A SCALING LIMIT FOR ADDITIVE FUNCTIONALS

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ABSTRACT. Inspired by models for synchronous spiking activity in neuroscience, we consider a scaling-limit framework for sequences of strong Markov processes. Within this framework, we establish the convergence of certain additive functionals toward Lévy subordinators, which are of interest in synchronous input drive modeling in neuronal models. After proving an abstract theorem in full generality, we provide detailed and explicit conclusions in the case of reflected one-dimensional diffusions. Specializing even further, we provide an in-depth analysis of the limiting behavior of a sequence of integrated Wright-Fisher diffusions. In neuroscience, such diffusions serve to parametrize synchrony in doubly-stochastic models of spiking activity. Additional explicit examples involving the Feller diffusion and the Brownian motion with drift are also given.

1. INTRODUCTION

1.1. **Neuroscientific motivation.** This work is concerned with constructing and characterizing scaling limits of certain additive functionals of reflected diffusions and more general strong Markov processes that feature in a variety of applied fields including statistical inference [TJ07; GG11; BJP12], economics and finance [Kru91; LS07; Lin05], queuing theory [Kin61; Har88; RR08] and mathematical biology [RS87].

Our primary motivation, however, stems from mathematical neuroscience. Within that field, a leading approach to modeling neural networks posits that neuronal state variables obey systems of coupled stochastic differential equations [GK02; Izh07]. In that approach, these neuronal state variables model membrane voltages and evolve continuously in time, whereas the interactions coupling these variables occur in an impulse-like fashion, by exchanging spikes among neurons. A core conundrum in mathematical neuroscience is understanding the relationship between the structure of neural networks and the regime of spiking activity that they support. Addressing this question hinges on producing a repertoire of probabilistic spiking models that is rich enough to reproduce realistic spiking activity, but also simple enough to drive stochastic dynamics that are amenable to analysis [GN09].

One such recently proposed stochastic model [Bec+24] captures synchrony in spiking neurons by using mixtures of iid distributions. More precisely, it models spiking configurations of K

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neurons as random vectors $(B_1, \dots, B_K) \in \{0,1\}^K$ with probability law

$$\mathbb{P}\left[B_1 = b_1, \dots, B_K = b_K\right] = \mathbb{E}\left[\prod_{k=1}^{K} Z^{b_k}(1-Z)^{1-b_k}\right], \tag{1.1}$$

where Z has the distribution $F(dz)$ (called the mixing measure) supported by $[0,1]$. Thus, F represents the probability distribution of the fraction Z of coactivating inputs. The more dispersed the distribution F, the more synchronous the spiking activity, a phenomenon that can be quantified by remarking that $\mathrm{Cov}[B_k, B_l] = \mathrm{Var}[Z]$, $k \neq l$, so that the pairwise spiking correlation satisfies

$$\rho = \mathrm{corr}\left[B_k, B_l\right] = \mathrm{Var}\left[Z\right] / (\mathbb{E}\left[Z\right](1 - \mathbb{E}\left[Z\right])) \text{ for } k \neq l.$$

As the model is exchangeable with respect to the neuron indices, it is enough to focus on the total number of spiking neurons $S = \sum_{k=1}^{K} B_k$. Moreover, to make the model dynamic, we replace S by a sequence $\{S_j\}_{j\in\mathbb{N}}$ of iid copies of S, where j plays the role of time.

Next, we pick a family of mixing distributions $\{F^\varepsilon\}_{\varepsilon>0}$ on $[0,1]$ whose means scale linearly with ε as $\varepsilon \searrow 0$, and construct the family $\{S_j^\varepsilon\}_{j\in\mathbb{N}}$, $\varepsilon > 0$ as above. The scaling limit

$$Y(t) = \lim_{\varepsilon\searrow 0} \sum_{j=1}^{\lfloor t/\varepsilon \rfloor} S_j^\varepsilon, t \geq 0, \tag{1.2}$$

can be shown to be a compound Poisson process whose jumps come at rate $\lim_{\varepsilon\searrow 0}(1-\mathbb{P}[S^\varepsilon = 0])/\varepsilon$, with the size J of each jump distributed as $\mathbb{P}[J = k] = \lim_{\varepsilon\searrow 0} \mathbb{P}[S^\varepsilon = k \mid S^\varepsilon > 0]$ for $k = 1, \dots, K$. The limiting spiking correlation can be backed out of this distribution as follows:

$$\lim_{\varepsilon\searrow 0} \rho^\varepsilon = \frac{\mathbb{E}[J(J-1)]}{(K-1)\mathbb{E}[J]}.$$

These compound Poisson processes can then serve as models for synchronous input drive to biophysical neuronal models, where the degree of synchrony is entirely parametrized by the jump distribution. Importantly, such compound-Poisson-process drives are simple enough to allow for the analysis of the resulting driven neuronal dynamics, e.g., via classical point-process techniques [BB13]. For instance, one can derive formulas quantifying the impact of synchrony on the mixed moments of the neuronal voltage responses [Bec+24].

There are two key limitations to the approach of [Bec+24]. First, being computationally oriented, the analysis of [Bec+24] focuses on a single exemplar, namely, a family of mixing measures with beta distributions $F_\varepsilon \sim \mathrm{Be}(\alpha_\varepsilon, \beta)$ with $\alpha_\varepsilon = \beta r\varepsilon/(1 - r\varepsilon)$ and for which $\rho = 1/(1 + \beta)$. In this narrow setting, it is possible to derive the exact jump distribution of the limiting compound Poisson process. However, it is unclear whether one can obtain similar exact results for a wider class of mixing measures, possibly corresponding to more realistic modeling choices. Second, the construction by which compound Poisson processes emerge asymptotically in [Bec+24] is rather unphysical as it assumes perfect independence across time at each step of the scaling process. Such an assumption is contrary to biophysically realistic models of spiking activity which are known to exhibit nonzero correlation time. It would therefore be desirable to obtain asymptotic independence via a limiting procedure involving physically relevant, temporally correlated spiking models rather than artificially independent ones.

A more realistic approach to modelling synchrony in discrete time uses doubly-stochastic models of spiking activity: total spiking counts are defined as random variables $\{S_j\}_{j\in\mathbb{N}}$ with

$$\mathbb{P}\left[S_1 = s_1, \dots, S_J = s_J\right] = \mathbb{E}\left[\prod_{j=1}^{J} \binom{K}{s_j} Z_j^{s_j} (1 - Z_j)^{1-s_j}\right], \tag{1.3}$$

for $J \in \mathbb{N}$ and $s_1, \dots, s_J \in \{1, \dots, K\}$, where $Z_j = \int_{j-1}^{j} X_t \, dt$, $j \in \mathbb{N}$ and $\{X_t\}_{t\geq 0}$ is a continuous-time process with values in $[0, 1]$, for instance the Wright-Fisher diffusion which will be treated in detail later in the paper.

Given this setting, the primary motivation for this work is to understand under which conditions more realistic doubly-stochastic models such as (1.3) yield compound Poisson scaling limits. To formulate this problem more generally, let us remark that for the independent spiking models (1.1) the process X is constant on intervals $[j-1, j)$ with iid F^ε-distributed values. In the properly chosen scaling limit, with F^ε taken to be a beta distribution parameterized as above, dZ^ε converges towards the random measure dA on $[0, \infty)$ as $\varepsilon \searrow 0$, such that A is the Lévy subordinator with jump measure $r\beta x^{-1} (1 - x)^{\beta-1} \, dx$. It is straightforward to show that given a discrete spiking generation mechanism, e.g., a binomial or Poisson random generator, the limit compound Poisson processes $Y(t)$ emerge as scaling limits if and only if the limit mixing process is a nondecreasing Lévy process. Since the mixing process of the doubly-stochastic model (1.3) is given by the additive functional $t \mapsto \int_0^t X_s \, ds$ associated to the underlying Markov process X, the general form of the problem at stake is to determine under which conditions scaling additive functionals of Markov processes admit a Lévy subordinator as a limit.

1.2. **Our contributions.** The main focus of this work is the derivation of a class of scaling limits which, in special cases, apply to models that address some of the above-mentioned limitations of [Bec+24]. Before we describe them, let us outline the major mathematical contributions of the paper.

1.2.1. *The abstract convergence theorem.* In their most general form, our results establish conditions under which scaling limits of nonnegative additive functionals of stationary reflected diffusions and, even more generally, stationary strong Markov processes converge to Lévy subordinators. While we give a formal definition at the beginning of Section 2, we remark here that, informally speaking, an additive functional $\{A_t\}_{t\in[0,\infty)}$ of a Markov process $\{X_t\}_{t\in[0,\infty)}$ is a generalization of the integral functional $t \mapsto \int_0^t g(X_s) \, ds$ where g is deterministic, and includes, e.g., the local time functional (which corresponds, formally, to the case $g = \delta_{\{0\}}$ when X is a Brownian motion and the local time is accumulated at 0).

Visit times of a recurrent point x_0 of a Markov process X split its trajectory into a sequence of independent excursions. This independence property then transfers to the increments of an additive functional A of X between visits to x_0. This observation suggests a limiting regime in which the visits to x_0 are “encouraged” by, for example, a time speedup. As the limit is approached, one has better and better mixing properties (in the sense of dynamical systems) which leads to the convergence of the additive functional to a process with independent increments, i.e., a Lévy process. These ideas are formalized in Theorem 2.1, with the most delicate task being the identification of an appropriate topology on the path space. Indeed, the standard Skorokhod's J_1-topology happens to be too strong as we expect sequences of continuous processes (such as integrals of diffusions) to converge towards a discontinuous process (a non-deterministic Lévy subordinator) in our setting. It turns out that another topology introduced by Skorokhod in [Sko56], namely, the M_1-topology, accomplishes the task. Indeed, our main, abstract, result establishes weak-M_1 convergence of a sequence of nondecreasing additive functionals of a sequence of general strong Markov processes towards a Lévy subordinator, provided that three conditions are met. The first one is of technical nature and requires a uniform bound on the moduli of continuity of the expectations of the additive functionals. The second makes sure that the recurrent point x_0 gets visited with higher and higher frequency, while the third concerns the convergence of the one-dimensional distributions of the additive functionals. In addition to the convergence result itself, our theorem gives an expression for the Laplace exponent of the limiting Lévy subordinator.

To the best of our knowledge, the convergence of additive functionals of Markov processes to subordinators has not been systematically studied in the literature. We do draw attention to the recent work [Bét23] where convergence towards a stable Lévy process for sequences of integrated one-dimensional diffusions is shown. In addition, related work on the convergence of additive functionals, by [JKO09] in the Markov chain setting and by [CCG21] on the functional central limit theorems for diffusions should be mentioned.

1.2.2. *Reflected diffusions.* Following our abstract result, we specialize to stationary one-dimensional reflected diffusions and give sufficient conditions (Theorem 3.2) on the sequence of characteristics of the diffusions (speed measures and scale functions) and the characteristics of the additive functionals (Revuz or representing measures). These are stated explicitly in terms of the sequence of fundamental solutions associated with the diffusions killed at "rates" dictated by the additive functionals. Through this killing operation we combine each diffusion in the sequence with its associated additive functional into a single killed diffusion which can than be analyzed by analytic means. As a result, we derive a purely analytic criterion for convergence in this framework, and give an expression for the Laplace exponent of the limiting subordinator.

