C^2-LUSIN APPROXIMATION OF STRONGLY CONVEX BODIES

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Abstract. We prove that, if $W \subset \mathbb{R}^n$ is a locally strongly convex body (not necessarily compact), then for any open set $V \supset \partial W$ and $\varepsilon > 0$, there exists a C^2 locally strongly convex body $W_{\varepsilon,V}$ such that $\mathcal{H}^{n-1}(\partial W_{\varepsilon,V} \triangle \partial W) < \varepsilon$ and $\partial W_{\varepsilon,V} \subset V$. Moreover, if W is strongly convex, then $W_{\varepsilon,V}$ is strongly convex as well.

1. Introduction

The aim of this note is to prove the following result.

Theorem 1.1. *Let $W \subset \mathbb{R}^n$ be a locally strongly convex body (not necessarily compact), $\varepsilon > 0$, and the set $V \supset \partial W$ be open. There exists a C^2 locally strongly convex body $W_{\varepsilon,V}$ such that $\mathcal{H}^{n-1}(\partial W_{\varepsilon,V} \triangle \partial W) < \varepsilon$ and $\partial W_{\varepsilon,V} \subset V$. Moreover, if W is strongly convex, then $W_{\varepsilon,V}$ is strongly convex as well.*

Here, $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure, and $A \triangle B$ is the symmetric difference of the sets A and B, that is, $A \triangle B := (A \setminus B) \cup (B \setminus A)$. Throughout this paper, we say that $W \subset \mathbb{R}^n$ is a *convex body* if it is closed, convex, and has nonempty interior; if its boundary ∂W can be represented locally (up to a suitable rotation) as the graph of a strongly convex function, then we say W is a *locally strongly convex body*. We say that W is a *strongly convex body* if it is a compact locally strongly convex body. One can prove that W is a strongly convex body if and only if it is the intersection of a family of closed balls of the same radius; see Proposition 2.4 for this and other equivalent characterizations of strongly convex bodies. Note that the epigraph of a strongly convex function is never a strongly convex body (though it is always a locally strongly convex body). However, if $u : \mathbb{R}^n \to \mathbb{R}$ is locally strongly convex and coercive, then for every $t > \min_{x \in \mathbb{R}^n}\{u(x)\}$ the level set $u^{-1}((-\infty, t])$ is compact and locally strongly convex, hence also strongly convex; again, see Proposition 2.4.

A function $u : U \to \mathbb{R}$ defined on an open convex set is *strongly convex* if there is $\eta > 0$, such that $u(x) - \frac{\eta}{2}|x|^2$ is convex (in which case we say that u is η-strongly convex). Note that, if u is of class C^2, then this is equivalent to saying that, for all x, the minimum eigenvalue of $D^2u(x)$ is greater than or equal to η. We say that u is locally strongly convex if for every $x \in U$ there is $r_x > 0$, such that the restriction of u to $B(x, r_x)$ is strongly convex.

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Theorem 1.1 was stated without proof in [1] as a corollary to the main result of that paper, which we recall next.

Let $\mathcal{G}_u$ represent the graph of a function $u : U \to \mathbb{R}$, where $U \subset \mathbb{R}^n$.

Theorem 1.2 (See [1]). *Let $U \subseteq \mathbb{R}^n$ be open and convex, and $u : U \to \mathbb{R}$ be locally strongly convex. Then for every $\varepsilon_o > 0$ and for every continuous function $\varepsilon : U \to (0, 1]$ there is a locally strongly convex function $v \in C^2(U)$, such that*

(a) $|\{x \in U : u(x) \neq v(x)\}| < \varepsilon_o$;
(b) $|u(x) - v(x)| < \varepsilon(x)$ *for all* $x \in U$;
(c) $\mathcal{H}^n(\mathcal{G}_u \triangle \mathcal{G}_v) < \varepsilon_o$.

Also, if u is η-strongly convex on U, then for every $\widetilde{\eta} \in (0, \eta)$ there exists such a function v which is $\widetilde{\eta}$-strongly convex on U.

Part (a) of this result says that we can approximate a locally strongly convex function by a C^2 locally strongly convex function in the Lusin sense. For motivation and background about this kind of approximation we refer the reader to the introductions of the papers [3, 1].

The rest of this note is organized as follows. In Section 2 we review some basic facts of convex analysis, provide multiple characterizations of strongly convex bodies, and detail useful technical estimates for the metric projection onto a compact convex body and onto the boundary of a $C^{1,1}$ convex body. For more details and omitted proofs regarding convex functions and convex bodies we refer to [7, 8, 9]. While most of the results of Section 2 are well known, some of the equivalent conditions in Proposition 2.4 are new, and Lemma 2.7 is new. In Section 3, we complete the proof of Theorem 1.1.

2. Preliminaries for the proof of Theorem 1.1.

Every closed convex set $W \subset \mathbb{R}^n$ is the intersection of all closed half-spaces that contain W. In fact, for every $x \in \partial W$ there is a half-space H_x such that $W \subset H_x$ and $x \in T_x \cap W$, where $T_x = \partial H_x$. The hyperplane T_x is called a *hyperplane supporting* W at x. For every $x \in \partial W$, there is a hyperplane supporting W at x, but such a hyperplane is not necessarily unique. We define the *normal cone* to W at x as the set of all vectors perpendicular to some supporting hyperplane of W at x and pointing outside W:

$$N_W(x) := \{\zeta \in \mathbb{R}^n : \langle \zeta, y - x \rangle \leq 0 \text{ for all } y \in W\}.$$

Because there must be a hyperplane supporting W at x for every $x \in \partial W$, we have $N_W(x) \neq \emptyset$ for every $x \in \partial W$.

It follows that given an open convex set $U \subset \mathbb{R}^n$ and a convex function $f : U \to \mathbb{R}$, we have that for every $x \in U$ there is $v \in \mathbb{R}^n$ such that $f(y) \geq f(x) + \langle v, y - x \rangle$ for all $y \in U$. Indeed, on the right hand side we have an equation of a supporting hyperplane of the convex *epigraph* $\text{epi}(f) = \{(x, t) \in U \times \mathbb{R} : x \in U,\ t \geq f(x)\}$. The (nonempty) set of all such v is denoted by $\partial f(x)$ and called the *subdifferential* of f at x:

$$\partial f(x) := \{v \in \mathbb{R}^n : f(y) \geq f(x) + \langle v, y - x \rangle \text{ for all } y \in U\}.$$

For every $\xi \in \partial f(x)$, we have that $n_\xi \in \mathbb{R}^{n+1}$ defined by

$$n_\xi := \frac{1}{\sqrt{1+|\xi|^2}}\,(\xi, -1) \tag{1}$$

is a unit normal vector to epi(f) at $(x, f(x))$ that points outside epi(f); that is,

$$n_\xi \in N_{\mathrm{epi}(f)}(x, f(x)) \cap \mathbb{S}^n.$$

A convex function f is differentiable at a point x_0 if and only if $\partial f(x_0)$ is a singleton, in which case we have $\partial f(x_0) = \{\nabla f(x_0)\}$, meaning that the tangent hyperplane to the graph of f at x_0 is the unique hyperplane supporting the epigraph of f at $(x_0, f(x_0))$. Convex functions are locally Lipschitz continuous, and it easily follows that if f is Lipschitz continuous with Lipschitz constant L in a neighborhood of x and $\xi \in \partial f(x)$, then

$$|\xi| \le L. \tag{2}$$

Also, it follows from Rademacher's theorem that convex functions are differentiable almost everywhere, so $\partial f(x) = \{\nabla f(x)\}$ for almost every $x \in U$.

