ and $\mathcal{F}_n$ denote the filtration for observations $i \in [n]$. Then,

$$\begin{aligned}\alpha_n &= \mathbb{E}\left[\left(v_1^T B_n U U^T B_n^T v_1\right)^2\right] \\ &= \mathbb{E}\left[\left(v_1^T B_{n-1} U U^T B_{n-1}^T v_1 + 2\eta v_1^T A_n B_{n-1} U U^T B_{n-1}^T v_1 + \eta^2 v_1^T A_n B_{n-1} U U^T B_{n-1}^T A_n v_1\right)^2\right] \\ &= \alpha_{n-1}\left(1 + 4\eta\lambda_1\right) + 4\eta^2\underbrace{\mathbb{E}\left[\left(v_1^T A_n B_{n-1} U U^T B_{n-1}^T v_1\right)^2\right]}_{T_1} \\ &\quad + 2\eta^2\underbrace{\mathbb{E}\left[\left(v_1^T B_{n-1} U U^T B_{n-1}^T v_1\right)\left(v_1^T A_n B_{n-1} U U^T B_{n-1}^T A_n v_1\right)\right]}_{T_2} \\ &\quad + 4\eta^3\underbrace{\mathbb{E}\left[\left(v_1^T A_n B_{n-1} U U^T B_{n-1}^T v_1\right)\left(v_1^T A_n B_{n-1} U U^T B_{n-1}^T A_n v_1\right)\right]}_{T_3} \\ &\quad + \eta^4\underbrace{\mathbb{E}\left[\left(v_1^T A_n B_{n-1} U U^T B_{n-1}^T A_n v_1\right)^2\right]}_{T_4} \qquad \text{(A.31)}\end{aligned}$$

For T_1,

$$
\begin{aligned}
T_1 &= \mathbb{E}\left[\left(v_1^T A_n B_{n-1} U U^T B_{n-1}^T v_1\right)^2\right] \\
&= \mathbb{E}\left[\left(X_n^T v_1 v_1^T X_n\right)\left(X_n^T B_{n-1} U U^T B_{n-1}^T v_1 v_1^T B_{n-1} U U^T B_{n-1}^T X_n\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(X_n^T v_1 v_1^T X_n\right)\left(X_n^T B_{n-1} U U^T B_{n-1}^T v_1 v_1^T B_{n-1} U U^T B_{n-1}^T X_n\right)\middle|\mathcal{F}_{n-1}\right]\right] \\
&\leq \mathbb{E}\left[\sqrt{\mathbb{E}\left[\left(X_n^T v_1 v_1^T X_n\right)^2\middle|\mathcal{F}_{n-1}\right]\mathbb{E}\left[\left(X_n^T B_{n-1} U U^T B_{n-1}^T v_1 v_1^T B_{n-1} U U^T B_{n-1}^T X_n\right)^2\middle|\mathcal{F}_{n-1}\right]}\right] \\
&\leq 4L^4\sigma^4 \operatorname{Tr}\left(\Sigma v_1 v_1^T\right)\mathbb{E}\left[\operatorname{Tr}\left(\Sigma B_{n-1} U U^T B_{n-1}^T v_1 v_1^T B_{n-1} U U^T B_{n-1}^T\right)\right], \text{ using Lemma } A.2.2 \text{ with } p=2 \\
&= 4L^4\sigma^4\lambda_1^2\alpha_{n-1} + 4L^4\sigma^4\lambda_1\mathbb{E}\left[\operatorname{Tr}\left(V_\perp \Lambda_2 V_\perp^T B_{n-1} U U^T B_{n-1}^T v_1 v_1^T B_{n-1} U U^T B_{n-1}^T\right)\right] \\
&\leq 4L^4\sigma^4\lambda_1^2\alpha_{n-1} + 4L^4\sigma^4\lambda_1\lambda_2\mathbb{E}\left[\operatorname{Tr}\left(V_\perp V_\perp^T B_{n-1} U U^T B_{n-1}^T v_1 v_1^T B_{n-1} U U^T B_{n-1}^T\right)\right] \\
&= 4L^4\sigma^4\lambda_1^2\alpha_{n-1} + 4L^4\sigma^4\lambda_1\lambda_2\mathbb{E}\left[v_1^T B_{n-1} U U^T B_{n-1}^T V_\perp V_\perp^T B_{n-1} U U^T B_{n-1}^T v_1\right] \\
&\leq 4L^4\sigma^4\lambda_1^2\alpha_{n-1} + 4L^4\sigma^4\lambda_1\lambda_2\mathbb{E}\left[\left(v_1^T B_{n-1} U U^T B_{n-1}^T v_1\right)\operatorname{Tr}\left(U^T B_{n-1}^T V_\perp V_\perp^T B_{n-1} U\right)\right] \\
&\leq 4L^4\sigma^4\lambda_1^2\alpha_{n-1} + 4L^4\sigma^4\lambda_1\lambda_2\sqrt{\alpha_{n-1}\beta_{n-1}}
\end{aligned}
$$

For T_2,

$$
\begin{aligned}
T_2 &= \mathbb{E}\left[\left(v_1^T B_{n-1} U U^T B_{n-1}^T v_1\right)\left(v_1^T A_n B_{n-1} U U^T B_{n-1}^T A_n v_1\right)\right] \\
&= \mathbb{E}\left[\left(v_1^T B_{n-1} U U^T B_{n-1}^T v_1\right)\mathbb{E}\left[\left(v_1^T A_n B_{n-1} U U^T B_{n-1}^T A_n v_1\right)|\mathcal{F}_{n-1}\right]\right] \\
&= \mathbb{E}\left[\left(v_1^T B_{n-1} U U^T B_{n-1}^T v_1\right)\mathbb{E}\left[\left(X_n^T v_1 v_1^T X_n\right)\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)|\mathcal{F}_{n-1}\right]\right] \\
&\leq \mathbb{E}\left[\left(v_1^T B_{n-1} U U^T B_{n-1}^T v_1\right)\sqrt{\mathbb{E}\left[\left(X_n^T v_1 v_1^T X_n\right)^2\middle|\mathcal{F}_{n-1}\right]\mathbb{E}\left[\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)^2|\mathcal{F}_{n-1}\right]}\right] \\
&\leq 4L^4\sigma^4\mathbb{E}\left[\left(v_1^T B_{n-1} U U^T B_{n-1}^T v_1\right)\operatorname{Tr}\left(\Sigma v_1 v_1^T\right)\operatorname{Tr}\left(\Sigma B_{n-1} U U^T B_{n-1}^T\right)\right], \text{ using Lemma } A.2.2 \text{ with } p=2 \\
&= 4L^4\sigma^4\lambda_1^2\alpha_{n-1} + 4L^4\sigma^4\lambda_1\mathbb{E}\left[\left(v_1^T B_{n-1} U U^T B_{n-1}^T v_1\right)\operatorname{Tr}\left(V_\perp \Lambda_2 V_\perp^T B_{n-1} U U^T B_{n-1}^T\right)\right] \\
&\leq 4L^4\sigma^4\lambda_1^2\alpha_{n-1} + 4L^4\sigma^4\lambda_1\lambda_2\mathbb{E}\left[\left(v_1^T B_{n-1} U U^T B_{n-1}^T v_1\right)\operatorname{Tr}\left(V_\perp V_\perp^T B_{n-1} U U^T B_{n-1}^T\right)\right] \\
&\leq 4L^4\sigma^4\lambda_1^2\alpha_{n-1} + 4L^4\sigma^4\lambda_1\lambda_2\sqrt{\alpha_{n-1}\beta_{n-1}}
\end{aligned}
$$

For T_3,

$$
\begin{aligned}
T_3 &= \mathbb{E}\left[\left(v_1^T A_n B_{n-1} U U^T B_{n-1}^T v_1\right)\left(v_1^T A_n B_{n-1} U U^T B_{n-1}^T A_n v_1\right)\right] \\
&= \mathbb{E}\left[\left(X_n^T v_1\right)^3\left(X_n^T B_{n-1} U U^T B_{n-1}^T v_1\right)\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)\right] \\
&\leq \mathbb{E}\left[\left(X_n^T v_1 v_1^T X_n\right)^{\frac{3}{2}}\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)^{\frac{3}{2}}\left\|U^T B_{n-1}^T v_1\right\|_2\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(X_n^T v_1 v_1^T X_n\right)^{\frac{3}{2}}\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)^{\frac{3}{2}}\middle|\mathcal{F}_{n-1}\right]\left\|U^T B_{n-1}^T v_1\right\|_2\right] \\
&\leq \mathbb{E}\left[\sqrt{\mathbb{E}\left[\left(X_n^T v_1 v_1^T X_n\right)^3\middle|\mathcal{F}_{n-1}\right]}\sqrt{\mathbb{E}\left[\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)^3\middle|\mathcal{F}_{n-1}\right]}\left\|U^T B_{n-1}^T v_1\right\|_2\right] \\
&\leq \left(3L^2\sigma^2\right)^3\lambda_1^{\frac{3}{2}}\mathbb{E}\left[\operatorname{Tr}\left(B_{n-1} U U^T B_{n-1}^T \Sigma\right)^{\frac{3}{2}}\left(v_1^T B_{n-1} U U^T B_{n-1}^T v_1\right)^{\frac{1}{2}}\right], \text{ using Lemma } A.2.2 \text{ with } p=3 \\
&\leq 2\left(3L^2\sigma^2\right)^3\lambda_1^{\frac{3}{2}}\left(\lambda_1^{\frac{3}{2}}\alpha_{n-1} + \lambda_2^{\frac{3}{2}}\mathbb{E}\left[\left(v_1^T B_{n-1} U U^T B_{n-1}^T v_1\right)^{\frac{1}{2}}\operatorname{Tr}\left(V_\perp^T B_{n-1} U U^T B_{n-1}^T V_\perp\right)^{\frac{3}{2}}\right]\right)
\end{aligned}
$$

$$= 2\left(3L^2\sigma^2\right)^3\lambda_1^3\alpha_{n-1} + 2\left(3L^2\sigma^2\right)^3 \times$$
$$\mathbb{E}\left[\frac{\sqrt{\lambda_1\lambda_2\left\|U^TB_{n-1}^Tv_1\right\|_2^2\operatorname{Tr}\left(V_\perp^TB_{n-1}UU^TB_{n-1}^TV_\perp\right)}}{\sqrt{3\eta L^2\sigma^2}}\sqrt{3\eta L^2\sigma^2}\lambda_1\lambda_2\operatorname{Tr}\left(V_\perp^TB_{n-1}UU^TB_{n-1}^TV_\perp\right)\right]$$
$$\leq 2\left(3L^2\sigma^2\right)^3\lambda_1^3\alpha_{n-1} + \left(3L^2\sigma^2\right)^3\mathbb{E}\left[\frac{\lambda_1\lambda_2v_1^TB_{n-1}UU^TB_{n-1}^Tv_1\operatorname{Tr}\left(V_\perp^TB_{n-1}UU^TB_{n-1}^TV_\perp\right)}{3\eta L^2\sigma^2}\right]$$
$$+\eta\left(3L^2\sigma^2\right)^4\mathbb{E}\left[\lambda_1^2\lambda_2^2\operatorname{Tr}\left(V_\perp^TB_{n-1}UU^TB_{n-1}^TV_\perp\right)^2\right]$$
$$= 2\left(3L^2\sigma^2\right)^3\lambda_1^3\alpha_{n-1} + \frac{\left(3L^2\sigma^2\right)^2\lambda_1\lambda_2}{\eta}\mathbb{E}\left[v_1^TB_{n-1}UU^TB_{n-1}^Tv_1\operatorname{Tr}\left(V_\perp^TB_{n-1}UU^TB_{n-1}^TV_\perp\right)\right]$$
$$+\eta\left(3L^2\sigma^2\right)^4\lambda_1^2\lambda_2^2\mathbb{E}\left[\operatorname{Tr}\left(V_\perp^TB_{n-1}UU^TB_{n-1}^TV_\perp\right)^2\right]$$
$$\leq 2\left(3L^2\sigma^2\right)^3\lambda_1^3\alpha_{n-1} + \frac{\left(3L^2\sigma^2\right)^2\lambda_1\lambda_2}{\eta}\sqrt{\alpha_{n-1}\beta_{n-1}} + \eta\left(3L^2\sigma^2\right)^4\lambda_1^2\lambda_2^2\beta_{n-1}$$