1.2.3. *The Wright-Fisher diffusion.* Next, we return to our original application and consider a model specified by (1.3) above, i.e., more precisely, the mixing process Z where the background process X is a reflecting diffusion. In an effort to retain the marginal beta distribution, as in the approach of [Bec+24], we choose the scaled stationary Wright-Fisher diffusion with three parameters α, β and τ as a model for our background parameter X. The first two, α and β dictate the shape of the marginal beta distribution, while τ plays a role in scaling and time correlation. The biophysically relevant regime where β is fixed, while $\alpha^\varepsilon \to 0$ and $\tau^\varepsilon \to 0$ at the same rate is adopted and the additive functionals are given by the integrals $\int_0^t X^\varepsilon(u)\,du$. Using the sufficient conditions of the previous section, we show that the limiting subordinator is a process whose Laplace functional can be expressed in terms of a quotient of modified Bessel functions of the first kind. We stress that our analysis is made significantly more complicated by the fact that explicit expressions for the fundamental solutions are not available for the killed Wright-Fisher diffusion. Our approach is via series expansions where we show that one can pass to the appropriate limit on the level of functions by passing to the limit coefficient-by-coefficient. This, in turn, can be accomplished by observing that the coefficients come as solutions to inhomogeneous but linear second-order recursive equations. The coefficients of these recursive relations admit simple limits which then serve as the coefficients of the limits recursion whose solution can be expressed in terms of hypergeometric and/or Bessel functions.

The limiting subordinator we obtain in this case has not been studied extensively in the literature, to the best of the authors' knowledge. Interestingly it has been featured in [PY03, eq. (48), p. 12] where it is shown to admit a representation in terms of a time-changed occupation time of a Bessel process. The last contribution of the section is a detailed study of various properties of this subordinator. There we determine the range of finite moments of the jump measure, give an explicit expression for its jump density in terms of the positive zeros of the Bessel function, provide a computationally efficient recursive formula for its moments, express its cumulants in terms of the Rayleigh function, and exhibit an unexpectedly simple continued-fraction expansion of the Laplace exponent.

1.2.4. *Additional examples.* Besides our main example, the Wright-Fisher diffusion, we treat explicitly two more. One involves Feller diffusions and the other the Brownian motion with drift in specific, interesting, limiting regimes. In both situations, the increasing and decreasing fundamental solutions of the killed processes admit explicit expressions in terms of special functions: the Kummer U-function for the Feller diffusion and the Airy A and B functions for the Brownian motion with drift. This simplifies the analysis and leads to thought-provoking findings; while the limit in the Feller case is the inverse-Gaussian subordinator, it degenerates to a determistic linear function even in the only natural scaling regime for the Brownian motion with drift.

2. Convergence towards a Lévy subordinator

Our first task is to prove an abstract convergence theorem for strong Markov processes. Even though all applications later in the paper fit into a diffusion framework, this level of generality allows us to better highlight the key properties we need, and provide a basis for eventual future applications.

For a metric space E, let $D(E)$ be the set of all right-continuous functions $\omega : [0,\infty) \to E$ which admit left limits at all $t > 0$. Such functions are commonly referred as càdlàg(short for "continue à droite, limite à gauche"). $D(E)$ comes naturally equipped with the σ-algebra $\mathcal{D}(E)$ generated by the evaluation maps $X(t) : D(E) \to E$, $X(t)(\omega) = \omega(t)$, as well as with the family $\{\theta(t)\}_{t\in[0,\infty)}$, of shift operators $\theta(t) : D(E) \to D(E)$ given by by $(\theta(t)(\omega))(u) = \omega(t+u)$ for $t, u \geq 0$.

Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of metric spaces, end let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points $x_n \in E_n$. Moreover, for $n \in \mathbb{N}$, let $\mathbb{P}_n$ be a measure on $D(E_n)$, and let $\{\mathcal{F}_n(t)\}_{t\in[0,\infty)}$ be the $\mathbb{P}_n$-completion of the natural filtration of the (canonical) process $X_n = \{X_n(t)\}_{t\in[0,\infty)}$ made up of evaluation maps. on $D(E_n)$.

We assume that X_n is a time-homogeneous strong Markov process under $\mathbb{P}_n$. More precisely, we assume that for each bounded random variable G on $D(E_n)$, there exists a bounded measurable function $\tilde{g}_n : E_n \to \mathbb{R}$ such that for each $\{\mathcal{F}_n(t)\}_{t\in[0,\infty)}$-stopping time τ, we have

$$\mathbb{E}_n[G \circ \theta_n(\tau) \mid \mathcal{F}_n(\tau)] = \tilde{g}_n(X_n(\tau)),\ \mathbb{P}_n\text{-a.s. on } \{\tau < \infty\}, \tag{2.1}$$

where $\mathbb{E}_n[\cdot]$ denotes the expectation operator with respect to $\mathbb{P}_n$. In fact, we only need the Markov property to hold on deterministic times and the following stopping times

$$T_n^{x_n,t} := \inf\{s \geq t \,:\, X_n(s) = x_n\}.$$

Next, let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence nondecreasing additive functionals on $D(E_n)$. More precisely, for $n \in \mathbb{N}$, A_n is an $\{\mathcal{F}_n(t)\}_{t\in[0,\infty)}$-adapted, càdlàg and nondecreasing process with the property that $A_n(0) = 0$ and, for each $s \geq 0$, we have

$$A_n(t + s) = A_n(t) + (A_n(s)) \circ \theta_t \text{ for all } t \geq 0, \mathbb{P}_n\text{-a.s.} \tag{2.2}$$

Before we state the main result of this section, we recall that for each Lévy subordinator (nondecreasing Lévy process) X there exists a nonnegative function Φ—called the Laplace exponent of X—such that $\mathbb{E}[\exp(-\mu X_t)] = \exp(-t\Phi(\mu))$. We refer the reader to [Whi02, Chapter 12] for the definition and the important properties of the Skorokhod's M_1-topology.

Theorem 2.1. *Suppose that the following conditions hold:*

(1) For each $t \geq 0$, $T_n^{x_n,t} \to t$ in distribution as $n \to \infty$.

(2) There exists a function $a : [0,\infty) \to [0,\infty)$, continuous at 0, such that $a(0) = 0$ and, for all $0 \leq s < t < \infty$ and $n \in \mathbb{N}$, we have

$$\mathbb{E}_n[A_n(t) - A_n(s)] \leq a(t - s)$$

(3) There exists a constant $\lambda > 0$ such that the limit

$$R^{\lambda,\mu} = \lim_n \mathbb{E}_n\left[\int_0^\infty \exp\Big(-\lambda t - \mu A_n(t)\Big)\,dt\right] \tag{2.3}$$

exists for all $\mu > 0$.

Then the sequence $\{A_n(t)\}_{t\in[0,\infty)}$ converges in law, under the Skorokhod's M_1-topology, to a Lévy subordinator whose Laplace exponent $\Phi(\mu)$ is given by

$$\Phi(\mu) = \frac{1}{R^{\lambda,\mu}} - \lambda.$$

Proof. For the sake of clarity, we divide the proof into four steps. As it will appear throughout the proof, we define the following shortcut:

$$\tau_n = T_n^{x_n,t},$$

where the dependence on t will always be clear from the context.

Step 1. For $n \in \mathbb{N}$, let $\mathbb{Q}_n$ denote the law of A_n on $D([0,\infty))$. Our first claim is that assumption *(2)* implies that the family $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$ is tight under the M_1 topology on $D([0,\infty))$. It will be enough to prove this fact for the restrictions of our processes to bounded intervals of the form $[0,T)$, $T>0$ (see [Whi02, section 12.9, pp. 414-416]). We base our approach on [Whi02, Theorem 12.12.3, p. 426] which gives two necessary and sufficient conditions, labeled (i) and (ii), for tightness under M_1 on $D([0,T])$. Condition (i) is easily seen to be satisfied in our case because our assumption *(2)* implies that $\sup_n \mathbb{E}_n[A_n(T)]<\infty$. Condition (ii) is related to the modulus of continuity w'_s, which is defined in [Whi02, eq. (12.2), p. 424] as a maximum of three terms. Since our processes are non-decreasing, the second term trivially vanishes. The third term can be safely ignored since it is used to control the behavior at T, a point we left out of our domain $[0,T)$ precisely for this reason. This leaves us with a single term, and the following, simplified version of condition (ii):

$$\forall\,\varepsilon,\eta>0,\ \exists\,\delta>0,\ \forall\,n\in\mathbb{N},\ \mathbb{P}_n[A_n(\delta)>\varepsilon]<\eta.$$

This, however, easily follows from assumption *(2)*, via Markov's inequality, thanks to the continuity of the function a at 0. Therefore, there exists an M_1-weakly convergent subsequence

$$\{\mathbb{Q}_{n_k}\}_{k\in\mathbb{N}} \text{ of } \{\mathbb{Q}_n\}_{n\in\mathbb{N}}, \tag{2.4}$$

and we denote its limit by $\mathbb{Q}$. To keep the notation manageable in the sequel, we do not relabel the convergent subsequence $\{\mathbb{Q}_{n_k}\}_{k\in\mathbb{N}}$ and proceed as if the original sequence $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$ converges. Lastly, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space on which a non-decreasing càdlàg process A, with law $\mathbb{Q}$, is defined.

Step 2. We start by transforming assumption *(1)* to a more useful form. Assumption *(1)* implies that, for each $t\geq 0$, there exists a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}_0}$ in $\mathbb{N}_0$ such that $n_0=0$ and for each $k\in\mathbb{N}$,

$$\mathbb{P}_n[\tau_n>t+(k+1)^{-1}]<(k+1)^{-1} \text{ for all } n>n_k.$$

We then define the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ (which may depend on t) by

$$\varepsilon_n=k^{-1} \text{ for } n_{k-1}<n\leq n_k,\ k\in\mathbb{N}, \tag{2.5}$$

so that $\varepsilon_n\to 0$, as $n\to\infty$. On the other hand, the inequality

$$\mathbb{P}_n[\tau_n>t+\varepsilon_n]=\mathbb{P}_n[\tau_n>t+k^{-1}]<k^{-1}=\varepsilon_n \text{ for } n_{k-1}<n\leq n_k,$$

implies that $\mathbb{P}_n[\tau_n>t+\varepsilon_n]<\varepsilon_n$ for all n. Consequently, we have shown that assumption *(1)* implies the existence of a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\epsilon_n\to 0$ such that

$$\mathbb{P}_n[\tau_n>t+\varepsilon_n]\xrightarrow{n\to\infty}0. \tag{2.6}$$

Step 3. By [Whi02, Theorem 2.5.1, (iv), p. 404] there exists a dense subset $\mathcal{T}$ of $[0,\infty)$, which includes 0, such $A_n\to A$ in the sense of finite-dimensional distributions on $\mathcal{T}$, i.e., such that for all $K\in\mathbb{N}$ and all $t_1,\dots,t_K\in\mathcal{T}$ we have

$$\Big(A_n(t_1),\dots,A_n(t_K)\Big)\xrightarrow{\mathcal{D}}\Big(A(t_1),\dots,A(t_K)\Big). \tag{2.7}$$

We pick $t,\delta\geq 0$ and define the sequence $\{F_n\}_{n\in\mathbb{N}}$ of random variables by

$$F_n=f(A_n(t_1),\dots,A_n(t_K)),\ \text{ for } K\in\mathbb{N} \text{ and } 0\leq t_1\leq\cdots\leq t_K\leq t,$$

where $t_1, \dots, t_K$, t, $t+\delta \in \mathcal{T}$ and $f : \mathbb{R}^K \to \mathbb{R}$ is continuous, bounded and bounded away from 0. For each n and each bounded Lipschitz function $g : \mathbb{R} \to \mathbb{R}$, we have

$$\mathbb{E}_n\Big[F_n g\Big(A_n(t+\delta) - A_n(t)\Big)\Big] = I_n^1 + I_n^2 + I_n^3,$$

where

$$I_n^1 = \mathbb{E}_n\big[F_n\, g\big(A_n(t+\delta) - A_n(t)\big) 1_{\{\tau_n > t+\varepsilon_n\}}\big],$$
$$I_n^2 = \mathbb{E}_n\Big[F_n\, \Big(g\big(A_n(t+\delta) - A_n(t)\big) - g\big(A_n(\tau_n+\delta) - A_n(\tau_n)\big)\Big) 1_{\{\tau_n \le t+\varepsilon_n\}}\Big]$$

and

$$I_n^3 = \mathbb{E}_n\Big[F_n\, \Big(g\big(A_n(\tau_n+\delta) - A_n(\tau_n)\big)\Big) 1_{\{\tau_n \le t+\varepsilon_n\}}\Big],$$

with $\{\varepsilon_n\}_{n\in\mathbb{N}}$ given by (2.5).