The following lemma is well known; for a proof see, for instance, [1, Lemma 3.4].

Lemma 2.1. *Let $u : U \to \mathbb{R}$ be a convex function defined on an open convex set $U \subseteq \mathbb{R}^n$. Then u is η-strongly convex if and only if*

$$u(y) \ge u(x) + \langle \xi, y - x\rangle + \frac{\eta}{2}|y-x|^2 \tag{3}$$

for all $x, y \in U$ and $\xi \in \partial u(x)$.

Remark 2.2. The proof of $(\Leftarrow)$ in [1, Lemma 3.4] also shows that if (3) holds for all $x, y \in U$ and *some* $\xi \in \partial u(x)$, then u is η-strongly convex, and therefore, by the proof of $(\Rightarrow)$, (3) is also true for *all* $\xi \in \partial u(x)$.

For any convex body $W \subset \mathbb{R}^n$ with $0 \in \mathrm{int}(W)$, the *Minkowski functional* (also known as *gauge*) of W is a map $\mu_W : \mathbb{R}^n \to [0, \infty)$ defined by

$$\mu_W(x) := \inf\left\{\lambda > 0 : \frac{1}{\lambda}x \in W\right\}.$$

The Minkowski functional is a positive homogeneous, subadditive convex function such that $\mu_W^{-1}([0,1]) = W$ and $\mu_W^{-1}(1) = \partial W$. Because $0 \in \mathrm{int}(W)$, there exists $\varepsilon > 0$ such that $B(0, \varepsilon) \subset W$; hence, for all $x \in \mathbb{R}^n$, $\frac{\varepsilon x}{2|x|} \in W$. Thus,

$$|\mu_W(x) - \mu_W(y)| \le \max\{\mu_W(x-y), \mu_W(y-x)\} \le \frac{2}{\varepsilon}|x-y|,$$

implying μ_W is Lipschitz.

As a consequence of the implicit function theorem and the positive homogeneity of μ, we have ∂W is a 1-codimension submanifold of class $C^k(\mathbb{R}^n)$ if and only if μ_W is C^k on $\mathbb{R}^n \setminus \mu_W^{-1}(0)$. Note that if W is compact, then $\mu_W^{-1}(0) = \{0\}$.

Lemma 2.3. *Given a convex body $W \subset \mathbb{R}^n$, for any selection*

$$\partial W \ni z \to \zeta(z) \in N_W(z) \cap \mathbb{S}^{n-1},$$

we have

$$W = \bigcap_{y\in\partial W} \{x \in \mathbb{R}^n : \langle \zeta(y), x-y\rangle \le 0\}. \tag{4}$$

Proof. Since $\zeta(y)$ is an outward unit normal vector to W at y, the halfspace $H_y^- := \{x : \langle \zeta(y), x-y\rangle \le 0\}$ contains W for every $y \in \partial W$, so we have that $W \subseteq V := \bigcap_{y\in\partial W} H_y^-$. If $W \neq V$, since both W and V are convex bodies, and we already know that $W \subseteq V$, we must have $y \in \mathrm{int}(V)$ for some $y \in \partial W$. But then $y \in \mathrm{int}(H_y^-) = \{x \in \mathbb{R}^n : \langle \zeta(y), x-y\rangle < 0\}$, which is absurd. □

Most of the equivalences provided by the following result are well known (see [10] for instance), except, perhaps, for condition (d) and the definition of a locally strongly convex body as a closed convex set whose boundary can be locally represented as the graph of a strongly convex function (which is not standard). We provide a complete proof for the reader's convenience.

Proposition 2.4. *Let $W \subset \mathbb{R}^n$ be a compact convex body satisfying $0 \in \mathrm{int}(W)$, and let $\mu : \mathbb{R}^n \to [0, \infty)$ denote the Minkowski functional of W. The following statements are equivalent:*

(a) W is strongly convex.
(b) There is $R > 0$, such that for every $x \in \partial W$, there is a closed ball $\overline{B}(y, R)$, such that $W \subset \overline{B}(y, R)$ and $x \in \partial\overline{B}(y, R)$.
(c) W is the intersection of a family of closed balls of the same positive radius.
(d) μ^2 is strongly convex.
(e) There exists a coercive, locally strongly convex function $g : \mathbb{R}^n \to \mathbb{R}$ such that $W = g^{-1}((-\infty, t])$ for some $t \in \mathbb{R}$ with $t > \min_{x\in\mathbb{R}^n} g(x)$.

Proof. (a) ⇒ (b):

Let $x \in \partial W$, because W is locally strongly convex, ∂W in a neighborhood of x is the graph of a strongly convex function. By translation and rotation if necessary, we can find $r_x > 0$ and a function $g_x : B^{n-1}(0, 2r_x) \to \mathbb{R}$ that satisfies $g_x(0) = 0$, g_x is η_x-strongly convex, L_x-Lipschitz and

$$W_{x,2r_x} := \{(t, g_x(t)) : t \in B^{n-1}(0, 2r_x)\} \subset \partial W,$$

where the coordinates on the right hand side for $\mathbb{R}^{n-1} \times \mathbb{R}$ depend on x. More generally, for $s \in (0, 2r_x]$ we define

$$W_{x,s} := \{(t, g_x(t)) : t \in B^{n-1}(0, s)\}.$$

Because ∂W is compact, $\partial W \subset \bigcup_{x\in\partial W} W_{x,r_x}$ has a finite subcover, that is $\partial W \subset \bigcup_{j=1}^m W_{x_j,r_{x_j}}$.

Let $z \in \partial W$. Then there exists $j \in \{1, ..., m\}$ and $x \in B^{n-1}(0, r_j)$ such that $z = (x, g_j(x))$, where we let g_j (resp. r_j) stand for g_{x_j} (resp. r_{x_j}). We will show there exists $R > 0$ such that for all $j \in \{1, ..., m\}$ and $x \in B^{n-1}(0, r_j)$, and any $\xi \in \partial g_j(x)$ we have

$$W \subseteq \overline{B}\left((x, g_j(x)) - R\, n_\xi, R\right), \tag{5}$$

where n_ξ is defined in (1), implying at once $W \subseteq \overline{B}\,(z - R\, n_\xi, R)$ and $z \in \partial B\,(z - R\, n_\xi, R)$, and thus (b) holds. Proceeding, our aim is to prove (5).

For $j \in 1, ..., m$, let $\eta_j := \eta_{x_j}$ and $L_j := L_{x_j}$. Let $L, \eta, r, r_0 > 0$ be

$$\begin{aligned} &L := \max\{L_1, ..., L_m\}, \\ &\eta := \min\{\eta_1, ..., \eta_m\}, \\ &r := \max\{r_1, ..., r_m\}, \text{ and} \\ &r_0 := \min\{r_1, ..., r_m\}. \end{aligned}$$

Let $R > 0$ be

$$R := \sqrt{1+L^2}\max\left\{\frac{1}{\eta}\left(1 + \frac{\eta^2}{4}r^2 + L^2 + \eta L r\right), \operatorname{diam}(W), \frac{\operatorname{diam}(W)^2}{\eta r_0^2}\right\}. \tag{6}$$

To verify (5) holds with R as in (6), fix $j \in \{1, ..., m\}$, $x \in B^{n-1}(0, r_j)$, and $\xi \in \partial g_j(x)$. Let $(y, s) \in W$ in coordinates provided by g_j. Then we want to show

$$|(y, s) - (x, g_j(x)) + R n_\xi|^2 \le R^2, \tag{7}$$

where n_ξ is defined in (1). We consider two cases.

