

THE TWINING CHARACTER FORMULA FOR REDUCTIVE GROUPS

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Abstract

Let $\widehat{G}$ be a connected reductive group over an algebraically closed field with a pinning-preserving outer automorphism σ . Jantzen’s twining character formula relates the trace of the action of σ on a highest-weight representation V_μ of $\widehat{G}$ to the character of a corresponding highest-weight representation $(V_\sigma)_\mu$ of a related group $\widehat{G^{\sigma, \circ}}$. This paper extends the methods of Hong’s geometric proof for the case $\widehat{G}$ is adjoint, to prove that the formula holds for all connected reductive groups, and examines the role of additional hypotheses. In the final section, it is explained how these results can be used to draw conclusions about quasi-split groups over a non-Archimedean local field. This paper thus provides a more general geometric proof of the Jantzen twining character formula and provides some apparently new results of independent interest along the way.

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Jantzen’s twining character formula is a twisted version of the Weyl character formula. Given a pinning-preserving outer automorphism σ on a connected, reductive group scheme $\widehat{G}$ over an algebraically closed field, the formula describes the twisted character of σ on a highest-weight representation of $\widehat{G}$ in purely combinatorial terms, and can be calculated using the σ -action on the root datum of $\widehat{G}$. It was first proved by Jantzen, [9], with alternative proofs provided by [11], [8], and [4].

Most of the above proofs share an assumption that the group $\widehat{G}$ is connected, semisimple, and adjoint, and some impose additional hypotheses. However, the proof of [4] holds for connected, reductive groups, following some cohomology calculations in [13]. In this paper I will make the same assumptions: that $\widehat{G}$ is a connected, and reductive group.

I will follow quite closely the geometric proof of Hong [8]. I will outline the structure, then reproduce the proof in a fairly self-contained way to make clear where the stronger hypotheses might be convenient—and why they are unnecessary. For one thing, Hong’s proof applies equally well as written whether $\widehat{G}$ is assumed to be adjoint or simply connected. For the more general case, Proposition 3.4 will be useful. Although Proposition 3.4 is not necessary to prove Theorem 0.1, as shown in [4], I believe it is interesting in its own right and have not seen it elsewhere in the literature.

Let $\widehat{G}$ be a connected, reductive group over an algebraically closed field K of characteristic 0, and fix a root datum of $\widehat{G}$. In particular, fix a maximal torus and Borel $\widehat{T} \subset \widehat{B} \subset \widehat{G}$. Let G be the complex group with dual root datum, and with a corresponding choice of maximal torus and Borel $T \subset B \subset G$. Let σ be an automorphism of $\widehat{G}$ preserving the root datum and a pinning, and consider its induced action on G , which preserves the dual root datum and a pinning. Let G^σ be the fixed-point subgroup of G , and let $G^{\sigma,\circ}$ be the neutral component of G^σ . Then $G^{\sigma,\circ}$ is a connected, reductive group [17] (see also [6]), and a closed subgroup of G , with maximal torus and Borel $T^{\sigma,\circ} \subset B^{\sigma,\circ} \subset G^{\sigma,\circ}$. The cocharacter lattice $X_*(T^{\sigma,\circ})$ is a subgroup of $X_*(T)$, and other components of the root datum of $G^{\sigma,\circ}$ can also be determined combinatorially. Let $\widehat{G^{\sigma,\circ}}$ be the dual of $G^{\sigma,\circ}$ over K .

Note that σ acts on the lattice of cocharacters of T (i.e. characters of $\widehat{T}$). If μ is any σ -invariant dominant cocharacter of G , then we are interested in the action of σ on the irreducible highest-weight representation V_μ of $\widehat{G}$, as well as the action of σ on weight spaces $V_\mu(\lambda)$, where λ is a nonzero weight of V_μ . The set of such weights is denoted $Wt(\mu)$. Up to a scalar, there is a unique vector space automorphism $\sigma : V_\mu \rightarrow V_\mu$ commuting with the action of $\widehat{G}$ on V_μ . We can normalize this automorphism by assuming σ not only stabilizes the highest-weight line $V_\mu(\mu)$, but fixes it pointwise, uniquely determining an action of σ . The irreducible representation of $\widehat{G^{\sigma,\circ}}$ of highest weight μ is denoted $(V_\sigma)_\mu$. Then we have the following theorem relating V_μ and $(V_\sigma)_\mu$.

Theorem 0.1 (Jantzen's twining character formula). *Let $\widehat{G}$, G , and σ be as above. Let μ be a σ -invariant dominant character in $X^*(\widehat{T})$ and let $\lambda \in Wt(\mu)$ be a σ -invariant weight of V_μ . Then σ preserves $V_\mu(\lambda)$, and we have the following equality:*

$$\mathrm{tr}(\sigma | V_\mu(\lambda)) = \dim((V_\sigma)_\mu(\lambda)). \quad (1)$$

The Weyl character formula for $\widehat{G^{\sigma,\circ}}$ thus implies a twining formula for the twisted character of σ :

$$\sum_{\substack{\lambda \in Wt(\mu) \\ \sigma(\lambda) = \lambda}} \mathrm{tr}(\sigma | V_\mu(\lambda)) e^\lambda = \mathrm{ch}((V_\sigma)_\mu) = \sum_{w \in W^\sigma} w \left(\prod_{\alpha \in N'_\sigma(\Phi)^+} \frac{1}{1 - e^{-\alpha}} \right) e^{w(\mu)}.$$

Here $N'_\sigma(\Phi)$ is a root system explicitly determined by the σ -action on Φ and is the root system of the group $\widehat{G^{\sigma,\circ}}$.

See Section 7 for details on how to determine $N'_\sigma(\Phi)$ using the root datum of $\widehat{G}$. Notation and proofs there are drawn from [7].

There are two apparent justifications for the stronger hypotheses taken in previous proofs of Theorem 0.1. First, unless $\widehat{G}$ is semisimple and either simply connected or

adjoint, G^σ may not be connected. This turns out to be immaterial, as the affine Grassmannian of G^σ (and the category of sheaves on it) “forgets” any disconnectedness of G^σ , cf. Proposition 3.4.

Second, the root lattice of $\widehat{G}$ is a strict sublattice of $X^*(\widehat{T})$ in the case $\widehat{G}$ is not semisimple and adjoint. In this case the affine Grassmannian of G is disconnected, and the dimensions of Schubert varieties vary in differing components. However, it turns out that, due to the use of Anderson’s polytope calculus and normalization to stable AMV cycles, disconnectedness of the affine Grassmannian is also immaterial.

In Section 1, I establish conventions used throughout the paper. In Section 2, I outline the proof of Theorem 0.1, omitting some details, to see how the stronger hypotheses are used in the literature. Sections 3 through 6 comprise a complete proof of Theorem 0.1: Section 3 describes the σ -action on subvarieties of the affine Grassmannian of G , relating them to corresponding varieties in the affine Grassmannian of Gr_{G^σ} ; Section 4 considers the action of σ on $\mathbf{i}$ -Lusztig strata, establishing a condition for invariance; Section 5 establishes a coweight-preserving bijection between the σ -invariant MV cycles of G and all MV cycles of $G^{\sigma, \circ}$; and Section 6 completes the proof of Theorem 0.1 by showing that σ fixes basis vectors corresponding to σ -invariant MV cycles, implying that the trace of σ is exactly the number of preserved MV cycles. Section 7 deals explicitly with root data and uses Theorem 0.1 to prove a Theorem 7.7, which is stated in [7] without reference to a fully general proof. Finally, Appendix A is a complete list of results from [14] used in this paper.

1 Notation

Here I will establish some notation and conventions. Throughout, $\widehat{G}$ is a connected, reductive group over an algebraically closed field K of characteristic 0. I primarily work with its complex dual group G . If I write “character,” “coweight,” or other similar term, without specifying which group I am referring to, I intend to refer to G .

Fix a maximal torus and Borel $T \subset B \subset G$. We also have the corresponding maximal unipotent subgroup $U \subset B$. Let the set of simple roots be denoted Π and the simple coroots denoted $\Pi^\vee$. Let the set of roots be denoted Φ and the set of coroots denoted $\Phi^\vee$. Let the character lattice be denoted $X^*(T)$ and the cocharacter lattice denoted $X_*(T)$.

Fix also a maximal torus and Borel $\widehat{T} \subset \widehat{B} \subset \widehat{G}$, with corresponding maximal unipotent $\widehat{U}$. Then the character lattice for $\widehat{G}$ is exactly $X^*(\widehat{T}) = X_*(T)$, and the cocharacter lattice is $X_*(\widehat{T}) = X^*(T)$; the set of roots of $\widehat{G}$ is $\Phi^\vee$, and the set of coroots is Φ ; the set of simple roots of $\widehat{G}$ is $\Pi^\vee$, and the set of simple coroots is Π .

Let G^{der} be the derived subgroup of G , and G^{sc} the universal cover of G^{der} . Then G^{sc} and G^{der} have the same set of coroots and simple coroots as G , and there is a natural embedding $X_*(T^{\mathrm{sc}}) \hookrightarrow X_*(T)$, with $X_*(T^{\mathrm{sc}})$ generated by $\Pi^\vee$. Given any two

cocharacters $\mu, \lambda \in X_*(T)$, we say $\lambda \leq \mu$ if and only if $\mu - \lambda \in \mathbb{Z}_{\geq 0}\Pi^\vee$. Let ρ be the half sum of positive roots of G and $\rho^\vee$ be the half sum of positive coroots.

Let W be the Weyl group $N_G(T)/T$. Then W acts on $X^*(T)$ and $X_*(T)$. For each $w \in W$, let $\lambda \leq_w \mu$ if and only if $w^{-1}(\lambda) \leq w^{-1}(\mu)$. Corresponding to the choice of simple roots Π , we have a set of simple reflections $S = \{s_\alpha\}_{\alpha \in \Pi}$ generating W , and (W, S) is a Coxeter system. Then there is a length function ℓ on elements of W ; let $w_0 \in W$ be the longest element.

Fix a pinning of G compatible with T and B , i.e. a collection of root homomorphisms $x_\alpha : \mathbb{G}_a \rightarrow U$ for each simple root $\alpha \in \Pi$. Each root homomorphism x_α also uniquely determines an opposite root homomorphism $y_\alpha : \mathbb{G}_a \rightarrow w_0 U w_0^{-1}$. Fix also pinning $\{x_{\alpha^\vee}\}_{\alpha^\vee \in \Pi^\vee}$ of $\widehat{G}$ compatible with $\widehat{T}$ and $\widehat{B}$.

Let σ be an automorphism of G preserving the pinning $\{x_\alpha\}_{\alpha \in \Pi}$, meaning that σ preserves T and B , and that $\sigma \circ x_\alpha = x_{\sigma(\alpha)}$ for all $\alpha \in \Pi$. Let G^σ be the fixed point subgroup, and $G^{\sigma, o}$ the neutral component of that fixed point subgroup.

Since σ acts on the constituents of the root datum of $\widehat{G}$, there is a unique action of σ on $\widehat{G}$ preserving its root datum and the pinning $\{x_{\alpha^\vee}\}$. Specifically, $\widehat{G}$ is generated by the images of cocharacters generating $X^*(T)$ and by the root and opposite root homomorphisms $x_{\alpha^\vee}$ and $y_{\alpha^\vee}$ for $\alpha^\vee \in \Pi^\vee$. It is thus sufficient to define σ on these images. Let R be a K -algebra, and suppose $g \in \widehat{G}(R)$. If $g = \lambda(t)$ for some $t \in \mathbb{G}_m(R)$ and $\lambda \in X^*(T)$, let $\sigma(g) = \sigma(\lambda)(t)$. And if $g = x_{\alpha^\vee}(u)$ for some $u \in \mathbb{G}_a(R)$, then let $\sigma(g) = x_{\sigma(\alpha^\vee)}(u)$; similarly, if $g = y_{\alpha^\vee}(u)$, let $\sigma(g) = y_{\sigma(\alpha^\vee)}(u)$.

For a complex, smooth, linear algebraic group H we have the loop group, positive loop group, negative loop group, and strictly negative loop group functors from $\mathbb{C}$ -algebras to sets given by $LH : R \mapsto H(R((\varpi)))$, $L^+H : R \mapsto H(R[[\varpi]])$, $L^-H : R \mapsto H(R[\varpi^{-1}]) \subset LH(R)$, and $L^{--}H : R \mapsto \ker(L^-H(R) \rightarrow H(R))$, respectively. The étale sheafification of the quotient functor $LH/L^+H : R \mapsto H(R((\varpi)))/H(R[[\varpi]])$ is known as Gr_H , the affine Grassmannian of H , and is representable by an ind-finite type (strict) ind-scheme over $\mathbb{C}$. The ind-scheme Gr_H is ind-projective if and only if H is reductive (see, for instance, [2] Theorem 4.5.1(iv)). For this reason, it is essential to this proof to assume G is reductive.

For each cocharacter $\nu \in X_*(T)$ we have by definition a homomorphism $\nu : \mathbb{G}_m \rightarrow T \subset G$, as well as a homomorphism $\nu : L\mathbb{G}_m \rightarrow LT$. Let $\varpi^\nu \in LT(\mathbb{C})$ be the image of ϖ under this homomorphism, and let $\varpi^\nu x_0$ be the image of ϖ under the composition

$$L\mathbb{G}_m(\mathbb{C}) \xrightarrow{\nu} LT(\mathbb{C}) \rightarrow LG(\mathbb{C}) \rightarrow \text{Gr}_G(\mathbb{C}),$$

where x_0 is the natural basepoint of $\text{Gr}_G(\mathbb{C})$, corresponding to the trivial coset in $LG(\mathbb{C})/L^+G(\mathbb{C})$.

Given a locally closed, reduced sub-ind-scheme $Y \subset \text{Gr}_G$, let $\overline{Y}$ be the reduced closure. If G is reduced, then $\overline{Y}$ is ind-projective. Gr_G has a Cartan stratification by L^+G -orbits. Given a cocharacter μ , let Gr_G^μ be the L^+G -orbit $\text{Gr}_G^\mu = L^+G \cdot \varpi^\mu x_0$. I refer to

these orbits as Schubert cells, and their closures as Schubert varieties. Schubert cells and Schubert varieties are reduced, finite-type, complex schemes. Typically μ will be taken dominant, since $\mathrm{Gr}_G^\mu = \mathrm{Gr}_G^{w(\mu)}$ for all $w \in W$. If μ is dominant, we have the following closure relations from the Cartan stratification:

$$\overline{\mathrm{Gr}_G^\mu} = \coprod_{\substack{\lambda \in X_*(T)^+ \\ \lambda \leq \mu}} \mathrm{Gr}_G^\lambda.$$

We also have, for each $w \in W$, an Iwasawa stratification. The strata of the Iwasawa stratification are known as semi-infinite cells. In contrast with Schubert varieties, semi-infinite cells and their closures are not representable by schemes. Given w and a cocharacter ν , let S_w^ν be the orbit $S_w^\nu = wLUw^{-1} \cdot \varpi^\nu x_0$. Then we have the following closure relations (see eg [14] Proposition 3.1(a)):

$$\overline{S_w^\nu} = \coprod_{\substack{\eta \in X_*(T) \\ \eta \leq_w \nu}} S_w^\eta$$

From a geometric description of the complex points, we have an intersection criterion (see eg [14] equation (3.5) in the proof of Theorem 3.2). That geometric description is

$$S_w^\nu(\mathbb{C}) = \{x \in \mathrm{Gr}_G(\mathbb{C}) \mid \lim_{s \rightarrow 0} w(\rho^\vee)(s) \cdot x = \varpi^\nu x_0\}, \quad (2)$$

where $w(\rho^\vee) : \mathbb{G}_m(\mathbb{C}) \rightarrow T(\mathbb{C})$ is a homomorphism of complex groups. As a consequence of this description, $S_w^\eta \cap S_{w'}^{\nu'} \neq \emptyset$ only if $\nu \leq_w \eta$ and $\eta \leq_{w'} \nu'$. Indeed, if $p \in (S_w^\eta \cap S_{w'}^{\nu'}) (\mathbb{C})$, then both $\varpi^\eta x_0$ and $\varpi^{\nu'} x_0$ are in the closure of the $T(\mathbb{C})$ -orbit of p . Since both S_w^η and $S_{w'}^{\nu'}$ are T -invariant, that means in particular that $\varpi^\eta x_0 \in \overline{S_{w'}^{\nu'}}(\mathbb{C})$ and $\varpi^{\nu'} x_0 \in \overline{S_w^\eta}(\mathbb{C})$, implying the inequalities.

Given a reduced, irreducible, projective subvariety $X \subset \mathrm{Gr}_G$, the sheaf $IC_X = j_{!*}(\mathbb{C}[\dim X])$ is the unique perverse sheaf restricting to constant coefficients on the non-singular locus of X . In the case $X = \overline{\mathrm{Gr}_G^\mu}$ for a dominant cocharacter μ , this sheaf is L^+G -equivariant and known simply as IC_μ . The category of L^+G -equivariant perverse sheaves on closed subvarieties of Gr_G consists of only direct sums of IC_μ for dominant μ and is referred to as $P_{L^+G}(\mathrm{Gr}_G)$.

2 Outline of proof

Here I will summarize the proof, adapted from [8], of Theorem 0.1. I do this primarily to see that the hypothesis that G is reductive is sufficient. Suppose G is a complex, semisimple, simply connected group, and let σ be a pinning-preserving automorphism of G .

The proof is geometric in nature, relying on the geometry of the affine Grassmannian Gr_G . Of particular importance is the geometric Satake equivalence, which constructs an

explicit and canonical bijection between certain varieties contained in Gr_G , called MV cycles, and basis vectors of highest-weight representations of $\widehat{G}$. For precise statements of the several theorems from [14] I am referring to when I say “the geometric Satake equivalence,” see the Appendix A. The most important results, stated according to conventions from Section 1, are summarized here:

Theorem 2.1. *Let μ be a dominant cocharacter, and let $\lambda \in \mathrm{Wt}(\mu)$.*

- i. $S_{w_0}^\lambda \cap \overline{\mathrm{Gr}_G^\mu}$ is equidimensional, and $\dim(S_{w_0}^\lambda \cap \overline{\mathrm{Gr}_G^\mu}) = \langle \rho, \mu - \lambda \rangle$
- ii. $S_{w_0}^\lambda \cap S_e^\mu$ is equidimensional, and $\dim(S_{w_0}^\lambda \cap S_e^\mu) = \langle \rho, \mu - \lambda \rangle$
- iii. $\mathbb{H}^\bullet(\mathrm{Gr}_G, IC_\mu) = \bigoplus_{\lambda \in \mathrm{Wt}(\mu)} H_c^{-2\langle \rho, \lambda \rangle}(S_{w_0}^\lambda, IC_\mu) = V_\mu$
- iv. $H_c^{-2\langle \rho, \lambda \rangle}(S_{w_0}^\lambda, IC_\mu) = \bigoplus_{A \in \mathrm{Irr}(S_{w_0}^\lambda \cap \overline{\mathrm{Gr}_G^\mu})} K[A] = V_\mu(\lambda)$.
- v. $P_{L+G}(\mathrm{Gr}_G, \mathbb{Z})$ is isomorphic as a tensor category to $\mathrm{Rep}_{\mathbb{Z}}(\widehat{G})$.