Let C denote a generic constant, independent of n, but possibly depending on f and g. As is customary, we allow C to change from occurrence to occurrence. The relation (2.6) above implies that

$$\big|I_n^1\big| \le C\, \mathbb{P}_n[\tau_n > t+\varepsilon_n] \to 0 \text{ as } n \to \infty. \tag{2.8}$$

Moving on to I_n^2, we use condition (2) together with the Lipschitz property of g and boundedness of f, to conclude that

$$\begin{aligned}
\big|I_n^2\big| &\le C\mathbb{E}_n\Big[\Big|g\Big(A_n(t+\delta) - A_n(t)\Big) - g\Big(A_n(\tau_n+\delta) - A_n(\tau_n)\Big)\Big| 1_{\{\tau_n \le t+\varepsilon_n\}}\Big] \\
&\le C\mathbb{E}_n\Big[\Big|A_n(t+\delta) - A_n(t) - A_n(\tau_n+\delta) + A_n(\tau_n)\Big| 1_{\{\tau_n \le t+\varepsilon_n\}}\Big] \\
&\le C\mathbb{E}_n\Big[\Big(|A_n(\tau_n) - A_n(t)| + |A_n(\tau_n+\delta) - A_n(t+\delta)|\Big) 1_{\{\tau_n \le t+\varepsilon_n\}}\Big] \\
&\le C\mathbb{E}_n\Big[\Big(A_n(t+\varepsilon_n) - A_n(t) + A_n(t+\delta+\varepsilon_n) - A_n(t+\delta)\Big)\Big] \\
&\le Ca(\varepsilon_n) \to 0.
\end{aligned}$$

Lastly, by (2.2) and the strong Markov property (2.1), for each $n \in \mathbb{N}$, there exists a bounded and measurable function $\tilde{g}_n : E_n \to \mathbb{R}$ such that

$$\begin{aligned}
I_n^3 &= \mathbb{E}_n\Big[F_n g\Big(A_n(\tau_n+\delta) - A_n(\tau_n)\Big) 1_{\{\tau_n \le t+\varepsilon_n\}}\Big] \\
&= \mathbb{E}_n\Big[F_n g\Big(A_n(\delta) \circ \theta_{\tau_n}\Big) 1_{\{\tau_n \le t+\varepsilon_n\}}\Big] \\
&= \mathbb{E}_n\Big[F_n \mathbb{E}_n\Big[g\Big(A_n(\delta) \circ \theta_{\tau_n}\Big) 1_{\{\tau_n \le t+\varepsilon_n\}} \;\Big|\; \mathcal{F}_n(\tau_n)\Big]\Big] \\
&= \mathbb{E}_n\big[F_n 1_{\{\tau_n \le t+\varepsilon_n\}}\big] \tilde{g}_n(x_n).
\end{aligned}$$

Thanks to (2.7),

$$\mathbb{E}_n[f(A_n(t_1),\dots,A_n(t_K))g(A_n(t+\delta)-A_n(t))] = \lim_n \mathbb{E}_n[Fg(A_n(t+\delta)-A_n(t))] =$$
$$= \lim_n (I_n^1+I_n^2+I_n^3) = \lim_n \mathbb{E}_n\big[F_n 1_{\{\tau_n \le t+\varepsilon_n\}}\big]\tilde{g}_n(x_n).$$

As in (2.8) above, we have $\mathbb{E}_n\big[F_n 1_{\{\tau_n > t+\varepsilon_n\}}\big] \to 0$ so that

$$\lim_n \mathbb{E}_n\big[F_n 1_{\{\tau_n \le t+\varepsilon_n\}}\big] = \mathbb{E}[f(A(t_1),\dots,A(t_K))].$$

Since f is bounded away from 0, we conclude that

$$\frac{\mathbb{E}[f(A(t_1),\dots,A(t_K))g(A(t+\delta)-A(t))]}{\mathbb{E}[f(A(t_1),\dots,A(t_K))]} = \lim_n \tilde{g}_n(x_n). \tag{2.9}$$

As the process A is càdlàg, (2.9) holds for all $K \in \mathbb{N}$, and all $0 \le t_1 \le \dots \le t_K \le t < \infty$, $\delta \ge 0$ — not only those in $\mathcal{T}$. Also, since the right-hand side depends neither on f nor on t, the random variable $A(t+\delta)-A(t)$ is independent of $\sigma(A_s, s \le t)$ and its distribution does not depend on t. In other words, A has stationary and independent increments. Since $A_n(0)=0$ for each n and M_1-convergence implies convergence in distribution at 0, we conclude that $A(0)=0$, as well. Being right-continuous and nondecreasing, A is, therefore, a Lévy subordinator.

Step 4. To close the loop and complete the proof, we use condition *(3)*. The space $D([0,\infty))$ is J_1-separable, where J_1 refers to Skorokhod's J_1 topology (see [Whi02, Section 3.3., p. 78]). Since the M_1 topology is weaker than J_1 (see [Whi02, Theorem 12.3.2, p. 398]), and $D([0,\infty))$ is separable under J_1 (see [Bil99, p. 112]), we conclude that $D([0,\infty))$ is M_1-separable, as well. Therefore, we can use the Skorokhod's representation theorem (see [Whi02, Theorem 3.2.2, p. 78]) to couple the laws of $\{A_n\}_{n\in\mathbb{N}}$ and A on the same probability space such that $A_n \to A$ in M_1, a.s. Next, we remember that, for right-continuous, nondecreasing functions, convergence on a dense set towards a right-continuous function implies convergence at every continuity point of the limit (see, e.g., [Kal21, proof of Theorem 6.20, p. 142], for the standard argument). From there, we conclude that for nondecreasing functions M_1-convergence implies convergence a.e., with respect to the Lebesgue measure. This is enough to establish that given nonnegative, continuous, and bounded function $f:\mathbb{R}^2\to\mathbb{R}$, integral functionals of the form

$$y \mapsto \int_0^t f(u,y_u)\,du, \tag{2.10}$$

are continuous in the M_1-topology when restricted to the set of nondecreasing functions in $D([0,\infty))$. The dominated convergence theorem yields

$$\mathbb{E}_n\Big[\int_0^\infty e^{-\lambda t}e^{-\mu A_n(t)}\,dt\Big] \to \mathbb{E}\Big[\int_0^\infty e^{-\lambda t}e^{-\mu A(t)}\,dt\Big],$$

for all $\lambda > 0$ and $\mu \ge 0$. Combined with condition (2.3), this implies that for some $\lambda > 0$ we have

$$\mathbb{E}\Big[\int_0^\infty e^{-\lambda t}e^{-\mu A(t)}\,dt\Big] = R^{\lambda,\mu} \text{ for all } \mu \ge 0.$$

On the other hand, since A is a Lévy subordinator, we have

$$R^{\lambda,\mu} = \int_0^\infty e^{-\lambda t}\mathbb{E}\Big[e^{-\mu A(t)}\Big]\,dt = \int_0^\infty e^{-\lambda t} e^{-t\Phi(\mu)}\,dt = \frac{1}{\lambda+\Phi(\mu)},$$

where Φ is the Laplace exponent of A. Since Φ completely characterizes the distribution of $A(1)$, and, thus, the law of the entire Lévy process A, we conclude that the limit is the same for each choice of a convergent subsequence in (2.4). This implies that the entire sequence $\{A_n\}_{n\in\mathbb{N}}$ converges in law, under M_1, towards A. □

3. Sequences of Stationary Reflected Diffusions

In this section we derive sufficient conditions for Theorem 2.1 to hold for sequences of stationary reflected one-dimensional diffusions. We refer the reader to [BS02, Chapter II] for a succinct but comprehensive summary of the terminology (such as "entrance" and "exit" boundaries, e.g.) and the standard properties of one-dimensional diffusions, or to the canonical book [IM74] for the complete treatment. In particular, for being Feller ($C_b \to C_b$) processes, diffusions are strong Markov processes and satisfy the pre-conditions for our abstract convergence result of Theorem 2.1.

3.1. **Preliminaries on one-dimensional diffusions.** Assume that X is a regular stationary one-dimensional diffusion X without explosion or killing, with the state space $I = [0, r)$, for some $r \in (0,\infty]$. We denote by $(\mathbb{P}^x)_{x\in I}$ the associated Markov family of probability measures on the canonical space, with $\mathbb{P}^x$ being the law of the process "started" at x at time 0. Non-deterministic initial conditions correspond in the usual way to the mixtures $\mathbb{P}^\nu := \int \mathbb{P}^x\,\nu(dx)$. All of these laws are fully determined by the speed measure m and the strictly increasing and continuous scale function s. We assume that the left endpoint 0 is nonsingular (both "entrance" and "exit") and that $m(\{0\}) = 0$ (instantaneous reflection at 0). The right endpoint may or may not be "entrance", but we do not allow it to be "exit".

Stationarity implies that m coincides with the unique stationary distribution up to a constant, and we assume from now on that this constant is 1, i.e., that m is a probability measure. This normalization immediately singles out a normalization for the increment $s(x) - s(0)$ of the scale function s. Since $0 \in I$, we may (and do) assume that $s(0) = 0$, which, then, completely determines s.

Let T^x denote the first hitting time of the level x for X. Given $\lambda > 0$, the *decreasing fundamental solution* (a.k.a. the decreasing function) φ^0 and the *increasing fundamental solution* (a.k.a. the increasing function) ψ^0 associated with X are defined by

$$\varphi^0(x) = \begin{cases} \mathbb{E}^x\big[e^{-\lambda T^{x_\varphi}}\big], & x \ge x_\varphi,\\ 1/\mathbb{E}^{x_\varphi}\big[e^{-\lambda T^x}\big], & x < x_\varphi,\end{cases} \qquad \psi^0(x) = \begin{cases} \mathbb{E}^x\Big[e^{-\lambda T^{x_\psi}}\Big], & x \le x_\psi,\\ 1/\mathbb{E}^{x_\psi}\big[e^{-\lambda T^x}\big], & x > x_\psi,\end{cases} \tag{3.1}$$

where $x_\varphi, x_\psi \in [0, r)$ are arbitrary, but fixed. In fact, we are only interested in the equivalence classes of φ^0 and ψ^0 modulo equality up to a multiplicative constant. In particular, this makes the choice of the constants x_φ and x_ψ irrelevant.