Case 1. Suppose $|y - x| < r_j$. Since $W \cap \{(y, s) : |y - x| < r_j\} \subset \{(y, s) : s \ge g_j(y)\}$ and g_j is η-strongly convex, Lemma 2.1 yields

$$g_j(x) + \langle \xi, y - x\rangle + \frac{\eta}{2}|y - x|^2 \le g_j(y) \le s. \tag{8}$$

Note that $s - g_j(x) \le \operatorname{diam}(W)$ and in light of (2), $|\xi| \le L$, so our choice of R in (6) yields

$$\frac{R}{\sqrt{1+|\xi|^2}} + g_j(x) - s \ge \frac{R}{\sqrt{1+L^2}} - \operatorname{diam}(W) \ge 0.$$

In combination with (8),

$$\left(\frac{R}{\sqrt{1+|\xi|^2}} + g_j(x) - s\right)^2 \le \left(\frac{R}{\sqrt{1+|\xi|^2}} - \langle \xi, y - x\rangle - \frac{\eta}{2}|y - x|^2\right)^2.$$

Using (1) we get

$$\begin{aligned} \left|(y, s) - (x, g_j(x)) + R n_\xi\right|^2 &= \left|(y, s) - (x, g_j(x)) + \frac{R}{\sqrt{1+|\xi|^2}}(\xi, -1)\right|^2 \\ &= \left|y - x + \frac{R\xi}{\sqrt{1+|\xi|^2}}\right|^2 + \left|s - g_j(x) - \frac{R}{\sqrt{1+|\xi|^2}}\right|^2 \\ &\le |y - x|^2 + \frac{2R}{\sqrt{1+|\xi|^2}}\langle \xi, y - x\rangle + \frac{R^2|\xi|^2}{1+|\xi|^2} \\ &\quad + \left(\frac{R}{\sqrt{1+|\xi|^2}} - \langle \xi, y - x\rangle - \frac{\eta}{2}|y - x|^2\right)^2. \end{aligned} \tag{9}$$

Expanding the square in (9) and noting by the Cauchy-Schwarz inequality $\langle \xi, y-x\rangle^2 \leq |\xi|^2|y-x|^2$, we have:

$$
\begin{aligned}
\big|(y,s)-&(x,g_j(x)) + Rn_\xi\big|^2 \\
&\leq |y-x|^2 + R^2\frac{1+|\xi|^2}{1+|\xi|^2} + \frac{\eta^2}{4}|y-x|^4 + \langle \xi, y-x\rangle^2 \\
&\quad + \eta\langle \xi, y-x\rangle |y-x|^2 - \frac{R\eta|y-x|^2}{\sqrt{1+|\xi|^2}} \\
&\leq R^2 + |y-x|^2\left(1 + \frac{\eta^2}{4}|x-y|^2 + |\xi|^2 + \eta\langle\xi, y-x\rangle - \frac{R\eta}{\sqrt{1+|\xi|^2}}\right). \qquad (10)
\end{aligned}
$$

To see the quantity in parentheses is bounded by 0, apply the Cauchy-Schwarz inequality to the inner product, the bounds $|\xi| \leq L$ and $|x-y| \leq r_j \leq r$, and the lower bound on R in (6) to estimate:

$$
1 + \frac{\eta^2}{4}|x-y|^2 + |\xi|^2 + \eta\langle \xi, y-x\rangle \leq 1 + \frac{\eta^2}{4}r^2 + L^2 + \eta L r \overset{(6)}{\leq} \frac{R\eta}{\sqrt{1+|\xi|^2}}.
$$

Subsituting this into (10), we see $\big|(y,s)-(x,g_j(x)) + Rn_\xi\big|^2 \leq R^2$, as desired.

Case 2. Now suppose $|y-x| \geq r_j$. Using (1) and $|n_\xi| = 1$, we estimate

$$
\begin{aligned}
\big|(y,s)-(x,g_j(x)) + Rn_\xi\big|^2 &= R^2 + |(y,s)-(x,g_j(x))|^2 + 2R\frac{\langle y-x,\xi\rangle + g_j(x) - s}{\sqrt{1+|\xi|^2}} \\
&\leq R^2 + \operatorname{diam}(W)^2 + 2R\frac{\langle y-x,\xi\rangle + g_j(x) - s}{\sqrt{1+|\xi|^2}} \\
&\leq R^2 + \frac{2R}{\sqrt{1+|\xi|^2}}\left(\frac{r_j^2\eta}{2} + \langle y-x,\xi\rangle + g_j(x) - s\right), \qquad (11)
\end{aligned}
$$

where the last inequality follows because the choice of R in (6) and $|\xi| \leq L$ ensure $\operatorname{diam}(W)^2 \leq r_j^2\eta\frac{R}{\sqrt{1+|\xi|^2}}$. Since $(y,s) \in W$ and $|y-x| \geq r_j$, we have

$$
s \geq g_j(x) + \langle \xi, y-x\rangle + \frac{\eta r_j^2}{2}. \qquad (12)
$$

To see this, let $u = r_j(y-x)/|y-x|$, so $x+u \in B^{n-1}(0,2r_j)$ and convexity of W along with Lemma 2.1 imply

$$
\frac{s-g_j(x)}{|y-x|} \geq \frac{g_j(x+u)-g_j(x)}{r_j} \geq \frac{\langle \xi, u\rangle + \frac{\eta}{2}|u|^2}{r_j} = \left\langle \xi, \frac{y-x}{|y-x|}\right\rangle + \frac{\eta}{2}r_j
$$

and (12) follows, because $|y-x| \geq r_j$.

Substituting (12) into (11), we conclude $\big|(y,s)-(x,g_j(x)) + Rn_\xi\big|^2 \leq R^2$. The proof of (7) and thus $(a) \Rightarrow (b)$ is complete.

$(b) \Rightarrow (c)$: By assumption, there is $R > 0$ such that for every $y \in \partial W$ there exists $x_y \in \mathbb{R}^n$ so that $y \in \partial B(x_y, R)$ and

$$
W \subseteq \overline{B}(x_y, R). \qquad (13)
$$

This implies that

$$\zeta(y) := \frac{1}{|y - x_y|}(y - x_y) \in N_W(y) \cap \mathbb{S}^{n-1}$$

for every $y \in \partial W$, and from Lemma 2.3 we deduce that

$$\bigcap_{y\in\partial W} \overline{B}(x_y, R) = \bigcap_{y\in\partial W} \overline{B}(y - R\zeta(y), R) \subseteq \bigcap_{y\in\partial W} \{x : \langle \zeta(y), x - y\rangle \le 0\} = W.$$

In combination with (13), we have $\bigcap_{y\in\partial W} \overline{B}(x_y, R) = W$.