The direct sum in statement iv. is indexed by irreducible components A of the variety $S_{w_0}^\lambda \cap \overline{\mathrm{Gr}_G^\mu}$. These irreducible components are the MV cycles. The heart of the proof of Theorem 0.1 is to establish a bijective correspondence between those MV cycles in Gr_G which are invariant under the action of σ and all MV cycles in Gr_{G^σ} .

This is done in three steps. First, the collection of MV cycles is generalized to a larger collection of what are here called AMV cycles (see Section 3 for definition), after Anderson’s polytope calculus [1]. Using Kamnitzer’s indexing of AMV cycles by $\mathbf{i}$ -Lusztig data (see Section 4), a convenient criterion for σ -invariance of an AMV cycle is found, as well as a procedure for finding the datum of a corresponding AMV cycle in Gr_{G^σ} . Second, a criterion of [1] for when an AMV cycle intersects generically with an MV cycle is applied to show that the restriction of this correspondence to MV cycles is also bijective. Finally, the eigenvalues of the σ -action on the $\widehat{G}$ -representation V_μ are examined to ensure that the twisted character of σ is exactly as expected, completing the proof of Theorem 0.1.

Based on the short description above, it is not obvious how the hypothesis that G is semisimple is used. However, much of the literature explicitly makes this and other assumptions.

The two complications introduced by relaxing the hypotheses from semisimple and simply connected to reductive come in the form of two different disconnected spaces. First, if G is not either semisimple or adjoint, then the fixed point subgroup G^σ is not necessarily connected. Using the classification of reductive groups, it is more convenient to work with $G^{\sigma, \circ}$ than G^σ whenever dealing with root data. However, it is easier, and in my opinion more natural, to relate the geometry of $(\mathrm{Gr}_G)^\sigma$ to Gr_{G^σ} than to $\mathrm{Gr}_{G^{\sigma, \circ}}$. Thankfully, the affine Grassmannians of G^σ and $G^{\sigma, \circ}$ are isomorphic for all reductive G

(see Proposition 3.4). So results including Proposition 3.5 and its consequences still go through without issue, as they may be applied to Gr_{G^σ} . But the group whose category of representations is isomorphic to $P_{L+G^\sigma}(\mathrm{Gr}_{G^\sigma})$ is $\widehat{G^{\sigma,\circ}}$, as needed for Theorem 0.1.

The second complication is that $\pi_0(\mathrm{Gr}_G) = \pi_1(G)$. So if G is not semisimple and simply connected, then Gr_G is not connected. However, in the framework of AMV cycles introduced by [1], this is not a complication at all. In fact, both Kamnitzer and Hong work primarily with stable AMV cycles, which are AMV cycles translated by $X_*(T)$ to be contained in $\overline{S_e^0}$, which is itself contained in the neutral component of Gr_G . This does not rule out possible sticking points in passing between MV cycles, AMV cycles, stable AMV cycles and back to MV cycles, but every result solely relating to stable AMV cycles holds automatically for all reductive groups.

And the result necessary to restrict the bijection on the level of AMV cycles to MV cycles is Theorem 3.7, which is proved again without a need for triviality of $\pi_1(G)$. More details are presented in Section 3.

In short, considering reductive groups, rather than semisimple groups—much less simply connected groups—is no more complicated for the proof of Theorem 0.1. Every potential obstacle is either immaterial or easily deflected. In particular, Proposition 3.4 and the preceding lemmas are the only results I had not previously found in the literature.

3 Action of σ on orbits

For the remainder of the paper, suppose G is a connected, reductive, complex group, and σ is a pinning-preserving automorphism on G .

Unlike in the case G is semisimple and simply connected, we cannot count on G^σ to be a connected group in general. This leads us to a choice: should we work with $G^{\sigma,\circ}$, to which the classification of connected, reductive groups applies, or should we work directly with G^σ , which has a simpler description relative to G ? Thanks to the upcoming Proposition 3.4, it is immaterial whether we work with G^σ or $G^{\sigma,\circ}$. I work primarily with G^σ for simplicity, and the final result will hold for $G^{\sigma,\circ}$. I will avoid referring to root datum of G^σ where possible. Note, however, that $\mathrm{Hom}(\mathbb{G}_m, T^\sigma) = \mathrm{Hom}(\mathbb{G}_m, T^{\sigma,\circ})$, so $X_*(T^\sigma) = X_*(T^{\sigma,\circ})$.

First, we will need a few lemmas relating loop groups of algebraic groups and their quotients. While the following lemmas will be applied only to group schemes over $\mathbb{C}$, their proofs hold in greater generality. In this section, the field k is only assumed to have characteristic 0. In particular, no assumption is made about its topology or algebraic closure.

Lemma 3.1. *Let G be an affine group scheme over a field k of characteristic 0, and suppose $H \subset G$ is a smooth normal subgroup with affine quotient G/H . There is a natural isomorphism of functors $L^+G/L^+H \rightarrow L^+(G/H)$, where L^+G/L^+H is the étale quotient.*

In particular, if $H \subset G$ is a normal subgroup and both groups are reductive, then L^+G/L^+H and $L^+(G/H)$ are canonically isomorphic group schemes.

Proof. Let the quotient map be denoted $q_0 : G \rightarrow G/H$. Note that $L^+G = \varprojlim G^{(n)}$, where $G^{(n)}$ is the n th jet group $R \mapsto G(R[\varpi]/(\varpi^{n+1}))$. Similarly, $L^+(G/H) = \varprojlim (G/H)^{(n)}$. Hence the map $q : L^+G \rightarrow L^+(G/H)$ corresponds to the inverse system of morphisms

$$\begin{array}{ccccccc} \cdots & \longrightarrow & G^{(n)} & \xrightarrow{i_n} & G^{(n-1)} & \xrightarrow{i_{n-1}} & \cdots \longrightarrow G^{(1)} \xrightarrow{i_1} G \\ & & \downarrow q_n & & \downarrow q_{n-1} & & \downarrow q_1 & \downarrow q_0 \\ \cdots & \longrightarrow & (G/H)^{(n)} & \xrightarrow{j_n} & (G/H)^{(n-1)} & \xrightarrow{j_{n-1}} & \cdots \longrightarrow (G/H)^{(1)} \xrightarrow{j_1} G/H \end{array}$$

The first step of the proof of surjectivity will proceed inductively. For each $n \geq 1$, I will use surjectivity of q_{n-1} and formal smoothness of q_0 to prove that q_n is surjective. Then I will use the surjectivity of each q_n to show that q is surjective, with kernel L^+H .

Suppose q_{n-1} is surjective, and let $g_n \in (G/H)^{(n)}(R)$, for a k -algebra R , with image $g_{n-1} \in (G/H)^{(n-1)}(R)$. By surjectivity of q_{n-1} , there is a lift $\tilde{g}_{n-1} \in G^{(n-1)}(S)$ lying over g_{n-1} , where $R \rightarrow S$ is an étale k -algebra homomorphism. Then by formal smoothness of q_0 , there is a simultaneous lift in S -points $\tilde{g}_n \in G^{(n)}(S)$ of both g_n and $\tilde{g}_{n-1}$. Indeed, g_n corresponds to a morphism $\text{Spec}(R[\varpi]/(\varpi^{n+1})) \rightarrow G/H$ (and by precomposition, to a morphism $\text{Spec}(S[\varpi]/(\varpi^{n+1})) \rightarrow G/H$), and $\tilde{g}_{n-1}$ corresponds to a morphism $\text{Spec}(S[\varpi]/(\varpi^n)) \rightarrow G$ such that $q_0 \circ \tilde{g}_{n-1}$ is equal to $j_n(g_n)$ as a morphism $\text{Spec}(S[\varpi]/(\varpi^n)) \rightarrow G/H$. Then by the infinitesimal lifting property of formally smooth morphisms, there is a lift $\tilde{g}_n$ in the diagram below. In particular, q_n is surjective.

$$\begin{array}{ccc} \text{Spec}(S[\varpi]/(\varpi^n)) & \xrightarrow{\tilde{g}_{n-1}} & G \\ \downarrow & \nearrow \tilde{g}_n & \downarrow q_0 \\ \text{Spec}(S[\varpi]/(\varpi^{n+1})) & \xrightarrow{g_n} & G/H \end{array}$$

Now suppose $g \in L^+(G/H)(R)$. For each n there is a corresponding element $g_n \in (G/H)^{(n)}(R)$. In particular, there is an element $g_0 \in (G/H)(R)$ with a lift in S -points $\tilde{g}_0 \in G(S)$, for some étale $R \rightarrow S$. By above, for each n there is also a lift $\tilde{g}_n \in (G/H)^{(n)}(S)$ of g_n . These $\tilde{g}_n$ form an inverse system, and thus correspond to an element $\tilde{g} \in L^+G(S)$ lifting g . Thus q is surjective.

Now consider the kernel of q . Let $g \in L^+G(R)$ for a k -algebra R , and suppose $q(g) = e \in L^+(G/H)(R)$. Then $g \in \ker(q_0)(R[\varpi]) = H(R[\varpi]) = L^+H(R)$. Similarly, for $g \in L^+H(R) = H(R[\varpi])$, we have $q(g)$ corresponds to the identity in $(G/H)(R[\varpi]) = L^+(G/H)(R)$, and so $g \in \ker q(R)$. $\square$

Lemma 3.2. *Let R be a k -algebra over a field k of characteristic 0. If $I \subset R((\varpi))$ is a set of nonzero idempotents, then $I \subset R$.*

As a result, given a decomposition of $R((\varpi))$ (respectively $R[[\varpi]]$) into nontrivial Cartesian factors, there is a corresponding decomposition of R into nontrivial factors, such that each factor of $R((\varpi))$ (or $R[[\varpi]]$) is a Laurent series ring (formal power series ring). Geometrically, we can say that the natural maps of k -schemes induced by inclusion of k -algebras

$$\mathrm{Spec} R((\varpi)) \rightarrow \mathrm{Spec} R[[\varpi]] \rightarrow \mathrm{Spec} R$$

are bijective on connected components.

Proof. First I will show that an idempotent $e \in R((\varpi))$ must be contained in $R[[\varpi]]$. For an element $x \in R((\varpi))$, let x_n be the degree- n coefficient. That is, $x = \sum_{n \in \mathbb{Z}} x_n \varpi^n$. Let n_0 be the minimal nonzero degree of e , so that $e = \sum_{n \geq n_0} e_n \varpi^n$.

Suppose, for contradiction, that there is some idempotent element $e \notin R[[\varpi]]$, so that $n_0 < 0$. I will show that $e_{n_0} = 0$, contradicting the claim that $e \notin R[[\varpi]]$.

Comparing coefficients of e and e^2 , we have

$$e_n = (e^2)_n = \sum_{i=n_0}^{n-n_0} e_i e_{n-i} \quad (3)$$

for all $n \geq n_0$. Then for each n , let

$$f_n := e_n - \sum_{i=n_0}^{n-n_0} e_i e_{n-i} \in R.$$

Note that, by equation (3), each $f_n = 0$ in R . If $n < n_0$, we have $e_n = 0$, so $f_n = -\sum e_i e_{n-i}$ *a priori*. If furthermore $n < 2n_0$, then every term of the sum is also 0 *a priori*. I will construct e_{n_0} as an R -linear combination of such f_n , to show that $e_{n_0} = 0$. I claim

$$e_{n_0} = f_{n_0} + \sum_{i=2n_0}^0 \left(4 - \frac{6}{n_0}i\right) e_{n_0-i} f_i. \quad (4)$$

Substituting the f_n with e_n , we can expand the right-hand side of equation (4) as

$$\left(e_{n_0} - \sum_{i=n_0}^0 e_i e_{n_0-i}\right) + \sum_{i=2n_0}^0 \left(4 - \frac{6}{n_0}i\right) e_{n_0-i} \left(e_i - \sum_{j=n_0}^{i-n_0} e_j e_{i-j}\right). \quad (5)$$

The expression in equation (5) is a non-constant polynomial in the various e_n . We can rewrite, ordering by total degree in e_n :

$$e_{n_0} - \left(\sum_{i=n_0}^0 e_i e_{n_0-i} - \sum_{i=2n_0}^0 \left(4 - \frac{6}{n_0}i\right) e_{n_0-i} e_i\right) - \left(\sum_{i=2n_0}^0 \left(4 - \frac{6}{n_0}i\right) e_{n_0-i} \sum_{j=n_0}^{i-n_0} e_j e_{i-j}\right) \quad (6)$$

Gathering like terms, we can rewrite equation (6) as

$$e_{n_0} + \sum_{n_0 \leq i \leq \frac{n_0}{2}} c_i e_i e_{n_0-i} + \sum_{n_0 \leq i \leq 0} \sum_{i \leq j \leq \frac{n_0-i}{2}} c_{i,j} e_i e_j e_{n_0-i-j} \quad (7)$$

for some finite collections of rational coefficients $\{c_i\}$ and $\{c_{i,j}\}$. Note that we can ignore the terms where $i < n_0$ (and thus also those where $j > 0$), as they are *a priori* equal to 0. I claim that in fact every coefficient c_i or $c_{i,j}$ of equation (7) is 0.

First consider the c_i . Suppose that $i < n_0 - i$. Then

$$c_i = -2 + (4 - \frac{6}{n_0}i) + (4 - \frac{6}{n_0}(n_0 - i)) = 0.$$

On the other hand, if $i = n_0 - i$, then $i = n_0/2$ (and n_0 is even), and

$$c_{n_0/2} = -1 + (4 - \frac{6}{n_0} \cdot \frac{n_0}{2}) = 0.$$

Now consider the $c_{i,j}$, with cases based on which of i, j , and $n_0 - i - j$ are equal. First suppose $i = j = n_0 - i - j = n_0/3$. Then

$$c_{n_0/3, n_0/3} = 4 - \frac{6}{n_0}(n_0 - \frac{n_0}{3}) = 0.$$

Now suppose $i = j < n_0 - 2i$. We can see that

$$c_{i,i} = 2(4 - \frac{6}{n_0}(n_0 - i)) + (4 - \frac{6}{n_0}2i) = 0.$$

Next suppose $i < j = n_0 - i - j$. Then $j = (n_0 - i)/2$. We can see that

$$c_{i, (n_0-i)/2} = (4 - \frac{6}{n_0}(n_0 - i)) + 2(4 - \frac{6}{n_0} \cdot \frac{n_0 + i}{2}) = 0.$$

Finally, suppose $i < j < n_0 - i - j$. Then we can see

$$c_{i,j} = 2(4 - \frac{6}{n_0}(n_0 - i)) + 2(4 - \frac{6}{n_0}(n_0 - j)) + 2(4 - \frac{6}{n_0}(i + j)) = 0.$$

We therefore have canceling of every term in equation (7) except for e_{n_0} . Thus equation (4) holds, and $e_{n_0} = 0$ in R . So if $e \in R((\varpi))$ is idempotent, then $e \in R[[\varpi]]$.

Now suppose $e \in R[[\varpi]]$ is idempotent. Then in particular e_0 is idempotent, as $e_0 = (e^2)_0 = e_0^2$. Suppose for contradiction that $e \notin R$, so there is some $n > 0$ such that $e_n \neq 0$. Let n_1 be the smallest such integer. Then

$$e_{n_1} = (e^2)_{n_1} = 2e_0 e_{n_1},$$

but also

$$e_{n_1} = (e^3)_{n_1} = 3e_0^2 e_{n_1} = 3e_0 e_{n_1} = \frac{3}{2} e_{n_1},$$

which is only possible if $e_{n_1} = 0$. Therefore $e = e_0 \in R$. $\square$

Lemma 3.3. *Suppose G is an étale group scheme over a field k of characteristic 0. Then there are canonical isomorphisms $LG \cong L^+G \cong G$.*

Proof. Let R be a k -algebra, and let $k[G] = \Gamma(G, \mathcal{O}_G)$; note $k[G]$ is a Cartesian product of finitely many fields, all finite and separable over k ; write $k[G] = k_1 \times \cdots \times k_r$. The respective R -point sets are equal to $\text{Hom}_{k\text{-Alg}}(k[G], R((\varpi)))$, $\text{Hom}_{k\text{-Alg}}(k[G], R[[\varpi]])$, and $\text{Hom}_{k\text{-Alg}}(k[G], R)$. It is sufficient to show that for all $f \in k[G]$ and all homomorphisms $\phi : k[G] \rightarrow R((\varpi))$, we have $\phi(f) \in R$. As a result, all $\phi : k[G] \rightarrow R[[\varpi]]$ factor through R as well.

First I would like to reduce the domain to k_i , a factor of $k[G]$ and a finite separable extension of k . Note that in general, for two k -algebras $A \cong \prod_{i \in I} A_i$ and $B \cong \prod_{j \in J} B_j$, we have

$$\text{Hom}_{k\text{-Alg}}(A, B) = \prod_{j \in J} \left(\prod_{i \in I} \text{Hom}_{k\text{-Alg}}(A_i, B_j) \right).$$

Thus it is sufficient to show, if $R((\varpi)) \cong \prod_{j \in J} B_j$, that for all homomorphisms $\phi_{i,j} : k_i \rightarrow B_j$ we have $\phi(k_i) \subset R \cap B_j$. By Lemma 3.2, we know that $B_j \cong R_j((\varpi))$ for a k -algebra R_j . In particular, $R_j = e_j R$ for an idempotent $e_j \in R$, and thus $R \cap R_j((\varpi)) = R_j$. Thus it is sufficient to assume $\phi : k_i \rightarrow R((\varpi))$ where k_i is a finite separable extension of k and $\text{Spec } R$ is connected, and show that the image of ϕ is contained in R .

Suppose k_i is a factor of $k[G]$, in particular a finite separable extension of k , and let $f \in k_i$, to show that $\phi(f) \in R$, where $\text{Spec } R$ is connected. Both f and $\phi(f)$ are invertible and satisfy a separable, irreducible polynomial P over k . For an element $x \in R((\varpi))$, write x_n for the degree- n coefficient of x , that is $x = \sum_{n \in \mathbb{Z}} x_n \varpi^n$. Then $\phi(f)_n = 0$ for all sufficiently small n ; let n_0 be the smallest integer such that $\phi(f)_{n_0} \neq 0$. Invertibility of $\phi(f)$ implies that $\phi(f)_{n_0}$ is invertible in R , and in particular not nilpotent. Then algebraicity of $\phi(f)$ over k implies that $n_0 \geq 0$. Therefore $\phi(f) \in R[[\varpi]]$.