Let A be a continuous and nondecreasing additive functional of X. More precisely, A denotes a continuous nondecreasing process, defined on the canonical space $D([0,\infty))$, with the property that for each $s \geq 0$ we have

$$A(t+s) = A(t) + A(s) \circ \theta_t, \text{ for all } t \geq 0, \ \mathbb{P}^n\text{-a.s.}$$

One of the main ideas of this section is to use A to "kill" the process X so as to be able to analyze the behavior of both A and X by studying a single, killed diffusion. With that in mind, we let X^μ be the process with the same dynamics as X, but killed at the "rate" $\mu\, dA(t)$, with $\mu > 0$. More precisely, with τ being an exponentially distributed random variable with rate 1, independent of $\{X(t)\}_{t\in[0,\infty)}$ under each $(\mathbb{P}^x)_{x\in I}$, we define

$$T^\mu = \inf\{t \geq 0 \,:\, \mu A(t) \geq \tau\},$$

and then

$$X^\mu(t) = \begin{cases} X(t), & t < T^\mu \\ \Delta, & t \geq T^\mu, \end{cases} \tag{3.2}$$

where Δ is an isolated "cemetery" state added to the state space I. The killed diffusion process X^μ can be equivalently characterized in terms of its killing measure k. This measure, also known as the *representing measure of* A. has the property (see [BS02, par. 23., p. 28]) that

$$A(t) = \int_0^t L(t,y)\, k(dy),$$

where $L(t,y)$ denotes the (diffusion) local time of X at the level y, accumulated up to time t. In the particular case for which we choose $A(t) = \int_0^t g(X_u)\, du$, the definition of the local time as an occupation density with respect to the speed measure m implies that $k(dy) = g(y)\, m(dy)$ (see [BS02, par. 23., p. 28]).

Even though the speed measure and the scale function of X and X^μ are the same, the decreasing and increasing fundamental solutions for the killed process X^μ defined by

$$\begin{aligned} \varphi^\mu(x) &= \begin{cases} \mathbb{E}^x\big[e^{-\lambda T^{x_\varphi}} 1_{\{T^{x_\varphi} < T^\mu\}}\big], & x \geq x_\varphi, \\ 1/\mathbb{E}^{x_\varphi}\big[e^{-\lambda T^x} 1_{\{T^x < T^\mu\}}\big], & x < x_\varphi, \end{cases} \\ \psi^\mu(x) &= \begin{cases} \mathbb{E}^x\Big[e^{-\lambda T^{x_\psi}} 1_{\{T^{x_\psi} < T^\mu\}}\Big], & x \leq x_\psi, \\ 1/\mathbb{E}^{x_\psi}\big[e^{-\lambda T^x} 1_{\{T^x < T^\mu\}}\big], & x > x_\psi, \end{cases} \end{aligned} \tag{3.3}$$

generally differ from their X-related counterparts defined in (3.1). Note that x_φ and x_ψ play the same role in (3.1) and (3.3). To prepare for the statement of Lemma 3.1 below, we remind the reader of the following notation:

$$\frac{d^+}{ds} f(x) = \lim_{\varepsilon \searrow 0} \frac{f(x+\varepsilon) - f(x)}{s(x+\varepsilon) - s(x)} \text{ and } \frac{d^+}{ds} f(x+) = \lim_{y \searrow x} \frac{d^+}{ds} f(y).$$

Lemma 3.1. *Let A be a continuous and nonnegative additive functional of X, let k be its representing measure. For each $x \in I$, $\mu \geq 0$ and $\lambda > 0$, we have*

$$0 \leq \mathbb{E}^x\left[\int_0^\infty e^{-\lambda t-\mu A(t)}\,dt\right] - \frac{\varphi^\mu(x)}{\varphi^\mu(0)}\frac{\int_I \varphi^\mu(y)\,m(dy)}{\mu k(\{0\})\varphi^\mu(0) - \frac{d^+}{ds}\varphi^\mu(0+)} \leq \frac{1}{\lambda}\left(1 - \frac{\varphi^0(x)}{\varphi^0(0)}\right). \tag{3.4}$$

Proof. For $x \in I$, we set

$$R(x) = \mathbb{E}^x\left[\int_0^\infty e^{-\lambda t-\mu A(t)}\,dt\right] \text{ and } E(x) = \mathbb{E}^x\left[\int_0^{T^0} e^{-\lambda t-\mu A(t)}\,dt\right],$$

where T^0 denotes the first-hitting time of level 0 for X. It follows directly from the definition of the fundamental solutions (3.1) of X that

$$0 \leq E(x) \leq \mathbb{E}^x\left[\int_0^{T^0} e^{-\lambda t}\,dt\right] = \frac{1}{\lambda}\Big(1 - \mathbb{E}^x\Big[e^{-\lambda T^0}\Big]\Big) = \frac{1}{\lambda}\left(1 - \frac{\varphi^0(x)}{\varphi^0(0)}\right).$$

On the other hand, we have

$$\begin{aligned} R(x) &= \mathbb{E}^x\left[\int_0^{T^0} e^{-\lambda t-\mu A(t)}\,dt + \int_{T^0}^\infty e^{-\lambda t-\mu A(t)}\,dt\right] \\ &= E(x) + \mathbb{E}^x\left[e^{-\lambda T^0-\mu A(T^0)}\int_{T^0}^\infty e^{-\lambda(t-T^0)-\mu\left(A(t)-A(T^0)\right)}\,dt\right] \\ &= E(x) + \mathbb{E}^x\Big[e^{-\lambda T^0-\mu A(T^0)}\Big]R(0) \end{aligned}$$

where the last equality follows from the strong Markov property. Since $\mathbb{E}^x\big[1_{\{t<T^\mu\}}\,\big|\,\mathcal{F}_t^X\big] = \mathbb{P}^x\big[T^\mu > t\,\big|\,\mathcal{F}_t^X\big] = \exp(-\mu A(t))$, the equality

$$\frac{\varphi^\mu(x)}{\varphi^\mu(0)} = \mathbb{E}^x\Big[e^{-\lambda T^0-\mu A(T^0)}\Big].$$

follows readily from the definition of the fundamental solutions (3.3) of X^μ. Let

$$\begin{aligned} U_\lambda f(x) &= \mathbb{E}^x\left[\int_0^\infty e^{-\lambda t}f(X_t^\mu)\,dt\right] = \mathbb{E}^x\left[\int_0^{T^\mu} e^{-\lambda t}f(X_t)\,dt\right] \\ &= \mathbb{E}^x\left[\int_0^\infty f(X_t)e^{-\lambda t-\mu A(t)}\,dt\right] \end{aligned}$$

be the resolvent operator associated to X^μ, so that $R(x) = U_\lambda f(x)$ for $f \equiv 1$. According to [BS02, Section II.1, par. 10, p. 18], the resolvent operator of the killed diffusion X^μ has a kernel that is absolutely continuous with respect to m. Moreover, we have the following expression for $R(0)$:

$$R(0) = \int_I \tilde{r}(y)\,m(dy), \text{ with } \tilde{r}(y) = \frac{1}{w}\psi^\mu(0)\varphi^\mu(y),$$

where the value of the Wronskian w given by

$$w = \varphi^\mu(x)\frac{d^+}{ds}\psi^\mu(x) - \psi^\mu(x)\frac{d^+}{ds}\varphi^\mu(x),$$

is independent of the choice of $x \in (0, r)$. Since 0 is a reflective (when $k(\{0\}) = 0$) or an elastic (when $k(\{0\}) > 0$) boundary, we have $\lim_{x\to 0} \frac{d^+}{ds}\psi^\mu(x) = \mu k(\{0\})\psi^\mu(0)$, which, in turn, implies that

$$\begin{aligned} w &= \lim_{x\searrow 0}\Big(\varphi^\mu(x)\frac{d^+}{ds}\psi^\mu(x) - \psi^\mu(x)\frac{d^+}{ds}\varphi^\mu(x)\Big) \\ &= \psi^\mu(0)\bigg(\mu k(\{0\})\varphi^\mu(0) - \frac{d^+}{ds}\varphi^\mu(0+)\bigg), \end{aligned}$$

completing the proof of the lemma. □

3.2. **Sufficient conditions for convergence.** Adopting the diffusion framework of the previous subsection, let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of diffusions, and $\{A_n\}_{n\in\mathbb{N}}$ be an associated sequence of continuous and nonnegative additive functionals. We keep all the notation from the previous subsection with an additional subscript $n \in \mathbb{N}$; in particular, the state space of X_n is $I_n = [0, r_n)$, its speed measure m_n, and the representing measure of A_n is k_n.

The main result of this section is Theorem 3.2 below which provides sufficient—and readily verifiable—conditions for Theorem 2.1 to hold within our diffusion framework. The notation $\nu \preceq_1 m$ refers to first-order stochastic dominance, i.e., to the fact that $\int f\, d\nu \leq \int f\, dm$, for all nondecreasing, nonnegative, measurable function f. We remind the reader that the speed measures m_n are always assumed to be finite and normalized so that $m_n(I_n) = 1$ for all n, and denote by T_n^0 the first hitting time of the level 0 by X_n.

Theorem 3.2. *Let $\{\nu_n\}_n$ be a sequence of probability measures on $\{I_n\}_n$ such that, $\nu_n \preceq_1 m_n$. Suppose that*

(1) $\lim_n \int \varphi_n^0(x)\, m_n(dx) = 1$,
(2) $\mathbb{E}_n^{\nu_n}[A_n(t) - A_n(s)] \leq a(t-s)$ for all $n \in \mathbb{N}$ and all $s < t$, for some function $a : [0,\infty) \to [0,\infty)$, continuous at 0,
(3) For each $\mu > 0$
   (a) $\lim_n \int \varphi_n^\mu(x)\, \nu_n(dx) = 1$ and
   (b) the limit $\Phi(\mu) := \lim_n \Phi_n(\mu)$ exists in $\mathbb{R}$, where

$$\Phi_n(\mu) := \mu\frac{\int \varphi_n^\mu(x)\, k_n(dx)}{\int \varphi_n^\mu(x)\, m_n(dx)}, \tag{3.5}$$

where k_n is the representing measure of A_n.

Then the conditions (1), (2) and (3) of Theorem 2.1 are satisfied when X_n is started from ν_n, $E_n = I_n$ and $x_n = 0$. Morover, Φ is the Laplace exponent of the limiting subordinator.

Proof. We start with the condition *(1)* of Theorem 2.1. Let ρ_1, ρ_2 be two probability distributions on $[0,\infty)$ such that ρ_2 dominates ρ_1 in the first-order stochastic sense. This is equivalent to saying that two random variables N_1 and N_2, with distributions ρ_1 and ρ_2 can be defined on the same probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that, a.s., we have $N_1 \leq N_2$. The same space can be extended so that it supports two processes X^{ρ_1} and X^{ρ_2}, with laws $\mathbb{P}_n^{\rho_1}$ and $\mathbb{P}_n^{\rho_2}$, respectively, and with the additional requirement that $X^{\rho_1}(0) = N_1$ and $X^{\rho_2}(0) = N_2$. Denoting the coupling time of X^{ρ_1} and X^{ρ_2} by $\tau = \inf\{t \geq 0 : X^{\rho_1}(t) = X^{\rho_2}(t)\}$, the strong Markov property implies that the process $\tilde{X}^{\rho_1}$ given by

$$\tilde{X}^{\rho_1}(t) = \begin{cases} X^{\rho_1}(t), & t < \tau \\ X^{\rho_2}(t), & t \geq \tau \end{cases},$$

has the same law as X^{ρ_1}, namely $\mathbb{P}_n^{\rho_1}$. Since the paths of X^{ρ_1} and X^{ρ_2} are continuous and $X^{\rho_1}(0) \leq X^{\rho_2}(0)$, we have

$$\tilde{X}^{\rho_1}(t) \leq X^{\rho_2}(t) \text{ for all } t \geq 0, \ \tilde{\mathbb{P}}\text{-a.s.} \tag{3.6}$$

Taking $\rho_1 = \delta_{x_1}$ and $\rho_2 = \delta_{x_2}$ with $x_1 \leq x_2$, allows us to conclude that the map $x \mapsto \mathbb{P}_n^x[T_n^0 \geq \varepsilon]$ is nondecreasing. We then combine this fact with (3.6), but now applied to $\rho_1 = \nu_n$ and $\rho_2 = m_n$, to obtain the following estimate:

$$\mathbb{P}_n^{\nu_n}[T^{x_n,t} \geq t + \varepsilon] = \mathbb{E}_n^0\Big[\mathbb{P}_n^{X_n(t)}[T_n^0 \geq \varepsilon]\Big] = \tilde{\mathbb{E}}\Big[\mathbb{P}_n^{\tilde{X}^{\nu_n}(t)}[T_n^0 \geq \varepsilon]\Big] \leq \tilde{\mathbb{E}}\Big[\mathbb{P}_n^{X^{m_n}(t)}[T_n^0 \geq \varepsilon]\Big]$$

We conclude by observing that applying Markov inequality leads to

$$\begin{aligned} \tilde{\mathbb{E}}\Big[\mathbb{P}_n^{X^{m_n}(t)}[T_n^0 \geq \varepsilon]\Big] &= \tilde{\mathbb{E}}\Big[\mathbb{P}_n^{X^{m_n}(t)}[1 - e^{-\lambda T_n^0} > 1 - e^{-\lambda\varepsilon}]\Big] \\ &\leq \frac{1}{1 - e^{-\lambda\varepsilon}} \tilde{\mathbb{E}}\Big[\tilde{\mathbb{E}}\Big[1 - e^{-\lambda T_n^0} \,\Big|\, X^{m_n}(t)\Big]\Big] \\ &= \frac{1}{1 - e^{-\lambda\varepsilon}} \int (1 - \varphi_n^0(x))\, m_n(dx) \to 0, \end{aligned}$$

where the convergence to 0 follows from assumption *(1)* above. This shows that assumption *(1)* of Theorem 3.2 implies condition *(1)* of Theorem 2.1

Assumption *(2)* in Theorem 3.2 is identical to condition *(2)* in Theorem 2.1.