For T_4,

$$T_4 = \mathbb{E}\left[\left(v_1^TA_nB_{n-1}UU^TB_{n-1}^TA_nv_1\right)^2\right]$$
$$= \mathbb{E}\left[\left(X_n^Tv_1v_1^TX_n\right)^2\left(X_n^TB_{n-1}UU^TB_{n-1}^TX_n\right)^2\right]$$
$$= \mathbb{E}\left[\mathbb{E}\left[\left(X_n^Tv_1v_1^TX_n\right)^2\left(X_n^TB_{n-1}UU^TB_{n-1}^TX_n\right)^2\middle|\mathcal{F}_{n-1}\right]\right]$$
$$\leq \left(4L^2\sigma^2\right)^4\mathbb{E}\left[\sqrt{\mathbb{E}\left[\left(v_1^TX_nX_n^Tv_1\right)^4\middle|\mathcal{F}_{n-1}\right]\mathbb{E}\left[\operatorname{Tr}\left(B_{n-1}UU^TB_{n-1}^T\Sigma\right)^4\middle|\mathcal{F}_{n-1}\right]}\right]$$
$$\leq \left(4L^2\sigma^2\right)^4\mathbb{E}\left[\sqrt{\left(v_1^T\Sigma v_1\right)^4\mathbb{E}\left[\operatorname{Tr}\left(B_{n-1}UU^TB_{n-1}^T\Sigma\right)^4\middle|\mathcal{F}_{n-1}\right]}\right], \text{ using Lemma } A.2.2 \text{ with } p=4$$
$$= \left(4L^2\sigma^2\right)^4\lambda_1^2\mathbb{E}\left[\sqrt{\left(\operatorname{Tr}\left(B_{n-1}UU^TB_{n-1}^T\Sigma\right)\right)^4}\right]$$
$$= \left(4L^2\sigma^2\right)^4\lambda_1^2\mathbb{E}\left[\left(\operatorname{Tr}\left(B_{n-1}UU^TB_{n-1}^TV_\perp\Lambda_2V_\perp^T\right) + \lambda_1\operatorname{Tr}\left(B_{n-1}UU^TB_{n-1}^Tv_1v_1^T\right)\right)^2\right]$$
$$\leq 2\left(4L^2\sigma^2\right)^4\lambda_1^2\left(\lambda_2^2\beta_{n-1} + \lambda_1^2\alpha_{n-1}\right)$$

Substituting in Eq A.31 along with using $\eta L^2\sigma^2\lambda_1 \leq \frac{1}{4}$ we have,

$$\alpha_n \leq \left(1 + 4\eta\lambda_1 + 100\eta^2L^4\sigma^4\lambda_1^2\right)\alpha_{n-1} + 100\eta^2L^4\sigma^4\lambda_1^2\sqrt{\alpha_{n-1}\beta_{n-1}} + 600\eta^4L^8\sigma^8\lambda_1^4\beta_{n-1}$$

Hence proved. □

Lemma A.2.10. *For all $t > 0$, under subgaussianity (Definition 2.1), let $U \in \mathbb{R}^{d\times m}$, $\alpha_n := \mathbb{E}\left[\left(v_1^TB_nUU^TB_n^Tv_1\right)^2\right]$, $\beta_n := \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^TB_nUU^TB_n^TV_\perp\right)^2\right]$ and $\eta L^2\sigma^2 \leq \frac{1}{4}\min\left\{\frac{1}{\lambda_1}, \frac{1}{\operatorname{Tr}(\Sigma)}, \frac{1}{\sqrt{\lambda_1\operatorname{Tr}(\Sigma)}}\right\}$ then*

$$\beta_n \leq \left(1 + 4\eta\lambda_2 + 100\eta^2\log(n)L^4\sigma^4\lambda_2\operatorname{Tr}(\Sigma)\right)\beta_{n-1} + 100\eta^2\log(n)L^2\sigma^2\lambda_1\operatorname{Tr}(\Sigma)\sqrt{\alpha_{n-1}\beta_{n-1}}$$
$$+ 600\eta^2\log^2(n)L^4\sigma^4\lambda_1^2\alpha_{n-1}$$

where B_n is defined in Eq 3.

Proof. Let $A_n := X_nX_n^T$ and $\mathcal{F}_n$ denote the filtration for observations $i \in [n]$.

Let

$$a_{n-1} := \operatorname{Tr}\left(V_\perp^TB_{n-1}UU^TB_{n-1}^TV_\perp\right), b_{n-1} := \operatorname{Tr}\left(V_\perp^TA_nB_{n-1}UU^TB_{n-1}^TV_\perp\right),$$
$$c_{n-1} := \operatorname{Tr}\left(V_\perp^TA_nB_{n-1}UU^TB_{n-1}^TA_nV_\perp\right)$$

Then,

$$
\begin{aligned}
&\beta_n \\
&= \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T B_n U U^T B_n^T V_\perp\right)^2\right] \\
&= \mathbb{E}\left[\left(a_{n-1} + 2\eta b_{n-1} + \eta^2 c_{n-1}\right)^2\right] \\
&\leq \beta_{n-1}\left(1 + 4\eta\lambda_2\right) + 4\eta^2 \underbrace{\mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T A_n B_{n-1} U U^T B_{n-1}^T V_\perp\right)^2\right]}_{T_1} \\
&\quad + 2\eta^2 \underbrace{\mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T B_{n-1} U U^T B_{n-1}^T V_\perp\right)\operatorname{Tr}\left(V_\perp^T A_n B_{n-1} U U^T B_{n-1}^T A_n V_\perp\right)\right]}_{T_2} \\
&\quad + 4\eta^3 \underbrace{\mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T A_n B_{n-1} U U^T B_{n-1}^T V_\perp\right)\operatorname{Tr}\left(V_\perp^T A_n B_{n-1} U U^T B_{n-1}^T A_n V_\perp\right)\right]}_{T_3} \\
&\quad + \eta^4 \underbrace{\mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T A_n B_{n-1} U U^T B_{n-1}^T A_n V_\perp\right)^2\right]}_{T_4}
\end{aligned} \tag{A.32}
$$

For T_1,

$$
\begin{aligned}
T_1 &= \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T A_n B_{n-1} U U^T B_{n-1}^T V_\perp\right)^2\right] \\
&= \mathbb{E}\left[\left(X_n^T B_{n-1} U U^T B_{n-1}^T V_\perp V_\perp^T X_n\right)^2\right] \\
&\leq \mathbb{E}\left[\left\|V_\perp^T B_{n-1} U U^T B_{n-1}^T X_n\right\|_2^2 \left\|V_\perp^T X_n\right\|_2^2\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left\|V_\perp^T B_{n-1} U U^T B_{n-1}^T X_n\right\|_2^2 \left\|V_\perp^T X_n\right\|_2^2 \middle| \mathcal{F}_{n-1}\right]\right] \\
&\leq \mathbb{E}\left[\sqrt{\mathbb{E}\left[\left\|V_\perp^T B_{n-1} U U^T B_{n-1}^T X_n\right\|_2^2 \middle| \mathcal{F}_{n-1}\right]\mathbb{E}\left[\left\|V_\perp^T X_n\right\|_2^2 \middle| \mathcal{F}_{n-1}\right]}\right] \\
&\leq \left(2L^2\sigma^2\right)^2 \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T B_{n-1} U U^T B_{n-1}^T \Sigma B_{n-1} U U^T B_{n-1}^T V_\perp\right)\operatorname{Tr}\left(V_\perp^T \Sigma V_\perp\right)\right], \text{ using Lemma } A.2.2 \text{ with } p = 2 \\
&\leq \left(2L^2\sigma^2\right)^2 \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T B_{n-1} U U^T B_{n-1}^T \left(\lambda_1 v_1 v_1^T + V_\perp \Lambda_2 V_\perp^T\right) B_{n-1} U U^T B_{n-1}^T V_\perp\right)\operatorname{Tr}\left(V_\perp^T \Sigma V_\perp\right)\right] \\
&\leq \left(2L^2\sigma^2\right)^2 \operatorname{Tr}\left(\Lambda_2\right)\left(\lambda_2\beta_{n-1} + \lambda_1\sqrt{\alpha_{n-1}\beta_{n-1}}\right)
\end{aligned}
$$

For T_2,

$$
\begin{aligned}
T_2 &= \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T B_{n-1} U U^T B_{n-1}^T V_\perp\right)\operatorname{Tr}\left(V_\perp^T A_n B_{n-1} U U^T B_{n-1}^T A_n V_\perp\right)\right] \\
&= \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T B_{n-1} U U^T B_{n-1}^T V_\perp\right)\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)\left(X_n^T V_\perp V_\perp^T X_n\right)\right] \\
&= \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T B_{n-1} U U^T B_{n-1}^T V_\perp\right)\mathbb{E}\left[\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)\left(X_n^T V_\perp V_\perp^T X_n\right) \middle| \mathcal{F}_{n-1}\right]\right] \\
&\leq \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T B_{n-1} U U^T B_{n-1}^T V_\perp\right)\sqrt{\mathbb{E}\left[\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)^2 \middle| \mathcal{F}_{n-1}\right]\mathbb{E}\left[\left(X_n^T V_\perp V_\perp^T X_n\right)^2 \middle| \mathcal{F}_{n-1}\right]}\right] \\
&\overset{(i)}{\leq} \left(2L^2\sigma^2\right)^2 \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T B_{n-1} U U^T B_{n-1}^T V_\perp\right)\operatorname{Tr}\left(U^T B_{n-1}^T \Sigma B_{n-1} U\right)\operatorname{Tr}\left(V_\perp^T \Sigma V_\perp\right)\right], \\
&= \left(2L^2\sigma^2\right)^2 \operatorname{Tr}\left(\Lambda_2\right)\mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T B_{n-1} U U^T B_{n-1}^T V_\perp\right)\operatorname{Tr}\left(U^T B_{n-1}^T \left(\lambda_1 v_1 v_1^T + V_\perp \Lambda_2 V_\perp^T\right) B_{n-1} U\right)\right] \\
&\leq \left(2L^2\sigma^2\right)^2 \operatorname{Tr}\left(\Lambda_2\right)\left(\lambda_2\beta_{n-1} + \lambda_1\sqrt{\alpha_{n-1}\beta_{n-1}}\right)
\end{aligned}
$$

where in (i) we used Lemma $A.2.2$ with $p=2$. For T_3,

$$
\begin{aligned}
T_3 &= \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T A_n B_{n-1} U U^T B_{n-1}^T V_\perp\right) \operatorname{Tr}\left(V_\perp^T A_n B_{n-1} U U^T B_{n-1}^T A_n V_\perp\right)\right] \\
&= \mathbb{E}\left[\left(X_n^T B_{n-1} U U^T B_{n-1}^T V_\perp V_\perp^T X_n\right)\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)\left(X_n^T V_\perp V_\perp^T X_n\right)\right] \\
&\leq \mathbb{E}\left[\left\|U^T B_{n-1}^T X_n\right\|_2^3 \left\|V_\perp^T X_n\right\|_2^3 \left\|U^T B_{n-1}^T V_\perp\right\|_2\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left\|U^T B_{n-1}^T X_n\right\|_2^3 \left\|V_\perp^T X_n\right\|_2^3 \middle| \mathcal{F}_{n-1}\right] \left\|U^T B_{n-1}^T V_\perp\right\|_2\right] \\
&\leq \mathbb{E}\left[\sqrt{\mathbb{E}\left[\left\|U^T B_{n-1}^T X_n\right\|_2^6 \middle| \mathcal{F}_{n-1}\right] \mathbb{E}\left[\left\|V_\perp^T X_n\right\|_2^6 \middle| \mathcal{F}_{n-1}\right]} \left\|U^T B_{n-1}^T V_\perp\right\|_2\right] \\
&= \mathbb{E}\left[\sqrt{\mathbb{E}\left[\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)^3 \middle| \mathcal{F}_{n-1}\right] \mathbb{E}\left[\left(X_n^T V_\perp V_\perp^T X_n\right)^3 \middle| \mathcal{F}_{n-1}\right]} \left\|U^T B_{n-1}^T V_\perp\right\|_2\right] \\
&\leq \left(3L^2\sigma^2\right)^3 \mathbb{E}\left[\operatorname{Tr}\left(U^T B_{n-1}^T \Sigma B_{n-1} U\right)^{\frac{3}{2}} \operatorname{Tr}\left(V_\perp^T \Sigma V_\perp\right)^{\frac{3}{2}} \left\|U^T B_{n-1}^T V_\perp\right\|_2\right], \text{ using Lemma } A.2.2 \text{ with } p=3 \\
&= \left(3L^2\sigma^2\right)^3 \operatorname{Tr}\left(\Lambda_2\right)^{\frac{3}{2}} \mathbb{E}\left[\operatorname{Tr}\left(U^T B_{n-1}^T \Sigma B_{n-1} U\right)^{\frac{3}{2}} \left\|U^T B_{n-1}^T V_\perp\right\|_2\right] \\
&\leq 2\left(3L^2\sigma^2\right)^3 \operatorname{Tr}\left(\Lambda_2\right)^{\frac{3}{2}} \mathbb{E}\left[\left(\lambda_1^{\frac{3}{2}} \left(v_1^T B_{n-1} U U^T B_{n-1}^T v_1\right)^{\frac{3}{2}} + \lambda_2^{\frac{3}{2}} \operatorname{Tr}\left(V_\perp^T B_{n-1} U U^T B_{n-1}^T V_\perp\right)^{\frac{3}{2}}\right) \left\|U^T B_{n-1}^T V_\perp\right\|_2\right] \\
&\leq 2\left(3L^2\sigma^2\right)^3 \lambda_1^{\frac{3}{2}} \operatorname{Tr}\left(\Lambda_2\right)^{\frac{3}{2}} \mathbb{E}\left[\left(v_1^T B_{n-1} U U^T B_{n-1}^T v_1\right)^{\frac{3}{2}} \operatorname{Tr}\left(V_\perp^T B_{n-1} U U^T B_{n-1}^T V_\perp\right)^{\frac{1}{2}}\right] \\
&\quad + 2\left(3L^2\sigma^2\right)^3 \lambda_2^{\frac{3}{2}} \operatorname{Tr}\left(\Lambda_2\right)^{\frac{3}{2}} \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T B_{n-1} U U^T B_{n-1}^T V_\perp\right)^2\right] \\
&= 2\left(3L^2\sigma^2\right)^3 \lambda_1^{\frac{3}{2}} \operatorname{Tr}\left(\Lambda_2\right)^{\frac{3}{2}} \mathbb{E}\left[\left(v_1^T B_{n-1} U U^T B_{n-1}^T v_1\right)^{\frac{3}{2}} \operatorname{Tr}\left(V_\perp^T B_{n-1} U U^T B_{n-1}^T V_\perp\right)^{\frac{1}{2}}\right] \\
&\quad + 2\left(3L^2\sigma^2\right)^3 \lambda_2^{\frac{3}{2}} \operatorname{Tr}\left(\Lambda_2\right)^{\frac{3}{2}} \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T B_{n-1} U U^T B_{n-1}^T V_\perp\right)^2\right] \\
&= 2\left(3L^2\sigma^2\right)^3 \lambda_1^{\frac{3}{2}} \operatorname{Tr}\left(\Lambda_2\right)^{\frac{3}{2}} \mathbb{E}\left[\sqrt{\left(v_1^T B_{n-1} U U^T B_{n-1}^T v_1\right) \operatorname{Tr}\left(V_\perp^T B_{n-1} U U^T B_{n-1}^T V_\perp\right)} \left(v_1^T B_{n-1} U U^T B_{n-1}^T v_1\right)\right] \\
&\quad + 2\left(3L^2\sigma^2\right)^3 \lambda_2^{\frac{3}{2}} \operatorname{Tr}\left(\Lambda_2\right)^{\frac{3}{2}} \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T B_{n-1} U U^T B_{n-1}^T V_\perp\right)^2\right] \\
&\leq 2\left(3L^2\sigma^2\right)^2 \lambda_1 \operatorname{Tr}\left(\Lambda_2\right)^2 \sqrt{\alpha_{n-1}\beta_{n-1}} + 2\left(3L^2\sigma^2\right) \lambda_1^2 \operatorname{Tr}\left(\Lambda_2\right) \alpha_{n-1} + 2\left(3L^2\sigma^2\right)^3 \lambda_2^{\frac{3}{2}} \operatorname{Tr}\left(\Lambda_2\right)^{\frac{3}{2}} \beta_{n-1}
\end{aligned}
$$