$(c) \Rightarrow (d)$: For some nonempty set $\mathcal{A} \subset \mathbb{R}^n$, we may write $W = \bigcap_{a\in\mathcal{A}} B_a$, where $B_a := \overline{B}(a, R)$, $R > 0$. Since $0 \in \mathrm{int}(W)$ we have $|a| < R$ for all $a \in \mathcal{A}$, and in fact there exists $r > 0$ such that

$$\overline{B}(0, r) \subseteq W \subseteq B_a \subseteq \overline{B}(0, 2R),$$

which implies that

$$|a| \le R - r \text{ for all } a \in \mathcal{A}, \tag{14}$$

and also that

$$\frac{1}{2R}|x| \le \mu_a(x) \le \frac{1}{r}|x| \text{ for all } x \in \mathbb{R}^n, \tag{15}$$

where, for any $a \in \mathcal{A}$,

$$\mu_a(x) := \inf\left\{\lambda > 0 : \frac{x}{\lambda} \in B_a\right\}$$

is the Minkowski functional of the ball B_a (with respect to the origin, not necessarily the center a of B_a). Since $W = \bigcap_{a\in\mathcal{A}} B_a$, we have for $x \in \mathbb{R}^n$,

$$\mu(x) = \inf\left\{\lambda > 0 : \frac{x}{\lambda} \in \bigcap_{a\in A} B_a\right\} \le \sup_{a\in A} \inf\left\{\lambda > 0 : \frac{x}{\lambda} \in B_a\right\} = \sup_{a\in A} \mu_a(x)$$

and $W \subset B_a$ implies $\mu_a(x) \le \mu(x)$ for all $a \in A$. Thus, for $x \in \mathbb{R}^n$, $\mu(x) = \sup_{a\in\mathcal{A}} \mu_a(x)$, and $\mu^2 : \mathbb{R}^n \to [0, \infty)$ satisfies

$$\mu^2(x) = \sup_{a\in\mathcal{A}} {\mu_a}^2(x).$$

Because the supremum of a family of η-strongly convex functions is η-strongly convex, to prove μ^2 is strongly convex, we need only show that there exists $\eta > 0$ such that μ_a^2 is η-strongly convex for all $a \in \mathcal{A}$.

A straightforward calculation yields

$$\mu_a(x) = \frac{-\langle x, a\rangle + \sqrt{\langle x, a\rangle^2 + k_a|x|^2}}{k_a},$$

where

$$k_a := R^2 - |a|^2 > 0.$$

Differentiating $\mu_a(x)$, for every $x \in \mathbb{R}^n \setminus \{0\}$ we obtain

$$\nabla\mu_a(x) = \frac{1}{k_a}\left[-a + \frac{\langle x, a\rangle a + k_a x}{(\langle x, a\rangle^2 + k_a|x|^2)^{1/2}}\right] = \lambda_a(x)(x - \mu_a(x)a),$$

where $\lambda_a : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is defined by

$$\lambda_a(x) := \frac{1}{\sqrt{\langle x, a\rangle^2 + k_a|x|^2}}.$$

Then we have

$$D^2\mu_a(x) = \nabla\lambda_a(x) \otimes (x - \mu_a(x)a) + \lambda_a(x)\left(I - a \otimes \nabla\mu_a(x)\right),$$

where I denotes the identity operator. Since $D^2\mu_a(x)$ is symmetric, we also have

$$D^2\mu_a(x) = (x - \mu_a(x)a) \otimes \nabla\lambda_a(x) + \lambda_a(x)\left(I - \nabla\mu_a(x) \otimes a\right).$$

Thus,

$$D^2\mu_a^2(x) = 2\nabla\mu_a(x) \otimes \nabla\mu_a(x) + 2\mu_a(x)D^2\mu_a(x).$$

Recalling that $\nabla\mu_a(x) = \lambda_a(x)(x - \mu_a(x)a)$, the above expressions tell us that

$$v_0 := \frac{\nabla\mu_a(x)}{|\nabla\mu_a(x)|} = \frac{x - \mu_a(x)a}{|x - \mu_a(x)a|}$$

is an eigenvector of both $D^2\mu_a(x)$ and $D^2\mu_a^2(x)$ (here we are using the easy facts that, for any vectors $b, c \in \mathbb{R}^n$, we have $b \otimes c(b) = \langle b, c\rangle b$; hence b is an eigenvector of $b \otimes c$, and that any vector is an eigenvector of the identity). Since μ is convex we have $D^2\mu_a \geq 0$, so we estimate

$$\begin{aligned} v_0^T D^2\mu_a^2(x)v_0 &\geq v_0^T\big(2\nabla\mu_a(x) \otimes \nabla\mu_a(x)\big)v_0 \\ &= 2\langle\nabla\mu_a(x), v_0\rangle^2 = 2|\nabla\mu_a(x)|^2 \geq 2\left(\frac{\mu_a(x)}{|x|}\right)^2 \geq \frac{1}{2R^2}, \end{aligned}$$

where in the two last inequalities we used convexity of μ_a and (15). On the other hand, for every $v \in \mathbb{S}^{n-1}$ with $\langle v_0, v\rangle = 0$ we also have $\langle x - \mu_a(x)a, v\rangle = 0 = \langle\nabla\mu_a(x), v\rangle$, hence

$$\begin{aligned} v^T D^2\mu_a^2(x)v &= v^T\big(2\mu_a(x)\lambda_a(x)I\big)v \\ &= \frac{2\mu_a(x)}{\sqrt{\langle x, a\rangle^2 + k_a|x|^2}} \geq \frac{2\mu_a(x)}{\sqrt{|a|^2|x|^2 + k_a|x|^2}} = \frac{2\mu_a(x)}{R|x|} \geq \frac{1}{R^2}. \end{aligned}$$

Let α_0 be the eigenvalue associated to the eigenvector v_0, and let $\alpha_1, ..., \alpha_{n-1}$ be the rest of eigenvalues of $D^2\mu_a^2(x)$ (possibly repeated). Because $D^2\mu_a^2(x)$ is symmetric and v_0 is an eigenvector of norm 1, we can find eigenvectors $v_1, ..., v_{n-1}$ of $D^2\mu_a^2(x)$ with associated eigenvalues $\alpha_1, ..., \alpha_{n-1}$ so that $\{v_0, v_1, ..., v_{n-1}\}$ is an orthonormal basis of $\mathbb{R}^n$. The last two inequalities imply

$$\alpha_j = v_j^T D^2\mu_a^2(x)v_j \geq \frac{1}{2R^2}$$

for all $j = 0, 1, ..., n$, $x \neq 0$. We deduce that the minimum eigenvalue of $D^2\mu_a^2(x)$ is greater than or equal to $\frac{1}{2R^2}$, and therefore μ_a^2 is $\frac{1}{2R^2}$-strongly convex on any convex subset of $\mathbb{R}^n \setminus \{0\}$. Finally, if $x = 0$, since μ_a is positive homogeneous, μ_a^2 is 2-homogeneous, and we compute

$$v^T D^2\mu_a^2(x)v = \frac{d^2}{dt^2}\mu_a(tv)^2|_{t=0} = \frac{d^2}{dt^2}t^2\mu_a(v)^2|_{t=0} = 2\mu_a(v)^2 \geq \frac{|v|^2}{2R^2} = \frac{1}{2R^2}$$

for every $v \in \mathbb{S}^{n-1}$. We conclude μ_a^2 is $\frac{1}{2R^2}$-strongly convex on all of $\mathbb{R}^n$, and thus $\mu^2(x) = \sup_{a\in A}\mu_a^2(x)$ is $\frac{1}{2R^2}$-strongly convex on $\mathbb{R}^n$.

$(d) \Rightarrow (e)$ is trivial (let $g = \mu_a^2$, $t = 1$).