Moving on to condition *(3)* in Theorem 2.1, we start by applying inequalities (3.4) from Lemma 3.1 to X_n and A_n. Integrating in x the resulting inequalities with respect to ν_n yields

$$0 \leq \mathbb{E}_n^{\nu_n}\left[\int_0^\infty e^{-\lambda t - \mu A_n}\, dt\right] - b_n R_n^{\lambda,\mu} \leq c_n,$$

where

$$b_n = \int \varphi_n^\mu(x) \nu_n(dx), \quad R_n^{\lambda,\mu} = \frac{\int \varphi_n^\mu(x) m_n(dx)}{k_n(\{0\})\mu\varphi_n^\mu(0) - \frac{d^+}{ds_n}\varphi_n^\mu(0+)}, \text{ and}$$

$$c_n = \frac{1}{\lambda}\left(1 - \int \varphi_n^0(x)\, \nu_n(dx)\right).$$

Assumption *(3a)* in Theorem 3.2 directly implies the convergence $b_n \to 1$. Since φ_n^0 is a non-increasing function that is bounded from above by 1 and since $\nu_n \preceq_1 m_n$, assumption (1) in

Theorem 3.2 also implies that

$$1 \geq \int \varphi_n^0(x)\,\nu_n(dx) \geq \int \varphi_n^0(x)\,m_n(dx) \to 1,$$

i.e., $c_n \to 0$. To show that the assumptions of Theorem 3.2 imply condition *(3)* of Theorem 3.2, it remains to check that the existence of $\lim_n R_n^{\lambda,\mu}$ is equivalent to that of $\lim_n \Phi_n(\mu)$. To check this, notice that since the right endpoint r_n is either natural or entrance but not exit, the function φ_n^μ has the following properties (see [BS02, Section II.1, par. 10., pp. 18-19]):

(1) For all $a < b$ with $a, b \in \operatorname{Int} I_n$

$$\lambda \int_a^b \varphi_n^\mu(x)\,m_n(dx) + \int_a^b \varphi_n^\mu(x)\mu k_n(dx) = \frac{d^+}{ds_n}\varphi_n^\mu(b) - \frac{d^+}{ds_n}\varphi_n^\mu(a),$$

(2)

$$\lim_{b\to r_n} \frac{d^+}{ds_n}\varphi_n^\mu(b) = 0.$$

Therefore, remembering that $m_n(\{0\}) = 0$ and letting $b \to r_n$ and then $a \to 0$, we obtain

$$\begin{aligned}
-\frac{d^+}{ds_n}\varphi_n^\mu(0+) &= \lambda \int_0^\infty \varphi_n^\mu(x)\,m_n(dx) + \mu \int_{0+}^\infty \varphi_n^\mu(x)\,k_n(dx)\\
&= \lambda \int_0^\infty \varphi_n^\mu(x)\,m_n(dx) + \mu \int_0^\infty \varphi_n^\mu(x)\,k_n(dx) - \mu k_n(\{0\})\varphi_n^\mu(0)
\end{aligned}$$

Bearing in mind the definitions of $R_n^{\lambda,\mu}$ and $\Phi_n(\mu)$, the above equality implies that $R_n^{\lambda,\mu} = (\lambda + \Phi_n(\mu))^{-1}$, which shows that the existence of $\lim_n R_n^{\lambda,\mu}$ is equivalent to the existence of $\lim_n \Phi_n(\mu)$ and completes the proof. □

Remark 3.3. For practical purpose, establishing the existence of the limit in (3.5) is the only "hard" condition of Theorem 3.2. Indeed,

(1) Condition *(1)* is equivalent to $\mathbb{P}_n^{m_n}[T_n^0 > \varepsilon] \to 0$. This will hold, in particular, if X^n is the sequence of time-dilations (speedup) of a single diffusion - subject to regularity conditions - which has 0 as a recurrent point.

(2) Condition *(2)* is easy to check if the functional A_n is of the form $A_n(t) = \int_0^t g(X_n(t))\,dt$, which will be the case of interest in most applications. A simple sufficient condition in that case is that the expectation $\mathbb{E}_n^{\nu_n}[g(X_n(t))]$ be bounded, uniformly in $n \in \mathbb{N}$ and t on compacts. This will clearly be the case when $\nu_n = m_n$ and g is uniformly integrable over all m_n. If we only have $\nu_n \preceq_1 m_n$, the coupling argument of the proof of Theorem 3.2, leading to (3.6) above, implies that the $\mathbb{P}_n^{\nu_n}$-distributions of $X_n(t)$ increase with t in the sense of the first-order stochastic dominance. Therefore, *(2)* will hold in that case too, if we additionally assume that g is nondecreasing.

(3) Finally, condition *(3b)* makes sure that the accumulation of the additive functional A_n by the time 0 is hit for the first time can be (asymptotically) ignored. A limiting theorem could be proven even if this condition is not satisfied, but the limiting process would have

a nontrivial independent initial jump, drawn from a possibly different jump distribution, before shifting into the subordinator dynamics.

The two following sections deal with applications of Theorem 3.2. In the first section, we focus on a sequence of Wright-Fisher diffusions, whose study is the practical motivation for this work. In the next section, we examine additional examples for which an analytical treatment is possible.

4. Wright-Fisher Diffusions

4.1. **The scaling regime.** Using the notation of Section 3 above, we consider a sequence $\{X_n\}_{n\in\mathbb{N}}$ of diffusions on the state space $I_n = [0,1)$, and generally parameterized by three sequences $\{\tau_n\}_{n\in\mathbb{N}}$, $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ of strictly positive numbers. Their infinitesimal generators are given by

$$\mathcal{G}_n f(x) = \frac{1}{\tau_n} x(1-x) f''(x) + \frac{1}{\tau_n}(\alpha_n(1-x) - \beta_n x) f'(x), \tag{4.1}$$

for $f \in C_c^2((0,1))$. We always assume that $X_n(0) = 0$, i.e., that the initial distribution ν_n is δ_0.

As our focus will be on the regime $\alpha_n \to 0$, the Feller condition at the left boundary point will not be satisfied, rendering it nonsingular. This implies, in particular, that information about the behavior there, additional to that contained in the generator (4.1), needs to be specified separately. We choose instantaneous reflection there as it is not only the most interesting choice mathematically, but it also best fits our intended application to neuroscience. The Feller condition at the right boundary, on the other hand, will be met since we always assume that $\beta_n > 1$. Thanks to [KT81, eq. (6.19), p. 240], this assumption will imply that the right boundary is "entrance" but not "exit". We have the following expressions for the derivatives of the scale functions and the densities of the speed measures

$$\begin{aligned} s_n'(x) &= \tau_n B(\alpha_n, \beta_n)\, x^{-\alpha_n}(1-x)^{-\beta_n}, \\ m_n'(x) &= \frac{1}{B(\alpha_n,\beta_n)}\, x^{\alpha_n - 1}(1-x)^{\beta_n - 1}, \end{aligned}$$

where $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ is the Beta function and $\Gamma(\cdot)$ is the Gamma function. We refer the reader to [KT81, Example 8, p. 239] for the details, as well as for a discussion of various properties and features of the Wright-Fisher diffusion.

The scaling regime adopted in this section is

$$\tau_n \to 0,\ \beta_n = \beta > 1 \text{ and } \frac{\alpha_n}{\tau_n} \to \gamma \text{ for some } \gamma \in (0,\infty), \tag{4.2}$$

with the sequence $\{A_n\}_{n\in\mathbb{N}}$ of additive functionals given by

$$A_n(t) = \frac{1}{\tau_n}\int_0^t X_n(u)\,du. \tag{4.3}$$

The particular choices made in (4.2) are partly dictated by modeling considerations, and partly by their mathematical interest. Moreover, this regime is essentially forced by the choice that

$\{\beta_n\}_{n\in\mathbb{N}}$ be constant, the assumptions of Theorem 3.2 and the requirement that the limit be nondeterministic. Indeed, as can be checked directly, we have

$$\mathbb{E}_n^{m_n}\left[\frac{1}{\tau_n}\int_0^1 X_n(t)\,dt\right] = \frac{\alpha_n}{\tau_n}\frac{1}{\alpha_n+\beta} \text{ and} \tag{4.4}$$

$$\mathrm{Var}_n^{m_n}\left[\frac{1}{\tau_n}\int_0^1 X_n(t)\,dt\right] = \frac{2\beta}{(\alpha_n+\beta)^3(1+\alpha_n+\beta)}\frac{e^{-\frac{\alpha_n+\beta}{\tau_n}}-1+\frac{\alpha_n+\beta}{\tau_n}}{\frac{\alpha_n+\beta}{\tau_n}} \tag{4.5}$$

From there, it follows that $1/\tau_n$ is, indeed, the proper scaling for $\int_0^1 X_n(t)\,dt$, and that, given that scaling, the limiting variance will be nontrivial only if the limit of α_n/τ_n exists in $(0,\infty)$.

4.2. **An application of Theorem 3.2.** Next, we turn to the decreasing fundamental solutions $\{\varphi_n^\mu\}_{n\in\mathbb{N}}$ of the killed diffusion defined in (3.3) above. We take the analytic approach and characterize φ_n^μ, up to a multiplicative constant, as a decreasing solution of the following second order ODE:

$$\mathcal{G}_n u(x) - \left(\lambda + \frac{\mu}{\tau_n}x\right)u(x) = 0,\ x\in(0,1). \tag{4.6}$$

Since the right boundary is singular, we impose no boundary conditions at all.