For T_4,

$$
\begin{aligned}
T_4 &= \mathbb{E}\left[\operatorname{Tr}\left(V_\perp^T A_n B_{n-1} U U^T B_{n-1}^T A_n V_\perp\right)^2\right] \\
&= \mathbb{E}\left[\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)^2 \left(X_n^T V_\perp V_\perp^T X_n\right)^2\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)^2 \left(X_n^T V_\perp V_\perp^T X_n\right)^2 \middle| \mathcal{F}_{n-1}\right]\right] \\
&\leq \mathbb{E}\left[\sqrt{\mathbb{E}\left[\left(X_n^T B_{n-1} U U^T B_{n-1}^T X_n\right)^4 \middle| \mathcal{F}_{n-1}\right] \mathbb{E}\left[\left(X_n^T V_\perp V_\perp^T X_n\right)^4 \middle| \mathcal{F}_{n-1}\right]}\right] \\
&\leq \left(4L^2\sigma^2\right)^4 \mathbb{E}\left[\operatorname{Tr}\left(U^T B_{n-1}^T \Sigma B_{n-1} U\right)^2 \operatorname{Tr}\left(V_\perp^T \Sigma V_\perp\right)^2\right] \\
&\leq \left(4L^2\sigma^2\right)^4 \operatorname{Tr}\left(\Lambda_2\right)^2 \mathbb{E}\left[\operatorname{Tr}\left(U^T B_{n-1}^T \left(\lambda_1 v_1 v_1^T + V_\perp \Lambda_2 V_\perp^T\right)\right) B_{n-1} U\right]^2 \\
&\leq 2\left(4L^2\sigma^2\right)^4 \operatorname{Tr}\left(\Lambda_2\right)^2 \left(\lambda_1^2 \alpha_{n-1} + \lambda_2^2 \beta_{n-1}\right)
\end{aligned}
$$

Substituting in Eq A.32 along with using $\eta L^2\sigma^2 \leq \frac{1}{4}\min\left\{\frac{1}{\lambda_1}, \frac{1}{\operatorname{Tr}(\Sigma)}, \frac{1}{\sqrt{\lambda_1 \operatorname{Tr}(\Sigma)}}\right\}$ we have,

$$
\begin{aligned}
\beta_n \leq &\left(1 + 4\eta\lambda_2 + 100\eta^2 L^4 \sigma^4 \lambda_2 \operatorname{Tr}\left(\Sigma\right)\right) \beta_{n-1} + 100\eta^2 L^2 \sigma^2 \lambda_1 \operatorname{Tr}\left(\Sigma\right) \sqrt{\alpha_{n-1}\beta_{n-1}} \\
&+ 600\eta^2 L^4 \sigma^4 \lambda_1^2 \alpha_{n-1}
\end{aligned}
$$

Hence proved. □

Lemma A.2.11. *Let $U \in \mathbb{R}^{d\times m}$ and $u_0 \sim \mathcal{N}(0, I_d)$, then for all $n \geq 0$ we have*

$$\mathbb{E}\left[u_0^T B_n^T U U^T B_n u_0\right] = \mathbb{E}\left[v_1^T B_n^T U U^T B_n v_1\right] + \mathbb{E}\left[\mathrm{Tr}\left(V_\perp^T B_n^T U U^T B_n V_\perp\right)\right]$$

Proof.

$$\begin{aligned}
\mathbb{E}\left[u_0^T B_n^T U U^T B_n u_0\right] &= \mathbb{E}\left[\mathrm{Tr}\left(u_0^T B_n^T U U^T B_n u_0\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathrm{Tr}\left(U^T B_n u_0 u_0^T B_n^T U\right)|B_n\right]\right] \\
&= \mathbb{E}\left[\mathrm{Tr}\left(U^T B_n \mathbb{E}\left[u_0 u_0^T|B_n\right] B_n^T U\right)\right] \\
&= \mathbb{E}\left[\mathrm{Tr}\left(U^T B_n \mathbb{E}\left[u_0 u_0^T\right] B_n^T U\right)\right] \\
&= \mathbb{E}\left[\mathrm{Tr}\left(U^T B_n B_n^T U\right)\right] \\
&= \mathbb{E}\left[\mathrm{Tr}\left(U^T B_n v_1 v_1^T B_n^T U\right)\right] + \mathbb{E}\left[\mathrm{Tr}\left(U^T B_n V_\perp V_\perp^T B_n^T U\right)\right] \\
&= \mathbb{E}\left[v_1^T B_n^T U U^T B_n v_1\right] + \mathbb{E}\left[\mathrm{Tr}\left(V_\perp^T B_n^T U U^T B_n V_\perp\right)\right]
\end{aligned}$$

□

Lemma A.2.12. *Let $U \in \mathbb{R}^{d\times m}$ and $u_0 \sim \mathcal{N}(0, I_d)$, then for all $n \geq 0$ we have*

$$\mathbb{E}\left[\left(u_0^T B_n^T U U^T B_n u_0\right)^2\right] \leq 6\mathbb{E}\left[\left(v_1^T B_n^T U U^T B_n v_1\right)^2\right] + 6\mathbb{E}\left[\mathrm{Tr}\left(V_\perp^T B_n^T U U^T B_n V_\perp\right)^2\right]$$

Proof. Let the eigendecomposition of $B_n^T U U^T B_n$ for a fixed B_n be given as $P\Lambda P^T$ such that $PP^T = P^T P = I$ and $\Lambda \succeq 0$. Denote $u_0 \equiv u$ and $y := P^T u$. Therefore,

$$\begin{aligned}
\mathbb{E}\left[\left(u_0^T B_n^T U U^T B_n u_0\right)^2\right] &= \mathbb{E}\left[\left(u^T P \Lambda P^T u\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^d \lambda_i y_i^2\right)^2\right] \\
&= \sum_{i=1}^d \lambda_i^2 \mathbb{E}\left[y_i^4\right] + \sum_{i\neq j} \lambda_i \lambda_j \mathbb{E}\left[y_i^2 y_j^2\right] \qquad \text{(A.33)}
\end{aligned}$$

Note that $\mathbb{E}[y] = P^T\mathbb{E}[u] = 0,\ \mathbb{E}\left[yy^T\right] = P^T\mathbb{E}\left[uu^T\right]P = P^T P = I$. Therefore, $y \sim \mathcal{N}(0, I_d)$. Therefore, $\mathbb{E}\left[y_i^4\right] = 3$ and $\mathbb{E}\left[y_i^2 y_j^2\right] = \mathbb{E}\left[y_i^2\right]\mathbb{E}\left[y_j^2\right] = 1$. Therefore,

$$\mathbb{E}\left[\left(u_0^T B_n^T U U^T B_n u_0\right)^2\right] = 3\sum_{i=1}^d \lambda_i^2 + \sum_{i\neq j} \lambda_i \lambda_j \qquad \text{(A.34)}$$

Substituting in Eq A.33, we have

$$\begin{aligned}
\mathbb{E}\left[\left(u_0^T B_n^T U U^T B_n u_0\right)^2\right] &= 3\sum_{i=1}^d \lambda_i^2 + \sum_{i\neq j} \lambda_i \lambda_j \\
&\leq 3\mathbb{E}\left[\left(\sum_{i=1}^d \lambda_i\right)^2\right] \\
&= 3\mathbb{E}\left[\mathrm{Tr}\left(B_n^T U U^T B_n\right)^2\right] \\
&= 3\mathbb{E}\left[\left(\mathrm{Tr}\left(v_1^T B_n^T U U^T B_n v_1\right) + \mathrm{Tr}\left(V_\perp^T B_n^T U U^T B_n V_\perp\right)\right)^2\right] \\
&\leq 6\left(\mathbb{E}\left[\left(v_1^T B_n^T U U^T B_n v_1\right)^2\right] + \mathbb{E}\left[\mathrm{Tr}\left(V_\perp^T B_n^T U U^T B_n V_\perp\right)^2\right]\right)
\end{aligned}$$

□

A.3 Proofs of entrywise deviation of Oja's vector

We first state some useful results here. Let $a_0 = v_1(i)^2, b_0 = \mathrm{Tr}\left(V_\perp^T e_i e_i^T V_\perp\right) = 1 - v_1(i)^2$. Let the learning rate, η, be set according to Lemma A.2.4. Note that $(1+x) \leq \exp(x),\ \forall x \in \mathbb{R}$. From Lemma A.2.7, we have

$$\begin{aligned}\mathbb{E}\left[\left(v_1^T B_n e_i\right)^2\right] &\leq \left(1 + 2\eta\lambda_1 + 8L^4\sigma^4\eta^2\lambda_1^2\right)^n \left[a_0 + \eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)(b_0+a_0)\right] \\ &\leq \exp\left(2\eta n\lambda_1 + 8L^4\sigma^4\eta^2 n\lambda_1^2\right)\left(a_0 + \eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)(b_0+a_0)\right),\end{aligned} \tag{A.35}$$

$$\begin{aligned}\mathbb{E}\left[\mathrm{Tr}\left(V_\perp^T B_n e_i e_i^T B_n^T V_\perp\right)\right] &\leq b_0\left(1 + 2\eta\lambda_2 + 4L^4\sigma^4\eta^2\lambda_2\,\mathrm{Tr}(\Sigma) + 4L^4\sigma^4\eta^2\lambda_1^2\right)^n \\ &\quad + \left[\eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)\left(a_0\frac{\mathrm{Tr}(\Sigma)}{\lambda_1} + b_0\right)\right]\left(1 + 2\eta\lambda_1 + 8L^4\sigma^4\eta^2\lambda_1^2\right)^n \\ &\leq b_0\exp\left(2\eta n\lambda_2 + 4L^4\sigma^4\eta^2 n\lambda_2\,\mathrm{Tr}(\Sigma) + 4L^4\sigma^4\eta^2 n\lambda_1^2\right) \\ &\quad + \left[\eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)\left(a_0\frac{\mathrm{Tr}(\Sigma)}{\lambda_1} + b_0\right)\right]\exp\left(2\eta n\lambda_1 + 8L^4\sigma^4\eta^2 n\lambda_1^2\right)\end{aligned} \tag{A.36}$$

Similarly, from Lemma A.2.8 and using $(a+b)^2 \leq 2a^2 + 2b^2$, we have

$$\begin{aligned}\mathbb{E}\left[\left(v_1^T B_n e_i\right)^4\right] &\leq \left(1 + 2\eta\lambda_1 + 100L^4\sigma^4\eta^2\lambda_1^2\right)^{2n}\left[a_0 + \eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)(b_0+a_0)\right]^2 \\ &\leq \exp\left(4\eta n\lambda_1 + 200L^4\sigma^4\eta^2 n\lambda_1^2\right)\left[a_0 + \eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)(b_0+a_0)\right]^2,\end{aligned} \tag{A.37}$$

$$\begin{aligned}&\mathbb{E}\left[\mathrm{Tr}\left(V_\perp^T B_n e_i e_i^T B_n^T V_\perp\right)^2\right] \\ &\quad\leq 2b_0^2\left(1 + 2\eta\lambda_2 + 50\eta^2L^4\sigma^4\lambda_2\,\mathrm{Tr}(\Sigma) + 50L^4\sigma^4\eta^2\lambda_1^2\right)^{2n} \\ &\quad + 2\left[\eta\lambda_1\left(\frac{2\lambda_1}{\theta(\lambda_1-\lambda_2)}\right)\left(a_0\frac{\mathrm{Tr}(\Sigma)}{\lambda_1} + b_0\left(\frac{1-\theta}{\theta}\right)\right)\right]^2\left(1 + 2\eta\lambda_1 + 100L^4\sigma^4\eta^2\lambda_1^2\right)^{2n} \\ &\quad\leq 2b_0^2\exp\left(4\eta n\lambda_2 + 100\eta^2 nL^4\sigma^4\lambda_2\,\mathrm{Tr}(\Sigma) + 100L^4\sigma^4\eta^2 n\lambda_1^2\right) \\ &\quad + 2\left[\eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)\left(a_0\frac{\mathrm{Tr}(\Sigma)}{\lambda_1} + b_0\right)\right]^2\exp\left(4\eta n\lambda_1 + 200L^4\sigma^4\eta^2 n\lambda_1^2\right)\end{aligned} \tag{A.38}$$

Finally, noting that $(1+x) \geq \exp\left(x - x^2\right)\ \forall x \geq 0$, we have

$$\mathbb{E}\left[\left(v_1^T B_n e_i\right)^2\right] \geq \left(\mathbb{E}\left[v_1^T B_n e_i\right]\right)^2 = v_1(i)^2(1+\eta\lambda_1)^{2n} \geq v_1(i)^2\exp\left(2\eta n\lambda_1 - 2\eta^2 n\lambda_1^2\right) \tag{A.39}$$

Now we are ready to provide proofs of Lemmas 3.11 and 3.12.