$(e) \Rightarrow (b)$: We assume the function $g : \mathbb{R}^n \to \mathbb{R}$ is locally strongly convex and coercive (and $W = g^{-1}((-\infty, t])$ for some $t > \min_{x\in\mathbb{R}^n} g(x)$), implying g attains a minimum at a unique $x_0 \in \mathbb{R}^n$. We show for $c > g(x_0)$, there exists $R(c) > 0$ such that the level set $K_c := g^{-1}((-\infty, c])$ satisfies (b), and, therefore, $W = g^{-1}((-\infty, t])$ satisfies (b) with $R = R(t)$.

Notice the set K_c is compact because g is coercive, and therefore g is L-Lipschitz for some $L > 0$ on an open set $U \supset K_c$. Since g is locally strongly convex, up to taking a smaller U we may assume that g is η-strongly convex on U. Let $R := \frac{L}{\eta}$; fix $y \in \partial K_c$ and $\zeta_y \in \partial g(y)$; then $g(y) = c$. Because g is coercive, x_0 must lie in the interior of K_c. Since $y \neq x_0$, we have $\zeta_y \neq 0$, and by the strong convexity of g described in (3), for $x \in K_c$,

$$c = g(y) \geq g(x) \geq g(y) + \langle \zeta_y, x - y \rangle + \frac{\eta}{2}|x - y|^2,$$

which implies

$$\left| x - y + \frac{1}{\eta}\zeta_y \right|^2 \leq \frac{|\zeta_y|^2}{\eta^2},$$

showing that

$$K_c \subseteq \overline{B}\left(y - \frac{1}{\eta}\zeta_y, \frac{|\zeta_y|}{\eta} \right) \subseteq \overline{B}\left(y - \frac{R}{|\zeta_y|}\zeta_y, R \right),$$

completing the proof that K_c satisfies (b), and therefore, W satisfies (b) with $R = R(t) = \frac{L}{\eta}$.

(b) $\Rightarrow$ *(a)*: Let $x \in \partial W$. Since $0 \in \operatorname{int}(W)$ there exists $r > 0$ such that $\overline{B}(0, r) \subset \operatorname{int}(W)$. Let T_x be a hyperplane supporting W at x and $\zeta_x \in N_W(x)$ satisfy ζ_x is perpendicular to T_x. Then the ray $\{x - t\zeta_x : t > 0\}$ intersects $\operatorname{int}(W)$, so there are $t_0 > 0$ and $r > 0$ such that $\overline{B}(x - t_0\zeta_x, r) \subset \operatorname{int}(W)$. For every $y \in T_x$ with $|y - x| < r$, the ray $\{y - t\zeta_x : t \geq 0\}$ passes through the ball $B(x - t_0\zeta_x, r)$ and, therefore, intersects ∂W at exactly two points; the first defines a function whose graph coincides with ∂W in a neighborhood of x. Precisely, we define the function $g : T_x \cap B(x, r) \to [0, \infty)$ by $g(y) := \min\{t \geq 0 : y + t\zeta_x \in \partial W\}$. This function is convex because its graph coincides with ∂W on U_x a neighborhood of x, and W is convex. We will show that g is strongly convex, proving (a).

By assumption, there exists $R > 0$ such that for every $y \in U_x \cap \partial W$ there is $v_y \in \mathbb{S}^{n-1}$ so that $W \subseteq \overline{B}(y - Rv_y, R)$. In particular $v_y \in N_W(y) \cap N_{\overline{B}(y - Rv_y, R)}(y)$, and $\partial B(y - Rv_y, R) \cap U_x$ is the graph of a C^∞ convex function $f_y : T_x \cap B(x, r) \to \mathbb{R}$ such that

$$f_y \leq g, \;\; f_y(y) = g(y), \;\; \nabla f_y(y) \in \partial g(y), \;\; \text{and} \;\; v_y = (\xi_y, s_y), \tag{16}$$

where

$$\xi_y := \frac{1}{\sqrt{1 + |\nabla f_y(y)|^2}} \nabla f_y(y), \;\; \text{and} \;\; s_y = \frac{-1}{\sqrt{1 + |\nabla f_y(y)|^2}}.$$

In the coordinates given by the hyperplane T_x and its normal vector $-\zeta_x$, we have

$$f_y(z) = g(y) - Rs_y - \sqrt{R^2 - |z - y + R\xi_y|^2},$$

and a straightforward calculation shows that

$$D^2 f_y(z) = \frac{(z - y + R\xi_y) \otimes (z - y + R\xi_y) + (R^2 - |z - y + R\xi_y|^2)\, I}{(R^2 - |z - y + R\xi_y|^2)^{3/2}},$$

where I is the identity operator. Clearly,

$$w_0 := \frac{1}{|z-y+R\xi_y|}\,(z-y+R\xi_y)$$

is an eigenvector of $D^2 f_y(z)$, and we have

$$w_0^T D^2 f_y(z) w_0 = \frac{R^2}{(R^2-|z-y+R\xi_y|^2)^{3/2}} \geq \frac{1}{R}.$$

On the other hand, for all $w\in\mathbb{S}^{n-1}$ with $\langle w, w_0\rangle = 0$ we have $\langle z-y+R\xi_y, w\rangle = 0$, so

$$w^T D^2 f_y(z) w = \frac{R^2-|z-y+R\xi_y|^2}{(R^2-|z-y+R\xi_y|^2)^{3/2}} = \frac{1}{(R^2-|z-y+R\xi_y|^2)^{1/2}} \geq \frac{1}{R}.$$

Hence

$$\min_{|w|=1} w^T D^2 f_y(z) w \geq \frac{1}{R},$$

and f_y is $\frac{1}{R}$-strongly convex. Now, by (16) and Lemma 2.1 it follows that

$$g(z) \geq f_y(z) \geq f_y(y) + \langle \nabla f_y(y), z-y\rangle + \frac{1}{2R}|z-y|^2 \tag{17}$$

$$= g(y) + \langle \nabla f_y(y), z-y\rangle + \frac{1}{2R}|z-y|^2, \tag{18}$$

so by Remark 2.2 we conclude that g is $\frac{1}{R}$-strongly convex too. The proof is complete. □

A self-evident local variant of the proofs of $(a) \Rightarrow (b)$ and $(b) \Rightarrow (a)$ in Proposition 2.4 shows the following:

Lemma 2.5. *For any (possibly unbounded) convex body $W\subset\mathbb{R}^n$ the following statements are equivalent:*

(1) W is locally strongly convex (in the sense that ∂W is locally, up to a rigid change of coordinates, the graph of a strongly convex function).
(2) For every $x\in\partial W$ there exist an open neighborhood $U_x \ni x$ and a number $R_x>0$ such that, for all $y\in U_x\cap\partial W$ there is $v_y\in\mathbb{S}^{n-1}$ such that $W\cap U_x\subset \overline{B}(y-R_x v_y, R)$.

Moreover, if (in appropriate coordinates) W is the epigraph of a convex function f and one of the conditions is satisfied, then f is locally strongly convex.