In order to pass to a limit in the following subsection we need a better understanding of the structure of the solution of (4.6) than is provided by the general theory. Given that we are working with a polynomial diffusion, i.e., a diffusion with an infinitesimal generator whose coefficients are polynomials, it is likely that the solutions to (4.6) admit power-series expansions amenable to further analysis. It turns out that this most direct approach is the most convenient one as well. To see this, let us consider a candidate solution u_n^μ specified as

$$u_n^\mu(x) = \sum_{k=0}^\infty a_n(k)(1-x)^k, \tag{4.7}$$

where the coefficient sequence $\{a_n(k)\}_{k\in\mathbb{N}_0}$ is defined by the following recursive relations:

$$a_n(0)=1,\ a_n(1)=\frac{\lambda\tau_n+\mu}{\beta} \text{ and} \tag{4.8}$$

$$a_n(k) = c_n(k-1)a_n(k-1) - c_n(k-2)a_n(k-2),\ \text{for } k\geq 0, \tag{4.9}$$

where

$$c_n(k-1) = \frac{\lambda\tau_n+\mu+(k-1)(k+\alpha_n+\beta-2)}{k(\beta+k-1)},\ \text{and} \tag{4.10}$$

$$c_n(k-2) = \frac{\mu}{k(\beta+k-1)}. \tag{4.11}$$

These recursions are obtained by coefficient matching when (4.7) is formally inserted in (4.6). Moreover, even though the equation (4.6) is of second order, the value of the coefficient $a_n(1)$ is completely determined by the equation due to degeneration of ellipticity at the right boundary. On the other hand, the choice $a_n(0)=1$ is only a normalization.

Lemma 4.1. *For each $\varepsilon > 0$ there exist constants $C_\varepsilon > 0$ and $N_\varepsilon \in \mathbb{N}$ such that*

$$|a_n(k)| \leq C_\varepsilon k^{-(2-\varepsilon)} \text{ for all } k \in \mathbb{N} \text{ and } n \geq N_\varepsilon. \tag{4.12}$$

Proof. Given $\varepsilon \in (0,1)$, we set $K^1_\varepsilon = 8\beta/\varepsilon$ and pick $N_\varepsilon \in N$, such that $\alpha_n < \varepsilon/4$ and $\tau_n < 1$ for $n \geq N_\varepsilon$. For $k \geq K^1_\varepsilon$ and $n \geq N_\varepsilon$, we have $\frac{2-\alpha_n}{k+\beta-1} \geq \frac{2-\varepsilon/2}{k}$, so that

$$0 \leq c_n(k-1) = 1 - \frac{(2-\alpha_n)}{k+\beta-1} + \frac{2+\lambda\tau_n - \alpha_n - \beta + \mu}{k(k+\beta-1)} \leq 1 - \frac{\eta+\varepsilon/2}{k} + \frac{\rho}{k^2},$$

where $\eta = 2-\varepsilon$, $\overline{\lambda} = \sup_n \lambda\tau_n < \infty$ and $\rho = 2 + \overline{\lambda} + \mu$. We also have

$$0 \leq c_n(k-2) \leq \frac{\mu}{k^2}.$$

Let $b_n(k) = |a_n(k)|/k^{-\eta}$, so that, for $k \geq K^1_\varepsilon$ and $n \geq N_\varepsilon$ we have

$$\begin{aligned} b_n(k) &= \Big(1 - \frac{\eta+\varepsilon/2}{k} + \frac{\rho}{k^2}\Big)\frac{b_n(k-1)(k-1)^{-\eta}}{k^{-\eta}} + \frac{\mu}{k^2}\frac{b_n(k-2)(k-2)^{-\eta}}{k^{\eta}} \\ &\leq \max\Big(b_n(k-1), b_n(k-2)\Big) f(1/k), \end{aligned}$$

where

$$f(x) = (1-x)^{-\eta}\Big(\rho x^2 - x(\eta+\varepsilon/2) + 1\Big) + \mu(1-2x)^{-\eta}x^2 \text{ for } x < 1/2.$$

Clearly, f is C^1 on $[0,1/2)$, $f(0) = 1$ and $f'(0) = -\varepsilon/2$, so there exists $x_0 > 0$ such that $f(x) \leq 1$ for $x \in [0, x_0]$, i.e.,

$$b_n(k) \leq \max(b_n(k-1), b_n(k-2)) \text{ for } k \geq K_\varepsilon := \max(K^1_\varepsilon, 1/x_0). \tag{4.13}$$

The absolute values of the coefficients $c_n(k)$ and the initial conditions $a_n(0), a_n(1)$ admit n-independent bounds, which implies that

$$B(k) := \sup_n b_n(k) \leq \sup_n k^\eta |a_n(k)| < \infty \text{ for each } k \in \mathbb{N}. \tag{4.14}$$

Combined with (4.13), the finiteness of $B(k)$ in (4.14) above implies that, for $n \geq N_\varepsilon$ we have

$$|a_n(k)|k^{-\eta} \leq C_\varepsilon := \max_{k \leq K_\varepsilon} B(k) < \infty.$$ □

Proposition 4.2. *The function u^μ_n is well-defined by (4.7) on $[0,2]$, real analytic on $(0,2)$, and we have $\varphi^\mu_n = u^\mu_n$ on $[0,1]$, up to a multiplicative constant.*

Proof. The bounds of (4.12), for $\varepsilon < 1$, imply immediately that the series (4.7) converges absolutely on $[0,2]$ and that it defines a continuous function there. Analyticity on $(0,2)$ then follows from the fact that the radius of convergence is at least 1. In particular, we can differentiate them term by term and then perform an easy calculation using (4.9) and (4.8) to conclude that u^μ_n solves (4.6) on $(0,1)$ and that $u^\mu_n(1) = 1$, $(u^\mu_n)'(1) = -(\lambda\tau_n + \mu)/\beta$.

Next, we show that u^μ_n is strictly decreasing. Arguing by contradiction, we assume, first, that $(u^\mu_n)'(x) \geq 0$ for some $x \in (0,1)$, and let $x_0 \in (0,1]$ be the supremum of all such x. Strict negativity of the derivative $(u^\mu_n)'(1)$ implies that $(u^\mu_n)' < 0$ in a neighborhood of 1, and so, $x_0 < 1$. Hence, $(u^\mu_n)'(x_0) = 0$ and $(u^\mu_n)'(x) < 0$ for $x \in (x_0, 1)$, which, in turn, implies that

$(u_n^\mu)''(x_0) \le 0$. Since u_n^μ satisfies (4.6), we must have $u_n^\mu(x_0) \le 0$. This is in contradiction with the fact that $u_n^\mu(1) = 1$ and $(u_n^\mu)'(x) \le 0$ for all $x \in [x_0, 1)$.

Finally, we appeal to the general general theory of one-dimensional diffusions (see [BS02, Section II.1, par. 10., pp. 18-19]), which states that φ_n^μ is the unique, up to a multiplicative constant, decreasing solution to (4.6) (no boundary conditions needed). Therefore, φ_n^μ and u^μ agree on $(0, 1)$, up to a multiplicative constant. By continuity, the same is true on $[0, 1]$. □

Next, we analyze the limiting behavior of the sequence $\{\varphi_n^\mu\}_n = \{u_n^\mu\}_{n\in\mathbb{N}}$. Since the coefficients in (4.9), as well as the initial conditions (4.8), converge to finite values as $n \to \infty$, the solutions converge too, and we set $a(k) := \lim_n a_n(k)$. Moreover, the limiting coefficients satisfy the following (limiting) recursive equation

$$a(0) = 1,\ a(1) = \frac{\mu}{\beta} \text{ and} \tag{4.15}$$

$$a(k) = \frac{\mu + (k-1)(k+\beta-2)}{k(k+\beta-1)} a(k-1) - \frac{\mu}{k(\beta+k-1)} a_n(k-2) \text{ for } k \ge 2. \tag{4.16}$$

It is easily checked that (4.15) with (4.16) above admit an explicit solution, namely,

$$a(k) = \frac{\mu^k}{k!(\beta)_k}, \tag{4.17}$$

where $(\beta)_k := \beta(\beta+1)\dots(\beta+k-1)$ is the Pochhammer symbol (also known as rising factorial). Therefore, we set

$$\varphi^\mu(x) := \sum_{k=0}^\infty \frac{\mu^k}{k!(\beta)_k}(1-x)^k,$$

with absolute convergence for all x, and note that $\varphi^\mu(x) = \Gamma(\beta)x^{-(1+\beta)/2} I_{\beta-1}(2\sqrt{x})$, where I_ν is the modified Bessel function of the first kind of order ν.

By Lemma 4.1 applied with $\varepsilon = 1/2$, we have $|a_n(k) - a(k)| \le Ck^{-3/2}$ some $C > 0$, all $k \in \mathbb{N}_0$, and large enough $n \in \mathbb{N}$. Therefore, we can use the dominated convergence to conclude that $\lim_n \sum_k |a_n(k) - a(k)| = 0$ so that

$$\sup_{x\in[0,1]} |\varphi_n^\mu(x) - \varphi^\mu(x)| \le \sum_{k=0}^\infty |a_n(k) - a(k)| \to 0 \text{ as } n \to \infty. \tag{4.18}$$

Since $m_n \to \delta_0$ weakly, where δ_0 denotes the Dirac measure concentrated at 0, the uniform convergence of (4.18) above implies that

$$\int \varphi_n^\mu(x)\, m_n(dx) \to \varphi^\mu(0) = \sum_{k=0}^\infty \frac{\mu^k}{k!(\beta)_k} = \Gamma(\beta)\mu^{-(1+\beta)/2} I_{\beta-1}(2\sqrt{\mu}). \tag{4.19}$$

To compute the limit $\int \varphi_n^\mu(x)\, k_n(dx)$, we first note that the density $k_n'(x)$ of k_n with respect to the Lebesgue measure satisfies

$$\frac{\tau_n}{\alpha_n}(\alpha_n+\beta)k_n'(x) = \frac{\Gamma(\alpha_n+1+\beta)}{\Gamma(\alpha_n+1)\Gamma(\beta)} x^{(\alpha+1)-1}(1-x)^{\beta-1}, \tag{4.20}$$

where the right-hand side above can be recognized as the probability density of the Beta distribution with parameters $\alpha_n + 1$ and β. As $n \to \infty$, these distributions converge weakly towards the Beta distribution with parameters 1 and β, and so, by (4.18), we have

$$\int \varphi_n^\mu(x)\, k_n(dx) \to \frac{\gamma}{\beta} \int \varphi^\mu(x)\, \beta(1-x)^{\beta-1}\, dx = \gamma \sum_{k=0}^{\infty} \frac{\mu^k}{k!(\beta)_k} \int_0^1 (1-x)^{\beta-1+k}\, dx$$
$$= \frac{\gamma}{\beta} \sum_{k=0}^{\infty} \frac{\mu^k}{k!(\beta+1)_k}\, dx = \gamma\Gamma(\beta) x^{-\beta/2} I_\beta(2\sqrt{x}) \tag{4.21}$$

We are now ready for the main result of this section.

Theorem 4.3. *Consider the sequence $\{X_n\}_{n\in\mathbb{N}}$ of Wright-Fisher diffusions on $[0,1)$ with generators given by (4.1), started at $X_n(0) = 0$, reflected at 0, and in the scaling regime (4.2). The sequence $\{A_n\}_{n\in\mathbb{N}}$ of scaled and integrated diffusions, given by*

$$A_n(t) = \frac{1}{\tau_n} \int_0^t X_n(u)\, du,\ \ t \geq 0,$$

converges weakly, under the Skorokhod's M_1-topology, towards a Lévy subordinator A with Laplace exponent given by

$$\Phi(\mu) = \gamma\sqrt{\mu} \frac{I_\beta(2\sqrt{\mu})}{I_{\beta-1}(2\sqrt{\mu})}, \tag{4.22}$$

where I_ν is the modified Bessel function of the first kind with index ν.

