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Kindergarteners' Strategies and Ideas When Reasoning with Function Tables and Graphs

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Abstract: This study investigated the seeds of algebraic thinking that Kindergarten students use when engaging with function tables and graphs. Through interviews with three Kindergarteners, we explored how they reasoned about functional relationships. Our results illustrate how the Kindergarteners used seeds of algebraic thinking when using function tables and graphs to represent and reason about functional relationships. Building on the seeds of algebraic thinking and Knowledge in Pieces frameworks, we categorized these seeds as either strategies (*classify*, *pair*, and *compare*) or ideas (*seeds of covariation*). Strategy seeds were goal-oriented, and *seeds of covariation* were elicited without any goal and reflected a broader understanding of change between quantities.

Introduction

In this study, we explored how students worked with functional relationships, representing them through tables and graphs. Our work is situated in the broader field of early algebra, as children engaged in algebraic thinking practices such as generalizing, representing, justifying, and reasoning with structure and relationships (Blanton et al., 2011). The fundamental assumption of early algebra is that engaging students in these practices earlier may better prepare them for formal algebra, a course often considered a gatekeeper to higher-level mathematics. Along with other researchers (e.g., Carraher & Schliemann, 2007; Stephens et al., 2017), we view functional thinking as an important entry point into algebra, offering opportunities to generalize and represent relationships between co-varying quantities using language, notation, diagrams, tables, and graphs (Stephens et al., 2017).Since we are interested in students' initial understandings, we draw from the seeds of algebraic thinking framework (Levin & Walkoe, 2022). The authors suggest that early algebraic ideas form from experience and later develop into more formal ones. For instance, children may understand concepts like balance, covariation, and comparison before formal instruction (Walkoe et al., 2022). While prior research has identified seeds as foundational resources, it has not clearly differentiated between those that are strategies and those that are emerging ideas (Levin & Walkoe, 2022). We aim to explore these differences in the context of functional thinking. Specifically, we examine how Kindergarteners use seeds of algebraic thinking when engaging in the practice of representation through tables and function graphs. Our research questions are: *What seeds of algebraic thinking do Kindergarten students use when working with tables and graphs to represent a functional relationship? What are characteristics of these seeds?*

Early algebraic thinking

This study took place in the context of an early algebra classroom teaching experiment (Steffe et al., 2012). We define early algebra using Kaput's (2008) conceptual analysis of algebra and frame it around four fundamental thinking practices: (1) generalizing mathematical relationships and structure; (2) representing generalized relationships in diverse ways; (3) reasoning with generalized relationships; and (4) justifying generalizations (Blanton et al., 2011; Kaput, 2008). In this study, we focus on representing because its role in early algebra learning remains relatively underexplored despite being foundational to algebraic thinking. Representing has been widely acknowledged as essential to mathematics learning (Goldin, 1998; Kaput, 1998), with significant emphasis placed on the importance of representational fluency (Fonger, 2019) and flexibility (Warner et al., 2009). Levin and Walkoe (2022) conceptualized the *seeds of algebraic thinking* framework by drawing on the Knowledge in Pieces framework (diSessa, 2018). These seeds are (1) formed in early experience, (2) small in grain size, and (3) used across different contexts (Levin & Walkoe, 2022). For instance, according to Walkoe et al. (2022), *balance* is a *seed of algebraic thinking* that children develop early on while learning to walk, ride a bike, or play on a teeter-totter. *Balance* is (1) formed in early experience, (2) sub-conceptual, and (3) can be applied (productively and unproductively) across diverse contexts, including algebraic contexts. The authors note that while adding one to both sides of an equation to maintain balance is a productive application of the idea, using the "constant difference" or "additive strategy" to reason about proportions, such as assuming that two cups of lemon juice to three cups of water is the same as three cups of lemon juice to four cups of water because "there is one more cup of water than lemon juice," is an unproductive application of *balance*. Thus, *balance* as a *seed of algebraic thinking* is neither right nor wrong but can be used productively or unproductively depending on the context. In this study, we provide an in-depth illustration of seeds by showcasing different moments in which students used seeds of algebraic thinking to reason with function tables and graphs. The seeds of the algebraic thinking framework and Piaget's (1970) view of knowledge construction as emerging through interaction with the environment have theoretical similarities. For example, the seed of balance can be seen as an extension of Piaget's (1970) observations of children constructing an understanding of equilibrium while interacting with physical systems, such as balancing blocks or navigating a seesaw. In addition, Levin and Walkoe (2022) note that seeds can be formed through repeated experience in the world, which is one way that Piaget describes the mental process of reflective abstraction (Ellis et al., 2024). Prior works have identified seeds of algebraic thinking, such as covariation schemes (Levin, 2018); replacement, inbetweeness, and closing-in (Levin & Walkoe, 2022); balance, boundedness, comparing (Walkoe et al., 2022), and sameness (Kieran & Martínez-Hernández, 2022). Some seeds, such as *closing-in* and *comparing,* can be described as actions, whereas seeds such as *replacement*, *inbetweeness*, and *covariation* can be characterized as ideas. Levin and Walkoe (2022) highlight the existence of this heterogeneity by explaining that a seed such as *closing-in* is a strategy that one uses which elicits the idea of *inbetweeness* where the latter can emerge without a specific goal.