Let $P(T) = \sum_{j=0}^d P_j T^j$, where $P(f) = 0$ in k_i and $P(\phi(f)) = \phi(P(f)) = 0$ in $R[[\varpi]]$, and all $P_j \in k$. Assume, for the sake of contradiction, that $\phi(f) \notin R$, so there is some integer $n \geq 1$ such that $\phi(f)_n \neq 0$; let n_1 be the smallest. I claim that $\phi(f)_{n_1} = 0$, contradicting the assumption and implying $\phi(f) \in R$. Indeed, $\phi(f)_0$ must satisfy $P(\phi(f)_0) = P(\phi(f))_0 = 0$. Now consider $\phi(f)_{n_1}$. Since $P(\phi(f)) = 0$, we have

$P(\phi(f))_{n_1} = 0$ in particular. We can expand this as

$$\begin{aligned}
P(\phi(f))_{n_1} &= \left(\sum_{j=0}^d P_j \phi(f)^j \right)_{n_1} \\
&= \sum_{j=0}^d P_j \left(\left(\sum_{n \geq 0} \phi(f)_n \varpi^n \right)^j \right)_{n_1} \\
&= \sum_{j=0}^d P_j \left(\sum_{k_1 + \dots + k_{j-1} + k_j = n_1} \phi(f)_{k_1} \cdots \phi(f)_{k_j} \right) \\
&= \sum_{j=0}^d P_j (j \phi(f)_0^{j-1} \phi(f)_{n_1}) \\
&= P'(\phi(f)_0) \phi(f)_{n_1} = 0,
\end{aligned}$$

where P' is the formal derivative of P . Since P is separable and irreducible, and since $P(\phi(f)_0) = 0$ and $\text{Spec } R$ is connected, $P'(\phi(f)_0) \in k(\phi(f)_0) \subset R$ and is in particular neither 0 nor a zero divisor. Thus the only way $P'(\phi(f)_0) \phi(f)_{n_1} = 0$ is if $\phi(f)_{n_1} = 0$. $\square$

Note that Lemma 3.2 is essential in the proof of Lemma 3.3. If there were a k -algebra R such that $R((\varpi))$ had any idempotents not contained in R , then for any nontrivial étale group G , we would have $LG(R) \supsetneq G(R)$. For instance, the square roots of unity group μ_2 is étale over k and has $k[\mu_2] \cong k \times k$. If there were a ring R with an idempotent $e \in R((\varpi)) \setminus R$, then $1 - e$ would also be an idempotent in $R((\varpi)) \setminus R$, and the map $\phi : k \times k \rightarrow R((\varpi))$ given by $\phi(1, 0) = e$ and $\phi(0, 1) = 1 - e$ would be a homomorphism such that $\phi(k[\mu_2]) \not\subset R$. However, by Lemma 3.3, we know $L\mu_2 \cong \mu_2$.

Proposition 3.4. *Let G^σ be a possibly disconnected, split, reductive group over a field k of characteristic 0, and let $G^{\sigma, \circ}$ be the identity component. Then the natural map of functors*

$$\text{Gr}_{G^{\sigma, \circ}} \xrightarrow{\eta} \text{Gr}_{G^\sigma}$$

is an isomorphism $\text{Gr}_{G^{\sigma, \circ}} \cong \text{Gr}_{G^\sigma}$ of étale sheaves over k .

Proof. Recall the affine Grassmannian is the étale sheafification of a presheaf $P\text{Gr}_{G^\sigma} : k\text{-Alg} \rightarrow \text{Sets}$ defined by $R \mapsto LG^\sigma(R)/L^+G^\sigma(R)$. In order to prove that η is an isomorphism of sheaves, it is sufficient to prove η is both injective and surjective (as a map of sheaves).

Injectivity of η follows from injectivity of the the presheaf map $\eta^P : P\text{Gr}_{G^{\sigma, \circ}} \rightarrow P\text{Gr}_{G^\sigma}$. Let R be an arbitrary k -algebra. Then the component map η_R^P is injective. Indeed, an element $x \in P\text{Gr}_{G^{\sigma, \circ}}(R)$ can be written as a coset $x = gL^+G^{\sigma, \circ}(R)$, where

$g \in LG^{\sigma,\circ}(R)$, and $\eta_R^P(x) = gL^+G^\sigma(R)$. This definition of η_R^P makes sense because $LG^{\sigma,\circ} \subset LG^\sigma$ and is well-defined since $L^+G^{\sigma,\circ} \subset L^+G^\sigma$. The map η_R^P is also injective. Indeed, if g and g' are two elements of $LG^{\sigma,\circ}(R)$ such that $gL^+G^\sigma(R) = g'L^+G^\sigma(R)$, let $h \in L^+G^\sigma(R)$ be any element such that $gh = g'$. Then in fact $h \in L^+G^{\sigma,\circ}(R)$, otherwise $gh \notin LG^{\sigma,\circ}(R)$.

In order to show that η is surjective, I find it convenient to sheafify; the aim is to show that for all k -algebras R , and for all $x \in \mathrm{Gr}_{G^\sigma}(R)$, there is some étale k -algebra morphism $R \rightarrow S$ such that x , viewed by restriction as a point in $\mathrm{Gr}_{G^\sigma}(S)$, lifts to a point $\tilde{x} \in \mathrm{Gr}_{G^{\sigma,\circ}}(S)$. To do so, consider the following diagram of étale sheaves, for which the rows are exact (as sheaves in pointed sets):

$$\begin{array}{ccccccc} 1 & \longrightarrow & L^+G^{\sigma,\circ} & \longrightarrow & LG^{\sigma,\circ} & \longrightarrow & \mathrm{Gr}_{G^{\sigma,\circ}} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \eta \\ 1 & \longrightarrow & L^+G^\sigma & \longrightarrow & LG^\sigma & \longrightarrow & \mathrm{Gr}_{G^\sigma} \longrightarrow 1 \end{array}$$

Let $x \in \mathrm{Gr}_{G^\sigma}(R)$. By surjectivity of $LG^\sigma \rightarrow \mathrm{Gr}_{G^\sigma}$, there is some lift $g \in LG^\sigma(S_1)$ of x , where $R \rightarrow S_1$ is étale. Then if we can find some $h \in L^+G^\sigma(S_2)$ such that $gh^{-1} \in LG^{\sigma,\circ}(S_2)$ (where $S_1 \rightarrow S_2$ is étale), then $\eta([gh^{-1}]) = x$. The reason we can find such h (and S_2) is that the étale quotient functors $L^+G^\sigma/L^+G^{\sigma,\circ}$ and $LG^\sigma/LG^{\sigma,\circ}$ are isomorphic—in fact, they are isomorphic to the algebraic group $G^\sigma/G^{\sigma,\circ}$, so we can take $h \in G^\sigma(S_2)$. In particular, if $h' \in LG^\sigma(S_1)$ is any point with $g(h')^{-1} \in LG^{\sigma,\circ}(S_1)$, then $[h'] \in (LG^\sigma/LG^{\sigma,\circ})(S_1) = (G^\sigma/G^{\sigma,\circ})(S_1)$ lifts to $h \in G^\sigma(S_2)$ for some S_2 .

Use the previous two lemmas to see that the sheaves $L^+G^\sigma/L^+G^{\sigma,\circ}$ and $LG^\sigma/LG^{\sigma,\circ}$ really are isomorphic. The group $G^{\sigma,\circ}$ is reductive and a normal subgroup of the reductive group G^σ , and the quotient $G^\sigma/G^{\sigma,\circ}$ is étale. Thus by Lemma 3.1, $L^+G^\sigma/L^+G^{\sigma,\circ} \cong L^+(G^\sigma/G^{\sigma,\circ})$, and by Lemma 3.3, $G^\sigma/G^{\sigma,\circ} \cong L^+(G^\sigma/G^{\sigma,\circ})$.

It remains to be seen that $LG^\sigma/LG^{\sigma,\circ} \cong L(G^\sigma/G^{\sigma,\circ})$. Note that we have a natural map $LG^\sigma \rightarrow L(G^\sigma/G^{\sigma,\circ})$ by applying the loop group functor L to the quotient map $q_0 : G^\sigma \rightarrow G^\sigma/G^{\sigma,\circ}$. By Lemma 3.3, $L(G^\sigma/G^{\sigma,\circ}) \cong G^\sigma/G^{\sigma,\circ} \cong L^+(G^\sigma/G^{\sigma,\circ})$. Then the map $LG^\sigma \rightarrow L(G^\sigma/G^{\sigma,\circ})$ is surjective, since the map $L^+G^\sigma \rightarrow L^+(G^\sigma/G^{\sigma,\circ})$ factors through it, and is itself surjective by Lemma 3.1. And the kernel is $LG^{\sigma,\circ}$, for reasons essentially identical to those in the proof of Lemma 3.1: if the quotient map kills $g \in LG^\sigma(R)$, then g corresponds to an element of $G^\sigma(R((\varpi)))$ also killed by quotient, and thus $g \in G^{\sigma,\circ}(R((\varpi))) = LG^{\sigma,\circ}(R)$. And if $g \in LG^{\sigma,\circ}(R)$, then the quotient map kills g when viewed as an $R((\varpi))$ -point of $G^{\sigma,\circ}$.

□

It follows from the same reasoning that the natural map $G^{\sigma,\circ} \hookrightarrow G^\sigma$ induces an isomorphism of categories

$$P_{L^+G^{\sigma,\circ}}(\mathrm{Gr}_{G^{\sigma,\circ}}) \xrightarrow{\sim} P_{L^+G^\sigma}(\mathrm{Gr}_{G^\sigma}). \quad (8)$$

Specifically, recall that the cocharacter lattices $X_*(T^\sigma)$ and $X_*(T^{\sigma,\circ})$ are isomorphic. And for each cocharacter $\mu \in X_*(T^\sigma)^+$, the map η restricts to an isomorphism $\mathrm{Gr}_{G^\sigma}^\mu \cong \mathrm{Gr}_{G^{\sigma,\circ}}^\mu$. I do not directly use this fact, but I think it is worth acknowledging.

For the remainder of the paper, the groups G , G^σ , and $G^{\sigma,\circ}$ will be assumed to lie over the field $k = \mathbb{C}$, and their dual groups will lie over the algebraically closed K field of characteristic 0.

I will prove equation (1) of Theorem 0.1 holds using G^σ . Then since the two categories in equation (8) are isomorphic, they share a Tannakian dual group, which by Theorem 2.1 v. has root datum dual to the connected reductive group $G^{\sigma,\circ}$, i.e. is isomorphic to the group $\widehat{G^{\sigma,\circ}}$.

Much of the proof follows from an understanding of the relationship between the σ -action on semi-infinite cells, the σ -fixed sub-ind-scheme of a σ -invariant semi-infinite cell, and the corresponding semi-infinite cell of the σ -fixed point affine Grassmannian.

Proposition 3.5. *i. There is a natural injective map of ind-schemes $\mathrm{Gr}_{G^\sigma} \rightarrow \mathrm{Gr}_G$, and the reduced ind-scheme $(\mathrm{Gr}_{G^\sigma})_{\mathrm{red}}$ may be identified with $(\mathrm{Gr}_G)_{\mathrm{red}}^\sigma$.*

ii. For $\mu \in X_(T)^+$, $\sigma(\mathrm{Gr}_G^\mu) = \mathrm{Gr}_G^{\sigma(\mu)}$*

iii. For $\mu \in X_(T)^{+,\sigma}$, we can identify $(\mathrm{Gr}_G^\mu)^\sigma = \mathrm{Gr}_{G^\sigma}^\mu$.*

iv. For $\nu \in X_(T)$ and $w \in W$, $\sigma(S_w^\nu) = S_{\sigma(w)}^{\sigma(\nu)}$*

v. For $\nu \in X_(T)^\sigma$ and $w \in W^\sigma$, we can identify $(S_w^\nu)^\sigma = (S_\sigma)_w^\nu$, where $(S_\sigma)_w^\nu$ is the semi-infinite cell $wL(U^\sigma)w^{-1} \cdot \varpi^\nu x_0 \subset \mathrm{Gr}_{G^\sigma}$.*

Note that, as perverse sheaves are defined using the étale topology, the identification of reduced ind-schemes in Statement i. implies an identification of Satake categories as well. Note also that Gr_{G^σ} may be reduced even if Gr_G is not.

Proof. Statements ii. and iv. are immediate.

Note that σ acts on LG , preserving L^+G . Thus we have, a priori, an action of σ on Gr_G and an injective map of functors $\mathrm{Gr}_{G^\sigma} \rightarrow (\mathrm{Gr}_G)^\sigma$.

Statements i. and iii. follow from statement v., along with the observation that for a sub-ind-scheme $X \subset \mathrm{Gr}_G$, $X^\sigma = X \cap (\mathrm{Gr}_G)^\sigma$. Note that statement i. is taken only on the level of reduced structures, since it follows from taking a union of locally closed sub-ind-schemes. The same is technically true for statement iii., but Schubert varieties are already reduced.

To see statement v., consider the action of $wLUw^{-1}$ on S_w^ν : there is a subgroup, $J_G(w, \nu) \subset wLUw^{-1}$, with a simply transitive action on S_w^ν . Define $J_G(w, \nu)$ as follows:

$$J_G(w, \nu) := wLUw^{-1} \cap \varpi^\nu L^{--}G\varpi^{-\nu}.$$

By construction of $J_G(w, \nu)$, it is clear that for σ -invariant w and ν , $J_G(w, \nu)^\sigma = J_{G^\sigma}(w, \nu)$. Therefore the σ -fixed points of S_w^ν are exactly those in the orbit of the σ -fixed subgroup $J_{G^\sigma}(w, \nu)$.

So let us see that the action of the subgroup $J_G(w, \nu)$ is simply transitive on S_w^ν , implying Statement v.. It is well known that LU has a decomposition $L^-U \cdot L^+U$. Since $L^-G \supset L^-U$ acts freely on Gr_G and $LG \supset LU$ has stabilizer $L^+G \supset L^+U$ at the basepoint x_0 , this decomposition implies that $J_G(e, 0) = L^-U$ acts simply transitively on S_e^0 . Similarly, we have $J_G(w, 0)$ acting simply transitively on S_w^0 for all $w \in W$.

For more general $J_G(w, \nu)$, consider the decomposition

$$wLUw^{-1} = J_G(w, \nu) \cdot (wLUw^{-1} \cap \varpi^\nu L^+ G \varpi^{-\nu}).$$

It is immediate both that this is a decomposition of $wLUw^{-1}$ (from normality of wUw^{-1} in wBw^{-1}), and also that $wLUw^{-1} \cap \varpi^\nu L^+ G \varpi^{-\nu}$ is the stabilizer of $\varpi^\nu x_0$ in $wLUw^{-1}$. Therefore $J_G(w, \nu)$ acts simply transitively on S_w^ν , as needed. $\square$

Note also that closure relations hold as expected, simply by intersection. Specifically, given a σ -invariant dominant cocharacter μ ,

$$\overline{\text{Gr}_G^\mu} = \overline{\text{Gr}_G^\mu} \cap \text{Gr}_{G^\sigma} = \left(\coprod_{\substack{\lambda \in X_*(T)^+ \\ \lambda \leq \mu}} \text{Gr}_G^\lambda \right) \cap \text{Gr}_{G^\sigma} = \coprod_{\substack{\lambda \in X_*(T)^+ \\ \lambda \leq \mu}} (\text{Gr}_G^\lambda \cap \text{Gr}_{G^\sigma}) = \coprod_{\substack{\lambda \in X_*(T)^+ \\ \lambda \leq \mu}} \text{Gr}_{G^\sigma}^\lambda,$$

and given σ -invariant $\nu \in X_*(T)$ and $w \in W^\sigma$,

$$\overline{(S_\sigma)_w^\nu} = \overline{S_w^\nu} \cap \text{Gr}_{G^\sigma} = \left(\coprod_{\substack{\eta \in X_*(T) \\ \eta \leq w\nu}} S_w^\eta \right) \cap \text{Gr}_{G^\sigma} = \coprod_{\substack{\eta \in X_*(T) \\ \eta \leq w\nu}} (S_w^\eta \cap \text{Gr}_{G^\sigma}) = \coprod_{\substack{\eta \in X_*(T)^\sigma \\ \eta \leq w\nu}} (S_\sigma)_w^\eta.$$

Either of these equalities implies that the cocharacter lattice $X_*(T^\sigma) \subset X_*(T)$ inherits the partial order $\leq$. This can also be seen combinatorially; see Section 7 for details.

The primary varieties in consideration in this proof are Mirković–Vilonen (MV) cycles and Anderson–Mirković–Vilonen (AMV) cycles. MV cycles of coweight (λ, μ) are irreducible components of the intersection $S_{w_0}^\lambda \cap \overline{\text{Gr}_G^\mu}$, and according to the geometric Satake correspondence they index a basis for $V_\mu(\lambda)$. On the other hand, AMV cycles of coweight (λ, μ) are irreducible components of the variety $\overline{S_{w_0}^\lambda \cap S_e^\mu}$. Many authors refer to AMV cycles as simply “MV cycles.” The following proposition will make clear how closely related they are, and why AMV cycles may be considered a generalization of MV cycles. Note that, for the purposes of the geometric Satake equivalence, it is not important whether we are dealing with an equi-dimensional variety or its closure. Indeed, we can use top-dimensional cohomology with compact support, which in this case depends only on dimension and number of components. However, it is (formally) convenient to require AMV cycles to be projective when defining their moment polytopes.

Proposition 3.6 ([1] Proposition 3). *If A is an irreducible component of $\overline{S_{w_0}^\lambda \cap S_e^\mu}$ and $A \subset \overline{\text{Gr}_G^\mu}$, then A is the closure of an MV cycle of coweight (λ, μ) . If A' is an MV cycle of coweight (λ, μ) , then $\overline{A'}$ is an irreducible component of $\overline{S_{w_0}^\lambda \cap S_e^\mu}$. Thus the closures of MV cycles of coweight (λ, μ) are exactly the AMV cycles of coweight (λ, μ) contained in $\overline{\text{Gr}_G^\mu}$.*

Proof. This result follows from dimension estimates in Theorem 2.1 i. and ii..

First suppose A is an irreducible component of $\overline{S_{w_0}^\lambda \cap S_e^\mu}$ and that $A \subset \overline{\text{Gr}_G^\mu}$. Of course we have $A \subset \overline{S_{w_0}^\lambda \cap \text{Gr}_G^\mu}$. Now the Iwasawa stratification implies

$$\overline{S_{w_0}^\lambda \cap \text{Gr}_G^\mu} = \coprod_{\substack{\nu \in X_*(T) \\ \nu \geq \lambda}} (S_{w_0}^\nu \cap \overline{\text{Gr}_G^\mu}) = (S_{w_0}^\lambda \cap \overline{\text{Gr}_G^\mu}) \cup X,$$

where $\dim X < \dim A$. And so $A' := A \cap (S_{w_0}^\lambda \cap \overline{\text{Gr}_G^\mu})$ is dense in A . Since A is irreducible, this implies A' is an MV cycle.