Proof. Since $\nu_n = \delta_0$, conditions $\nu_n =\preceq_1 m_n$ and *(3a)* are trivially satisfied. For *(1)*, we note that $\varphi^0(x) = 1$ and that $\varphi_n^0 \to \varphi^0$ uniformly, so $\int \varphi_n^0\, dm_n \to 1$ as $m_n \to \delta_0$ weakly. Next, the explicit formula (4.4) above implies that $\mathbb{E}_n^{m_n}[X_n(t)/\tau_n]$ is bounded in n and t. By part *(2)* of Remark 3.3, this - together with the trivial fact that $g(x) = x/\tau_n$ is nondecreasing - is enough to satisfy condition *(2)*. Lastly, to establish the existence of and get an expression for the limiting exponent Φ, it is enough to take the quotient of (4.19) and (4.21). □

4.3. **Properties of the limiting subordinator.** We conclude this section with some facts about the limiting Laplace functional Φ and the limiting subordinator, which we denote by A in Theorem 4.3 above. Given that it is only a scaling parameter, we assume throughout that $\gamma = 1$, for simplicity.

(1) It has been shown in [PY03, eq. (48), p. 12] that a subordinator with the Laplace exponent Φ can be realized as

$$A(t) = 2 \int_0^{\tau(t)} 1_{\{X_u \leq 1\}}\, du,$$

where X is a Bessel process of index $\beta - 1$ and τ is the inverse local time of X at level 1, i.e.,

$$\tau(t) = \inf\{s \geq 0 \,:\, L^1(s) = t\},$$

where L^1 denotes the local time of X at level 1.

(2) Since Φ is a Laplace exponent of an infinitely-divisible distribution supported by $[0,\infty)$, it admits a Lévy-Khinchine representation of the form:

$$\Phi(\mu) = b\mu + \int_0^\infty (1-e^{-\mu x})\Pi(dx) \text{ for } \mu > 0, \tag{4.23}$$

where $d \geq 0$ and Π is a measure on $(0,\infty)$ such that $\int \min(1,x)\,\Pi(dx) < \infty$. By [DLMF, (10.30.4)], we have $\lim_{x\to\infty} \sqrt{2\pi x} e^{-x} I_\nu(x) = 1$, so

$$\lim_{\mu\to\infty} \frac{1}{\mu}\Phi(\mu) = 2 \lim_{x\to\infty} \frac{1}{x}\frac{I_{\beta-1}(x)}{I_\beta(x)} = 0,$$

which implies that $b = 0$, i.e., that A has no drift.

(3) According to [IK79, Theorem 1.9, p. 886], the function

$$\Psi(\mu) = \frac{2\beta}{\sqrt{\mu}}\frac{I_\beta(\sqrt{\mu})}{I_{\beta-1}(\sqrt{\mu})} \text{ with } \mu > 0,$$

is a Laplace transform of the infinitely divisible distribution with density

$$f(y) = 4\beta \sum_n \exp\big(-j^2_{\beta-1,n} y\big), y \geq 0,$$

where $\{j_{\nu,n}\}_{n\in\mathbb{N}}$ is an enumeration of the set of strictly positive zeros of the Bessel function J_ν of index ν. We have

$$\Psi(\mu) = \frac{4\beta}{\mu}\Phi\Big(\frac{\mu}{4}\Big)$$

so that

$$\begin{aligned}\int_0^\infty e^{-\mu y} f(y)\,dy &= 4\beta \int_0^\infty \frac{1-e^{-\frac{\mu}{4}x}}{\mu}\Pi(dx)\\ &= \beta\int_0^\infty \int_0^x e^{-\frac{\mu}{4}y}\,dy\,\Pi(dx)\\ &= \int_0^\infty e^{-\mu z}\,4\beta\,\Pi\Big(\Big[\frac{1}{4}z,\infty\Big)\Big)\,dz,\end{aligned}$$

for all $\mu > 0$, we conclude that the Lévy measure Π is absolutely continuous with respect to the Lebesgue measure, with density

$$\pi(x) = \frac{1}{\beta}f'(4x) = \sum_n (2j_{\beta-1,n})^2 e^{-(2j_{\beta-1,n})^2 x}, \quad x > 0. \tag{4.24}$$

(4) Thanks to (4.24) above, we have

$$\int x^r \Pi(x) = \sum_n \int_0^\infty x^r (2j_{\beta-1,n})^2 e^{-(2j_{\beta-1,n})^2 x}\,dx = 4^{-r}\Gamma(1+r)\sum_n j^{-2r}_{\beta-1,n} \tag{4.25}$$

Since the zeros of the Bessel functions grow approximately linearly, or, more precisely (see [DLMF, (10.21.19)]),

$$j_{\beta-1,n} \sim \pi\big(n + \tfrac{1}{2}(\beta - 3/2)\big) + O(1/n)$$

for each $T \in [0,\infty)$, we have

$$\mathbb{E}\left[\sum_{t\leq T} (\Delta A_t)^r\right] = \begin{cases} +\infty, & r \leq 1/2, \text{ and} \\ < +\infty, & r > 1/2. \end{cases}$$

(5) When the Lévy exponent Φ is analytic in a neighborhood of 0, as in our case, the sequence $\{\kappa_n\}_{n\in\mathbb{N}}$ of *cumulants* is defined using the Maclaurin expansion

$$\Phi(\mu) = \sum_{n=0}^{\infty} (-1)^n \kappa_n \frac{\mu^n}{n!},$$

of the function Φ. Their importance stems from the fact that they are the moments of the jump measure, i.e.,

$$\kappa_n = \int_0^\infty x^n \Pi(dx), \text{ for } n \in \mathbb{N}.$$

The explicit expression (4.25) reveals that, in our case, we have

$$\kappa_n = 4^{-n} n!\, \sigma_n(\beta - 1) \text{ where } \sigma_n(\nu) = \sum_m (j_{\nu,n})^{-2m}.$$

The function σ_n is known as the *Rayleigh function*, and satisfies the following simple convolution identity (see [Kis63, Eq. (20), p. 531]), useful for efficient computation of cumulants and moments:

$$\sigma_n(\nu) = \frac{1}{\nu + n} \sum_{k=1}^{n-1} \sigma_k(\nu)\sigma_{n-k}(\nu), \quad \sigma_1(\nu) = \frac{1}{4(\nu+1)}.$$

Once the cumulants are known, the moments $m_n = \mathbb{E}[A(t)^n]$, $n \in \mathbb{N}$, of the distribution of $A(t)$ can be efficiently computed by using the following well-known recursive relationship, which is, in turn, a direct consequence of the formula of Faà-di-Bruno:

$$m_{n+1} = t \sum_{i=0}^{n} (-1)^i \binom{n}{i} \kappa_{i+1} m_{n-i}, \ m_0 = 1.$$

In particular, as is the case with any Lévy process, m_n is a polynomial in t of order at most n.

(6) We have the following simple continued-fraction expansion of Laplace exponent Φ (see [JT81, Theorem 6.3, p. 206]):

$$\Phi(\mu) = \cfrac{\mu}{\beta + \cfrac{\mu}{(\beta+1) + \cfrac{\mu}{(\beta+2) + \cfrac{\mu}{(\beta+3) + \ddots}}}}$$

5. Additional examples

Additional examples illustrating Theorem 3.2 are given in this section. In all of them we take $\nu_n = \delta_0$, i.e., $X_n(0) = 0$.

5.1. **Feller (CIR) diffusions.** Each process in the sequence $\{X_n\}_{n\in\mathbb{N}}$ has $I_n = [0,\infty)$ as its state space and the infinitesimal generator $\mathcal{G}_n$ given by

$$\mathcal{G}_n f = nxf''(x) + n(\alpha_n - \beta x)f'(x) \text{ for } f \in C_c^2((0,\infty)).$$

The speed-up factor n takes the place of the (formally) more general $1/\tau_n$ for simplicity. Since the Feller condition will not be satisfied for large enough n, instantaneously reflective behavior at 0 is assumed. The normalized speed measure (stationary distribution) of X_n is the $\Gamma(\alpha_n, \beta)$-distribution, i.e., $m_n(dx) = m_n'(x)\,dx$, where

$$m_n'(x) = \frac{\beta^{\alpha_n}}{\Gamma(\alpha_n)} x^{\alpha_n - 1} e^{-\beta x}, x \in [0,\infty)$$

We set

$$A_n(t) = n\int_0^\infty X_n(u)\,du,$$

and consider the regime $\alpha_n \to 0$ with $n\alpha_n \to \gamma \in (0,\infty)$.

Let $U(a,b,\cdot)$ be Kummer's U-function (see [DLMF, (13.2.6)]) so that $u(x) = U(a,b,x)$ solves Kummer's differential equation

$$xu''(x) + (b-x)u'(x) - a\,u(x) = 0 \text{ for } x \in (0,\infty). \tag{5.1}$$

A direct computation shows that for $\mu > 0$, the function

$$\varphi_n^\mu(x) = \frac{1}{\Gamma(A_n)} e^{Lx} U(A_n, \alpha, Sx)$$

where

$$L = \frac{\beta - \sqrt{\beta^2 + 4\mu}}{2},\ R = \frac{\beta + \sqrt{\beta^2 + 4\mu}}{2},\ S = R - L,\ A_n = \frac{\lambda/n - \alpha_n L}{S}$$

satisfies

$$\mathcal{G}_n \varphi_n^\mu(x) - (\lambda + \mu n x) = 0. \tag{5.2}$$

For $a > 0$ and $x > 0$, the following integral representation (see [DLMF, (13.4.4)]))

$$U(a, b, x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tx} t^{a-1} (1+t)^{b-a-1} \, dt \text{ for } a, x \in (0, \infty),$$

can be used to justify the identity

$$\varphi_n^\mu(x) = \frac{1}{\Gamma(A_n)} \int_0^\infty e^{-x(St-L)} t^{A_n - 1} (1+t)^{\alpha - A_n - 1} \, dt. \tag{5.3}$$

Since $S > 0$ and $L < 0$, we conclude immediately that φ^μ is positive and strictly decreasing. This is enough (see [BS02, Section II.1, par. 10., pp. 18-19]) to identify φ^μ out of all solutions of (5.2) as the decreasing fundamental solution, up to a multiplicative constant.

The representation (5.3) yields

$$\begin{aligned}
\frac{\Gamma(A_n)}{\Gamma(\alpha_n)} \int \varphi_n^\mu(x) \, m_n(dx) &= \int t^{A_n - 1} (1+t)^{\alpha_n - A_n - 1} \frac{1}{\Gamma(\alpha_n)} \int x^{\alpha_n - 1} e^{-(R+St)x} dx \, dt \\
&= \int (St + R)^{-\alpha_n} t^{A_n - 1} (1+t)^{\alpha_n - A_n - 1} \, dt \\
&= \int_0^1 r^{-1+A_n} (R(1-r) + Sr)^{-\alpha_n} \, dr,
\end{aligned}$$

where we use the substitution $r \leftarrow t/(1+t)$ to get the last equality. Similarly

$$\frac{\Gamma(A_n)}{\Gamma(\alpha_n + 1)} \int \varphi_n^\mu(x) x \, m_n(dx) = \int_0^1 r^{-1+A_n} (1-r) (R(1-r) + Sr)^{-\alpha_n - 1} \, dr$$

Combining the integral representations given above allows one to express

$$\Phi_n(\mu) = \frac{\int \varphi^\mu(x) \mu \, n x m_n(dx)}{\int \varphi^\mu(x) \, m_n(dx)} = \frac{n \alpha_n \mu (1 + A_n) \int_0^1 \frac{r^{-1+A_n}(1-r)}{B(A_n, 2)} (R(1-r) + Sr)^{-\alpha_n - 1} \, dr}{\int_0^1 \frac{r^{-1+A_n}}{B(A_n, 1)} (R(1-r) + Sr)^{-\alpha_n} \, dr}.$$

To apply Theorem 3.2, there remains to study the limit behavior of the above ratio when $n \to \infty$. Since $A_n \to 0$, the sequence of beta distributions with parameters (A_n, B) for any $B > 0$ converge weakly towards the Dirac mass at 0. Moreover, since $R, S > 0$, we have

$$(R(1-r) + Sr)^{-\alpha_n} \to 1 \text{ and } (R(1-r) + Sr)^{-\alpha_n - 1} \to (R(1-r) + Sr)^{-1}$$

uniformly on $[0, 1]$. Since $n\alpha_n \to \gamma$, it follows directly that

$$\Phi(\mu) = \lim_n \frac{\int \varphi^\mu(x) \mu \, n x m_n(dx)}{\int \varphi^\mu(x) \, m_n(dx)} = \frac{2\gamma\mu}{\beta + \sqrt{\beta^2 + 4\mu}} = \gamma\beta \left(\sqrt{1 + \frac{2}{\beta}\mu} - 1 \right).$$

In the case $\mu = 0$, we have,

$$\varphi_n^0(x) = U\left(\frac{\lambda}{n\beta}, \alpha_n, \beta x \right), x \geq 0.$$

and a similar analysis to the one discussed above shows that $\int \varphi_n^0(x) \, m_n(dx) \to 1$. The choice of $\nu_n = \delta_0$ and of α_n so that $\sup_n \mathbb{E}_n^{m_n}[A_n(t)] < \infty$ makes sure that all remaining assumptions of Theorem 3.2 are met. Consequently, the processes A_n converge weakly under M_1 to a subordinator with the Laplace exponent Φ given above. We recognize it as the Laplace exponent of the inverse-Gaussian distribution with the mean γ/β (typically denoted by μ) and the scale parameter γ^2/β (typically denoted by λ).