Lemma 3.11 (Tail bound in support)**.** *Fix a $\delta \in (0.1, 1)$. Define the event $\mathcal{G} := \left\{|v_1^T u_0| \geq \frac{\delta}{\sqrt{e}}\right\}$ and threshold $\tau_n := \frac{\delta}{\sqrt{2e}}\min_{i\in S}|v_1(i)|(1+\eta\lambda_1)^n$. Let the learning rate be set as in Lemma 3.1. Then, for an absolute constant $C_H > 0$,*

$$\forall i \in S, \quad \mathbb{P}\left(|r_i| \leq \tau_n \middle| \mathcal{G}\right) \leq C_H\left[\eta\lambda_1\log(n) + \eta\lambda_1\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\frac{1}{v_1(i)^2}\right]$$

Proof of Lemma 3.11. Let $u_0 = av_1 + bv_\perp$ for $v_\perp^T v_1 = 0$ and $a = u_0^T v_1$, and $b = u_0^T v_\perp \in \mathbb{R}$ for some vector $v_\perp$ orthogonal to v_1. Then $\forall i \in S$,

$$\begin{aligned} |r_i| \leq \tau_n &\iff r_i^2 \leq \tau_n^2 \iff \left(e_i^T B_n y_0\right)^2 \leq \tau_n^2 \\ &\iff a^2 \left(e_i^T B_n v_1\right)^2 + b^2 \left(e_i^T B_n v_\perp\right)^2 \leq \tau_n^2 \\ &\implies a^2 \left(e_i^T B_n v_1\right)^2 \leq \tau_n^2 \end{aligned} \tag{A.40}$$

Then,

$$\begin{aligned} \mathbb{P}\left(|r_i| \leq \tau_n \middle| \mathcal{G}\right) &\leq \mathbb{P}\left(a^2 \left(e_i^T B_n v_1\right)^2 \leq \tau_n^2 \middle| \mathcal{G}\right), \text{ using Eq } A.40 \\ &\leq \mathbb{P}\left(\left(e_i^T B_n v_1\right)^2 \leq \frac{e\tau_n^2}{\delta^2} \middle| \mathcal{G}\right) \\ &= \mathbb{P}\left(\left(e_i^T B_n v_1\right)^2 \leq \frac{e\tau_n^2}{\delta^2}\right) \end{aligned} \tag{A.41}$$

For convenience of notation, define $\gamma_n := \frac{\sqrt{e}}{\delta}\tau_n$ and $q_i := |e_i^T B_n v_1|$. Then,

$$\begin{aligned} \mathbb{P}\left(\left(e_i^T B_n v_1\right)^2 \leq \frac{e\tau_n^2}{\delta^2}\right) &= \mathbb{P}\left(q_i \leq \gamma_n\right) \\ &\leq \mathbb{P}\left(|q_i - \mathbb{E}\left[q_i\right]| \geq |\mathbb{E}\left[q_i\right]| - \gamma_n\right), \\ &= \mathbb{P}\left(|q_i - \mathbb{E}\left[q_i\right]| \geq |\mathbb{E}\left[q_i\right]| - \gamma_n\right), \\ &\leq \frac{\mathbb{E}\left[q_i^2\right] - \mathbb{E}\left[q_i\right]^2}{\left(|\mathbb{E}\left[q_i\right]| - \gamma_n\right)^2}, \text{ using Chebyshev's inequality} \qquad \text{(A.42)} \\ &= \underbrace{\frac{\mathbb{E}\left[\left(v_1^T B_n^T e_i e_i^T B_n v_1\right)\right] - \mathbb{E}\left[v_1^T B_n^T e_i\right]^2}{\left(|\mathbb{E}\left[v_1^T B_n^T e_i\right]| - \frac{1}{\sqrt{2}}\left(\min_{i\in S}|v_1(i)|\right)(1+\eta\lambda_1)^n\right)^2}}_{T_i} \qquad \text{(A.43)} \end{aligned}$$

We now bound T_i using Eq A.35 and Eq A.39 as -

$$\begin{aligned} T_i &\leq \frac{\exp\left(2\eta n\lambda_1 + 8L^4\sigma^4\eta^2 n\lambda_1^2\right)\left[v_1(i)^2 + \eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)\right] - v_1(i)^2(1+\eta\lambda_1)^{2n}}{\left(|v_1(i)| - \frac{1}{2}\min_{i\in S}|v_1(i)|\right)^2(1+\eta\lambda_1)^{2n}} \\ &\leq \frac{\exp\left(2\eta n\lambda_1 + 8L^4\sigma^4\eta^2 n\lambda_1^2\right)\left[v_1(i)^2 + \eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)\right] - v_1(i)^2\exp\left(2\eta n\lambda_1 - 2\eta^2 n\lambda_1^2\right)}{\left(|v_1(i)| - \frac{1}{2}\min_{i\in S}|v_1(i)|\right)^2\exp\left(2\eta n\lambda_1 - 2\eta^2 n\lambda_1^2\right)} \\ &= \frac{\exp\left(2\left(4L^4\sigma^4+1\right)\eta^2 n\lambda_1^2\right)\left[v_1(i)^2 + \eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)\right] - v_1(i)^2}{\left(|v_1(i)| - \frac{1}{2}\min_{i\in S}|v_1(i)|\right)^2} \end{aligned}$$

The second inequality follows from the fact that $1 + x \geq \exp(x - x^2)$, for $x \geq 0$. Note that for $x \in (0,1)$, $\exp x \leq 1 + 2x$. Therefore, for $\left(8L^4\sigma^4 + 2\right)\eta^2 n\lambda_1^2 \leq \frac{1}{4}$, we have

$$\begin{aligned} T_i &\leq \frac{\left(\exp\left(2\left(4L^4\sigma^4+1\right)\eta^2 n\lambda_1^2\right) - 1\right)v_1(i)^2 + 3\eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)}{\left(|v_1(i)| - \frac{1}{2}\min_{i\in S}|v_1(i)|\right)^2} \\ &\leq 4\left(\frac{\left(4L^4\sigma^4+1\right)\eta^2 n\lambda_1^2 v_1(i)^2 + 3\eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)}{v_1(i)^2}\right) \\ &= 4\left(4\left(4L^4\sigma^4+1\right)\eta^2 n\lambda_1^2 + \frac{3\eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)}{v_1(i)^2}\right) \end{aligned}$$

The result then follows by using Claim(2) in Lemma A.2.4. □

We next provide the proof of Lemma 3.12.

Lemma 3.12 (Tail bound outside support)**.** *Fix a $\delta \in (0.1, 1)$. Let the learning rate be set as in Lemma 3.1 and define the threshold $\tau_n := \frac{\delta}{\sqrt{2e}} \min_{i\in S}|v_1(i)|(1+\eta\lambda_1)^n$. Then, for $\min_i |v_1(i)| = \widetilde{\Omega}\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\left(\frac{d}{n^2}\right)^{\frac{1}{4}}\right)$ and an absolute constant $C_T > 0$ we have,*

$$\forall i \notin S, \quad \mathbb{P}\left(|r_i| > \tau_n\right) \leq C_T \left[\eta^2\lambda_1^2 \left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)^2 \left(\frac{1}{\delta^2 \min_{i\in S_{hi}} v_1(i)^2}\right)^2\right]$$

Proof of Lemma 3.12. Note that for $i \notin S$, $a_0 = 0, b_0 = 1$. Therefore, we have,

$$\mathbb{E}\left[\left(v_1^T B_n^T e_i e_i^T B_n v_1\right)^2\right] \leq \exp\left(4\eta n\lambda_1 + 200L^4\sigma^4\eta^2 n\lambda_1^2\right)\left[\eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)\right]^2, \text{ using Eq } A.37, \tag{A.44}$$

$$\begin{aligned}\mathbb{E}\left[\mathrm{Tr}\left(V_\perp^T B_n^T e_i e_i^T B_n V_\perp\right)^2\right] &\leq 2\exp\left(4\eta n\lambda_2 + 100\eta^2 n\log(n) L^4\sigma^4\lambda_2 \mathrm{Tr}(\Sigma) + 100L^4\sigma^4\eta^2 n\lambda_1^2\right) \\ &\quad + 2\left[\eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)\right]^2 \exp\left(4\eta n\lambda_1 + 200L^4\sigma^4\eta^2 n\lambda_1^2\right), \text{ using Eq } A.38\end{aligned} \tag{A.45}$$

$$\mathbb{E}\left[v_1^T B_n^T e_i e_i^T B_n v_1\right] \leq \exp\left(2\eta n\lambda_1 + 8L^4\sigma^4\eta^2 n\lambda_1^2\right)\left(\eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)\right), \text{ using Eq } A.35 \tag{A.46}$$

$$\begin{aligned}\mathbb{E}\left[\mathrm{Tr}\left(V_\perp^T B_n^T e_i e_i^T B_n V_\perp\right)\right] &\leq \exp\left(2\eta n\lambda_2 + 4L^4\sigma^4\eta^2 n\lambda_2 \mathrm{Tr}(\Sigma) + 4L^4\sigma^4\eta^2 n\lambda_1^2\right) \\ &\quad + \left[\eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)\right]\exp\left(2\eta n\lambda_1 + 8L^4\sigma^4\eta^2 n\lambda_1^2\right), \text{ using Eq } A.36\end{aligned} \tag{A.47}$$

$$\frac{\delta^2}{2e}\left(\min_{i\in S} v_1(i)^2\right)(1+\eta\lambda_1)^n \geq \frac{\delta^2}{2e}\left(\min_{i\in S} v_1(i)^2\right)\exp\left(2\eta n\lambda_1 - 2\eta^2 n\lambda_1^2\right) \text{ using Eq } A.39 \tag{A.48}$$