Now suppose $A = \overline{A'}$ where A' is an MV cycle of coweight (λ, μ) . It is sufficient to see that $A' \subset \overline{S_{w_0}^\lambda \cap S_e^\mu}$. Note that by Theorem A.1(a),

$$\dim(S_e^\mu \cap \overline{\text{Gr}_G^\mu}) = \dim(\overline{\text{Gr}_G^\mu}) = 2\langle \rho, \mu \rangle,$$

implying that $S_e^\mu \cap \overline{\text{Gr}_G^\mu}$ is dense in $\overline{\text{Gr}_G^\mu}$, and in particular, $\overline{\text{Gr}_G^\mu} \subset \overline{S_e^\mu}$. So $A' \subset S_{w_0}^\lambda \cap \overline{S_e^\mu}$. Again using the Iwasawa stratification, we have

$$S_{w_0}^\lambda \cap \overline{S_e^\mu} = \coprod_{\substack{\nu \in X_*(T) \\ \nu \leq \mu}} (S_{w_0}^\lambda \cap S_e^\nu) = (S_{w_0}^\lambda \cap S_e^\mu) \cup Y,$$

where again $\dim Y < \dim A'$. And so $A' \cap (S_{w_0}^\lambda \cap S_e^\mu)$ is dense in A' , and $A' \subset \overline{S_{w_0}^\lambda \cap S_e^\mu}$. $\square$

Working with AMV cycles rather than MV cycles is convenient. The primary reason is that they are defined as components of the intersection of a pair of semi-infinite cells, rather than components of a semi-infinite cell and a Schubert variety. One useful consequence is that $X_*(T)$ acts on the set of AMV cycles by translation, where such a translation of an MV cycle is no longer necessarily an MV cycle.

Given an AMV cycle A and a cocharacter $\nu \in X_*(T)$, let $\nu \cdot A = \varpi^\nu A$. We have $\varpi^\nu S_w^\lambda = S_w^{\lambda+\nu}$ by normality of wUw^{-1} in wBw^{-1} , so if A has coweight (λ, μ) , then $\nu \cdot A$ is an AMV cycle and has coweight $(\lambda+\nu, \mu+\nu)$. Given an AMV cycle A of coweight (λ, μ) , the $X_*(T)$ -orbit of A has one AMV cycle of coweight $(\lambda-\mu, 0)$. This AMV cycle is called the stable AMV cycle representing A , and denoted A_0 . For many purposes, I

will work with stable AMV cycles only, equivalent to assuming the second coweight is 0. In these cases I will use the subscript $_0$. This is especially convenient for the consideration of non-simply-connected groups G , since stable AMV cycles are contained in the neutral component of the affine Grassmannian.

The following theorem of Anderson is useful for determining which AMV cycles are MV cycles.

Theorem 3.7 ([1] Theorem 1 (1)). *Let G be a semisimple group over $\mathbb{C}$. There exists a family of polytopes $\mathcal{MV} = (P_A)_{A \in \mathbb{B}}$ in $X_*(T)_{\mathbb{R}}$ with parameter set $\mathbb{B}$ graded by Λ^- (i.e. $\mathbb{B} = \bigcup_{\nu \in \Lambda^-} \mathbb{B}_{\nu}$) such that weight multiplicities may be calculated according to the following rule: If V_{μ} is an irreducible representation of $\hat{G}$ with highest weight μ , then the multiplicity of the weight λ in V_{μ} equals the number of $A \in \mathbb{B}_{\lambda-\mu}$ for which $P_A + \mu \subset \text{Conv}(W \cdot \mu)$.*

Above, $X_*(T)_{\mathbb{R}} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, $\mathbb{B}_{\nu}$ is the set of irreducible components of $\overline{X(\nu, 0)}$, and Λ^- is the set of negative coweights in $X_*(T^{\text{sc}})$, i.e. the negative coroot semilattice of G . And P_A is the moment polytope of the MV cycle A , defined as follows:

Definition 3.8 (Moment polytope). *Suppose X is an irreducible, projective, T -invariant subvariety $X \subset \text{Gr}_G$. Then define the moment polytope of X as*

$$P_X := \text{Conv}(\nu \in X_*(T) \mid \varpi^{\nu} x_0 \in X).$$

This definition is inspired by the image of the moment map $\Phi : \text{Gr}_G \rightarrow X_(T)_{\mathbb{R}}$ of the action of T on Gr_G . However, for our purposes, there is no need to define Φ , only the image of T -invariant subvarieties.*

Note that Schubert varieties, semi-infinite cells, and AMV cycles are all T -invariant. It is helpful to note some properties of the moment map and polytopes it produces:

Proposition 3.9 ([1] Proposition 4 and proof). *i. The T -fixed points of Gr_G are the $\varpi^{\nu} x_0$. Those in Gr_G^{μ} are the $\varpi^{W \cdot \mu} x_0$. Those in $\overline{\text{Gr}_G^{\mu}}$ are the $\varpi^{\nu} x_0$ where*

$$\nu \in \text{Conv}(W \cdot \mu) \cap (\mu + X_*(T^{\text{sc}})).$$

The one in S_w^{ν} is $\varpi^{\nu} x_0$. Those in $\overline{S_w^{\nu}}$ are the $\varpi^{\eta} x_0$ where $\eta \leq_w \mu$.

ii. If X is a one-dimensional T -orbit, then $P_{\overline{X}}$ is a line segment in a coroot direction joining two coweights in a common coset modulo $X_(T^{\text{sc}})$.*

iii. If X is any projective, irreducible, T -invariant variety, then P_X is the convex hull of the images of its T -fixed points.

Proof. Statement i. is well known. Statement iii. is an immediate consequence of the definition of the moment polytope used here.

Suppose $X \subset \text{Gr}_G$ is a one-dimensional T -orbit. Let x be a complex point $x \in X(\mathbb{C})$. By the Iwasawa stratification, for all $w \in W$ there is a unique ν_w for which $x \in S_w^{\nu_w}$. For each w , the set of T -fixed points in $\overline{S_w^{\nu_w}}$ is $\{\varpi^\eta x_0 \mid \eta \leq_w \nu_w\}$. So the set of T -fixed points in the intersection of $S_w^{\nu_w}$ for all $w \in W$ is contained in $\text{Conv}(\nu_w \mid w \in W)$. In particular, $P_{\overline{X}} \subset \text{Conv}(\nu_w \mid w \in W)$.

In fact, $P_{\overline{X}} = \text{Conv}(\nu_w \mid w \in W)$. Recall the geometric description of the semi-infinite cells (2): $x \in S_w^\nu(\mathbb{C})$ if and only if

$$\lim_{t \rightarrow 0} w(\rho^\vee)(t) \cdot x = \varpi^\nu x_0.$$

The $\varpi^{\nu_w} x_0$ for $w \in W$ are therefore limit points for the torus action on X , and contained in $\overline{X}$.

Since X is a one-dimensional quotient of T , it must be isomorphic to $\mathbb{G}_m$, and so $X(\mathbb{C})$ has at most two distinct limit points in $\text{Gr}_G(\mathbb{C})$. But if there is only one, call it ν , then $X(\mathbb{C}) \subset (S_{w_0}^\nu \cap S_e^\nu)(\mathbb{C})$, which consists of the single point $\varpi^\nu x_0$, violating the assumption that X is one-dimensional. So let the two distinct cocharacters in the set $\{\nu_w \mid w \in W\}$ be denoted ν and η . Suppose $\nu = \nu_e$, so that $\nu \geq \eta$. Then also we have $\eta = \nu_{w_0}$.

Find some $w \in W$ and simple reflection $s_i \in W$ such that $\nu \geq_w \eta$ but $\nu \leq_{ws_i} \eta$, and note that then $\ell(w) < \ell(ws_i)$. Such w and s_i can be found by choosing a reduced word $\mathbf{i} = (i_1, i_2, \dots, i_{\ell(w_0)})$ for w_0 and comparing ν and η under the order $\leq_{s_{i_1} \dots s_{i_k}}$ for each $0 \leq k \leq \ell(w_0)$. Then $w^{-1}(\nu) \leq w^{-1}(\eta)$, so $w^{-1}(\nu - \eta)$ is a sum of positive coroots; and $s_i w^{-1}(\nu) \geq s_i w^{-1}(\eta)$, so $s_i w^{-1}(\nu - \eta)$ is a sum of negative coroots. This is only possible when $w^{-1}(\nu - \eta) = n\alpha_i^\vee$ for some integer $n > 0$. And so $\nu - \eta$ is a multiple of the positive coroot $w(\alpha_i^\vee)$. $\square$

I will sketch Anderson's proof of Theorem 3.7 below, to see that it does not depend on the assumption G is semisimple.

Although T -invariance is the primary consideration for the moment map, it is convenient to also consider the "dilation" action of $\mathbb{G}_m$ on Gr_G defined by $\varpi \mapsto c\varpi \in R((\varpi))$ for $c \in \mathbb{G}_m(R)$, since we can usefully describe a fixed point in the closure of a $(\mathbb{G}_m \times T)$ -orbit. Note also that Schubert varieties, semi-infinite cells, and AMV cycles are dilation-invariant as well as T -invariant. The following statement about the fixed points of torus actions is well-known. The proof in the current case is taken from Anderson.

Lemma 3.10 ([1] Lemma 6). *Every $(\mathbb{G}_m \times T)$ -orbit $X \subset \text{Gr}_G$ has a T -fixed point $\varpi^\eta x_0 \in \overline{X}$, such that $X \subset \text{Gr}_G^\eta$.*

Note that in the statement above, η is not required to be dominant.

Proof. Suppose X is a $(\mathbb{G}_m \times T)$ -orbit in Gr_G . By the Cartan stratification, X must be contained in some Gr_G^μ . There is a point $x \in \overline{X}(\mathbb{C}) \cap (G(\mathbb{C}) \cdot \varpi^\mu x_0)$, found as a

limit of the dilation action. Then there is some fixed point $\varpi^\eta x_0$ in the closure of the $T(\mathbb{C})$ -orbit of x . Since $T \subset G$, we still have $\varpi^\eta x_0 \in G(\mathbb{C}) \cdot \varpi^\mu x_0$; in particular, η is in the Weyl orbit $W \cdot \mu$. And since X is both $\mathbb{G}_m$ - and T -invariant, both x and $\varpi^\eta x_0$ are contained in $\overline{X}(\mathbb{C})$. $\square$

Now we are ready to prove Theorem 3.7.

Proof of Theorem 3.7. First suppose A is an MV cycle of coweight (λ, μ) for some dominant cocharacter μ and some $\lambda \in \text{Wt}(\mu)$. Then $A \subset \overline{\text{Gr}}_G^\mu$, and so $P_A \subset P_{\overline{\text{Gr}}_G^\mu} = \text{Conv}(W \cdot \mu)$.

Now suppose A is an AMV cycle of coweight (λ, μ) , and suppose $P_A \subset \text{Conv}(W \cdot \mu)$. Then for every vertex ν of P_A and each $w \in W$, we have $\nu \leq_w w(\mu)$. Since A is $(\mathbb{G}_m \times T)$ -invariant, it is a union of $(\mathbb{G}_m \times T)$ -orbits. So every complex point $a \in A(\mathbb{C})$ is contained in such an orbit $a \in X(\mathbb{C}) \subset A(\mathbb{C})$, and by Lemma 3.10 there is a cocharacter η such that $\varpi^\eta x_0 \in \overline{X}(\mathbb{C}) \subset A(\mathbb{C})$ and $X \subset \text{Gr}_G^\eta$. In particular, $a \in \text{Gr}_G^\eta(\mathbb{C})$. By the assumption that $P_A \subset \text{Conv}(W \cdot \mu)$, the difference $\mu - w(\eta)$ must be a nonnegative real combination of simple roots for each $w \in W$. And since $\varpi^\eta x_0 \in A(\mathbb{C})$, and A is irreducible, it follows that all $w(\eta)$ are in the same coset modulo $X_*(T^{\text{sc}})$ as the vertices ν of P_A —so in fact $\eta \in \text{Wt}(\mu)$. Therefore $a \in \overline{\text{Gr}}_G^\mu(\mathbb{C})$. $\square$

The following observation summarizes the convenience of moment polytopes in studying the geometry of the affine Grassmannian via semi-infinite cells. It is a direct consequence of Anderson's work.

Lemma 3.11. *Let X be a T -invariant, projective, irreducible subvariety of Gr_G , and let $\nu \in X_*(T)$. Then $X \cap S_w^\nu$ is dense in X if and only if ν is $\leq_w$ -maximal among vertices of the moment polytope P_X .*

Proof. First suppose $X \cap S_w^\nu$ is dense in X . Then $X = X \cap \overline{S_w^\nu}$. So if $\eta \not\leq_w \nu$, then $X \cap S_w^\eta = \emptyset$. In particular, $\varpi^\eta x_0 \notin X$. And so $\eta \notin P_X$, unless $\eta \notin \nu + X_*(T^{\text{sc}})$, in which case η cannot be a vertex of P_X .

Now suppose ν is $\leq_w$ -maximal among vertices of P_X . By the Iwasawa stratification, $X \subset \overline{S_w^\nu}$. In particular,

$$X = \coprod_{\eta \leq_w \nu} (X \cap S_w^\eta)$$

and so

$$X = \overline{\coprod_{\eta \leq_w \nu} (X \cap S_w^\eta)} = \bigcup_{\eta \leq_w \nu} \overline{(X \cap S_w^\eta)},$$

with the last equality holding since there are only finitely many semi-infinite cells intersecting X . By irreducibility and completeness of X , there is thus some $\eta \leq_w \nu$ such

that $X = \overline{X \cap S_w^\eta}$. But the only $\eta \leq_w \nu$ with $\varpi^\nu x_0 \in S_w^\eta$ is $\eta = \nu$. Therefore $X = \overline{X \cap S_w^\nu}$. $\square$

These results suggest the construction of GGMS strata: small, T -invariant subvarieties stratifying Gr_G , whose closures are sometimes AMV cycles. A GGMS stratum, named for Gelfand, Goresky, MacPherson, and Serganova, is an intersection of a sequence of semi-infinite cells indexed by W . A sequence $\nu_\bullet = (\nu_w)_{w \in W}$ of cocharacters such that $\nu_w \geq_w \nu_{w'}$ for all $w, w' \in W$ is known as a GGMS datum, and specifies the GGMS stratum

$$A(\nu_\bullet) := \bigcap_{w \in W} S_w^{\nu_w}.$$

GGMS data are in bijection with pseudo-Weyl polytopes, or convex polytopes in $X_*(T)_\mathbb{R}$ whose edges are in root directions, and whose vertices are cocharacters in a common coset modulo $X_*(T^{sc})$. Combining the information in Proposition 3.9, we see that if A is an AMV cycle, then P_A is a pseudo-Weyl polytope, and the vertices of P_A form a GGMS datum $\nu_\bullet^A$. By Lemma 3.11, the GGMS stratum $GGMS(A) := A(\nu_\bullet^A)$ is dense in A , and is the minimal intersection of semi-infinite cells with this property.

As a consequence of their construction as the intersections of semi-infinite cells, the σ -action on the set of GGMS strata adheres to the following dichotomy:

Lemma 3.12. *i. A GGMS stratum $A(\nu_\bullet)$ is σ -invariant if and only if the sequence of cocharacters $\nu_\bullet$ is σ -invariant, meaning $\sigma(\nu_w) = \nu_{\sigma(w)}$ for all $w \in W$.*

ii. If $A(\nu_\bullet)$ is not σ -invariant, then $A(\nu_\bullet)^\sigma = \emptyset$.

Proof. First, note that Gr_G is stratified by GGMS strata. Intersecting the Iwasawa stratifications of Gr_G for all $w \in W$, we have

$$\text{Gr}_G = \coprod_{\nu_\bullet} A(\nu_\bullet).$$

where the disjoint union runs over all GGMS data $\nu_\bullet$.

Then let $\nu_\bullet = (\nu_w)_{w \in W}$. We have

$$\sigma(A(\nu_\bullet)) = \sigma\left(\bigcap_{w \in W} S_w^{\nu_w}\right) = \bigcap_{w \in W} S_{\sigma(w)}^{\sigma(\nu_w)} = A(\sigma(\nu_\bullet)),$$

where $\sigma(\nu_\bullet) = (\sigma(\nu_{\sigma^{-1}(w)}))_{w \in W}$. Then σ permutes GGMS strata, implying ii. And i. follows from observing that $\sigma(\nu_\bullet) = \nu_\bullet$ if and only if $\nu_w = \sigma(\nu_\bullet)_w$ for each $w \in W$. $\square$

4 Indexing using $\mathbf{i}$ -Lusztig data

Unfortunately, not all GGMS strata are dense in some AMV cycle. In order to work with an indexing set where we can be sure the result is an AMV cycle, we use $\mathbf{i}$ -Lusztig data.

Let A be an AMV cycle with GGMS datum $\nu_\bullet$. If we walk along the edges of the moment polytope P_A from ν_e to ν_{w_0} such that the vertices are indexed by Weyl elements of strictly increasing lengths, by recording the length of each edge we produce a sequence of $\ell(w_0)$ nonnegative integers $n_\bullet \in \mathbb{Z}_{\geq 0}^{\ell(w_0)}$ called the $\mathbf{i}$ -Lusztig datum of A . This $\mathbf{i}$ -Lusztig datum uniquely determines the stable AMV cycle A_0 , where $\mathbf{i}$ is the corresponding reduced word for w_0 . The $\mathbf{i}$ -Lusztig strata are in a useful bijection with stable AMV cycles:

Theorem 4.1 ([10] Theorem 4.2). *Given fixed reduced word $\mathbf{i}$ for w_0 , the set of $\mathbf{i}$ -Lusztig data are in bijective correspondence with stable AMV cycles.*

$$\mathbb{Z}_{\geq 0}^{\ell(w_0)} \leftrightarrow \{\text{stable AMV cycles}\}$$

Note that since Theorem 4.1 concerns stable AMV cycles, there is no question that it applies to connected, reductive groups, and not just to simply connected, semisimple groups.

Proof. From the construction of the $\mathbf{i}$ -Lusztig datum of an AMV cycle, it is sufficient to find an inverse mapping of $\mathbf{i}$ -Lusztig data to stable AMV cycles. This can be done explicitly.

Note $\text{Conv}(W \cdot \rho^\vee)$ is a pseudo-Weyl polytope, and has $\mathbf{i}$ -Lusztig datum $(1, \dots, 1)$ for every reduced word $\mathbf{i}$ for w_0 ; indeed, for any two neighboring vertices $w(\rho^\vee)$ and $ws_i(\rho^\vee)$, the difference is a single coroot $w(\rho^\vee - s_i(\rho^\vee)) = \pm w(\alpha_i^\vee)$. This polytope is sometimes known as the permutahedron.

One way to construct a subvariety of Gr_G whose GGMS datum has a given $\mathbf{i}$ -Lusztig datum $n_\bullet$ is to intersect semi-infinite cells corresponding to the cocharacters encountered in the path from ν_e to ν_{w_0} corresponding to $\mathbf{i}$, the directions of which are determined by $\mathbf{i}$, and which can all be seen as the directions of edges of the permutahedron. It will turn out that this construction produces a collection of irreducible varieties in bijection with the $\mathbf{i}$ -Lusztig data.