For any closed convex set $C\subset\mathbb{R}^n$ the metric projection $\pi_C:\mathbb{R}^n\to C$ (defined, for every $x\in\mathbb{R}^n$, as the unique point $\pi(x)\in C$ such that $\operatorname{dist}(x,\pi(x)) = \operatorname{dist}(x,C)$) is 1-Lipschitz; see [7, (3.1.6)] for a proof. Clearly, $\pi(x)\in\partial C$ if $x\notin \operatorname{int}(C)$. When the boundary ∂C is of class $C^{1,1}$, a bit more is true: the metric projection onto the (not necessarily convex) boundary ∂C is also well defined and Lipschitz on an open neighborhood of ∂C.

For $W\subset\mathbb{R}^n$ satisfying ∂W is of class $C^{1,1}$, let $n_{\partial W}:\partial W\to\mathbb{S}^{n-1}$ be the outward unit normal vector to ∂W. Recall,

$$\operatorname{Lip}(n_{\partial W}) := \sup\left\{\frac{|(n_{\partial W}(x)-n_{\partial W}(y)|}{|x-y|} : x,y\in\partial W, x\neq y\right\}.$$

Lemma 2.6. *Let $W \subset \mathbb{R}^n$ be a closed convex set with nonempty interior such that ∂W is of class $C^{1,1}$. Then the metric projection $\pi : \Omega \to \partial W$ is well defined and 2-Lipschitz, where*

$$\Omega := \left\{ x \in \mathbb{R}^n : d(x, \partial W) < \frac{1}{2\,\mathrm{Lip}(n_{\partial W})} \right\} \cup W^c.$$

Proof. See, for instance, [4, Theorem 2.4], or the references therein. □

We intend to apply the following lemma when $W, V \subset \mathbb{R}^n$ are compact but include the more general result:

Lemma 2.7. *Let W, V be (possibly not bounded) convex bodies such that $W \subset V \subsetneq \mathbb{R}^n$, and $\mathcal{H}^{n-1}(\partial V \setminus \partial W) < \infty$. Then the projection $\pi_W : \mathbb{R}^n \to W$ maps ∂V onto ∂W.*

Proof. We consider two cases:

Case 1. Suppose that ∂V does not contain any lines.[1] We will show for all $x \in \partial W$, there is $z \in \partial V$ such that $\pi_W(z) = x$. Let $\nu \in N_W(x)$. It suffices to show that the ray $R_x := \{x + t\nu : t \geq 0\}$ intersects ∂V at some point z, implying $\pi_W(z) = x$.

Suppose not; then $R_x \cap \partial V = \emptyset$, implying $R_x \subset \operatorname{int} V$. Let $T_x \subset \mathbb{R}^n$ be a supporting hyperplane for W at $x \in \partial W$:

$$T_x := \{x + v : \langle v, \nu \rangle = 0\}$$

Then the open half space $H_x := \{x + v : \langle v, \nu \rangle > 0\}$ satisfies $H_x \cap \partial W = \emptyset$. Because $x \in \mathrm{int}(V)$, there exists $\delta > 0$ such that $B(x, 2\delta) \cap T_x \subset \mathrm{int}(V)$. Further, $(B(x, 2\delta) \cap T_x) \cup R_x \subset \mathrm{int}(V)$ and V is convex, so $C_x \subset \mathrm{int}(V)$, where $C_x := \{p + t\nu : p \in \partial B(x, \delta) \cap T_x,\ t > 0\}$ is the side surface of a half-cylinder. Since ∂V does not contain a line, the set V does not contain a line. Therefore, for $p \in R_x$, $v \in S^{n-1}$ satisfying v is parallel to T_x, the line $L_{p,v} := \{p + tv : t \in \mathbb{R}\}$ must intersect ∂V. Let $A \subset H_x$ be

$$A := \bigcup_{p \in R_x, \langle v, \nu \rangle = 0} L_{p,v} \cap \partial V;$$

Let π be the radial projection of A onto C_x along lines $L_{p,v}$; π is 1-Lipschitz and hence $\mathcal{H}^{n-1}(A) \geq \mathcal{H}^{n-1}(\pi(A)) \geq \mathcal{H}^{n-1}(C_x)/2 = \infty$. Since $H_x \cap \partial W = \emptyset$, we have $A \subset \partial V \setminus \partial W$, implying $\mathcal{H}^{n-1}(\partial V \setminus \partial W) = \infty$, a contradiction.

Case 2. Suppose that ∂V contain at least one line. Because $V \neq \mathbb{R}^n$, V must have a cylindrical structure: up to isometry, $V = V_1 \times E_0$, where V_1 is line-free, convex, and at least 1-dimensional, and E_0 is a linear subspace. By an argument similar to the proof of [3, Proposition 1.10], we deduce that $\partial V = \partial W$ because $\mathcal{H}^{n-1}(\partial V \setminus \partial W) < \infty$. □

Let us conclude our preliminaries with a restatement of [1, Cor. 3.10].

Lemma 2.8. *If $u : \mathbb{R}^n \to \mathbb{R}$ is η-strongly convex, then for every $0 < \widetilde{\eta} < \eta$ and every $\varepsilon > 0$, there is a $\widetilde{\eta}$-strongly convex function $v \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, such that $v \geq u$ and $|\{x \in \mathbb{R}^n : u(x) \neq v(x)\}| < \varepsilon$.*

[1] This case was shown in an argument inside the proof of [2, Theorem 1.6]; we reproduce it here for completeness.

3. Proof of Theorem 1.1.

We are now fully equipped to proceed with the proof of Theorem 1.1. We begin with an auxiliary $C^{1,1}$ version of it.

Lemma 3.1. *Let W be a compact convex body in $\mathbb{R}^n$, and V be an open set containing ∂W. Then for every $\varepsilon > 0$ there exists a compact convex body $W_\varepsilon \subseteq W$ of class $C^{1,1}$ such that $\mathcal{H}^{n-1}(\partial W \triangle \partial W_\varepsilon) < \varepsilon$ and $\partial W_\varepsilon \subset V$. Moreover, if W is a strongly convex body, then W_ε is a strongly convex body as well.*

Proof. Next, we recall and adapt the proof of [3, Corollary 1.7], or [2, Theorem 1.4], to our context, showing the bound on the $(n-1)$-dimensional Hausdorff measure of the symmetric difference $\partial W \triangle \partial W_\varepsilon$, that W_ε is a strongly convex body if W is a strongly convex body, and $\partial W_\varepsilon \subset V$.

We assume that $0 \in \operatorname{int}(W)$; recall the Minkowski functional of W, $\mu : \mathbb{R}^n \to [0, \infty)$ defined by

$$\mu(x) := \inf\{\lambda > 0 : \frac{x}{\lambda} \in W\},$$

satisfies μ is convex and Lipschitz. Let L be the Lipschitz constant of μ. By Lemma 2.8, there exists a convex function $g = g_\varepsilon \in C^{1,1}_{\text{loc}}(\mathbb{R}^n)$ such that

$$|\{x \in 2W \ : \ \mu(x) \neq g(x)\}| < \frac{\varepsilon}{L}.$$

Let $C_{1,2}, A \subset \mathbb{R}^n$ be

$$C_{1,2} := 2W \setminus W = \{x \in \mathbb{R}^n : 1 < \mu(x) \leq 2\}, \text{ and}$$
$$A := \{x \in C_{1,2} : \mu(x) \neq g(x)\}.$$