Method

This paper reports on one part of a more extensive study on students' understandings and uses of representations. The study took place at an elementary school in the Northeastern United States in a classroom of 17 Kindergarten students (ages 5-6). The school serves a demographically diverse population, with 68% minority (non-white) students, 16.3% of the students below the poverty level, and 35% English Language Learners. In the larger study, we taught fourteen weekly lessons: seven on generalized arithmetic and seven on functional thinking. A teacher-researcher led the lessons and received occasional aid from the classroom teacher. Each lesson lasted approximately 30 minutes. We also conducted three individual interviews with three Kindergarten students before, during, and after the lessons. In this paper, we focus on the individual interviews. The seeds we illustrate in the results section are from the second and third interviews, where we asked Kindergarteners to represent the relationship $y = 2x$ using function tables and graphs. In the second interview, we asked the Kindergarteners about the relationship between the number of birds and bird wings. We gave them a preconstructed table and then asked them to interpret the information. We asked them about the different parts of the table, such as the headings, what each number represented, and the meaning of each column and row. Later, we asked the Kindergarteners to explore the relationship by having them identify and describe patterns, consider if these patterns would always hold, and use the table to examine how changes in one quantity affected the other. We began the third interview by asking the Kindergarteners to work with the same table as the second interview, and then we presented a graph of the relationship. We wanted the Kindergarteners to interpret the points that showed the number of birds and bird wings in a Cartesian graph. Two team members reviewed the interview transcripts alongside the video, checking for accuracy and adding information about gestures and notations made by the students. After multiple reviews by two team members, we identified *seeds of algebraic thinking* when we observed evidence of students' utterances and written work that were characteristic of the three attributes of seeds (i.e., (1) formed in early experience, (2) small in grain size, and (3) applicable in different contexts). We inferred that the student used a seed based on what they did (i.e., what they wrote or gestured) and said and asked ourselves, "Is this way of thinking formed early in experience, small in grain size, and can it be applied in other contexts? For example, we inferred the student was using a seed of covariation when they described a relationship between the number of birds and the number of bird wings, "as they get smaller, they get bigger." The team collaboratively and iteratively reviewed the interview transcripts, until no new instances of *seeds of algebraic thinking* were identified. Some of our observations aligned with existing literature on seeds that students use in algebraic contexts and students' understandings of mathematical representations. If that was the case, we used the same language used by those researchers to be explicit about the similarities between our observations and prior research. Otherwise, we described our findings using language that best captured the characteristics of the seed. To distinguish between ideas and strategies, we examined whether the seed was task-oriented and classified it as a strategy or represented a broader conceptual understanding and classified it as an idea. We coded students' utterances with the seed's name when we identified a seed. Once we coded the interview transcripts, we reviewed the coded utterances. We then selected those that best exemplified a student using a seed of algebraic thinking to reason with tables and graphs.