Fix $\mathbf{i} = (i_1, \dots, i_{\ell(w_0)})$, a reduced word for w_0 . Let $(w_k^\mathbf{i})_{0 \leq k \leq \ell(w_0)}$ be the sequence of Weyl group elements corresponding to the the initial (as in leftmost, assuming W acts from the left) k letters of $\mathbf{i}$: $w_0^\mathbf{i} = e$, $w_k^\mathbf{i} = w_{k-1}^\mathbf{i} s_{i_k}$, and $w_{\ell(w_0)}^\mathbf{i} = w_0$.

For $1 \leq k \leq \ell(w_0)$, let $\beta_k^{\mathbf{i}, \vee}$ be the difference $w_k^\mathbf{i}(\rho^\vee) - w_{k-1}^\mathbf{i}(\rho^\vee)$, which is the direction of an edge of the permutahedron. Then $\beta_k^{\mathbf{i}, \vee}$ is a negative coroot; specifically, $\beta_k^{\mathbf{i}, \vee} =$

$-w_{k-1}^{\mathbf{i}}(\alpha_{i_k}^\vee):$

$$\begin{aligned} w_k^{\mathbf{i}}(\rho^\vee) - w_{k-1}^{\mathbf{i}}(\rho^\vee) &= w_{k-1}^{\mathbf{i}}(s_{i_k}(\rho^\vee) - \rho^\vee) \\ &= w_{k-1}^{\mathbf{i}}(s_{i_k}(\frac{1}{2} \sum_{\alpha^\vee \in \Phi^{\vee,+}} \alpha^\vee) - \frac{1}{2} \sum_{\alpha^\vee \in \Phi^{\vee,+}} \alpha^\vee) \\ &= w_{k-1}^{\mathbf{i}}(-\alpha_{i_k}^\vee), \end{aligned}$$

since s_{i_k} permutes all positive coroots except for $\alpha_{i_k}^\vee$, which it transposes with $-\alpha_{i_k}^\vee$. Let $\nu_0 = 0$, and for each $k \geq 1$ let $\nu_k = \nu_{k-1} + \beta_k^{\mathbf{i},\vee}$. Each ν_k is $\leq_{w_k^{\mathbf{i}}}$ -maximal in the sequence (ν_k) , since $w_k^{\mathbf{i}}(\rho^\vee)$ is $\leq_{w_k^{\mathbf{i}}}$ -maximal in $W \cdot \rho^\vee$. Then let

$$A_0^{\mathbf{i}}(n_\bullet) := \bigcap_{0 \leq k \leq \ell(w_0)} S_{w_k^{\mathbf{i}}}^{\nu_k}.$$

Then $\overline{A_0^{\mathbf{i}}(n_\bullet)}$ is a projective subvariety of $\overline{X(\nu_{\ell(w_0)}, 0)} = \overline{S_{w_0}^{\nu_{\ell(w_0)}} \cap S_e^0}$. Note also that the map

$$\{n_\bullet \mid \mathbf{i}\text{-Lusztig data}\} \rightarrow \{A_0 \mid \text{stable AMV cycles}\}$$

given by $n_\bullet \mapsto \overline{A_0^{\mathbf{i}}(n_\bullet)}$ is injective.

Suppose that $A_0^{\mathbf{i}}(n_\bullet)$ is irreducible. Then it is not hard to see that $\overline{A_0^{\mathbf{i}}(n_\bullet)}$ is a stable AMV cycle, proving the theorem.

Indeed, if A_0 is a stable MV cycle with GGMS datum $\nu_\bullet$ and $\mathbf{i}$ -Lusztig datum $n_\bullet$, we have containment

$$A_0(\nu_\bullet) \subset A_0^{\mathbf{i}}(n_\bullet) \subset \overline{X(\nu_{w_0}, 0)}, \quad A_0(\nu_\bullet) \subset A_0 \subset \overline{X(\nu_{w_0}, 0)}.$$

Then by Lemma 3.11, $A_0^{\mathbf{i}}(n_\bullet) \cap A_0$ is dense in A_0 . Since A_0 is projective, the result follows from irreducibility of $A_0^{\mathbf{i}}(n_\bullet)$.

And all stable AMV cycles are constructed in this way: the number of stable AMV cycles of a given coweight $(\nu, 0)$ is the Kostant partition function of ν , defined as the number of ways ν can be written as a combination of negative coroots. The reason for this is that the Kostant partition function is a sharp upper bound on the dimension of the weight space $V_\mu(\mu + \nu)$, and by Theorem 3.7 every stable AMV cycle A_0 represents an MV cycle $n\rho^\vee \cdot A_0$ for sufficiently large n . And since every negative coroot is the direction of exactly one edge in the paths defined by $\mathbf{i}$, the Kostant partition function of ν is exactly the number of distinct $\mathbf{i}$ -Lusztig data of coweight $(\nu, 0)$.

Irreducibility is somewhat difficult. The heart of the proof of irreducibility is the construction of surjective maps

$$\mathbf{y}_{\mathbf{i}} : B(n_\bullet) \rightarrow A_0^{\mathbf{i}}(n_\bullet),$$

where $B(n_\bullet)$ is the subfunctor of $LG_a^{\ell(w_0)}$ defined on R -points as

$$B(n_\bullet)(R) = \{(x_1, \dots, x_{\ell(w_0)}) \in R((\varpi))^{\ell(w_0)} \mid \text{val}_\varpi(x_k) = n_k \text{ for } 1 \leq k \leq \ell(w_0)\}.$$

Clearly $B(n_\bullet)$ is an irreducible ind-scheme, so the existence of a surjective map of sheaves $\mathbf{y}_i$ implies that $A_0^i(n_\bullet)$ is irreducible as well. For details and construction of $\mathbf{y}_i$, see [10] Theorem 4.5. This construction is explicit, but somewhat complicated. Note that it does not depend in any way on simply-connectedness or semisimplicity of G . $\square$

It is also the case that, fixing a reduced word $\mathbf{i}$, the variety $X(\nu, 0)$ is stratified by $\mathbf{i}$ -Lusztig strata:

$$X(\nu, 0) = S_{w_0}^\nu \cap S_e^0 = \coprod_{\substack{n_\bullet \in \mathbb{Z}_{\geq 0}^{\ell(w_0)} \\ \nu_{\ell(w_0)} = \nu}} A_0^i(n_\bullet). \quad (9)$$

Definition 4.2 (σ -compatibility). *A reduced word $\mathbf{i}$ for the longest element $w_0 \in W$ is said to be σ -compatible if there is a (uniquely determined) reduced word $\mathbf{i}_\sigma$ for the longest element $w_{0,\sigma} \in W^\sigma$ such that $\mathbf{i}$ is an expansion of $\mathbf{i}_\sigma$.*

The group W^σ is a Coxeter group whose simple reflections are the longest elements in the subgroup generated by the simple reflections in a single σ -orbit (see Proposition 7.2). If $\mathbf{i}_\sigma$ is a sequence of orbits $(\eta_1, \dots, \eta_{\ell(w_{0,\sigma})})$, we say $\mathbf{i}$ is an expansion of $\mathbf{i}_\sigma$ if it can be partitioned into consecutive subsequences of letters corresponding to the orbits of $\mathbf{i}_\sigma$. If this is the case, then each such consecutive subsequence will be the longest word in the Coxeter subgroup generated by the simple reflections in the corresponding orbit.

Example 4.3. *To illustrate the concept of σ -compatibility, consider the involution on the standard pinning of SL_5 , with fixed-point subgroup $(SL_5)^\sigma \cong SO(5)$. On the set of simple roots, σ acts by (14)(23), with two orbits: $\eta_1 = \{1, 4\}$ and $\eta_2 = \{2, 3\}$. Then*

$$W = \langle s_1, s_2, s_3, s_4 \mid m_{13} = m_{14} = m_{24} = 2, m_{12} = m_{23} = m_{34} = 3 \rangle$$

$$W^\sigma = \langle s_{\eta_1}, s_{\eta_2} \mid m_{\eta_1 \eta_2} = 4 \rangle,$$

so one (of two) possible reduced word for $w_{0,\sigma}$ is $\mathbf{i}_\sigma = (\eta_1, \eta_2, \eta_1, \eta_2)$. Since $m_{14} = 2$, the longest words on the letters in η_1 are (1, 4) and (4, 1). Since $m_{23} = 3$, the longest words on the letters in η_2 are (2, 3, 2) and (3, 2, 3). There are 16 reduced words for w_0 expanding $\mathbf{i}_\sigma$, including

$$\mathbf{i} = (1, 4, 2, 3, 2, 1, 4, 2, 3, 2).$$

Proposition 4.4. *If $A_0^i(n_\bullet)$ is an $\mathbf{i}$ -Lusztig stratum, then $\sigma(A_0^i(n_\bullet)) = A_0^{\sigma(\mathbf{i})}(n_\bullet)$.*

Proof. Let $\mathbf{i}$ be a reduced word for w_0 , and $n_\bullet$ an $\mathbf{i}$ -Lusztig datum. Consider

$$\sigma(A_0^i(n_\bullet)) = \sigma \left(\bigcap_{0 \leq k \leq \ell(w_0)} S_{w_k^i}^{\nu_k} \right) = \bigcap_{0 \leq k \leq \ell(w_0)} S_{\sigma(w_k^i)}^{\sigma(\nu_k)}.$$

Now $\sigma(w_k^{\mathbf{i}}) = w_k^{\sigma(\mathbf{i})}$, and each $\sigma(\nu_k) = \sigma(\nu_{k-1}) + n_k \sigma(\beta_k^{\mathbf{i}, \vee})$. We have

$$\sigma(\beta_k^{\mathbf{i}, \vee}) = \sigma(w_k^{\mathbf{i}}(\rho^\vee) - w_{k-1}^{\mathbf{i}}(\rho^\vee)) = w_k^{\sigma(\mathbf{i})}(\rho^\vee) - w_{k-1}^{\sigma(\mathbf{i})}(\rho^\vee) = \beta_k^{\sigma(\mathbf{i}), \vee}.$$

By induction the $\sigma(\mathbf{i})$ -Lusztig datum for $\sigma(A_0^{\mathbf{i}}(n_\bullet))$ is thus also $n_\bullet$. $\square$

Suppose $\mathbf{i}$ and $\mathbf{i}'$ are distinct reduced words for w_0 that both expand a common reduced word $\mathbf{i}_\sigma$ for $w_{0,\sigma}$. Let $n_\bullet$ be an $\mathbf{i}$ -Lusztig datum. Then there is an explicit procedure for producing the $\mathbf{i}'$ -Lusztig datum $n'_\bullet$ for $A_0^{\mathbf{i}'}(n_\bullet)$.

Lemma 4.5 ([10] Proposition 5.2). *Let $\mathbf{i}$ and $\mathbf{i}'$ be two reduced words for w_0 related by a braid move corresponding to a pair of simple roots either disconnected or connected by an edge of weight 1. Specifically, let i_k and i_{k+1} be the indices of a pair of simple roots, and let $m_{i_k, i_{k+1}} \leq 3$ be the order of $s_{i_k} s_{i_{k+1}}$ in W . Then suppose*

$$\begin{aligned} \mathbf{i} &= (i_1, \dots, i_{k-1}; i_k, i_{k+1}, i_k, \dots; i_{k+m_{i_k, i_{k+1}}}+1, \dots, i_{\ell(w_0)}), \\ \mathbf{i}' &= (i_1, \dots, i_{k-1}; i_{k+1}, i_k, i_{k+1}, \dots; i_{k+m_{i_k, i_{k+1}}}+1, \dots, i_{\ell(w_0)}). \end{aligned}$$

Then define a function $R_{\mathbf{i}}^{\mathbf{i}'} : \mathbb{Z}_{\geq 0}^{\ell(w_0)} \rightarrow \mathbb{Z}_{\geq 0}^{\ell(w_0)}$ as follows:

1. For $k' < k$ or $k' > k + m_{i_k, i_{k+1}}$, let $R_{\mathbf{i}}^{\mathbf{i}'}(n_\bullet)_{k'} = n_{k'}$.
2. Suppose $m_{i_k, i_{k+1}} = 2$. Then let $R_{\mathbf{i}}^{\mathbf{i}'}(n_\bullet)_k = n_{k+1}$ and let $R(n_\bullet)_{k+1} = n_k$.
3. Suppose $m_{i_k, i_{k+1}} = 3$, and let $p = \min\{n_k, n_{k+2}\}$. Then let $R_{\mathbf{i}}^{\mathbf{i}'}(n_\bullet)_k = n_{k+1} + n_{k+2} - p$, $R_{\mathbf{i}}^{\mathbf{i}'}(n_\bullet)_{k+1} = p$, and $R_{\mathbf{i}}^{\mathbf{i}'}(n_\bullet)_{k+2} = n_k + n_{k+1} - p$.

If $n_\bullet$ is the $\mathbf{i}$ -Lusztig datum of a stable AMV cycle A_0 , then $R_{\mathbf{i}}^{\mathbf{i}'}(n_\bullet)$ is the $\mathbf{i}'$ -Lusztig datum of A_0 .

Since any two simple roots in a σ -orbit are either disconnected or connected by an edge of weight 1 in the Dynkin diagram, we are only interested in the cases above. However, [10] proves the proposition in all cases. Since all reduced words for w_0 are related by sequences of braid moves, Kamnitzer's full proposition implies the existence of a function $R_{\mathbf{i}}^{\mathbf{i}'}$ for all pairs of reduced words $(\mathbf{i}, \mathbf{i}')$. As a consequence of the cases above, we can define $R_{\mathbf{i}}^{\mathbf{i}'}$ whenever $\mathbf{i}$ and $\mathbf{i}'$ expand a common word $\mathbf{i}_\sigma$. Thus we have justification for the following definition.

Definition 4.6 (σ -invariant $\mathbf{i}$ -Lusztig datum). *Fix a reduced word $\mathbf{i}$ for w_0 which expands a reduced word $\mathbf{i}_\sigma$ for $w_{0,\sigma}$. Then an $\mathbf{i}$ -Lusztig datum $n_\bullet$ is σ -invariant if it is constant on each σ -orbit. If $n_\bullet$ is a σ -invariant $\mathbf{i}$ -Lusztig datum, then let $\bar{n}_\bullet$ be the corresponding $\mathbf{i}_\sigma$ -Lusztig datum.*

Consider Example 4.3. In this case, an $\mathbf{i}$ -Lusztig datum $n_\bullet$ is σ -invariant if and only if $n_1 = n_2$, $n_3 = n_4 = n_5$, $n_6 = n_7$, and $n_8 = n_9 = n_{10}$. If this is the case, then $\bar{n}_\bullet = (n_1, n_3, n_6, n_8)$.

Note that $\mathbf{i}$ and $\sigma(\mathbf{i})$ are two different reduced words for w_0 expanding $\mathbf{i}_\sigma$, and $R_{\mathbf{i}}^{\sigma(\mathbf{i})}(n_\bullet) = n_\bullet$ if and only if $n_\bullet$ is σ -invariant. Thus, given a σ -compatible reduced word $\mathbf{i}$ for w_0 , an $\mathbf{i}$ -Lusztig datum $n_\bullet$ is σ -invariant if and only if the stable MV cycle $\overline{A_0^{\mathbf{i}}(n_\bullet)}$ is σ -invariant.

Lemma 4.7. *Let $\mathbf{i}$ be a σ -compatible reduced word for w_0 , and let $n_\bullet$ be a σ -invariant $\mathbf{i}$ -Lusztig datum. Then we can identify the fixed-point subvariety*

$$(A_0^{\mathbf{i}}(n_\bullet))^\sigma = A_0^{\mathbf{i}_\sigma}(\bar{n}_\bullet).$$

Proof. Recall the surjective map

$$\mathbf{y}_{\mathbf{i}} : B(n_\bullet) \rightarrow A_0^{\mathbf{i}}(n_\bullet)$$

from the proof of Theorem 4.1, which is defined explicitly in the proof of [10] Theorem 4.5. Consider also the map

$$\mathbf{y}_{\mathbf{i}_\sigma} : B_\sigma(\bar{n}_\bullet) \rightarrow A_0^{\mathbf{i}_\sigma}(\bar{n}_\bullet),$$

defined analogously for G^σ . It is clear from the explicit definition of $\mathbf{y}_{\mathbf{i}}$ that in the square

$$\begin{array}{ccc} B_\sigma(\bar{n}_\bullet) & \xrightarrow{\text{diag}} & B(n_\bullet) \\ \mathbf{y}_{\mathbf{i}_\sigma} \downarrow & & \downarrow \mathbf{y}_{\mathbf{i}} \\ A_0^{\mathbf{i}_\sigma}(\bar{n}_\bullet) & \xrightarrow{\iota} & A_0^{\mathbf{i}}(n_\bullet) \end{array}$$

the map ι is well-defined and injective, and the square commutes, for σ -invariant $n_\bullet$. In particular, $A_0^{\mathbf{i}_\sigma}(\bar{n}_\bullet) \subset (A_0^{\mathbf{i}}(n_\bullet))^\sigma = A_0^{\mathbf{i}}(n_\bullet) \cap \text{Gr}_{G^\sigma}$.

The reverse inclusion follows from the $\mathbf{i}_\sigma$ - and $\mathbf{i}$ -Lusztig stratifications from equation (9) of the varieties

$$(X(\nu_{\ell(w_0)}, 0))^\sigma = (S_{w_0}^{\nu_{\ell(w_0)}} \cap S_e^0) \cap \text{Gr}_{G^\sigma} = (S_\sigma)^{\nu_{\ell(w_0)}}_{w_0, \sigma} \cap (S_\sigma)_e^0$$

and

$$X(\nu_{\ell(w_0)}, 0) = S_{w_0}^{\nu_{\ell(w_0)}} \cap S_e^0.$$

Note also the semi-infinite cells $S_{w_0}^\nu$ and S_e^0 are σ -invariant, since σ -invariance of the $\mathbf{i}$ -Lusztig datum $n_\bullet$ implies $\nu_{\ell(w_0)}$ is a σ -invariant cocharacter. $\square$

Lemma 4.8. *Let $n_\bullet$ be a σ -invariant $\mathbf{i}$ -Lusztig datum, where $\mathbf{i}$ expands $\mathbf{i}_\sigma$. Suppose $A_0^{\mathbf{i}}(n_\bullet)$ is of coweight $(\nu, 0)$. Then $A_0^{\mathbf{i}_\sigma}(\bar{n}_\bullet)$ is also of coweight $(\nu, 0)$.*

Proof. Let $\pi : \{1, \dots, \ell(w_0)\} \rightarrow \{1, \dots, \ell(w_{0,\sigma})\}$ be the surjection of indices of the words. It is sufficient to prove that, for each $1 \leq k' \leq \ell(w_{0,\sigma})$, we have

$$\bar{n}_{k'} \beta_{k'}^{\mathbf{i}_\sigma, \vee} = \sum_{k \in \pi^{-1}(k')} n_k \beta_k^{\mathbf{i}, \vee}. \quad (10)$$

Then since $\bar{n}_{k'} = n_k$ for all $k \in \pi^{-1}(k')$, equation (10) is equivalent to

$$\beta_{k'}^{\mathbf{i}_\sigma, \vee} = \sum_{k \in \pi^{-1}(k')} \beta_k^{\mathbf{i}, \vee} \quad (11)$$

for all k' . For each k' there are two possibilities. Either $\pi^{-1}(k') = \{k, k+1, k+2\}$, where $\alpha_{i_k}^\vee = \alpha_{i_{k+2}}^\vee$ and $\alpha_{i_{k+1}}^\vee$ are two distinct simple coroots connected by an edge of weight 1 in the Dynkin diagram of G ; or all simple coroots corresponding to $k \in \pi^{-1}(k')$ are pairwise disconnected. In the first case we can say the orbit is of type A_2 , and in the second we can say it is of type $A_1 \times \dots \times A_1$.