Define

$$g_i := \tau_n^2 - \mathbb{E}\left[r_i^2\right] \tag{A.49}$$

Note that using Assumptions 2,

$$\frac{16e\eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)}{\min_{i\in S} v_1(i)^2} \leq \frac{3\log(n)}{n(\lambda_1-\lambda_2)} \times \frac{128e\lambda_1}{\min_{i\in S} v_1(i)^2} \leq \frac{768e}{\min_{i\in S} v_1(i)^2} \times \frac{\log(n)}{n} \leq \delta^2$$

where the last inequality follows for sufficiently large n mentioned in the theorem statement. Therefore, using Eq A.46 and A.47 along with Claim (3) from Lemma A.2.4, g_i is bounded as

$$\begin{aligned} g_i &\geq \left(\frac{\delta^2}{4e}\left(\min_{i\in S} v_1(i)^2\right)\exp\left(2\eta n\lambda_1 - 2\eta^2 n\lambda_1^2\right) - \exp\left(2\eta n\lambda_2 + 4L^4\sigma^4\eta^2 n\lambda_2 \mathrm{Tr}(\Sigma) + 4L^4\sigma^4\eta^2 n\lambda_1^2\right)\right) \\ &\geq \exp\left(2\eta n\lambda_1 - 2\eta^2 n\lambda_1^2\right)\left(\frac{\delta^2}{4e}\left(\min_{i\in S} v_1(i)^2\right) - \exp\left(-\theta\eta n(\lambda_1-\lambda_2)\right)\right) \\ &\geq \frac{2\delta^2}{9e^2}\left(\min_{i\in S} v_1(i)^2\right)\exp\left(2\eta n\lambda_1 - 2\eta^2 n\lambda_1^2\right) \end{aligned} \tag{A.50}$$

where the last inequality used Claim (4) from Lemma A.2.4. Therefore, we have,

$$\begin{aligned}
\mathbb{P}\left(|r_i| > \tau_n\right) &= \mathbb{P}\left(r_i^2 > \tau_n^2\right) \\
&= \mathbb{P}\left(r_i^2 - \mathbb{E}\left[r_i^2\right] > \tau_n^2 - \mathbb{E}\left[r_i^2\right]\right), \\
&\leq \mathbb{P}\left(|r_i^2 - \mathbb{E}\left[r_i^2\right]| > \tau_n^2 - \mathbb{E}\left[r_i^2\right]\right), \text{ since from Eq A.50 } g_i \geq 0 \\
&\leq \frac{\mathbb{E}\left[\left(r_i^2 - \mathbb{E}\left[r_i^2\right]\right)^2\right]}{\left(\tau_n^2 - \mathbb{E}\left[r_i^2\right]\right)^2} \\
&= \frac{\mathbb{E}\left[r_i^4\right] - \mathbb{E}\left[r_i^2\right]^2}{\left(\tau_n^2 - \mathbb{E}\left[r_i^2\right]\right)^2} \\
&= \frac{\mathbb{E}\left[\left(y_0^T B_n^T e_i e_i^T B_n y_0\right)^2\right] - \mathbb{E}\left[y_0^T B_n^T e_i e_i^T B_n y_0\right]^2}{\left(\frac{\delta^2}{2e}\left(\min_{i\in S} v_1\left(i\right)^2\right)\left(1+\eta\lambda_1\right)^n - \left(\mathbb{E}\left[y_0^T B_n^T e_i e_i^T B_n y_0\right]\right)\right)^2} =: R_i \qquad \text{(A.51)}
\end{aligned}$$

We now bound R_i. Therefore,

$$\begin{aligned}
R_i &\leq \frac{\mathbb{E}\left[\left(y_0^T B_n^T e_i e_i^T B_n y_0\right)^2\right]}{\left(\frac{\delta^2}{2e}\left(\min_{i\in S} v_1\left(i\right)^2\right)\left(1+\eta\lambda_1\right)^n - \left(\mathbb{E}\left[y_0^T B_n^T e_i e_i^T B_n y_0\right]\right)\right)^2} \\
&\overset{(i)}{\leq} \frac{6\left(\mathbb{E}\left[\left(v_1^T B_n^T e_i e_i^T B_n v_1\right)^2\right] + \mathbb{E}\left[\mathrm{Tr}\left(V_\perp^T B_n^T e_i e_i^T B_n V_\perp\right)^2\right]\right)}{\left(\frac{\delta^2}{2e}\left(\min_{i\in S} v_1\left(i\right)^2\right)\left(1+\eta\lambda_1\right)^n - \mathbb{E}\left[v_1^T B_n^T e_i e_i^T B_n v_1\right] - \mathbb{E}\left[\mathrm{Tr}\left(V_\perp^T B_n^T e_i e_i^T B_n V_\perp\right)\right]\right)^2}
\end{aligned}$$

where (i) uses Lemmas A.2.11 and A.2.12. Denote the numerator and denominator of R_i as $N\left(R_i\right)$ and $D\left(R_i\right)$. For the numerator $N\left(R_i\right)$ using Eq A.44 and A.45, we have

$$\begin{aligned}
N\left(R_i\right) &\leq 18\left[\eta\lambda_1\left(\frac{4\lambda_1}{\left(\lambda_1-\lambda_2\right)}\right)\right]^2 \exp\left(4\eta n\lambda_1 + 200L^4\sigma^4\eta^2 n\lambda_1^2\right)\left(1+\frac{2}{3}\exp\left(-\theta\eta n\left(\lambda_1-\lambda_2\right)\right)\right) \\
&\leq 20\left[\eta\lambda_1\left(\frac{4\lambda_1}{\left(\lambda_1-\lambda_2\right)}\right)\right]^2 \exp\left(4\eta n\lambda_1 + 200L^4\sigma^4\eta^2 n\lambda_1^2\right)
\end{aligned}$$

where the last inequality follows from Claim (4) in Lemma A.2.4. For the denominator $D\left(R_i\right)$, using Eq A.50

$$D\left(R_i\right) = g_i^2 \geq \frac{\delta^4}{20e^2}\left(\min_{i\in S} v_1\left(i\right)^2\right)^2 \exp\left(4\eta n\lambda_1 - 4\eta^2 n\lambda_1^2\right)$$

Recall that for $x \in (0,1)$, $\exp\left(x\right) \leq 1+2x$. Therefore, for $\left(100L^4\sigma^4+2\right)\eta^2 n\lambda_1^2 \leq \frac{1}{4}$ which holds due to Claim (2) from Lemma A.2.4, substituting in Eq A.51, we have

$$\mathbb{P}\left(|r_i| > \tau_n\right) \leq \left(\frac{400e^2\left(4\lambda_1\right)^2}{\delta^4\left(\min_{i\in S} v_i^4\right)\left(\lambda_1-\lambda_2\right)^2}\right)\eta^2\lambda_1^2 \qquad \text{(A.52)}$$

□

A.4 Proof of convergence for support recovery (Lemma 3.1,Theorem 3.2)

We start with the proof of Lemma 3.1.

Lemma 3.1 (s-Agnostic Recovery)**.** *Under Assumptions 1,2, for* $\min_i |v_1(i)| = \widetilde{\Omega}\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\left(\frac{d}{n^2}\right)^{\frac{1}{4}}\right)$, $\widehat{S} \leftarrow \mathsf{OjaSupportRecovery}\left(\left\{X_i\right\}_{i\in[n]}, k, \eta := \frac{3\log(n)}{n(\lambda_1-\lambda_2)}\right)$ *with* $k \geq s$ *satisfies,* $\mathbb{P}\left(S \subseteq \widehat{S}\right) \geq 0.9$.

Proof. Let $\mathcal{E} := \left\{S \subseteq \widehat{S}\right\}$ and set $\delta := \frac{1}{50}$ for this proof. We upper bound $\mathbb{P}\left(\mathcal{E}^c\right)$. Define $r_i := e_i^T B_n u_0$, $i \in [d]$. Observe that

$$\mathcal{E} \iff \exists \tau_n > 0 \text{ such that } \left\{\forall i \in S, |r_i| \geq \tau_n\right\} \bigcap \left\{\left|\left\{i : i \notin S, |r_i| \geq \tau_n\right\}\right| \leq k-s\right\}$$

or equivalently,

$$\mathcal{E}^c \iff \forall \tau_n > 0, \{\exists i \in S, |r_i| \leq \tau_n\} \bigcup \{|\{i : i \notin S, |r_i| \geq \tau_n\}| > k - s\}$$

Therefore, for any fixed $\tau_n > 0$

$$\mathcal{E}^c \implies \{\exists i \in S, |r_i| \leq \tau_n\} \bigcup \{|\{i : i \notin S, |r_i| \geq \tau_n\}| > k - s\}$$

Let $\mathcal{G} := \left\{|v_1^T u_0| \geq \frac{\delta}{\sqrt{e}}\right\}$ and threshold $\tau_n := \frac{\delta}{\sqrt{2e}} \min_{i \in S} |v_1(i)| (1 + \eta\lambda_1)^n$. Using a union-bound,

$$\begin{aligned}
\mathbb{P}(\mathcal{E}^c) &\leq \mathbb{P}(\{\exists i \in S, |r_i| \leq \tau_n\}) + \mathbb{P}(|\{i : i \notin S, |r_i| \geq \tau_n\}| > k - s) \\
&\leq \mathbb{P}(\{\exists i \in S, |r_i| \leq \tau_n\}) + \frac{\sum_{i \notin S} \mathbb{P}(|r_i| \geq \tau_n)}{k - s + 1}, \text{ using Markov's inequality} \\
&= \mathbb{P}(\mathcal{G}) \mathbb{P}\left(\exists i \in S, |r_i| \leq \tau_n \middle| \mathcal{G}\right) + \mathbb{P}(\mathcal{G}^c) \mathbb{P}\left(\exists i \in S, |r_i| \leq \tau_n \middle| \mathcal{G}^c\right) + \\
&\quad + \frac{\sum_{i \notin S} \mathbb{P}(|r_i| \geq \tau_n)}{k - s + 1} \\
&\leq \mathbb{P}(\mathcal{G}^c) + \mathbb{P}\left(\exists i \in S, |r_i| \leq \tau_n \middle| \mathcal{G}\right) + \frac{\sum_{i \notin S} \mathbb{P}(|r_i| \geq \tau_n)}{k - s + 1} \\
&\leq \mathbb{P}(\mathcal{G}^c) + \sum_{i \in S} \mathbb{P}\left(|r_i| \leq \tau_n \middle| \mathcal{G}\right) + \frac{\sum_{i \notin S} \mathbb{P}(|r_i| \geq \tau_n)}{k - s + 1} \\
&\leq \mathbb{P}(\mathcal{G}^c) + \underbrace{\sum_{i \in S} \mathbb{P}\left(|r_i| \leq \tau_n \middle| \mathcal{G}\right)}_{T_1} + \underbrace{\frac{\sum_{i \notin S} \mathbb{P}(|r_i| \geq \tau_n)}{k - s + 1}}_{T_2}
\end{aligned}$$

Using Lemma A.2.1, we have $\mathbb{P}(\mathcal{G}^c) \leq \delta$. We bound T_1 and T_2 using Lemmas 3.11 and 3.12 respectively. Therefore,

$$\begin{aligned}
\mathbb{P}(\mathcal{E}^c) &\leq \delta + C_H \left[\eta\lambda_1 s \log(n) + \eta\lambda_1 \left(\frac{\lambda_1}{(\lambda_1 - \lambda_2)}\right) \sum_{i \in S} \frac{1}{v_1(i)^2}\right] \\
&\quad + C_T \left[\eta^2 \lambda_1^2 \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\right)^2 \left(\frac{1}{\delta^2 \min_{i \in S_{\text{hi}}} v_1(i)^2}\right)^2\right] (d - s) \\
&\leq 5\delta
\end{aligned}$$

where the last inequality follows by using the bound on n. □

Next, using Lemma 3.1, we prove Theorem 3.2.

Theorem 3.2 (High probability support recovery)**.** *Let Assumptions 1, 2 hold. For dataset $\mathcal{D} := \{X_i\}_{i \in [n]}$, let $\mathcal{A}$ be the randomized algorithm which computes $\widehat{S} \leftarrow \mathsf{OjaSupportRecovery}\left(\{X_i\}_{i \in [n]}, k, \eta\right)$, where $\eta := \frac{3\log(n)}{n(\lambda_1 - \lambda_2)}$ and $k = s$. Then, for $\delta \in (0, 1)$, $\min_i |v_1(i)| = \widetilde{\Omega}\left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\left(\frac{d}{n^2}\right)^{\frac{1}{4}}\right)$, $\tilde{S} \leftarrow \mathsf{SuccessBoost}\left(\{X_i\}_{i \in [n]}, \mathcal{A}, \delta\right)$ satisfies,*

$$\mathbb{P}\left(\tilde{S} = S\right) \geq 1 - \delta$$

Proof. Consider the set $\mathcal{T} := \{\mathcal{S} : \mathcal{S} \subseteq [d], |\mathcal{S}| = s\}$ with the associated metric $\rho(\mathcal{S}, \mathcal{S}') := \mathbb{1}(\mathcal{S} \neq \mathcal{S}')$.

Then, Lemma 3.1 shows that the randomized algorithm, $\mathcal{A}$, is a $\mathsf{ConstantSuccessOracle}(\mathcal{D}, \theta, \mathcal{T}, \rho, S, 0)$ (Definition 3.9).

Therefore, the result follows from Lemma 3.10.

□

A.5 Proof of convergence for sparse PCA (Theorems 3.5,3.7)

We start by providing the proof of Theorem 3.5 in Section A.5.1, and then provide the proof of Theorem 3.7 in Section A.5.2.

A.5.1 Proof of theorem 3.5

Let $\widehat{v} := \frac{B_n w_0}{\|B_n w_0\|_2}$. Then, for any subset $\hat{S} \subseteq S$ (obtained from a support recovery procedure such as Algorithm 1), the corresponding output of a truncation procedure with respect to $\widehat{S}$ is given as:

$$v_{\text{trunc}} := \frac{\lfloor \widehat{v} \rfloor_{\widehat{S}}}{\left\| \lfloor \widehat{v} \rfloor_{\widehat{S}} \right\|_2} = \frac{I_{\widehat{S}} \widehat{v}}{\left\| I_{\widehat{S}} \widehat{v} \right\|_2} = \frac{I_{\widehat{S}} B_n w_0}{\left\| I_{\widehat{S}} B_n w_0 \right\|_2} \tag{A.53}$$

We first prove a general and flexible result that bounds the $\sin^2$ error as a function of B_n (see Eq 3) and analyze the performance of v_{trunc} by viewing it as a power method on w_0 followed by a truncation using the set $\widehat{S}$ in the following result.