Results

We identified four types of seeds of algebraic thinking in our analysis: *compare*, *pair*, *classify*, and *seeds of covariation.* Compare, pair, and classify are strategies formed in early everyday experiences. These strategies helped students make sense of quantities and relationships in tables and graphs. In contrast, covariation is an idea, an intuitive understanding that changes in one quantity correspond to changes in another, such as recognizing that more birds mean more wings. Each seed was observed in table and graph contexts, offering insight into how Kindergarteners used them to interpret and represent functional relationships. In the following sections, we present some examples to illustrate how three students, Zoe, Alice, and Liam, used the different seeds of algebraic thinking. These seeds were not used in isolation; we observed Kindergarteners use more than one seed in any given task, but to clarify, we describe each seed separately. While we observed students using these seeds in both tables and graphs, we do not illustrate every instance in this paper due to space limitations.

Compare

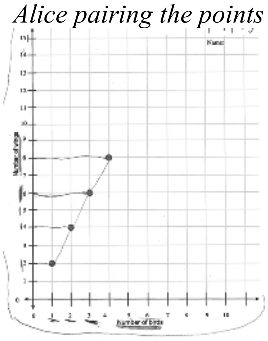
Students *compared* when determining whether two or more values were greater than, less than, or equal to another value. This strategy revealed the Kindergarteners' ability to identify differences or similarities and establish relationships between quantities. For example, when the interviewer asked Zoe about what the different parts of the graphs meant in the third interview, Zoe suggested *comparing* the two axes, "This one (points to the y-axis) has a bigger number than the last one. This one (points to the x-axis) has a little number in the last one." Here, Zoe *compared* to evaluate the relative sizes of the values along the *y*-axis and *x*-axis. By *comparing* the "bigger" and "little" numbers, we infer that Zoe began to understand that each axis accounted for a different quantity.

Pair

We observed students *pairing* when they connected or grouped two related items, such as numerical values or corresponding elements in tables and graphs. For example, when Alice was asked what the points in the graph represented, she said that they matched the numbers in the *y*-axis while making lines that connected the point to the number (see Figure 1), "This dot (1, 2) is matching to the two. This one (2, 4) is matching to the four. This one (3, 6) is matching to the six. This one (4, 8) is matching to the eight." In this case, Alice was *pairing* each point on the graph with a corresponding value on the *y*-axis. For her, the points are not isolated data markers but are understood as connected to numerical values on the *y*-axis. By drawing lines from each point to the corresponding number on the *y*-axis, Alice visually reinforced the connection, which she used to interpret the graph's points.

Figure 1

Alice pairing the points with the numbers in the y-axis



Classify

Students classified when they were working with the function table. For example, in the second interview, Alice classified the quantities in the table into two groups: birds and bird wings. When asked, "What's it showing us?" Alice responded, "It is showing us one, two; two, four; three, six; four, eight." The interviewer followed up, "One what?" to which Alice replied, "One bird. Two bird. Another two birds. Four birds." The interviewer then pointed to the label "number of birds" and the column for "number of bird wings" and asked, "What do you think these numbers show us?" Alice answered, "How many wings there are." When asked whether the numbers in the bird wings column were wings or birds, Alice confidently replied, "Wings." The interviewer then asked, "Then what is this now? These numbers?" while pointing to the birds column. Alice responded, "Birds." To clarify, the interviewer said, "Birds. So the left side is birds and the right side is wings?" Alice corrected, "Actually I said this one is birds" (pointing to the wings column) "and this one is wings" (pointing to the birds column). Even though Alice did not correctly identify which numbers referred to which group, she still classified to distinguish between two groups. This example shows how the table's labels prompted Alice to classify the quantities. The table's structure, with distinct columns for birds and wings, provided a visual prompt for Alice to classify according to their respective categories.