First suppose k' corresponds to an orbit of type A_2 . Recall

$$\begin{aligned} \beta_{k+j}^{\mathbf{i}, \vee} &= w_{k+j}^{\mathbf{i}}(\rho^\vee) - w_{k+j-1}^{\mathbf{i}}(\rho^\vee) = w_{k+j-1}^{\mathbf{i}}(s_{i_{k+j}}(\rho^\vee) - \rho^\vee) \quad (\text{for } j = 0, 1, 2), \text{ and} \\ \beta_{k'}^{\mathbf{i}_\sigma, \vee} &= w_{k'}^{\mathbf{i}_\sigma}(\rho_\sigma^\vee) - w_{k'-1}^{\mathbf{i}_\sigma}(\rho_\sigma^\vee) = w_{k'-1}^{\mathbf{i}_\sigma}(s_{i_{k'}}(\rho_\sigma^\vee) - \rho_\sigma^\vee). \end{aligned}$$

Then the right hand side of (11) is

$$\begin{aligned} \beta_k^{\mathbf{i}, \vee} + \beta_{k+1}^{\mathbf{i}, \vee} + \beta_{k+2}^{\mathbf{i}, \vee} &= w_{k+2}^{\mathbf{i}}(\rho^\vee) - w_{k-1}^{\mathbf{i}}(\rho^\vee) \\ &= w_{k-1}^{\mathbf{i}}(s_{i_k} s_{i_{k+1}} s_{i_{k+2}}(\rho^\vee) - \rho^\vee) \\ &= w_{k'-1}^{\mathbf{i}_\sigma}(s_{i_{k'}}(\rho_\sigma^\vee) - \rho_\sigma^\vee). \end{aligned} \quad (12)$$

The equality of line (12) follows from Propositions 7.2 and 7.3: $s_{i_{k'}} = s_{i_k} s_{i_{k+1}} s_{i_{k+2}}$ is the longest element of the Coxeter subgroup generated by letters in the orbit η , and so by induction $w_{k-1}^{\mathbf{i}} = w_{k'-1}^{\mathbf{i}_\sigma}$.

The case k' corresponds to an orbit of type $A_1 \times \dots \times A_1$ is similar, except the orbit can be of any order, and each index in the orbit appears only once. Then in this case $s_{i_{k'}} = s_{i_k} s_{i_{k+1}} \dots s_{i_{k+|\eta|-1}}$, and so equation (11) holds. $\square$

5 The bijection for AMV cycles

Theorem 5.1. *Let (λ, μ) be a coweight with μ dominant, $\lambda \leq \mu$, and both fixed by σ . Then taking σ -fixed points induces a bijection between σ -invariant MV cycles in Gr_G of coweight and (λ, μ) and MV cycles in Gr_{G^σ} of coweight (λ, μ) .*

Proof. First I will establish the bijection between AMV cycles. The action of $-\mu$ on the set of AMV cycles gives a bijection between AMV cycles of coweight (λ, μ) and stable AMV cycles of coweight $(\lambda - \mu, 0)$.

Fix a σ -compatible reduced word $\mathbf{i}$ for w_0 expanding the reduced word $\mathbf{i}_\sigma$ for $w_{0,\sigma}$. Theorem 4.1 gives a bijection between $\mathbf{i}$ -Lusztig data and all stable AMV cycles, and restricts to a bijection between those of coweight $(\lambda - \mu, 0)$. And by Lemma 4.7, there is a bijection between σ -invariant $\mathbf{i}$ -Lusztig strata of coweight $(\lambda - \mu, 0)$ and $\mathbf{i}_\sigma$ -Lusztig strata of coweight $(\lambda - \mu, 0)$, given by taking fixed points.

Then by composing the corresponding bijection between $\mathbf{i}_\sigma$ -Lusztig strata of coweight $(\lambda - \mu, 0)$ and AMV cycles of coweight (λ, μ) , we have the desired bijection on the level of AMV cycles.

It remains to see that the bijection on AMV cycles restricts to one on MV cycles. Recall Proposition 3.6: MV cycles of coweight (λ, μ) correspond bijectively to those AMV cycles of coweight (λ, μ) contained in $\overline{\text{Gr}_G^\mu}$. Clearly, if A is an AMV cycle and $A^\sigma \not\subset \overline{\text{Gr}_{G^\sigma}^\mu}$, then $A \not\subset \overline{\text{Gr}_G^\mu}$. However, the converse is more subtle.

Let A be a σ -invariant AMV cycle as in the hypotheses, and suppose $A \not\subset \overline{\text{Gr}_G^\mu}$; i.e. A is not an MV cycle. Let $(\nu_w)_{w \in W}$ be the GGMS datum of A , the W -indexed sequence of vertices in the moment polytope P_A . Then the GGMS stratum of A is

$$GGMS(A) := \bigcap_{w \in W} S_w^{\nu_w}.$$

Note that $\overline{\text{Gr}_G^\mu}$ is an AMV cycle of coweight $(-\mu, \mu)$, and thus the closure of its GGMS stratum:

$$\overline{\text{Gr}_G^\mu} = \overline{\bigcap_{w \in W} S_w^{w(\mu)}}$$

Now if $A \not\subset \overline{\text{Gr}_G^\mu}$, then its GGMS stratum does not intersect with $\overline{\text{Gr}_G^\mu}$ at all. Indeed, suppose there exists a complex point

$$\begin{aligned} p \in (GGMS(A) \cap \overline{\text{Gr}_G^\mu})(\mathbb{C}) &= \left(\bigcap_{w \in W} S_w^{\nu_w}(\mathbb{C}) \right) \cap \left(\overline{\bigcap_{w \in W} S_w^{w(\mu)}(\mathbb{C})} \right) \\ &\subset \bigcap_{w \in W} \left(S_w^{\nu_w}(\mathbb{C}) \cap \overline{S_w^{w(\mu)}(\mathbb{C})} \right). \end{aligned}$$

Then for each $w \in W$ the closure relations for semi-infinite cells imply $\nu_w \leq_w w(\mu)$, so $P_A \subset \text{Conv}(W \cdot \mu)$. By Theorem 3.7, this implies A is in fact an MV cycle, contrary to assumption.

By Lemma 3.11, $GGMS(A)$ is dense in A . So if $(A \cap \overline{\text{Gr}_G^\mu})^\sigma$ were dense in A^σ , there would have to be some complex point

$$p \in (GGMS(A)^\sigma \cap \overline{\text{Gr}_{G^\sigma}^\mu})(\mathbb{C}) \subset (GGMS(A) \cap \overline{\text{Gr}_G^\mu})(\mathbb{C}),$$

but the last intersection is empty.

As a result, the map

$$\left\{ A \subset \text{Gr}_G \mid \begin{array}{c} \sigma\text{-invariant MV cycles} \\ \text{of coweight } (\lambda, \mu) \end{array} \right\} \longrightarrow \left\{ A \subset \text{Gr}_{G^\sigma} \mid \begin{array}{c} \text{MV cycles of} \\ \text{coweight } (\lambda, \mu) \end{array} \right\}$$

$$A \longmapsto \overline{A}^\sigma \cap (S_\sigma)_{w_0, \sigma}^\lambda$$

is a bijection. $\square$

6 Eigenvalues

I would like to prove that equation (1) holds for all σ -invariant (λ, μ) . It is known that the number of σ -invariant MV cycles of coweight (λ, μ) in Gr_G is the same as the number of MV cycles of coweight (λ, μ) in Gr_{G^σ} , and that the bijection is given by taking the σ -fixed points. From a naive understanding of the geometric Satake equivalence, it is thus clear that there is a linear map $\sigma' : V_\mu(\lambda) \rightarrow V_\mu(\lambda)$ such that equation (1) holds, replacing σ with σ' . Indeed, one may simply choose a basis $\{e_A\}$ for $V_\mu(\lambda)$ indexed by MV cycles A , and let σ' be any map such that $\sigma'(e_A)$ is a scalar multiple of $e_{\sigma(A)}$ for all A and $\sigma'(e_A) = e_A$ for σ -invariant A . However, it is not clear at this point that the map constructed from the action of σ on $\widehat{G}$ shares these properties. We need to take a more careful look at the geometric Satake equivalence to see that there is indeed a basis $\{e_A\}$ for which σ satisfies these properties.

But first, we need to more carefully define the operator we are considering, as well as construct an alternative operator for comparison. In the end, we will need to make some (limited) choice to identify the operators on $V_\mu(\lambda)$, so from here I will start decorating them to keep them distinct.

On the one hand, we have constructed $\widehat{G}$ as the group dual to G using the root datum, and given it an arbitrary pinning (although there is a canonical choice one could make, to me it is arbitrary since I will make an identification that is not canonical in any case). This pinning uniquely determines an automorphism of $\widehat{G}$, which I will now call $\hat{\sigma}$, preserving $\widehat{B}$ and $\widehat{T}$, and compatible with the root homomorphisms $x_{\alpha^\vee}$. It is now straightforward to construct an action of $\hat{\sigma}$ on irreducible highest-weight representations V_μ of $\widehat{G}$ when μ is $\hat{\sigma}$ -invariant. Since $\hat{\sigma}^*$ is a tensor auto-equivalence on $\text{Rep}_{\mathbb{C}}(\widehat{G})$, we have $\hat{\sigma}^*V_\mu$ is an irreducible highest-weight representation. Furthermore, given a vector $v \in V_\mu$ of weight λ and $t \in \widehat{T}$, we have $\hat{\sigma}(t) \cdot v = \sigma^{-1}(\lambda)(t)v$, so as an element of $\hat{\sigma}^*V_\mu$, v has weight $\sigma^{-1}(\lambda)$. In particular, the highest weight of $\hat{\sigma}^*V_\mu$ is $\sigma^{-1}(\mu)$. In the case μ is σ -invariant, we thus have $\hat{\sigma}^*V_\mu \cong V_\mu$. By Schur's lemma, there is up to scalar a unique isomorphism of representations $\hat{\sigma}^*V_\mu \rightarrow V_\mu$. However, since $\hat{\sigma}^*V_\mu$ and V_μ share underlying vector spaces, we may canonically choose the isomorphism which fixes

the highest-weight line $V_\mu(\mu)$ pointwise. This automorphism on the underlying vector space is the action of σ on V_μ , and will be known as $\hat{\sigma}_\mu : V_\mu \rightarrow V_\mu$. It is difficult, from this construction, to directly deduce precise eigenvalues of $\hat{\sigma}_\mu$ for vectors outside of the highest-weight line $V_\mu(\mu)$ (and other extreme-weight lines $V_\mu(w(\mu))$).

On the other hand, we have a construction of the dual group of G using the Tannakian formalism: $\tilde{G}$ is the group of fiber functor automorphisms of $P_{L+G}(\mathrm{Gr}_G)$. I will construct a linear isomorphism $\tilde{\sigma}_\mu : \mathbb{H}^\bullet(\mathrm{Gr}_G, IC_\mu) \rightarrow \mathbb{H}^\bullet(\mathrm{Gr}_G, IC_\mu)$ for σ -invariant cocharacters μ of G , and identify $\tilde{G} \cong \hat{G}$ in such a way that (using uniqueness from Schur's lemma) the automorphisms $\hat{\sigma}_\mu$ and $\tilde{\sigma}_\mu$ must be equal. However, the identification of $\tilde{G}$ with $\hat{G}$ is non-canonical, and in particular is sensitive to the pinning on G . In order to identify $\tilde{G}$ with $\hat{G}$, I will construct a pinning on $\tilde{G}$ that is preserved by an automorphism $\tilde{\sigma}$ of $\tilde{G}$.

The advantage of considering these “tilde” constructions is that, as an induced map on cohomology groups, it will be much more straightforward to prove that $\tilde{\sigma}_\mu$ fixes all basis vectors corresponding to σ -invariant MV cycles, and thus satisfies equation (1).

I will henceforth refer to the functor $\mathbb{H}^\bullet(\mathrm{Gr}_G, -)$ as F , as in “fiber.” Similarly, for cocharacters μ and λ , let $F_\lambda IC_\mu$ be the group $H_c^{-2\langle\rho, \mu-\lambda\rangle}(S_{w_0}^\lambda \cap \mathrm{Gr}_G^\mu, \mathbb{C})$.

The map $\tilde{\sigma}_\mu$ will be constructed from an identification $IC_\mu \cong \sigma^! IC_\mu$. (Note that, since σ is an étale morphism of varieties, $\sigma^* \mathcal{A}$ is canonically isomorphic to $\sigma^! \mathcal{A}$ for all perverse sheaves $\mathcal{A}$ on Gr_G . Similarly, since σ is proper, $\sigma_! \mathcal{A}$ is canonically isomorphic to $\sigma_* \mathcal{A}$.) Suppose we have an isomorphism $\phi_\mu : IC_\mu \rightarrow \sigma^! IC_\mu$. Then, using the counit ϵ of the adjunction $(\sigma_!, \sigma^!)$ we have an isomorphism

$$\sigma_! IC_\mu \xrightarrow{\sigma_! \phi_\mu} \sigma_! \sigma^! IC_\mu \xrightarrow{\epsilon} IC_\mu$$

Since Gr_G is an ind-proper ind-scheme, F is canonically isomorphic to a direct sum of compactly supported cohomology functors, implying $F IC_\mu \cong F \sigma_! IC_\mu$. Thus, we can compose arrows in the following (commutative) diagram

$$\begin{array}{ccccc} F IC_\mu & \xrightarrow{F \phi_\mu} & F \sigma^! IC_\mu & & \\ \downarrow \text{can} & & \downarrow \text{can} & & \\ F \sigma_! IC_\mu & \xrightarrow{F \sigma_! \phi_\mu} & F \sigma_! \sigma^! IC_\mu & \xrightarrow{F \epsilon} & F IC_\mu \end{array}$$

to produce a map $\tilde{\sigma}_\mu := F \epsilon \circ \text{can} \circ F \phi_\mu$, given an isomorphism ϕ_μ .

Similarly to the construction of $\hat{\sigma}_\mu$, we can construct ϕ_μ canonically. For all x in the smooth locus Gr_G^μ of $\overline{\mathrm{Gr}_G^\mu}$, the stalks of both IC_μ and $\sigma^! IC_\mu$ at x are one-dimensional and concentrated in degree $-2\langle\rho, \mu\rangle$. Furthermore, the basepoint $\varpi^\mu x_0$ is preserved by σ if μ is. So let ϕ_μ be the unique isomorphism $IC_\mu \rightarrow \sigma^! IC_\mu$ that restricts to the identity on the stalk at $\varpi^\mu x_0$.

Lemma 6.1. *Suppose μ is a σ -invariant dominant cocharacter. As constructed above, the map $\tilde{\sigma}_\mu$ is a direct sum of natural maps on top-dimensional cohomology groups with compact support, with constant coefficients, induced by the morphisms of varieties $\sigma : S_{w_0}^\lambda \cap \mathrm{Gr}_G^\mu \rightarrow S_{w_0}^{\sigma(\lambda)} \cap \mathrm{Gr}_G^\mu$. As a result, for all irreducible components $A \subset S_{w_0}^\lambda \cap \mathrm{Gr}_G^\mu$, the fundamental class*

$$[A] \in H_c^{2\langle\rho, \mu - \lambda\rangle}(S_{w_0}^\lambda \cap \mathrm{Gr}_G^\mu, \mathbb{C})$$

satisfies $\tilde{\sigma}_\mu([A]) \in \mathbb{C}[\sigma(A)]$. And in particular, if $\sigma(A) = A$, then $\tilde{\sigma}_\mu([A]) = [A]$ exactly.

Proof. Recall by the geometric Satake correspondence, specifically 2.1 iii. and iv., that there is a natural, canonical isomorphism

$$GSE_\mu : FIC_\mu \xrightarrow{\sim} \bigoplus_{\lambda \in X_*(T)} H_c^{-2\langle\rho, \lambda\rangle}(S_{w_0}^\lambda, IC_\mu) \xrightarrow{\sim} \bigoplus_{\lambda \in Wt(\mu)} H_c^{2\langle\rho, \mu - \lambda\rangle}(S_{w_0}^\lambda \cap \mathrm{Gr}_G^\mu, \mathbb{C}).$$

So the goal will be to show that a natural map on cohomology groups induced by the morphism $\sigma : S_{w_0}^\lambda \cap \mathrm{Gr}_G^\mu \rightarrow S_{w_0}^{\sigma(\lambda)} \cap \mathrm{Gr}_G^\mu$ commutes with GSE_μ and fixes pointwise the cohomology groups corresponding to σ -invariant MV cycles.

Cohomology with compact support is a covariant functor. That is, given a map of sheaves $\phi : \mathcal{F} \rightarrow \mathcal{G}$ on a variety Y , we have for each i a map on cohomology with compact support

$$H_c^i(\phi) : H_c^i(Y, \mathcal{F}) \rightarrow H_c^i(Y, \mathcal{G})$$

Given a morphism of varieties $f : X \rightarrow Y$, we have naturally induced such a map on sheaves $\epsilon : f_! f^! \mathbb{C}_Y \rightarrow \mathbb{C}_Y$, where ϵ is the counit of the adjunction $(f_!, f^!)$. Furthermore, $H_c^i(Y, f_! f^! \mathbb{C}_Y)$ is canonically isomorphic to $H_c^i(X, f^! \mathbb{C}_Y)$. If f is an isomorphism of varieties, this is further isomorphic to $H_c^i(X, \mathbb{C}_X)$ as $f^! \mathbb{C}_Y \cong \mathbb{C}_X$. Thus, after choosing an appropriate isomorphism $\mathbb{C}_X \rightarrow f^! \mathbb{C}_Y$, we have a map

$$\mathrm{Tr}_f^i : H_c^i(X, \mathbb{C}) \rightarrow H_c^i(Y, \mathbb{C}).$$

This map is constructed in [16, 0GJY] and referred to as the trace map of f .

Let $i = 2\langle\rho, \mu - \lambda\rangle$, where μ is a σ -invariant cocharacter and $\lambda \in Wt(\mu)$. Then the cohomology group $H_c^i(S_{w_0}^\lambda \cap \mathrm{Gr}_G^\mu, \mathbb{C})$ is spanned by fundamental classes $[A]$, where A is an MV cycle of coweight (λ, μ) , and $H_c^i(S_{w_0}^{\sigma(\lambda)} \cap \mathrm{Gr}_G^\mu, \mathbb{C})$ is spanned by fundamental classes $[\sigma(A)]$. Looking on stalks, we see that $\mathrm{Tr}_\sigma^i([A])$ is a scalar multiple of $[\sigma(A)]$. Supposing λ is σ -invariant and choosing the normalization $\mathbb{C} \rightarrow \sigma^! \mathbb{C}$ corresponding to that for $\tilde{\sigma}_\mu$, which is identity on stalks preserved by σ , we get that in the case $\sigma(A) = A$, $\mathrm{Tr}_\sigma^i([A]) = [A]$ exactly.