Lemma A.5.1. *Let B_n and v_{trunc} be defined as in Eq 3 and Eq A.53 respectively. For $\widehat{S} \subseteq [d]$ such that $\widehat{S} \perp\!\!\!\perp B_n, w_0$, then with probability at least $1 - \delta$*

$$\sin^2(v_{trunc}, v_1) \leq \frac{C \log\left(\frac{1}{\delta}\right)}{\delta^2} \frac{\operatorname{Tr}\left(B_n^T \left(I_{\widehat{S}} - I_{\widehat{S}} v_1 v_1^T I_{\widehat{S}}\right) B_n\right)}{v_1^T B_n^T I_{S \bigcap \widehat{S}} B_n v_1}$$

where C is an absolute constant and $\delta \in (0, 1)$.

Proof. Using the definition of $\sin^2$ error,

$$\sin^2(v_{\text{trunc}}, v_1) = 1 - \left(\frac{v_1^T I_{\widehat{S}} B_n w_0}{\left\| I_{\widehat{S}} B_n w_0 \right\|_2} \right)^2 = \frac{w_0^T B_n^T \left(I_{\widehat{S}} - I_{\widehat{S}} v_1 v_1^T I_{\widehat{S}}\right) B_n w_0}{w_0^T B_n^T I_{\widehat{S}} B_n w_0} \tag{A.54}$$

For the denominator, with probability at least $(1 - \delta)$, we have

$$w_0^T B_n^T I_{\widehat{S}} B_n w_0 \geq w_0^T B_n^T I_{S \cap \widehat{S}} B_n w_0 \overset{(i)}{\geq} \frac{\delta^2}{e} \operatorname{Tr}\left(B_n^T I_{S \cap \widehat{S}} B_n\right) \geq \frac{\delta^2}{e} v_1^T B_n^T I_{S \cap \widehat{S}} B_n v_1 \tag{A.55}$$

where (i) follows from Lemma A.2.1. For the numerator, using ζ_2 from Lemma 3.1 of [JJK$^+$16], with probability at least $(1 - \delta)$, we have

$$w_0^T B_n^T \left(I_{\widehat{S}} - I_{\widehat{S}} v_1 v_1^T I_{\widehat{S}}\right) B_n w_0 \leq C' \log\left(\frac{1}{\delta}\right) \operatorname{Tr}\left(B_n^T \left(I_{\widehat{S}} - I_{\widehat{S}} v_1 v_1^T I_{\widehat{S}}\right) B_n\right) \tag{A.56}$$

Combining Eq A.55 and Eq A.56 with Eq A.54 completes our proof. □

Lemma A.5.1 provides an intuitive sketch of our proof strategy. Following the recipe proposed in [JJK$^+$16], we show how to upper-bound $\epsilon_n := \frac{C \log\left(\frac{1}{\delta}\right)}{\delta^2} \frac{\operatorname{Tr}\left(B_n^T \left(I_{\widehat{S}} - I_{\widehat{S}} v_1 v_1^T I_{\widehat{S}}\right) B_n\right)}{v_1^T B_n^T I_{S \cap \widehat{S}} B_n v_1}$. For upper-bounding the numerator, we bound $\mathbb{E}\left[\operatorname{Tr}\left(B_n^T \left(I_{\widehat{S}} - I_{\widehat{S}} v_1 v_1^T I_{\widehat{S}}\right) B_n\right)\right]$ and use Markov's inequality. To lower-bound the denominator, we lower-bound $\mathbb{E}\left[v_1^T B_n^T I_{S \cap \widehat{S}} B_n v_1\right]$, upper-bound the variance $\mathbb{E}\left[\left(v_1^T B_n^T I_{S \cap \widehat{S}} B_n v_1\right)^2\right]$ and finally use Chebyshev's inequality. A formal analysis is provided in the following theorem -

Theorem A.5.2 (Convergence of Truncated Oja's Algorithm)**.** *Let $\widehat{S} \subseteq [d]$ be the estimated support set, such that $\widehat{S} \perp\!\!\!\perp B_n, w_0$ (see Algorithm 2). Consider any event $\mathcal{E}$ solely dependent on the randomness of $\widehat{S}$. Define:*

$$W_{\widehat{S}} := \mathbb{E}\left[I_{\widehat{S}} - I_{\widehat{S}} v_1 v_1^T I_{\widehat{S}} \mid \mathcal{E}\right],\ G_{\widehat{S}} := \mathbb{E}\left[I_{S \cap \widehat{S}} \mid \mathcal{E}\right]$$

$$\alpha_0 := v_1^T W_{\widehat{S}} v_1,\ \beta_0 := \operatorname{Tr}\left(V_\perp^T W_{\widehat{S}} V_\perp\right),\ p_0 := v_1^T G_{\widehat{S}} v_1,\ q_0 := \operatorname{Tr}\left(V_\perp^T G_{\widehat{S}} V_\perp\right)$$

Fix $\delta \in (0.1, 1)$. Set the learning rate as $\eta := \frac{3\log(n)}{n(\lambda_1-\lambda_2)}$. Then, under Assumption 2, for $n = \Omega\left(s\left(\frac{\lambda_1\log(n)}{\lambda_1-\lambda_2}\right)^2\right)$ and $p_0\left(1+\frac{\delta}{16}\right) \leq 1 + 2\eta\lambda_1 s\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)$, we have with probability at least $1-\delta-\mathbb{P}\left(\mathcal{E}^c\right)$,

$$\sin^2\left(v_{trunc}, v_1\right) \leq \frac{C'\log\left(\frac{1}{\delta}\right)}{\delta^3}\frac{\lambda_1}{\lambda_1-\lambda_2}\frac{\alpha_0\left(1+2\eta\,\mathrm{Tr}\left(\Sigma\right)\right)+2\eta\lambda_1\beta_0}{p_0}$$

where v_{trunc} is defined in Eq A.53 and $C' > 0$ is an absolute constant.

Proof of Theorem A.5.2. We first note that from Lemma A.5.1, with probability at least $1-\delta$,

$$\sin^2\left(v_{\text{trunc}}, v_1\right) \leq \frac{C\log\left(\frac{1}{\delta}\right)}{\delta^2}\frac{\mathrm{Tr}\left(B_n^T\left(I_{\widehat{S}} - I_{\widehat{S}}v_1v_1^TI_{\widehat{S}}\right)B_n\right)}{v_1^TB_n^TI_{S\bigcap\widehat{S}}B_nv_1} =: \chi \tag{A.57}$$

Next, we bound χ, conditioned on the event $\mathcal{E}$. Using Markov's inequality, we have with probability at least $1-\delta$,

$$\begin{aligned}
&\mathrm{Tr}\left(B_n^T\left(I_{\widehat{S}} - I_{\widehat{S}}v_1v_1^TI_{\widehat{S}}\right)B_n\right)\\
&\leq \frac{\mathbb{E}\left[\mathrm{Tr}\left(B_n^T\left(I_{\widehat{S}} - I_{\widehat{S}}v_1v_1^TI_{\widehat{S}}\right)B_n\right)\middle|\mathcal{E}\right]}{\delta}\\
&= \frac{\mathbb{E}\left[v_1^TB_n^T\left(I_{\widehat{S}} - I_{\widehat{S}}v_1v_1^TI_{\widehat{S}}\right)B_nv_1\middle|\mathcal{E}\right] + \mathbb{E}\left[\mathrm{Tr}\left(V_\perp^TB_n^T\left(I_{\widehat{S}} - I_{\widehat{S}}v_1v_1^TI_{\widehat{S}}\right)B_nV_\perp\right)\middle|\mathcal{E}\right]}{\delta}\\
&= \frac{\mathbb{E}\left[v_1^TB_n^TW_{\widehat{S}}B_nv_1\right] + \mathbb{E}\left[\mathrm{Tr}\left(V_\perp^TB_n^TW_{\widehat{S}}B_nV_\perp\right)\right]}{\delta}, \text{ using } \widehat{S} \perp\!\!\!\perp B_n
\end{aligned} \tag{A.58}$$

Note that $(1+x) \leq \exp\left(x\right) \forall x \in \mathbb{R}$. From Lemma A.2.7, we have

$$\mathbb{E}\left[\mathrm{Tr}\left(v_1^TB_n^TW_{\widehat{S}}B_nv_1\right)\right] \leq \left(1+2\eta\lambda_1+8L^4\sigma^4\eta^2\lambda_1^2\right)^n\left[\alpha_0+\eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)\left(\beta_0+\alpha_0\right)\right] \tag{A.59}$$

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{Tr}\left(V_\perp^TB_n^TW_{\widehat{S}}B_nV_\perp\right)\right]\\
&\leq \beta_0\left(1+2\eta\lambda_2+4L^4\sigma^4\eta^2\lambda_2\,\mathrm{Tr}\left(\Sigma\right)+4L^4\sigma^4\eta^2\lambda_1^2\right)^n\\
&+\left[\eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)\left(\alpha_0\frac{\mathrm{Tr}\left(\Sigma\right)}{\lambda_1}+\beta_0\right)\right]\left(1+2\eta\lambda_1+8L^4\sigma^4\eta^2\lambda_1^2\right)^n\\
&\leq \exp\left(2\eta n\lambda_1+8L^4\sigma^4\eta^2n\lambda_1^2\right)\left[\eta\lambda_1\left(\alpha_0\frac{\mathrm{Tr}\left(\Sigma\right)}{\lambda_1}+\beta_0\right)\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)+\beta_0\exp\left(-2\theta\eta n\left(\lambda_1-\lambda_2\right)\right)\right]\\
&\leq 2\exp\left(2\eta n\lambda_1+8L^4\sigma^4\eta^2n\lambda_1^2\right)\left[\eta\lambda_1\left(\alpha_0\frac{\mathrm{Tr}\left(\Sigma\right)}{\lambda_1}+\beta_0\right)\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)\right]
\end{aligned} \tag{A.60}$$

where the last inequality follows due to Lemma A.2.4.

Substituting Eq A.59 and Eq A.60 in Eq A.58, we have with probability at least $(1-\delta)$, conditioned on the event $\mathcal{E}$,

$$\mathrm{Tr}\left(B_n^T\left(I_{\widehat{S}} - I_{\widehat{S}}v_1v_1^TI_{\widehat{S}}\right)B_n\right) \leq \frac{\left(\alpha_0\left(1+2\eta\,\mathrm{Tr}\left(\Sigma\right)\right)+2\eta\lambda_1\beta_0\right)\left(\frac{12\lambda_1}{(\lambda_1-\lambda_2)}\right)\exp\left(2\eta n\lambda_1+8L^4\sigma^4\eta^2n\lambda_1^2\right)}{\delta} \tag{A.61}$$

Similarly, for the denominator we have with probability at least $1-\delta$ using Chebyshev's inequality, conditioned on the event $\mathcal{E}$,

$$v_1^T B_n I_{S\bigcap \widehat{S}} B_n^T v_1 \geq \mathbb{E}\left[v_1^T B_n I_{S\bigcap \widehat{S}} B_n^T v_1 \middle| \mathcal{E}\right]\left[1-\frac{1}{\sqrt{\delta}}\sqrt{\frac{\mathbb{E}\left[\left(v_1^T B_n I_{S\bigcap \widehat{S}} B_n^T v_1\right)^2 \middle| \mathcal{E}\right]}{\mathbb{E}\left[v_1^T B_n I_{S\bigcap \widehat{S}} B_n^T v_1 \middle| \mathcal{E}\right]^2}-1}\right] \tag{A.62}$$

Recall that $p_0 := v_1^T \mathbb{E}\left[I_{S\bigcap \widehat{S}} \middle| \mathcal{E}\right] v_1$. Using the argument from Lemma 11 from [JJK$^+$16] and $\widehat{S} \perp\!\!\!\perp B_n$,

$$\mathbb{E}\left[v_1^T B_n I_{S\bigcap \widehat{S}} B_n^T v_1 \middle| \mathcal{E}\right] = \mathbb{E}\left[v_1^T B_n \mathbb{E}\left[I_{S\bigcap \widehat{S}} \middle| \mathcal{E}\right] B_n^T v_1\right] \geq p_0 \exp\left(2\eta n\lambda_1 - 4\eta^2 n\lambda_1^2\right) \tag{A.63}$$

This is since the base case of their recursion, [JJK$^+$16] has $v_1^T I v_1$ which is 1, but we have $v_1^T \mathbb{E}\left[I_{S\bigcap \widehat{S}} \middle| \mathcal{E}\right] v_1$ which is defined as p_0.