Seeds of covariation

Seeds of covariation were elicited when we asked the students to articulate the relationship between the number of birds and bird wings. For example, in the third interview, we saw Liam elicit a seed of covariation when he noticed that the number of bird wings was increasing while working with the graph. Referring to the point (1, 2), Liam remarked, "This means little goes to bigger." When prompted to explain, he continued, "Did you know this little number—it makes a bigger number?" He elaborated by pointing to the number 2 on the y-axis and said, "Like, this gets a bigger number." When asked if the numbers on the y-axis were getting bigger, Liam confirmed, "Yeah." His observations in the graph context were tied to one variable, and his comment "little goes to bigger" explicitly pointed out how the bird wings increased with each point.

Discussion and conclusion

We observed students use five seeds of algebraic thinking, which we categorize into two types: strategies and ideas. These seeds align with Piaget's (1970) framing of knowledge as constructed through action and interaction with the environment. Seeds such as *classify*, *pair*, and *compare* closely resemble core concepts and descriptions in Piaget's theory of development. For instance, classifying aligns with Piaget's (1970) observations of children sorting objects based on shared attributes. Regarding strategies, we observed that Kindergarteners used the *classify*, *pair*, and *compare* seeds to answer questions regarding both tables and function graphs. Our results indicate that strategy seeds such as classify, pair, and compare are foundational, goal-oriented actions that young learners employ to navigate early algebra tasks that involve the practice of representing (Blanton et al., 2011). These seeds are characterized by being goal-oriented because students use them to complete immediate objectives within the given context. For instance, when *pairing*, students connected values across representations. In one example, Alice paired points on a graph with their corresponding *y*-axis values by drawing lines, stating that each point "matched" to a specific number on the *y*-axis. This action allowed her to focus on linking elements directly related to the task at hand, supporting her understanding of the representation through a goal-oriented approach. We believe this may be a seed associated with a multiplicative object; a concept initially introduced by Piaget (1970) and then later developed in the context of functional relationships (Thompson & Carlson, 2017). While strategy seeds like *pair* are formed through early experiences and can be flexibly applied across various contexts (Levin & Walkoe, 2022), their goal-oriented nature lies in how students activate them to address a particular task's demands. In our study, strategy seeds enabled students to engage with representations. This goal-oriented application is a key characteristic of strategy seeds, highlighting their role in supporting students' problem-solving processes within specific contexts. The other seeds we observed were *seeds of covariation* (Levin & Walkoe, 2022). Unlike the strategy seeds we described, the *seeds of covariation* were not goal-oriented actions to complete a specific task. Instead, they represented an idea that the students elicited when engaging with the function tables and graphs while interacting with the interviewer. C*ovariation seeds* are foundational ideas that can be connected to more formal covariational reasoning (Carlson et al., 2002) and functional thinking (Stephens et al., 2017). Our study explored the seeds of algebraic thinking that Kindergarteners used when working with tables and graphs to represent a functional relationship. In response to our first research question, we identified that Kindergarten students employ both strategies (i.e., *classify*, *pair*, and *compare*) and ideas (i.e., "*y becomes bigger*"*)*. Regarding the nature of the seeds, strategy seeds allowed students to navigate specific tasks, while *seeds of covariation* were elicited without a specific goal. Strategy seeds were often task-specific, helping students address immediate objectives. In contrast, the *seeds of covariation* reflected an idea that transcended the immediate goal of solving the task (e.g., interpreting a row in a table). In future studies, we plan to explore students' use of *compare*, *classify*, *pair* seeds and *seeds of covariation*, including the design of different tasks which would elicit the same seeds with varying contexts. Specifically, we might relate students' use of *compare*, *classify*, and *pair to* their foundational ideas of covariation, similar to how Levin and Walkoe (2022) describe *closing-in* evoking *boundedness*. Examining the interplay between strategies and ideas and better understanding the role of context could deepen our understanding of how young learners' early algebraic reasoning develops.

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