By the similarity in construction of the two horizontal maps, the following diagram

commutes:

$$\begin{array}{ccc}
 F IC_\mu & \xrightarrow{\tilde{\sigma}_\mu} & F IC_\mu \\
 \downarrow GSE_\mu & & \downarrow GSE_\mu \\
 \bigoplus_{\lambda \in Wt(\mu)} H_c^{2\langle \rho, \mu - \lambda \rangle}(S_{w_0}^\lambda \cap \text{Gr}_G, \mathbb{C}) & \xrightarrow{\text{Tr}_\sigma^i} & \bigoplus_{\lambda \in Wt(\mu)} H_c^{2\langle \rho, \mu - \lambda \rangle}(S_{w_0}^{\sigma(\lambda)} \cap \text{Gr}_G, \mathbb{C})
 \end{array}$$

Therefore $\tilde{\sigma}_\mu$ acts on fundamental classes as expected. $\square$

Note that $\tilde{\sigma}_\mu$ thus satisfies (1).

In order to identify the actions $\tilde{\sigma}$ and $\hat{\sigma}$, we extend the construction of $\tilde{\sigma}_\mu$ to cases where μ is not σ -invariant, constructing an automorphism $\tilde{\sigma} : \tilde{G} \rightarrow \tilde{G}$. Given such an automorphism $\tilde{\sigma}$, we can verify that the linear map $\tilde{\sigma}^* V_\mu \rightarrow V_\mu$ on induced on σ -invariant highest-weight representations of $\tilde{G}$ by the automorphism $\tilde{\sigma}$ is equal to the linear operator $\tilde{\sigma}_\mu$ defined above. Then, in order to prove that $\hat{\sigma}_\mu$ satisfies equation (1), it will be sufficient to verify that $\tilde{G}$ may be identified with $\hat{G}$ in such a way that the automorphisms $\tilde{\sigma}$ and $\hat{\sigma}$ commute with the identification $\tilde{G} \rightarrow \hat{G}$.

Consider the following proposition, an immediate consequence of definitions in the first chapter of [5]:

Proposition 6.2. *Let $\hat{G}$ be a reductive group, and let $F : \text{Rep}_\mathbb{C}(\hat{G}) \rightarrow \text{Vec}_\mathbb{C}$ be the natural fiber functor. Given a tensor auto-equivalence $T : (\text{Rep}_\mathbb{C}(\hat{G}), \otimes) \rightarrow (\text{Rep}_\mathbb{C}(\hat{G}), \otimes)$ and an isomorphism of fiber functors $\phi : FT \rightarrow F$, there is a corresponding automorphism τ of $\hat{G}$, given by $\tau(g) = \phi \circ g \circ \phi^{-1}$. In the other direction, an automorphism τ can be used to construct such a pair, using the functor $\tau^*(\rho, V) = (\rho \circ \tau, V)$, and the isomorphism $F\tau^* \rightarrow F$ which takes identity on objects in $\text{Vec}_\mathbb{C}$.*

$$\begin{array}{ccc}
 \text{Aut}(\hat{G}) & \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} & \left\{ \begin{array}{l} \text{tensor auto-equivalences with} \\ \text{an isomorphism of fiber functors} \\ T : (\text{Rep}_\mathbb{C}(\hat{G}), \otimes) \rightarrow (\text{Rep}_\mathbb{C}(\hat{G}), \otimes) \\ \phi : FT \rightarrow F \end{array} \right\} \\
 \tau & \longmapsto & (\tau^*, \text{id}) \\
 (g \mapsto \phi \circ g \circ \phi^{-1}) & \longleftarrow & (T, \phi)
 \end{array}$$

We could impose an equivalence relation on the right hand side above and tautologically the pairs (T, ϕ) modulo equivalence would be in bijection with automorphisms. All we need is a sufficient condition for construction of an automorphism $\tilde{\sigma} : \tilde{G} \rightarrow \tilde{G}$.

We already have a tensor auto-equivalence of functors $\sigma^! : P_{L+G}(\text{Gr}_G) \rightarrow P_{L+G}(\text{Gr}_G)$, so we want an isomorphism of fiber functors $\phi : F\sigma^! \rightarrow F$. As before, we can use the counit $\epsilon : \sigma_! \sigma^! IC_\mu \rightarrow IC_\mu$.

$$F\sigma^! IC_\mu \xrightarrow{\text{can}} F\sigma_! \sigma^! IC_\mu \xrightarrow{F\epsilon} F IC_\mu$$

So we let $\phi = F\epsilon \circ \text{can}$; then $\tilde{\sigma}_\mu(g) = \phi \circ g \circ \phi^{-1}$, as needed.

Finally, we need to compare $\tilde{\sigma}$ with $\hat{\sigma}$. Consider that, for σ -invariant μ , $\tilde{\sigma}$ satisfies a commutative diagram

$$\begin{array}{ccccc} FIC_\mu & \xrightarrow{\tilde{\sigma}_\mu} & FIC_\mu & \xleftarrow{F\epsilon \circ \text{can}} & F\sigma^!IC_\mu \\ g \downarrow & & \downarrow \tilde{\sigma}(g) & & \downarrow g \\ FIC_\mu & \xrightarrow{\tilde{\sigma}_\mu} & FIC_\mu & \xleftarrow{F\epsilon \circ \text{can}} & F\sigma^!IC_\mu \end{array} \quad (13)$$

In fact $\tilde{\sigma}_\mu$ factors through $F\sigma^!IC_\mu$, implying the linear operator on FIC_μ induced by the automorphism $\tilde{\sigma}$ is exactly $\tilde{\sigma}_\mu$.

Now we turn our attention to the question of identifying $\tilde{G}$ and $\hat{G}$. They are by GSE 2.1 v. abstractly isomorphic. Since a pinning-preserving automorphism is determined uniquely by its action on the Dynkin diagram and the pinning it preserves, it is sufficient to prove that there is *some* pinning preserved by $\tilde{\sigma}$; any choice of such a preserved pinning will imply $\tilde{\sigma} = \hat{\sigma}$ after identification $\tilde{G} \cong \hat{G}$. Note that the pinning preserved by $\tilde{\sigma}$ depends on the isomorphism $\sigma : \text{Gr}_G \rightarrow \text{Gr}_G$, which ultimately depends on the pinning on G . It is for this reason that the identification $\tilde{G} \cong \hat{G}$ is not canonical.

To begin with, consider that $\tilde{G}$ has a natural choice of maximal torus and Borel $\tilde{T} \subset \tilde{B} \subset \tilde{G}$, implying that identification with $\hat{G}$ may only vary by conjugation by $\tilde{T}$. Indeed, let $\tilde{T} \subset \tilde{G}$ consist of those fiber functor automorphisms that preserve weight spaces of all representations, and $\tilde{B} \subset \tilde{G}$ consist of those fiber functor automorphisms that preserve the positive cone of weight spaces. In particular, $\tilde{B}$ is generated by $\tilde{T}$ and $\tilde{U}_{\alpha^\vee}$ for $\alpha^\vee \in \Phi^{\vee,+}$, where we have $\tilde{U}_{\alpha^\vee}$ defined by the property

$$g(F_\lambda IC_\mu) \subset F_{\lambda+\alpha^\vee} IC_\mu. \quad (14)$$

Since the action of $\tilde{\sigma}$ on V_μ permutes weight spaces according to the action of σ on $X_*(T)$, the group automorphism $\tilde{\sigma}$ preserves $\tilde{T}$ and $\tilde{B}$. And $\tilde{T}$ can be identified with $\hat{T}$, such that $X_*(\tilde{T}) = X_*(\hat{T})$. In particular, $\tilde{\sigma}|_{\tilde{T}} = \hat{\sigma}|_{\hat{T}}$. Similarly, $\tilde{\sigma}$ permutes root subgroups according to the action of σ on $\Phi^\vee$. Indeed, equations (13) and (14) imply that $g \in \tilde{U}_{\alpha^\vee}$ maps to $\tilde{\sigma}(g) \in \tilde{U}_{\sigma(\alpha^\vee)}$.

It remains to be seen that there exists a pinning $\{x_{\alpha_i^\vee}\}_{\alpha_i^\vee \in \Pi^\vee}$ preserved by $\tilde{\sigma}$. However, if such a pinning exists, in each σ -orbit $\eta \subset \Pi^\vee$, one root homomorphism $x_{\alpha_i^\vee} : \mathbb{G}_a \rightarrow \tilde{U}_{\alpha_i^\vee}$ may be determined arbitrarily, with $x_{\sigma(\alpha_i^\vee)} := \tilde{\sigma} \circ x_{\alpha_i^\vee}$. In the case σ acts freely on a simple root, i.e. $|\eta|$ is equal to the order of σ , there is no further obstruction: an arbitrary choice of pinning for one $\alpha_i^\vee \in \eta$ will determine a root homomorphism respected by $\tilde{\sigma}$ for all $\alpha_j^\vee \in \eta$. However, if $\sigma^n(\alpha_i^\vee) = \alpha_i^\vee$ for some n less than the order of σ , we need to know σ^n preserves $\tilde{U}_{\alpha_i^\vee}$ pointwise. It is sufficient to consider the case $n = 1$, as in the following lemma of Hong.

Lemma 6.3 ([8] Lemma 4.3). *Suppose some simple $\alpha_i^\vee \in \Pi^\vee$ is σ -invariant, i.e. $\sigma(\alpha_i^\vee) = \alpha_i^\vee$. Then $\tilde{\sigma}$ fixes $\tilde{U}_{\alpha_i^\vee}$ pointwise.*

Proof. We may assume $\tilde{G}$ is semisimple and almost simple. Indeed, $\tilde{G}^{der} = \tilde{G}_1 \times \cdots \times \tilde{G}_m$ where each $\tilde{G}_j$ is semisimple and almost simple, and the inclusion $\tilde{U}_{\alpha_i^\vee} \hookrightarrow \tilde{G}$ factors through some $\tilde{G}_j \rightarrow \tilde{G}^{der} \rightarrow \tilde{G}$. If σ preserves $\alpha_i^\vee$, this inclusion commutes with $\tilde{\sigma}$. So suppose $\tilde{G} = \tilde{G}_j$.

We compare two different actions of σ on $\tilde{\mathfrak{g}}$, the Lie algebra of $\tilde{G}$. The first, $d\tilde{\sigma}$, comes from differentiating the automorphism $\tilde{\sigma}$ at the identity. The second, $\tilde{\sigma}_{\gamma^\vee}$, comes from viewing $\tilde{\mathfrak{g}}$ as the representation $V_{\gamma^\vee}$, where $\gamma^\vee \in \Phi^\vee$ is the highest coroot. As noted earlier, $\tilde{\sigma}_{\gamma^\vee}$ fixes the weight space $F_{\alpha_i^\vee} IC_{\gamma^\vee} = \tilde{\mathfrak{g}}_{\alpha_i^\vee}$ pointwise, so it is sufficient to prove that $d\tilde{\sigma} = \tilde{\sigma}_{\gamma^\vee}$, as tangent space isomorphisms, implying the automorphism $\tilde{\sigma}$ fixes $\tilde{U}_{\alpha_i^\vee}$ pointwise.

For each $\alpha^\vee \in \Phi^\vee$, let $e_{\alpha^\vee}$ be the fundamental class of the (unique) MV cycle of coweight $(\alpha^\vee, \gamma^\vee)$ in Gr_G . Note that for σ -invariant $\alpha^\vee$, we have $\tilde{\sigma}_{\gamma^\vee}(e_{\alpha^\vee}) = e_{\alpha^\vee}$. We can also identify $\tilde{\mathfrak{h}} = \text{Lie}(\tilde{T})$ with $X_*(\tilde{T}) \otimes_{\mathbb{Z}} \mathbb{C}$, to understand the action of $d\tilde{\sigma}$ on $\tilde{\mathfrak{h}}$.

Schur's lemma implies that the two maps may only differ by a constant scalar: let $d\tilde{\sigma} = c \cdot \tilde{\sigma}_{\gamma^\vee}$. Furthermore, by commutativity of diagram (13), differentiating the adjoint action of $\tilde{G}$ on $\tilde{\mathfrak{g}}$, we have $\tilde{\sigma}_{\gamma^\vee}([a, b]) = [d\tilde{\sigma}(a), \tilde{\sigma}_{\gamma^\vee}(b)]$.

The highest root $\gamma \in \Phi$ is σ -invariant, so $\tilde{\sigma}_{\gamma^\vee}$ fixes the image of $\gamma : \mathbb{G}_m \rightarrow \tilde{T}$ pointwise and $d\tilde{\sigma}$ fixes $\mathbb{C} \cdot \gamma \subset \tilde{\mathfrak{h}}$ as well. So $d\tilde{\sigma}([e_{\gamma^\vee}, e_{-\gamma^\vee}]) = [e_{\gamma^\vee}, e_{-\gamma^\vee}] \in \mathbb{C} \cdot \gamma$. Since $d\tilde{\sigma}$ is a Lie algebra homomorphism, we also have

$$d\tilde{\sigma}([e_{\gamma^\vee}, e_{-\gamma^\vee}]) = [d\tilde{\sigma}(e_{\gamma^\vee}), d\tilde{\sigma}(e_{-\gamma^\vee})] = c^2 \cdot [\tilde{\sigma}_{\gamma^\vee}(e_{\gamma^\vee}), \tilde{\sigma}_{\gamma^\vee}(e_{-\gamma^\vee})] = c^2 \cdot [e_{\gamma^\vee}, e_{-\gamma^\vee}].$$

The last equality holds by σ -invariance of $\gamma^\vee$ and $-\gamma^\vee$. And so $c^2 = 1$, and we must have $c = \pm 1$.

Now by comparing trace of $\tilde{\sigma}_{\gamma^\vee}$ and $d\tilde{\sigma}$ on $\tilde{\mathfrak{h}}$, we see $c \neq -1$. In particular, $\tilde{\sigma}_{\gamma^\vee}$ preserves $\tilde{\mathfrak{h}} = F_0 IC_{\gamma^\vee}$, so equation (1) implies that $\text{tr}(\tilde{\sigma} | \tilde{\mathfrak{h}}) \geq 0$. Similarly, since σ acts on $X^*(T) = X_*(\tilde{T})$ by permutation of characters forming a basis, we have $\text{tr}(d\tilde{\sigma} | X_*(\tilde{T}) \otimes_{\mathbb{Z}} K) \geq 0$ as well. But by assumption, there is a σ -invariant simple coroot $\alpha_i^\vee$, and so $\text{tr}(d\tilde{\sigma} | X_*(\tilde{T}) \otimes_{\mathbb{Z}} K) > 0$. So the ratio of those two traces, $c = \text{tr}(d\tilde{\sigma}) / \text{tr}(\tilde{\sigma}_{\gamma^\vee})$, must be nonnegative, hence $c = 1$. $\square$

7 Root data and an application

Here I will be explicit about the root datum and pinning of $\widehat{G}$ and $\widehat{G^{\sigma, \circ}}$, both in terms of the pinned root datum tuple, and more simply the root system. I will also prove Theorem 7.7 of [7], which was originally stated without proof.

Suppose G is a quasi-split, connected, reductive group over a non-Archimedean local field F . Then let Σ be the root system of G and let $\check{\Sigma}$ be the échelonnage root system of $G_{\check{F}}$, as defined in [7]. By Corollary 5.3 in [7], $\check{\Sigma}^\vee$ is the root system for $\widehat{G}^{I,\circ}$, where I is the inertia group of F . Then, in light of Theorem 0.1, to prove Theorem 7.7 of [7] it is sufficient to prove that $N'_\tau(\check{\Sigma}^\vee)$ is equal to the set of roots of $\widehat{G}^{I,\circ}_{\tau,\circ}$.

We will work with pinned root data. According to the classification of connected reductive groups (see, for instance, [15]), a group H over an algebraically closed field is determined up to isomorphism by its root datum: a quadruple $(X, \Phi, X^\vee, \Phi^\vee)$, where X and $X^\vee$ are dual, finitely generated, free abelian groups; and $\Phi \subset X$ and $\Phi^\vee \subset X^\vee$ are dual reduced root systems. The group H is determined up to inner automorphism if the root datum is associated to a particular choice of maximal torus $T \subset H$. If the root datum is based, or endowed with a system of simple roots and coroots $\Pi \subset \Phi$ and $\Pi^\vee \subset \Phi^\vee$ corresponding to a choice of Borel $T \subset B \subset H$, then the datum determines H up to inner automorphism by an element of T . Since the systems of roots and coroots can be constructed from the quadruple $(X, \Pi, X^\vee, \Pi^\vee)$, there is no need to give the entire 6-tuple. Finally, H is determined up to unique automorphism by a pinning: a collection of root homomorphisms $x_{\alpha_i} : \mathbb{G}_a \rightarrow SL_2 \rightarrow H$ for $\alpha_i \in \Pi$.

When we say σ preserves a pinning of a connected reductive group H over an algebraically closed field K , we mean that if H has pinned root datum $(X, \Pi, X^\vee, \Pi^\vee, \{x_{\alpha_i}\})$, then σ preserves $T \subset B \subset H$, and $\sigma \circ x_{\alpha_i} = x_{\sigma(\alpha_i)}$ for each $\alpha_i \in \Pi$. Then σ also acts on X , Π , $X^\vee$, and $\Pi^\vee$, preserving Dynkin diagram edges and abelian group structure.

Proposition 7.1. *Let G be a complex, connected, reductive group with pinning $T \subset B \subset G$ and $\{x_{\alpha_i}\}_{\alpha_i \in \Pi}$. Let the corresponding pinned root datum be denoted $(X, \Pi, X^\vee, \Pi^\vee, \{x_{\alpha_i}\})$. Let σ be an automorphism of G preserving its pinning. Then the fixed point subgroup $G^\sigma \subset G$ is a closed subgroup, and is reductive. The neutral component $G^{\sigma,\circ} \subset G^\sigma \subset G$ is also a closed subgroup and a connected, reductive group. The root datum of $G^{\sigma,\circ}$ is $(X_\sigma/\text{tor}, \text{res}_\sigma(\Pi), (X^\vee)^\sigma, N'_\sigma(\Pi^\vee), \{x_{\alpha_\eta}\})$, where*

- i. X_σ is the group of σ -coinvariants $X/\langle x - \sigma(x) \rangle$. X_σ/tor is the quotient by all torsion elements.
- ii. For every orbit of simple roots $\eta \subset \Pi$, there is a single root α_η equal to the image of any $\alpha_i \in \eta$ under the quotient map $X \rightarrow X_\sigma/\text{tor}$. Then $\text{res}_\sigma(\Pi) = \{\alpha_\eta \mid \eta \in \Pi/\sigma\}$.
- iii. $(X^\vee)^\sigma \subset X^\vee$ is the subgroup of σ -invariant cocharacters.
- iv. For every σ -orbit $\eta \subset \Pi^\vee$, we have $\alpha_\eta^\vee = \sum_{\alpha_i^\vee \in \eta} \alpha_i^\vee$ in the case η consists of pairwise disconnected simple roots, and $\alpha_\eta^\vee = 2(\sum_{\alpha_i^\vee \in \eta} \alpha_i^\vee)$ in the case η consists of a pair of simple roots connected by an edge. Then $N'_\sigma(\Pi^\vee) = \{\alpha_\eta^\vee \mid \eta \in \Pi^\vee/\sigma\}$.