Next, using Lemma A.2.8 and noting that $v_1^T I_{S\bigcap \widehat{S}} v_1 + \mathrm{Tr}\left(V_\perp^T I_{S\bigcap \widehat{S}} V_\perp\right) = \mathrm{Tr}\left(I_{S\bigcap \widehat{S}}\right)$ we have,

$$\begin{aligned}
&\mathbb{E}\left[\left(v_1^T B_n I_{S\bigcap \widehat{S}} B_n^T v_1\right)^2 \middle| \mathcal{E}\right] \\
&\quad \leq \left(1+2\eta\lambda_1+100L^4\sigma^4\eta^2\lambda_1^2\right)^{2n} \mathbb{E}\left[\left(v_1^T I_{S\bigcap \widehat{S}} v_1 + \eta\lambda_1\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)\mathrm{Tr}\left(I_{S\bigcap \widehat{S}}\right)\right)^2 \middle| \mathcal{E}\right] \\
&\quad \leq \left(1+2\eta\lambda_1+100L^4\sigma^4\eta^2\lambda_1^2\right)^{2n} \mathbb{E}\left[\left(v_1^T I_{S\bigcap \widehat{S}} v_1 + \eta\lambda_1 s\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)\right)^2 \middle| \mathcal{E}\right] \\
&\quad \leq \exp\left(4\eta n\lambda_1+200L^4\sigma^4\eta^2 n\lambda_1^2\right)\left[p_0+2\eta\lambda_1 s p_0\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)+\left(\eta\lambda_1 s\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)\right)^2\right],
\end{aligned} \tag{A.64}$$

where in the last inequality, we used $\left(v_1^T I_{S\bigcap \widehat{S}} v_1\right)^2 \leq v_1^T I_{S\bigcap \widehat{S}} v_1 \leq 1$. For convenience of notation, we define

$$\phi := p_0 + 2\eta\lambda_1 s p_0\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)+\left(\eta\lambda_1 s\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)\right)^2 \leq 4$$

where we used $\eta\lambda_1 s\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right) \leq \frac{1}{2}$ (due to Lemma A.2.4) and $p_0 \leq 1$.

Substituting Eq A.63 and Eq A.64 in Eq A.62, and we have with probability at least $(1-\delta)$,

conditioned on $\mathcal{E}$,

$$
\begin{aligned}
& v_1^T B_n G_{\widehat{S}} B_n^T v_1 \\
& \geq p_0 \exp\left(2\eta n\lambda_1 - 4\eta^2 n\lambda_1^2\right)\left(1 - \frac{1}{\sqrt{\delta}}\sqrt{\exp\left(\left(100L^4\sigma^4+4\right)\eta^2 n\lambda_1^2\right)\frac{\phi}{p_0^2} - 1}\right), \\
& \overset{(i)}{\geq} p_0 \exp\left(2\eta n\lambda_1 - 4\eta^2 n\lambda_1^2\right)\left(1 - \frac{1}{\sqrt{\delta}}\sqrt{\left(1+2\left(100L^4\sigma^4+4\right)\eta^2 n\lambda_1^2\right)\frac{\phi}{p_0^2} - 1}\right) \\
& \geq p_0 \exp\left(2\eta n\lambda_1 - 4\eta^2 n\lambda_1^2\right)\left(1 - \frac{1}{\sqrt{\delta}}\sqrt{\frac{\phi - p_0^2}{p_0^2} + 2\frac{\phi}{p_0^2}\left(100L^4\sigma^4+4\right)\eta^2 n\lambda_1^2}\right) \\
& \overset{(ii)}{\geq} \frac{p_0}{2}\exp\left(2\eta n\lambda_1 - 4\eta^2 n\lambda_1^2\right)
\end{aligned} \tag{A.65}
$$

where in (i) we used $\left(100L^4\sigma^4+4\right)\eta^2 n\lambda_1^2 \leq 1$ and $x \in (0,1)$, $\exp(x) \leq 1+2x$. For (ii), it suffices to have

$$
\frac{\phi - p_0^2}{p_0^2} \leq \frac{\delta}{8}, \;\; \frac{\phi}{p_0^2}\left(100L^4\sigma^4+4\right)\eta^2 n\lambda_1^2 \leq \frac{\delta}{16}
$$

which is further ensured by,

$$
p_0 \geq \max\left\{\frac{1+2\eta\lambda_1 s\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)}{1+\frac{\delta}{16}}, \;\; \frac{4\eta\lambda_1 s\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)}{\sqrt{\delta}}, \;\; 8\sqrt{\frac{\left(100L^4\sigma^4+4\right)\eta^2 n\lambda_1^2}{\delta}}\right\}
$$

Note that for the choice of η, and $\delta \geq \frac{1}{10}$, we have using Lemma A.2.4,

$$
\frac{1+2\eta\lambda_1 s\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)}{1+\frac{\delta}{16}} \geq \max\left\{\frac{4\eta\lambda_1 s\left(\frac{4\lambda_1}{(\lambda_1-\lambda_2)}\right)}{\sqrt{\delta}}, \;\; 8\sqrt{\frac{\left(100L^4\sigma^4+4\right)\eta^2 n\lambda_1^2}{\delta}}\right\}
$$

for sufficiently large n. Therefore, we only ensure that p_0 is greater than the first term in the theorem statement. Finally, let

$$
\xi := \frac{C'\log\left(\frac{1}{\delta}\right)}{\delta^3}\frac{\lambda_1}{\lambda_1-\lambda_2}\frac{\alpha_0\left(1+2\eta\operatorname{Tr}(\Sigma)\right)+2\eta\lambda_1\beta_0}{p_0}
$$

Using Eq A.61 and Eq A.65 and substituting in Eq A.57, we have with probability at least $1-2\delta$, conditioned on $\mathcal{E}$, $\chi \leq \xi$, or equivalently $\mathbb{P}\left(\chi \geq \xi \middle| \mathcal{E}\right) \leq 2\delta$. Therefore,

$$
\mathbb{P}(\chi \geq \xi) = \mathbb{P}(\mathcal{E})\,\mathbb{P}\left(\chi \geq \xi \middle| \mathcal{E}\right) + \mathbb{P}(\mathcal{E}^c)\,\mathbb{P}\left(\chi \geq \xi \middle| \mathcal{E}^c\right) \leq \mathbb{P}\left(\chi \geq \xi \middle| \mathcal{E}\right) + \mathbb{P}(\mathcal{E}^c) \leq 2\delta + \mathbb{P}(\mathcal{E}^c)
$$

The proof follows by making δ smaller by a constant factor. □

With Theorem A.5.2 in place, we are ready to finally provide the proof of one of our main results, Theorem 3.5.

Theorem 3.5 (Vector Truncation)**.** *Let Assumptions 1 and 2 hold and $k \geq s$. For dataset $D := \{X_i\}_{i\in[n]}$ and $w_0 \sim \mathcal{N}(0, I)$, let $\mathcal{A}$ be the randomized algorithm which computes $\hat{v}_{\mathsf{truncvec}} \leftarrow \mathsf{TruncateOja}\left(\{X_i\}_{i\in\left(\frac{n}{2},n\right]}, \widehat{S}, \mathsf{Oja}, \{\eta, w_0\}\right)$, where $\eta := \frac{3\log(n)}{n(\lambda_1-\lambda_2)}$. Then, for $\min_i |v_1(i)| = \widetilde{\Omega}\left(\left(\frac{d}{n^2}\right)^{\frac{1}{8}}\right)$, $\tilde{v} \leftarrow \mathsf{SuccessBoost}\left(\{X_i\}_{i\in[n]}, \mathcal{A}, d^{-10}\right)$ satisfies,*

$$
\sin^2(\tilde{v}, v_1) \leq C''\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)^2\frac{k\log^2(d)}{n}
$$

with probability at least $1-d^{-10}$, where $C'' \geq 0$ is an absolute constant.

Proof. Let $\mathcal{E} := \left\{S_{\mathsf{hi}} \subseteq \widehat{S}\right\}$ and set $\delta := \frac{1}{4}$ for this proof. Consider the following variables from Theorem A.5.2:

$$W_{\widehat{S}} := \mathbb{E}\left[I_{\widehat{S}} - I_{\widehat{S}} v_1 v_1^T I_{\widehat{S}} \mid \mathcal{E}\right],\ G_{\widehat{S}} := \mathbb{E}\left[I_{S \bigcap \widehat{S}} \mid \mathcal{E}\right]$$
$$\alpha_0 := v_1^T W_{\widehat{S}} v_1,\ \beta_0 := \mathrm{Tr}\left(V_\perp^T W_{\widehat{S}} V_\perp\right),\ p_0 := v_1^T G_{\widehat{S}} v_1,\ q_0 := \mathrm{Tr}\left(V_\perp^T G_{\widehat{S}} V_\perp\right)$$

Since $|\widehat{S}| = k$, therefore,

$$\beta_0 = \mathrm{Tr}\left(V_\perp^T W_{\widehat{S}} V_\perp\right) \leq \mathrm{Tr}\left(W_{\widehat{S}}\right) \leq \mathrm{Tr}\left(\widehat{S}\right) = k \tag{A.66}$$

Furthermore, under event $\mathcal{E}$, $S_{\mathsf{hi}} \subseteq \left\{S \bigcap \widehat{S}\right\}$. Therefore,

$$p_0 = v_1^T G_{\widehat{S}} v_1 \geq \sum_{i \in S_{\mathsf{hi}}} v_1(i)^2 = 1 - \sum_{i \notin S_{\mathsf{hi}}} v_1(i)^2 \geq 1 - \frac{s \log(n)}{n},\ \text{using definition of } S_{\mathsf{hi}} \tag{A.67}$$

To verify the assumption on p_0 mentioned in Theorem A.5.2, it is sufficient to ensure

$$\eta \lambda_1 s \left(\frac{20\lambda_1}{\lambda_1 - \lambda_2}\right) \leq \left(1 - \frac{s \log(n)}{n}\right)\left(1 + \frac{\delta}{4}\right) - 1$$

which is true by the definition of η and n (see Lemma A.2.4). Lastly,

$$\begin{aligned}
\alpha_0 = v_1^T W_{\widehat{S}} v_1 &= \mathbb{E}\left[v_1^T I_{\widehat{S}} v_1 - \left(v_1^T I_{\widehat{S}} v_1\right)^2 \mid \mathcal{E}\right] \\
&\leq 1 - \mathbb{E}\left[v_1^T I_{\widehat{S}} v_1 \mid \mathcal{E}\right],\ \text{using } v_1^T I_{\widehat{S}} v_1 \leq 1 \\
&\leq 1 - \sum_{i \in S_{\mathsf{hi}}} v_1(i)^2,\ \text{since } S_{\mathsf{hi}} \subseteq \widehat{S} \\
&= \sum_{i \notin S_{\mathsf{hi}}} v_1(i)^2 \leq \frac{s \log(n)}{n}
\end{aligned} \tag{A.68}$$

Therefore, using bounds on β_0, p_0 and α_0 from Eqs A.66, A.67 and A.68 respectively, in conjunction with Theorem A.5.2, with probability at least $1 - \delta - \mathbb{P}(\mathcal{E}^c)$,

$$\sin^2(v_{\text{oja}}, v_1) \leq \frac{C' \log\left(\frac{1}{\delta}\right)}{\delta^3} \frac{5\lambda_1}{\lambda_1 - \lambda_2} \eta \lambda_1 k \tag{A.69}$$

Using Lemma 3.1, $\mathbb{P}(\mathcal{E}^c) \leq 5\delta$. The result then follows using Eq A.69 and setting δ smaller by a constant. □

A.5.2 Proof of theorem 3.7

We first state the result from [Lia23] achieving the optimal $\sin^2$ error rate for Oja's Algorithm.

Proposition A.5.3 (Optimal Rate for Oja's Algorithm with Subgaussian Data (Theorem 3.1, [Lia23]))**.** *Let $\{X_i\}_{i \in [n]}$ be i.i.d samples from a subgaussian distribution (Definition 2.1) with covariance matrix, Σ, leading eigenvector v_1 and eigengap, $\lambda_1 - \lambda_2 > 0$. Then, there exists an algorithm* OptimalOja *which operates in $O(d)$ space, $O(nd)$ time, processes one datapoint at a time, and returns an estimate $\hat{v}$ which satisfies, with probability at least $1 - \delta$, $\delta \in (0, 1)$*

$$\sin^2(\hat{v}, v_1) \leq C \frac{\lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)^2} \frac{d \log\left(\frac{1}{\delta}\right)}{n}$$

where $C > 0$ is an absolute constant.