By the coarea formula for Lipschitz functions (see [6, Section 3.4.2], for instance) we have

$$\varepsilon > L\,|A| \geq \int_A |\nabla \mu(x)|\, dx = \int_1^2 \mathcal{H}^{n-1}\left(A \cap \mu^{-1}(t)\right)\, dt,$$

implying $|\{s \in (1,2] : \mathcal{H}^{n-1}(A \cap \mu^{-1}(s)) > \varepsilon\}| < 1$. Because $g \in C^{1,1}_{\text{loc}}(\mathbb{R}^n)$ is convex and does not attain a minimum in $g^{-1}((1,2])$, we have $|\nabla g(x)| > 0$ for all $x \in C_{1,2}$. Together, these results imply that there exists a regular value of g, $t_0 \in (1,2)$, where

$$\mathcal{H}^{n-1}\left(A \cap \mu^{-1}(t_0)\right) < \varepsilon. \tag{19}$$

Then, we define

$$W_\varepsilon = \frac{1}{t_0} g^{-1}((-\infty, t_0]).$$

Because g is convex and $C^{1,1}_{\text{loc}}$, and t_0 is a regular value of this function, W_ε is a convex body of class $C^{1,1}_{\text{loc}}$ with boundary

$$\partial W_\varepsilon = \frac{1}{t_0} g^{-1}(t_0),$$

implying

$$t_0(\partial W \setminus \partial W_\varepsilon) = A \cap \mu^{-1}(t_0).$$

With inequality (19), this yields

$$\mathcal{H}^{n-1}(\partial W \setminus \partial W_\varepsilon) \le t_0^{n-1}\mathcal{H}^{n-1}\left(\partial W \setminus \partial W_\varepsilon\right) = \mathcal{H}^{n-1}\left(A \cap \mu^{-1}(t_0)\right) < \varepsilon.$$

Since $g \ge \mu$, we have $W_\varepsilon \subset W$. In particular, W_ε is compact and, therefore, of class $C^{1,1}$. Because the metric projection $\pi : \mathbb{R}^n \to W_\varepsilon$, is 1-Lipschitz and maps ∂W onto ∂W_ε, we also have

$$\mathcal{H}^{n-1}\left(\partial W_\varepsilon \setminus \partial W\right) = \mathcal{H}^{n-1}\left(\pi\left(\partial W \setminus \partial W_\varepsilon\right)\right) \le \mathcal{H}^{n-1}\left(\partial W \setminus \partial W_\varepsilon\right) < \varepsilon.$$

Therefore $\mathcal{H}^{n-1}\left(\partial W \triangle \partial W_\varepsilon\right) < 2\varepsilon$.

If we further assume that W is a strongly convex body, then by Proposition 2.4, μ^2 is a strongly convex function, and applying Lemma 2.8, we obtain a strongly convex function $g \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ such that $\mu^2 \le g$, and

$$\left|\{x \in 2W \,:\, \mu^2(x) \neq g(x)\}\right| < \frac{\varepsilon}{L},$$

where $L = \mathrm{Lip}(\mu)$. Thus,

$$\left|\{x \in 2W \,:\, \mu(x) \neq h(x)\}\right| < \frac{\varepsilon}{L},$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is defined by $h(x) := |g(x)|^{1/2}$. Because $|\nabla g(x)| > 0$ for $x \in C_{1,2}$, we have $h \in C^{1,1}(C_{1,2})$. There exists a regular value of h, $t_0 \in (1,2)$ satisfying an analog of (19). Let $W_\varepsilon \subset \mathbb{R}^n$ be

$$W_\varepsilon := \frac{1}{t_0} h^{-1}((-\infty, t_0]);$$

then, $\frac{1}{t_0} g^{-1}(t_0^2) = \frac{1}{t_0} h^{-1}(t_0) = \partial W_\varepsilon$. Because h is coercive, by Proposition 2.4 (e) $\Rightarrow$ (a), we deduce W_ε is a strongly convex body. Further, the inequality $\mu^2 \le g$ implies that $W_\varepsilon \subset W$. The proof that $\mathcal{H}^{n-1}\left(\partial W \triangle \partial W_\varepsilon\right) < 2\varepsilon$ is completed exactly as above.

Finally, given an open set $V \supset \partial W$, we want to show $\partial W_\varepsilon \subset V$ if ε is small enough. Suppose not; then there exists a sequence of $C^{1,1}$ (strongly) convex bodies $(U_k)_{k\in\mathbb{N}}$ such that

$$\mathcal{H}^{n-1}\left(\partial W \triangle \partial U_k\right) < 1/k \text{ and } U_k \subseteq W \text{ for all } k \in \mathbb{N}.$$

Because W is compact, $V \supset \{x \in \mathbb{R}^n : \mathrm{dist}(x, \partial W) \le 2r\}$ for some $r > 0$. Thus, there is sequence $(z_k)_{k\in\mathbb{N}}$ with $z_k \in \partial U_k$ for each $k \in \mathbb{N}$ such that

$$\mathrm{dist}(z_k, \partial W) \ge 2r > 0$$

Since $(z_k)_{k\in\mathbb{N}} \subset W$, up to taking a subsequence, we may assume that $(z_k)_{k\in\mathbb{N}}$ converges to some $z_0 \in W$, and, necessarily, $\mathrm{dist}(z_0, \partial W) \ge 2r > 0$. Hence, there exists $k_0 \in \mathbb{N}$ such that for $k \ge k_0$, we have $B(z_k, r) \subset B(z_0, 2r) \subset W$. Let H_k denote the tangent hyperplane to ∂U_k at z_k, and H_k^- and H_k^+ denote the open halfspaces with common boundary H_k. Suppose $U_k \subset \overline{H_k^-}$; observe that the metric projection $\pi : \partial W \to \partial B(z_k, r)$ is 1-Lipschitz and maps $\partial W \cap H_k^+$ onto $\partial B(z_k, r) \cap H_k^+$. We deduce

$$\begin{aligned}\frac{1}{2}\mathcal{H}^{n-1}(\partial B(0,r)) &= \mathcal{H}^{n-1}\left(\partial B(z_k, r) \cap H_k^+\right) \\ &\le \mathcal{H}^{n-1}\left(\partial W \cap H_k^+\right) \le \mathcal{H}^{n-1}\left(\partial W \triangle \partial U_k\right) \le 1/k\end{aligned}$$

for all $k \ge k_0$, which is absurd. Thus for ε small enough, we must have $\partial W_\varepsilon \subset V$. □

Proof of Theorem 1.1. Let $W \subset \mathbb{R}^n$ be a locally strongly convex body, $\varepsilon > 0$, and the set $V \supset \partial W$ be open. We want to show there exists a C^2 locally strongly convex body $W_{\varepsilon,V}$ such that $\mathcal{H}^{n-1}(\partial W_{\varepsilon,V} \triangle \partial W) < \varepsilon$ and $\partial W_{\varepsilon,V} \subset V$. Moreover, if W is a strongly convex body, then $W_{\varepsilon,V}$ can be chosen to be a strongly convex body as well. We consider two cases:

Case 1. Suppose that W is <u>not</u> bounded. Because W is locally strongly convex, ∂W can be regarded, up to a suitable rotation, as the graph of a convex function $f : U \subseteq \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\lim_{y\in U, |y|\to\infty} f(y) = \infty$ (if U is not bounded) and $\lim_{y\to x} f(y) = \infty$ for every $x \in \partial U$ (if $U \neq \mathbb{R}^{n-1}$); see [5] for instance.[2] According to Lemma 2.5 the function f is locally strongly convex. Hence the result is a straightforward consequence of Theorem 1.2. (Notice (b) of Theorem 1.2 can be used to ensure $\partial W_{\varepsilon,V} \subset V$.)