Proof. See [17] chapters 7–8. See also [6] and [7]. □

Note that if $\pi : X \rightarrow X_\sigma/\text{tor}$ and we have a character $\lambda \in X_\sigma/\text{tor}$ and cocharacter $\mu \in (X^\vee)^\sigma$, then $\langle \lambda, \mu \rangle$ is given by $\langle \tilde{\lambda}, \mu \rangle$, where $\tilde{\lambda}$ is any lift $\pi(\tilde{\lambda}) = \lambda$.

Proposition 7.2. *The group of σ -invariants W^σ is the Coxeter group generated by s_η , where for each orbit η of simple roots, s_η is the longest element of the group generated by simple reflections in η .*

Furthermore, W^σ is the Weyl group for $G^{\sigma, \circ}$, i.e. $W^\sigma = N_G(T^{\sigma, \circ})/T^{\sigma, \circ}$.

Proof. See [17] and [6]. □

We define $\text{res}_\sigma(\Phi)$ and $N'_\sigma(\Phi^\vee)$ for the root system and coroot system of $G^{\sigma, \circ}$ as follows:

$$\text{res}_\sigma(\Phi) := W^\sigma \cdot \text{res}_\sigma(\Pi), \quad \text{and} \quad N'_\sigma(\Phi^\vee) := W^\sigma \cdot N'_\sigma(\Pi^\vee).$$

Proposition 7.3. *The half sum of positive coroots is equal for G and $G^{\sigma, \circ}$: $\rho^\vee = \rho_\sigma^\vee$.*

Proof. Recall $\rho^\vee$ is the sum of fundamental coweights, or dual vectors to the simple roots under the natural perfect pairing $\langle \cdot, \cdot \rangle$. For $\alpha_i \in \Pi$, let λ^i be the corresponding fundamental coweight. Similarly, for $\alpha_\eta \in \text{res}_\sigma(\Pi)$, let λ^η be the corresponding fundamental coweight. It is sufficient to show that for each η ,

$$\lambda^\eta = \sum_{i \in \eta} \lambda^i.$$

For any two orbits ζ and η ,

$$\langle \alpha_\zeta, \sum_{i \in \eta} \lambda^i \rangle = \sum_{i \in \eta} \langle \alpha_j, \lambda^i \rangle = \delta_{\zeta, \eta},$$

where α_j is any of the simple roots mapping to α_ζ . This works regardless of type of orbits, since res_σ treats all simple roots uniformly. □

Note that $\rho_\sigma \neq \rho$. In particular, if λ is a σ -invariant cocharacter, then in general $|\langle \rho_\sigma, \lambda \rangle| \leq |\langle \rho, \lambda \rangle|$. As a result, by Theorem 2.1 i., $\dim(A^\sigma) \leq \dim A$ for a σ -invariant AMV cycle A .

Now we can prove the following theorem, which appears in [7] as Theorem 7.7, although it is not proved in full generality there.

Theorem 7.4. *Let G be a quasi-split, connected reductive group over a non-Archimedean local field F with inertia group I and geometric Frobenius τ . Let $\widehat{G}$ be the complex dual of G , and let $\widehat{G}^I$ be the fixed point subgroup of $\widehat{G}$, and $\widehat{G}^{I, \circ}$ the neutral component. Let the root system of $\widehat{G}^{I, \circ}$ be denoted $\Sigma^\vee$. Then τ is an outer automorphism of $\widehat{G}^{I, \circ}$ preserving the natural pinning. Let $V_{\lambda, \xi}$ be the highest-weight representation of $\widehat{G}^{I, \circ} \rtimes \langle \tau \rangle$ where τ acts by the scalar $\xi \in \mathbb{C}^\times$ on weight spaces associated to weights $\nu \in W^\tau \cdot \lambda$.*

Let $\lambda \in X^*(\widehat{T}^{I,\circ})^{+,\tau}$ be a dominant, τ -invariant character of $\widehat{T}^{I,\circ}$. There is an equality

$$\sum_{\nu \in \text{Wt}(\lambda)^\tau} \text{tr}(\tau | V_{\lambda,1}(\nu)) e^\nu = \sum_{w \in W^\tau} w \left(\prod_{\alpha \in N'_\tau(\check{\Sigma}^\vee)^+} \frac{1}{1 - e^{-\alpha}} \right) e^{w(\lambda)}.$$

Proof. The group $\widehat{G}^{I,\circ}$ is connected and reductive. And since G is quasi-split over F , τ acts on $\widehat{G}^{I,\circ}$, preserving the natural pinning. Then the theorem follows from Theorem 0.1 by two observations. First, $V_{\lambda,1} = V_\lambda$ as a vector space and carries the same normalized τ -action as that described in Section 6.

And second, $N'_\tau(\check{\Sigma}^\vee)$ is the set of roots of $\widehat{\widehat{G}^{I,\circ}}^{\tau,\circ}$. Indeed, $\check{\Sigma}^\vee$ is the set of roots of $\widehat{G}^{I,\circ}$, so it is the set of coroots of the complex dual group $\widehat{\widehat{G}^{I,\circ}}$. Then by Proposition 7.1 iv., $N'_\tau(\check{\Sigma}^\vee)$ is the set of coroots of the neutral component of the τ -fixed subgroup $\widehat{\widehat{G}^{I,\circ}}^{\tau,\circ} \subset \widehat{\widehat{G}^{I,\circ}}$. And finally, by again taking complex dual, $N'_\tau(\check{\Sigma}^\vee)$ is the set of roots of $\widehat{\widehat{\widehat{G}^{I,\circ}}^{\tau,\circ}}$. $\square$

Note that in Theorem 7.4, G is assumed to be quasi-split over a non-Archimedean local field. In particular, G may be ramified. Then the root system $\check{\Sigma}^\vee$ for $\widehat{G}^{I,\circ}$ may be determined combinatorially from the absolute root datum of G , along with the action of the Galois group. In particular, as shown in [7] Theorem 6.8, $N'_\tau(\check{\Sigma}^\vee)$ is equal to the root system $\check{\Sigma}_0^\vee$ appearing in the Lusztig character formula of [12]. This is a necessary ingredient in the proof of Theorem D in [7].

A Geometric Satake equivalence

I use several theorems of [14], along with one from [3], summarized in Theorem 2.1. Collectively, I refer to these statements as the geometric Satake equivalence. In wording more similar to that used by the original sources, I have the following:

Theorem A.1 ([14] Theorem 3.2). *a) The intersection $S_e^\nu \cap \text{Gr}_G^\mu$ is nonempty precisely when $\varpi^\nu \in \overline{\text{Gr}_G^\mu}$ and then $S_e^\nu \cap \overline{\text{Gr}_G^\mu}$ is of pure dimension $\langle \rho, \mu + \nu \rangle$, if μ is chosen dominant.*

b) The intersection $S_{w_0}^\nu \cap \text{Gr}_G^\mu$ is nonempty precisely when $\varpi^\nu \in \overline{\text{Gr}_G^\mu}$ and then $S_{w_0}^\nu \cap \overline{\text{Gr}_G^\mu}$ is of pure dimension $-\langle \rho, \mu + \nu \rangle$, if μ is chosen anti-dominant.

Theorem A.2 ([14] Theorem 3.5). *For all $\mathcal{A} \in P_{L+G}(\text{Gr}_G, K)$ there is a canonical isomorphism*

$$H_c^k(S_e^\nu, \mathcal{A}) \xrightarrow{\sim} H_{S_{w_0}^\nu}^k(\text{Gr}_G, \mathcal{A})$$

and both sides vanish for $k \neq 2\langle \rho, \nu \rangle$.

In particular, the functors $F_\nu : P_{L+G}(\mathrm{Gr}_G, K) \rightarrow \mathrm{Mod}_K$, defined by

$$F_\nu := H_c^{2\langle \rho, \nu \rangle}(S_e^\nu, -) = H_{S_{w_0}^\nu}^{2\langle \rho, \nu \rangle}(\mathrm{Gr}_G, -),$$

are exact.

Theorem A.3 ([14] Theorem 3.6).

$$\mathbb{H}^\bullet \cong \bigoplus_{\nu \in X_*(T)} F_\nu = \bigoplus_{\nu \in X_*(T)} H_c^{2\langle \rho, \nu \rangle}(S_e^\nu, -) : P_{L+G}(\mathrm{Gr}_G, K) \rightarrow \mathrm{Vec}_K$$

Proposition A.4 ([14] Proposition 3.10). *Let R be a Noetherian ring of finite global dimension. There is a canonical identification*

$$H_c^{2\langle \rho, \lambda \rangle}(S_e^\lambda, IC_\mu(R)) \cong H_c^{2\langle \rho, \mu - \lambda \rangle}(S_e^\lambda \cap \mathrm{Gr}_G^\mu, R) \cong R[\mathrm{Irr}(S_e^\lambda \cap \overline{\mathrm{Gr}_G^\mu})],$$

here $R[\mathrm{Irr}(S_e^\lambda \cap \overline{\mathrm{Gr}_G^\mu})]$ stands for the free R -module generated by the irreducible components of $S_e^\lambda \cap \overline{\mathrm{Gr}_G^\mu}$.

Note that [14] Theorem 3.10 is proved using a constant sheaf on the smooth variety $S_e^\lambda \cap \mathrm{Gr}_G^\mu$, as written above. The second isomorphism thus follows from the bijection between $\mathrm{Irr}(S_e^\lambda \cap \mathrm{Gr}_G^\mu)$ and $\mathrm{Irr}(S_e^\lambda \cap \overline{\mathrm{Gr}_G^\mu})$.

Theorem A.5 ([14] Theorem 12.1). *The group scheme $\tilde{G}_\mathbb{Z}$ is the split reductive group scheme over $\mathbb{Z}$ whose root datum is dual to that of G .*

Here $\tilde{G}_\mathbb{Z}$ is the group over $\mathbb{Z}$ whose category of representations is isomorphic as a tensor category to $P_{L+G}(\mathrm{Gr}_G, \mathbb{Z})$. Mirković and Vilonen construct $\tilde{G}_\mathbb{Z}$ as a $\mathbb{Z}$ -scheme so a result analogous to Theorem A.5 will hold, by base change, for coefficients in any Noetherian ring R of finite global dimension. In particular, the complex group with dual root datum is $\widehat{G}$, and so $\widehat{G} \cong \tilde{G} := \mathrm{Spec} \mathbb{C} \times_{\mathbb{Z}} \tilde{G}_\mathbb{Z}$ has a representation category isomorphic to $P_{L+G}(\mathrm{Gr}_G, \mathbb{C})$. Furthermore, $\widehat{G}^{\sigma, \circ} \cong \widetilde{G^{\sigma, \circ}} := \mathrm{Spec} \mathbb{C} \times_{\mathbb{Z}} \widetilde{G^{\sigma, \circ}_\mathbb{Z}}$.

Note that $\tilde{G}$ is naturally endowed with a maximal torus and Borel $\tilde{T} \subset \tilde{B} \subset \tilde{G}$, identifiable using representations of $\tilde{G}$. In particular, consider the representation

$$\tilde{\mathfrak{g}}^{ss} := \mathrm{Lie}([\tilde{G}, \tilde{G}]) = \bigoplus_i \mathbb{H}^\bullet(\mathrm{Gr}_G, IC_{\gamma_i^\vee}),$$

where i runs through the components of the Dynkin diagram of G , and $\gamma_i^\vee$ is the highest coroot in $\Phi_i^\vee$. Then we have a decomposition into weight spaces

$$\tilde{\mathfrak{g}}^{ss} = \bigoplus_i \left(H_c^0(S_{w_0}^0, IC_{\gamma_i^\vee}) \oplus \bigoplus_{\alpha^\vee \in \Phi_i^\vee} H_c^{-2\langle \rho, \alpha^\vee \rangle}(S_{w_0}^{\alpha^\vee}, IC_{\gamma_i^\vee}) \right) = \tilde{\mathfrak{g}}^{ss}(0) \oplus \bigoplus_{\alpha^\vee \in \Phi^\vee} \tilde{\mathfrak{g}}^{ss}(\alpha^\vee).$$

For $g \in \tilde{G}$, we can say $g \in \tilde{T}$ if g preserves all weight spaces of $\tilde{\mathfrak{g}}^{ss}$, and $g \in \tilde{B}$ if g preserves the vector subspace

$$\tilde{\mathfrak{g}}^{ss}(0) \oplus \bigoplus_{\alpha^\vee \in \Phi^\vee, +} \tilde{\mathfrak{g}}^{ss}(\alpha^\vee).$$

Thus if we fix pinnings of $\tilde{G}$ and $\hat{G}$, we can identify the two groups uniquely.

Theorem A.6 ([14] Corollary 13.2). *Let R be a Noetherian ring of finite global dimension. The λ -weight spaces $S_\mu(\lambda)$ and $W_\mu(\lambda)$ of S_μ and W_μ , respectively, can both be canonically identified with the free R -module spanned by the irreducible components of $S_e^\lambda \cap \overline{\text{Gr}_G^\mu}$. In particular, the ranks of these modules can be given by the number of irreducible components of $S_e^\lambda \cap \overline{\text{Gr}_G^\mu}$.*

In the corollary above, S_μ and W_μ are canonical R -representations of $\hat{G}$. In particular, taking coefficients in $R = \mathbb{C}$ (or any other field), we have a natural map $S_\mu \rightarrow W_\mu$, bijective on underlying vector spaces, factoring through the irreducible highest weight representation V_μ . As a result, the underlying vector space of V_μ has a basis indexed by $\text{Irr}(S_e^\lambda \cap \overline{\text{Gr}_G^\mu})$.

Lemma A.7 ([3] Proposition 5 (iii)). *Let $\nu \in X_*(T)$ be such that $\nu \geq 0$. If $\mu \in X_*(T)^+$ is sufficiently dominant, then $S_{w_0}^{\mu-\nu} \cap S_e^\mu = S_{w_0}^{\mu-\nu} \cap \text{Gr}_G^\mu$.*

Here a dominant cocharacter μ may be considered “sufficiently dominant” if, for all simple roots $\alpha \in \Pi$, we have $\langle \alpha, \mu \rangle \geq N$, where N is some positive integer depending on the group G and the cocharacter ν .

These are all the statements necessary to state and prove Theorem 2.1:

Theorem A.8. *Let μ be a dominant cocharacter, and let $\lambda \in \text{Wt}(\mu)$.*

- i. $S_{w_0}^\lambda \cap \overline{\text{Gr}_G^\mu}$ is equidimensional, and $\dim(S_{w_0}^\lambda \cap \overline{\text{Gr}_G^\mu}) = \langle \rho, \mu - \lambda \rangle$
- ii. $S_{w_0}^\lambda \cap S_e^\mu$ is equidimensional, and $\dim(S_{w_0}^\lambda \cap S_e^\mu) = \langle \rho, \mu - \lambda \rangle$
- iii. $\mathbb{H}^\bullet(\text{Gr}_G, IC_\mu) = \bigoplus_{\lambda \in \text{Wt}(\mu)} H_c^{-2\langle \rho, \lambda \rangle}(S_{w_0}^\lambda, IC_\mu) = V_\mu$
- iv. $H_c^{-2\langle \rho, \lambda \rangle}(S_{w_0}^\lambda, IC_\mu) = \bigoplus_{A \in \text{Irr}(S_{w_0}^\lambda \cap \overline{\text{Gr}_G^\mu})} \mathbb{C}[A] = V_\mu(\lambda)$.
- v. $P_{L+G}(\text{Gr}_G, \mathbb{Z})$ is isomorphic as a tensor category to $\text{Rep}_{\mathbb{Z}}(\hat{G})$.

Proof. Statement i. follows immediately from A.1 (b). Note that $-\mu$ is anti-dominant exactly when μ is dominant.

Statement ii. is not a direct consequence of any statement in [14], but it is well-known. One way to see it follows from Lemma A.7 and statement i.. Indeed, note that for $\nu \in X_*(T)$, the translation of semi-infinite cells $\nu : S_w^\eta \rightarrow S_w^{\eta+\nu}$ is an isomorphism of ind-schemes. In particular,

$$\dim(S_{w_0}^\lambda \cap S_e^\mu) = \dim(S_{w_0}^{\lambda+\nu} \cap S_e^{\mu+\nu}).$$

So $\dim(S_{w_0}^\lambda \cap S_e^\mu) = \dim(S_{w_0}^{\lambda+n\rho^\vee} \cap S_e^{\mu+n\rho^\vee})$ for all integers n . For sufficiently large n , the character $\mu + n\rho^\vee$ is sufficiently dominant, with respect to G and $\mu - \lambda$, to satisfy the hypotheses of Lemma A.7. Therefore

$$\dim(S_{w_0}^\lambda \cap S_e^\mu) = \dim(S_{w_0}^{\lambda+n\rho^\vee} \cap S_e^{\mu+n\rho^\vee}) = \dim(S_{w_0}^{\lambda+n\rho^\vee} \cap \mathrm{Gr}_G^{\mu+n\rho^\vee}) = \langle \rho, \mu - \lambda \rangle.$$

Statements iii. and iv. are closely tied together. Theorem A.2 tells us that the global cohomology of L^+G -equivariant perverse sheaves decomposes canonically as a direct sum of cohomology groups with compact support, taken on semi-infinite cells. Specifically, for $\mathcal{A} \in P_{L^+G}(\mathrm{Gr}_G)$,

$$\mathbb{H}^\bullet(\mathrm{Gr}_G, \mathcal{A}) = \bigoplus_{\nu \in X_*(T)} H_c^{2\langle \rho, \nu \rangle}(S_e^\nu, \mathcal{A}).$$

By symmetry between our choice of Borel and its opposite, we have

$$H_c^{2\langle \rho, \nu \rangle}(S_e^\nu, IC_\mu) = H_c^{-2\langle \rho, \nu \rangle}(S_{w_0}^\nu, IC_\mu). \quad (15)$$

And of course by emptiness of the intersection of $S_{w_0}^\lambda \cap \mathrm{Gr}_G^\mu$ for $\lambda \notin Wt(\mu)$, the only λ appearing in the direct sum for $\mathbb{H}^\bullet$ are those contained in $Wt(\mu)$. Thus the first equality of iii. follows. And the first equality of iv. follows from Proposition A.4.

The second equality of iv. follows from Theorem A.6 and equation (15), which in turn implies the second equality of iii..

Statement v. is a restatement of Theorem A.5. □

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