Theorem 3.7 (Data Truncation)**.** *Let Assumptions 1 and 2 hold and $k \geq s$. For dataset $\mathcal{D} := \{X_i\}_{i \in [n]}$ and $w_0 \sim \mathcal{N}(0, I)$, let $\mathcal{A}$ be the randomized algorithm which computes $\hat{v}_{\mathsf{truncvec}} \leftarrow$*

$\mathsf{TruncateOja}\left(\left\{\lfloor X_i \rfloor_{\widehat{S}}\right\}_{i\in\left(\frac{n}{2},n\right]}, \widehat{S}, \mathsf{OptimalOja}, \{\{\eta_t\}_{t\in\left[\frac{n}{2}\right]}, w_0\}\right)$. *Then for* $\min_i |v_1(i)| = \widetilde{\Omega}\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\left(\frac{d}{n^2}\right)^{\frac{1}{4}}\right)$, $\tilde{v} \leftarrow \mathsf{SuccessBoost}\left(\{X_i\}_{i\in[n]}, \mathcal{A}, d^{-10}\right)$ *satisfies,*

$$\sin^2(\tilde{v}, v_1) \leq C'' \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)^2}\frac{k\log(d)}{n}$$

with probability at least $1-d^{-10}$*, where* $C'' \geq 0$ *is an absolute constant.*

Proof. Let $\delta := \frac{1}{3}$. Define the event $\mathcal{E} = \left\{S \subseteq \widehat{S}\right\}$. Using Lemma 3.1, we have that

$$\mathbb{P}(\mathcal{E}) \geq 1-\delta$$

Let $\chi := \sin^2(\hat{v}_{\mathsf{truncvec}}, v_1)$ and $\xi := C\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)^2}\frac{k\log\left(\frac{1}{\delta}\right)}{n}$ for an absolute constant $C>0$. Therefore,

$$\begin{aligned}
\mathbb{P}(\chi \geq \xi) &= \mathbb{P}(\mathcal{E})\,\mathbb{P}\left(\chi \geq \xi \middle| \mathcal{E}\right) + \mathbb{P}(\mathcal{E}^c)\,\mathbb{P}\left(\chi \geq \xi \middle| \mathcal{E}^c\right) \\
&\leq \mathbb{P}\left(\chi \geq \xi \middle| \mathcal{E}\right) + \mathbb{P}(\mathcal{E}^c) \\
&\leq \mathbb{P}\left(\chi \geq \xi \middle| \mathcal{E}\right) + \delta \qquad \text{(A.70)}
\end{aligned}$$

Therefore, next we bound $\mathbb{P}\left(\sin^2(\hat{v}_{\mathsf{truncvec}}, v_1)\middle|\mathcal{E}\right)$. Therefore, we seek to bound the $\sin^2$ error after truncating the data using the true support, S. Note that

$$\mathbb{E}\left[I_S X X^\top I_S\right] = \lambda_1 v_1 v_1^\top + I_S V_\perp \Lambda_2 V_\perp^\top I_S$$

Therefore, after truncation, the leading eigenvector and eigenvalue are preserved, and the second largest eigenvalue is at most λ_2. Furthermore, the truncated distribution is still subgaussian, and therefore Proposition A.5.3 is applicable here and we have with probability at least $1-\delta$,

$$\sin^2(\hat{v}_{\mathsf{truncvec}}, v_1) \leq \xi \qquad \text{(A.71)}$$

Eq (A.70) and (A.71) show that $\mathcal{A}$ is a $\mathsf{ConstantSuccessOracle}\left(\mathcal{D}, (\eta,k), \mathcal{T}, \rho, v_1, O\left(k\log(d)/n\right)\right)$ (Definition 3.9) for the set $\mathcal{T} = \left\{u : u\in\mathbb{R}^d, \|u\|_2 = 1\right\}$ with the metric $\rho(u,v) := \left\|uu^\top - vv^\top\right\|_F = \frac{1}{2}|\sin(u,v)|$. The result then follows from Lemma 3.10. □

A.6 Alternate method for truncation

In this section, we present another algorithm for truncation, based on a value-based thresholding, complementary to the technique described in Section 3. The proof technique uses the same tools as the ones described in Section 3. Both Algorithm 2 and 4 may be of independent interest depending on the particular use-case and constraints of the particular problem. Theorem A.6.1 provides the convergence guarantees for Algorithm 4. Note that compared to Theorem 3.5, Theorem A.6.1 provides a better guarantee for the sample size. However, this comes at the cost of the sparsity of the returned vector, $\hat{v}_{\text{oja-thresh}}$, not being a controllable parameter. We can however show that the support size of $\hat{v}_{\text{oja-thresh}}$ is $O(s)$ in expectation. For the purpose of this proof, let $S_{\text{hi}} := \left\{i : i\in S, |v_1(i)| \geq \sqrt{\frac{\log(d)}{n}}\right\}$.

Theorem A.6.1 (Convergence of Oja-Thresholded)**.** *Let* $\hat{v}_{\textit{oja-thresh}}$, $\widehat{S}$ *be obtained from Algorithm 4. Set the learning rate as* $\eta := \frac{3\log(n)}{n(\lambda_1-\lambda_2)}$. *Define threshold* $\gamma_n := \frac{3}{4\sqrt{2e}}\min_{i\in S_{hi}}|v_1(i)|(1+\eta\lambda_1)^n$. *Then for* $n = \widetilde{\Omega}\left(\frac{1}{s\min_{i\in S_{hi}} v_1(i)^4}\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)^2\right)$, *we have* $\mathbb{E}\left[|\widehat{S}|\right] \leq C's$ *and with probability at least* $\frac{3}{4}$,

$$\sin^2\left(\hat{v}_{\textit{oja-thresh}}, v_1\right) \leq C''\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)^2\frac{\max\{s,\log(d)\}\log(d)}{n}, \quad S_{\textit{hi}} \subseteq \widehat{S}$$

where $C', C'' > 0$ *are absolute constants.*

Algorithm 4 Oja-Thresholded $\left(\{X_i\}_{i\in[n]}, \gamma_n, \eta\right)$

Input : Dataset $\{X_i\}_{i\in[n]}$, learning rate $\eta > 0$, truncation threshold γ_n
Set $b_n \leftarrow 0$ and choose $y_0, w_0 \sim \mathcal{N}(0, I)$ independently
for t in range[1, $\frac{n}{2}$] **do**
    $y_t \leftarrow (I + \eta X_t X_t^T) y_{t-1}$
    $b_n \leftarrow b_n + \log\left(\|y_t\|_2\right)$
    $y_t \leftarrow \frac{y_t}{\|y_t\|_2}$
end for
$\widehat{S} \leftarrow$ Set of indices, $i \in [d]$, such that $\log\left(|e_i^T y_n|\right) + b_n - \log(\gamma_n) \geq 0$.
$\widehat{v} \leftarrow \mathsf{Oja}\left(\{X_i\}_{i\in\{n/2+1,\ldots,n\}}, \eta, w_0\right)$
$\hat{v}_{\text{oja-thresh}} \leftarrow \frac{\left[\widehat{v}\right]_{\widehat{S}}}{\left\|\left[\widehat{v}\right]_{\widehat{S}}\right\|_2}$
return $\left[\hat{v}_{\text{oja-thresh}}, \widehat{S}\right]$

Proof. Consider the setting of Theorem A.5.2. Set $\delta := \frac{1}{4}$ for this proof and let $\mathcal{E}$ be the event $\left\{|v_1^T y_0| \geq \frac{\delta}{\sqrt{e}}\right\}$. By Lemma A.2.1, $\mathbb{P}(\mathcal{E}) \geq 1 - \delta$. Recall the definitions,

$$W_{\widehat{S}} := \mathbb{E}\left[I_{\widehat{S}} - I_{\widehat{S}} v_1 v_1^T I_{\widehat{S}} \middle| \mathcal{E}\right],\ G_{\widehat{S}} := \mathbb{E}\left[I_{S\bigcap\widehat{S}} \middle| \mathcal{E}\right]$$
$$\alpha_0 := v_1^T W_{\widehat{S}} v_1,\ \beta_0 := \mathrm{Tr}\left(V_\perp^T W_{\widehat{S}} V_\perp\right),\ p_0 := v_1^T G_{\widehat{S}} v_1$$

We upper bound α_0, β_0 and lower bound p_0 under the setting of Algorithm 4. Define $r_i := e_i^T B_n y_0, i \in [d]$. For α_0, p_0, we have

$$\begin{aligned}
\alpha_0 = v_1^T W_{\widehat{S}} v_1 &= \mathbb{E}\left[v_1^T I_{\widehat{S}} v_1 - \left(v_1^T I_{\widehat{S}} v_1\right)^2 \middle| \mathcal{E}\right] \\
&= \mathbb{E}\left[v_1^T I_{\widehat{S}} v_1 \left(1 - v_1^T I_{\widehat{S}} v_1\right) \middle| \mathcal{E}\right] \\
&\leq 1 - \mathbb{E}\left[v_1^T I_{\widehat{S}} v_1 \middle| \mathcal{E}\right], \text{ using } v_1^T I_{\widehat{S}} v_1 \leq 1 \\
&= 1 - \sum_{i\in S} v_1(i)^2\, \mathbb{P}\left(i \in \widehat{S}; i \in S \middle| \mathcal{E}\right), \\
&= \sum_{i\in S} v_1(i)^2\, \mathbb{P}\left(i \notin \widehat{S}; i \in S \middle| \mathcal{E}\right) \qquad \text{(A.72)}
\end{aligned}$$

$$\begin{aligned}
p_0 &= v_1^T G_{\widehat{S}} v_1 \\
&= v_1^T \mathbb{E}\left[I_{S\bigcap\widehat{S}}\right] v_1 \\
&= \sum_{i\in S} v_1(i)^2\, \mathbb{P}\left(i \in \widehat{S}; i \in S \middle| \mathcal{E}\right) \\
&= 1 - \sum_{i\in S} v_1(i)^2\, \mathbb{P}\left(i \notin \widehat{S}; i \in S \middle| \mathcal{E}\right) \qquad \text{(A.73)}
\end{aligned}$$

Therefore, for both α_0, p_0, we seek to upper bound $\sum_{i\in S} v_1(i)^2 \mathbb{P}\left(i \notin \widehat{S}; i \in S \middle| \mathcal{E}\right)$. We have

$$\begin{aligned}
\sum_{i\in S} v_1(i)^2 \mathbb{P}\left(i \notin \widehat{S}; i \in S \middle| \mathcal{E}\right) &= \sum_{i\in S_{\text{hi}}} v_1(i)^2 \mathbb{P}\left(i \notin \widehat{S}; i \in S \middle| \mathcal{E}\right) + \sum_{i\in S\setminus S_{\text{hi}}} v_1(i)^2 \mathbb{P}\left(i \notin \widehat{S}; i \in S \middle| \mathcal{E}\right) \\
&\leq \frac{s\log(n)}{n} + \sum_{i\in S\setminus S_{\text{hi}}} v_1(i)^2 \mathbb{P}\left(i \notin \widehat{S}; i \in S \middle| \mathcal{E}\right) \\
&= \frac{s\log(n)}{n} + \sum_{i\in S\setminus S_{\text{hi}}} v_1(i)^2 \mathbb{P}\left(|r_i| < \gamma_n; i \in S \middle| \mathcal{E}\right) \\
&\overset{(i)}{\leq} \frac{s\log(n)}{n} + C_H \sum_{i\in S\setminus S_{\text{hi}}} v_1(i)^2 \left[\eta\lambda_1 \log(n) + \eta\lambda_1 \left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\frac{1}{v_1(i)^2}\right] \\
&= \frac{s\log(n)}{n} + C_H\eta\lambda_1 \log(n) + C_H\eta\lambda_1 s'\left(\frac{\lambda_1}{(\lambda_1-\lambda_2)}\right) \\
&\leq C_H\eta\lambda_1\{s, \log(n)\} \leq \frac{1}{2}, \text{ using } |v_1(i)| \geq \sqrt{\frac{\log(n)}{n}}, i \in S_{\text{hi}}
\end{aligned}$$

For β_0 we have

$$\begin{aligned}
\beta_0 &\leq \mathbb{E}\left[\text{Tr}\left(W_{\widehat{S}}\right)\middle|\mathcal{E}\right] = \sum_{i\in[d]} \mathbb{P}\left(i \in \widehat{S}\middle|\mathcal{E}\right) - \sum_{i\in S} v_1(i)^2 \mathbb{P}\left(i \in \widehat{S}\middle|\mathcal{E}\right) \\
&\leq \sum_{i\notin S} \mathbb{P}\left(i \in \widehat{S}\middle|\mathcal{E}\right) + \sum_{i\in S}\left(1 - v_1(i)^2\right)\mathbb{P}\left(i \in \widehat{S}\middle|\mathcal{E}\right) = \sum_{i\notin S} \mathbb{P}\left(i \in \widehat{S}\middle|\mathcal{E}\right) + s - 1 \\
&\leq \sum_{i\notin S} \mathbb{P}\left(|r_i| \geq \gamma_n \middle|\mathcal{E}\right) + s - 1 = \sum_{i\notin S} \frac{\mathbb{P}(|r_i| \geq \gamma_n)}{\mathbb{P}(\mathcal{E})} + s - 1 \\
&\leq 2\sum_{i\notin S} \mathbb{P}(|r_i| \geq \gamma_n) + s - 1, \text{ since } \mathbb{P}(\mathcal{G}) \geq 1 - \delta \\
&\leq 2C_T\left[\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)^2 \left(\frac{1}{\delta^2 \min_{i\in S_{\text{hi}}} v_1(i)^2}\right)^2\right]\eta^2\lambda_1^2(d-s) + s - 1, \text{ using Lemma 3.12} \\
&\leq 2s, \text{ using bound on } n
\end{aligned}$$

The result then follows using Theorem A.5.2 and substituting the bounds on α_0, β_0 and p_0. Finally, note that using a similar argument as Theorem 3.5, we have

$$\begin{aligned}
\mathbb{P}\left(S_{\text{hi}} \not\subseteq \widehat{S}|\mathcal{E}\right) &\leq \sum_{i\in S_{\text{hi}}} \mathbb{P}\left(i \notin \widehat{S}; i \in S_{\text{hi}} \middle|\mathcal{E}\right) \\
&= \sum_{i\in S_{\text{hi}}} \mathbb{P}\left(|r_i| < \gamma_n; i \in S \middle|\mathcal{E}\right) \leq C_H \sum_{i\in S_{\text{hi}}} \eta\lambda_1 \log(n) + \eta\lambda_1\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\frac{1}{v_1(i)^2} \\
&\leq C_H\eta\lambda_1 s\log(n) + C_H\eta\lambda_1\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\sum_{i\in S_{\text{hi}}}\frac{1}{v_1(i)^2} \leq \delta
\end{aligned}$$

using the sample size bound on n. □

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