Case 2. Suppose that W is bounded. Then W is compact and thus a strongly convex body. By Lemma 3.1, there exists a strongly convex body $W_{\varepsilon/2} \subseteq W$ of class $C^{1,1}$ such that $\mathcal{H}^{n-1}(\partial W \triangle \partial W_{\varepsilon/2}) < \varepsilon/2$. We will prove there exists a C^2 strongly convex body $W_{\varepsilon/2,V}$, satisfying $\mathcal{H}^{n-1}(\partial W_{\varepsilon/2} \triangle \partial W_{\varepsilon/2,V}) < \varepsilon/2$. Then because

$$\partial W \triangle \partial W_{\varepsilon/2,V} \subset (\partial W \triangle \partial W_{\varepsilon/2}) \cup (\partial W_{\varepsilon/2} \triangle \partial W_{\varepsilon/2,V}),$$

we deduce

$$\mathcal{H}^{n-1}\left(\partial W \triangle \partial W_{\varepsilon/2,V}\right) \leq \mathcal{H}^{n-1}\left(\partial W \triangle \partial W_{\varepsilon/2}\right) + \mathcal{H}^{n-1}\left(\partial W_{\varepsilon/2} \triangle \partial W_{\varepsilon/2,V}\right) < \varepsilon$$

Hence, from now on, we assume W is a $C^{1,1}$ strongly convex body.

By Lemma 2.6 we know that there exists an open neighborhood Ω of ∂W such that the metric projection $\pi : \Omega \to \partial W$ is well defined and 2-Lipschitz. Without loss of generality we may assume that $V \subset \Omega$ and $0 \in \operatorname{int}(W)$. Let $\mu : \mathbb{R}^n \to [0, \infty)$ be the Minkowski functional of W; recall

$$\mu(x) = \inf\{\lambda \geq 0 \, : \, \frac{1}{\lambda} x \in W\}.$$

The function μ is convex and Lipschitz on $\mathbb{R}^n$, and of class $C^{1,1}$ on $\mathbb{R}^n \setminus B(0, r)$ for every $r > 0$. Let L be the Lipschitz constant of μ, and let $R > 0$ be large enough so that

$$2W \subseteq B(0, R).$$

We may assume our given ε is in $(0, 1/4)$ and small enough so that

$$\mu^{-1}\left([1 - 5\varepsilon, 1 + 5\varepsilon]\right) \subset V \subset \Omega.$$

Applying Lemma 2.4 (a) $\Rightarrow$ (d) to W, we deduce μ^2 is strongly convex on $\mathbb{R}^n$. By Theorem 1.2 there exists a strongly convex function $g \in C^2(\mathbb{R}^n)$ such that

$$\left|\{x \in B(0, R) \, : \, \mu(x)^2 \neq g(x)\}\right| < \frac{\varepsilon^2}{(8L^2 R + 4\varepsilon/R)2^n} \tag{20}$$

and for all $x \in \mathbb{R}^n$,

$$|\mu^2(x) - g(x)| < \varepsilon. \tag{21}$$

Because μ is L-Lipschitz, we have

$$-\varepsilon \leq g(x) \leq \mu(x)^2 + \varepsilon \leq 4(LR)^2 + \varepsilon \quad (x \in B(0, 2R)).$$

[2]We warn the reader that what in this paper we call a locally strongly convex function is called a strongly convex function in [5].

Applying [1, Lemma 3.3], we deduce

$$\operatorname{Lip}\left(g_{|_{B(0,R)}}\right) \leq \frac{4(LR)^2 + 2\varepsilon}{R}.$$

Let $h : \mathbb{R}^n \to \mathbb{R}$ be defined as $h(x) := |g(x)|^{1/2}$; then for $x \in h^{-1}([1, 1+\varepsilon]) \subset g^{-1}([1, 1+\varepsilon])$,

$$\begin{aligned} |\nabla h(x)| &= \frac{|\nabla g(x)|}{2|g(x)|^{1/2}} \\ &\leq 2L^2 R + \varepsilon/R. \end{aligned}$$

Further from (21), for $x \in h^{-1}([1, 1+\varepsilon])$, we have

$$\begin{aligned} &h^2(x) - \varepsilon \leq \mu^2(x) \leq h^2(x) + \varepsilon, \text{ implying} \\ &1 - \varepsilon \leq \mu^2(x) \leq 1 + 4\varepsilon, \text{ and thus,} \\ &1 - \varepsilon \leq \mu(x) \leq 1 + 4\varepsilon \quad \big(x \in h^{-1}([1, 1+\varepsilon])\big). \end{aligned}$$

This shows that

$$h^{-1}([1, 1+\varepsilon]) \subset \mu^{-1}([1 - 5\varepsilon, 1 + 5\varepsilon]) \subset V \subset \Omega.$$

Now consider the set

$$A := \{x \in h^{-1}([1, 1+\varepsilon]) : \mu(x)^2 \neq g(x)\} = \{x \in h^{-1}([1, 1+\varepsilon]) : \mu(x) \neq h(x)\}.$$

By the coarea formula for Lipschitz functions (see [6, Theorem 3.10, Section 3.4.2] for instance) we have

$$\frac{\varepsilon^2}{2^{n+2}} > \left(2L^2 R + \varepsilon/R\right)|A| \geq \int_A |\nabla h(x)|\, dx = \int_1^{1+\varepsilon} \mathcal{H}^{n-1}\left(A \cap h^{-1}(t)\right)\, dt.$$

This inequality implies that there exists $t_0 \in (1, 1+\varepsilon)$ such that

$$\mathcal{H}^{n-1}\left(A \cap h^{-1}(t_0)\right) < \varepsilon/2^{n+2},$$

and because g is convex and cannot have a minimum in $g^{-1}((1, 2])$, the number ${t_0}^2$ is a regular value of g. Then, we define

$$W_\varepsilon := \frac{1}{t_0} h^{-1}((-\infty, t_0]).$$

Since $\partial W_\varepsilon = \frac{1}{t_0} h^{-1}(t_0) = \frac{1}{t_0} g^{-1}({t_0}^2)$ is a hypersurface of class C^2, and h is coercive, we apply Proposition 2.4 to deduce that W_ε is a strongly convex body of class C^2, and

$$t_0(\partial W_\varepsilon \setminus \partial W) = A \cap h^{-1}(t_0).$$

This yields

$$\mathcal{H}^{n-1}(\partial W_\varepsilon \setminus \partial W) \leq t_0^{n-1} \mathcal{H}^{n-1}(\partial W_\varepsilon \setminus \partial W) = \mathcal{H}^{n-1}\left(A \cap h^{-1}(t_0)\right) < \varepsilon/2^{n+2}.$$

Further,

$$\partial W_\varepsilon \subset \mu^{-1}([1 - \varepsilon, 1 + \varepsilon])) \subset V \subset \Omega,$$

and, consequently, the metric projection $\pi : \partial W_\varepsilon \to \partial W$ is well-defined and 2-Lipschitz. Hence,

$$\mathcal{H}^{n-1}(\partial W \setminus \partial W_\varepsilon) \leq 2^{n-1} \mathcal{H}^{n-1}(\partial W_\varepsilon \setminus \partial W) < \varepsilon/4.$$

Therefore, we conclude $\mathcal{H}^{n-1}(\partial W \triangle \partial W_\varepsilon) < \varepsilon$. □

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