5.2. **Reflected Brownian Motion with Drift.** In this subsection X_n is the sped-up Brownian motion with negative drift on $I_n = [0, \infty)$, reflected at 0. More precisely, its infinitesimal generator $\mathcal{G}_n$ is given by

$$\mathcal{G}_n u = n u''(x) - n\beta_n u'(x)$$

The normalized speed measure (stationary distribution) is the exponential distribution with parameter β_n, i.e., $m_n(dx) = m_n'(x)\,dx$ where

$$m_n'(x) = \beta_n e^{-\beta_n x}.$$

The decreasing fundamental solution φ_n^0 without killing, solves $\mathcal{G}_n u - \lambda u = 0$ and is easily seen to be given by

$$\varphi_n^0(x) = e^{\frac{1}{2}x\left(\beta_n - \sqrt{\beta_n^2 + 4\lambda/n}\right)},$$

Therefore

$$\int \varphi_n^0(x)\, m_n(dx) = 2\Big(1 + \sqrt{1 + 4\lambda/(\beta_n^2 n)}\Big)$$

and, in order for condition *(1)* of Theorem 3.2 to be satisfied, we need to impose the condition:

$$\beta_n^2 n \to \infty. \tag{5.4}$$

The random variable $\beta_n X_n(t)$ is exponentially distributed under m_n, with parameter 1, so the choice

$$A_n(t) = \beta_n \int_0^t X_n(u)\, du,$$

is essentially the only one which makes the condition *(2)* of Theorem 3.2 (nontrivially) satisfied.

We turn, next, to the computation of the decreasing fundamental solution φ_n^μ, for $\mu > 0$. Two independent solutions, u_1 and u_2 of the equation $\mathcal{G}_n u(x) - (\lambda + \beta_n x\mu)u(x) = 0$, are given by

$$u_1(x) = e^{\beta_n/2x}\,\mathrm{Ai}(f_n(x)), \quad u_2(x) = e^{\beta_n/2x}\,\mathrm{Bi}(f_n(x)),$$

where

$$f_n(x) = \frac{1}{4}(\beta_n^2 n)^{2/3}\mu^{-2/3} + \lambda(\beta_n^2 n)^{-1/3}\mu^{-2/3} + x n^{-1/3}\beta_n^{1/3}\mu^{1/3}, \tag{5.5}$$

and Ai and Bi are the AiryA and AiryB functions, respectively (see [DLMF, Chapter 9]). Since Bi is unbounded (see, e.g., [DLMF, (9.7.7)]), we must have

$$\varphi_n^\mu(x) = e^{\beta/2x}\,\mathrm{Ai}(f_n(x)),$$

up to a multiplicative function.

Lemma 5.1. *If* $\alpha > 0, \gamma_n > 0, \delta_n > 0$, $c_n \to \infty$, *and* $c_n^{1/2}\gamma_n/\delta_n \to \rho \in (0,\infty)$, *then*

$$\lim_n \frac{1}{\operatorname{Ai}(c_n)} \int_0^\infty \frac{\delta_n^{1+\alpha}}{\Gamma(1+\alpha)} e^{-\delta_n x} x^\alpha \operatorname{Ai}(c_n + \gamma_n x)\, dx = (1+\rho)^{-1-\alpha}.$$

Proof. We will need the following tail bounds for the Airy function (see [DLMF, (9.7.5)], [DLMF, (9.7.iii)])

$$(1 - r(y))q(y)e(y) \le \operatorname{Ai}(y) \le q(y)e(y), \text{ for } y > (3/2)^{2/3}. \tag{5.6}$$

where

$$r(y) = \frac{3}{2}y^{-3/2},\ e(y) = \frac{e^{-\frac{2}{3}y^{3/2}}}{\sqrt{4\pi}} \text{ and } q(y) = y^{-1/4}.$$

Moreover, we will use the following three simple inequalities: valid for all $y, b, c > 0$,

$$-c^{1/2}y \ge \frac{2}{3}\Big(c^{3/2} - (c+y)^{3/2}\Big) \ge -c^{1/2}y - \frac{1}{4}c^{-1/2}y^2, \tag{5.7}$$

$$1 \ge \left(\frac{c}{c+y}\right)^{1/4} \ge \exp\Big(-\frac{y}{4c}\Big) \text{ and} \tag{5.8}$$

$$1 \ge \int_0^\infty \frac{b^{1+\alpha}}{\Gamma(1+\alpha)} y^\alpha e^{-by-cy^2}\, dy \ge e^{-cb^{1/2}}\left(1 - \frac{\Gamma(1+\alpha, b^{5/4})}{\Gamma(1+\alpha)}\right), \tag{5.9}$$

where $\Gamma(1+\alpha, y) = \int_y^\infty \xi^\alpha e^{-\xi}\, d\xi$ is the incomplete Gamma function. Inequalities in (5.7) are a consequence of convexity bounds for the function $y \mapsto y^{3/2}$, those in (5.8) follows from the standard bounds on the exponential function, while (5.9) is obtained by estimating the integral from below by $e^{-cy_0^2}\int_0^{y_0} b^{1+\alpha}y^\alpha e^{-by}\, dy$ for $y_0 = b^{1/4}$ and from above by $\int b^{1+\alpha}y^\alpha e^{-by}\, dy = \Gamma(1+\alpha)$.

Using the upper bounds from (5.6) - (5.9) we obtain

$$\frac{\operatorname{Ai}(c_n + y)}{\operatorname{Ai}(c_n)} \le e^{-\frac{2}{3}\left((c_n+y)^{3/2} - c_n^{3/2}\right)} \left(\frac{c_n}{c_n+y}\right)^{1/4} \frac{1}{1 - r(c_n)} \le \frac{e^{-c_n^{1/2}y}}{1 - r(c_n)},$$

so that, with $\tilde\delta_n := \delta_n/\gamma_n$, we have

$$\int \frac{\delta_n^{\alpha+1}}{\Gamma(\alpha+1)} e^{-\delta_n x} x^\alpha \frac{\operatorname{Ai}(c_n + \gamma_n x)}{\operatorname{Ai}(c_n)}\, dx = \int \frac{\tilde\delta_n^{\alpha+1}}{\Gamma(\alpha+1)} e^{-\tilde\delta_n y} y^\alpha \frac{\operatorname{Ai}(c_n+y)}{\operatorname{Ai}(c_n)}\, dy \le$$

$$\le \frac{1}{1 - r(c_n)} \int \frac{\tilde\delta_n^{\alpha+1}}{\Gamma(\alpha+1)} y^\alpha e^{-(\tilde\delta_n + c_n^{-1/2})y}\, dy \le \underline{\eta}_n \left(\frac{\tilde\delta_n}{\tilde\delta_n + c_n^{1/2}}\right)^{\alpha+1},$$

where $\underline{\eta}_n = (1 - r(c_n))^{-1}$.

Lower bounds of (5.6) - (5.8) yield

$$\frac{\mathrm{Ai}(c_n+y)}{\mathrm{Ai}(c_n)} \geq e^{-\frac{2}{3}\left((c_n+y)^{3/2}-c_n^{3/2}\right)}\left(\frac{c_n}{c_n+y}\right)^{1/4}(1-r(c_n+y))$$
$$\geq \exp\left(-(c_n^{1/2}+\frac{1}{4}c_n^{-1})y-\frac{1}{4}c_n^{-1/2}y^2\right)(1-r(c_n)),$$

so that, by the lower bound of (5.9), we have

$$\int \frac{\delta_n^{\alpha+1}}{\Gamma(\alpha+1)}e^{-\delta_n x}x^\alpha \frac{\mathrm{Ai}(c_n+\gamma_n x)}{\mathrm{Ai}(c_n)}\,dx$$
$$\geq (1-r(c_n))\int \frac{\tilde{\delta}_n^{\alpha+1}}{\Gamma(\alpha+1)}y^\alpha e^{-b_n y-\frac{1}{4}c_n^{-1/2}y^2}\,dy \geq \overline{\eta_n}\left(\frac{\tilde{\delta}_n}{b_n}\right)^{\alpha+1}$$

where $b_n = \tilde{\delta}_n + c_n^{1/2} + \frac{1}{4}c_n^{-1}$, $\tilde{c}_n = b_n/c_n$ and

$$\overline{\eta_n} = (1-r(c_n))\exp\left(-\frac{1}{4}\tilde{c}_n^{1/2}\right)\left(1-\frac{\Gamma(1+\alpha, b_n^{5/4})}{\Gamma(1+\alpha)}\right).$$

It remains to note, under the assumed conditions, we have $b_n \to \infty$, $c_n \to \infty$ and $\tilde{c}_n \to 0$. Hence, $\underline{\eta}_n \to 1$, $\overline{\eta_n} \to 1$, $r(c_n) \to 1$ and $c_n^{1/2}/\tilde{\delta}_n \to \rho$, which implies the both the lower and the upper bounds converge to the same quantity, namely $(1+\rho)^{-1-\alpha}$. □

We use Lemma 5.1 above, with $\delta_n = \beta_n/2$, $c_n = \mu^{-2/3}(\beta_n^2 n)^{2/3}/4 + \lambda\mu^{-2/3}(\beta_n^2 n)^{-1/3}$, $\gamma_n = \mu^{1/3}\beta_n^{1/3}n^{-1/3}$ and $\rho = 1$:

$$\Phi(\mu) = \mu \lim_n \frac{\int \varphi_n^\mu(x)\,k_n(x)}{\int \varphi_n^\mu(x)\,m_n(x)} = \lim_n \mu \frac{\int \beta_n^2 x e^{-\beta_n}\,\mathrm{Ai}(c_n+\gamma_n x)\,dx}{\int \beta_n e^{-\beta_n}\,\mathrm{Ai}(c_n+\gamma_n x)\,dx}$$
$$= \mu \lim_n \frac{\frac{1}{\mathrm{Ai}(c_n)}\int \beta_n^2 x e^{-\beta_n}\,\mathrm{Ai}(c_n+\gamma_n x)\,dx}{\frac{1}{\mathrm{Ai}(c_n)}\int \beta_n e^{-\beta_n}\,\mathrm{Ai}(c_n+\gamma_n x)\,dx} = \mu$$

Therefore,

$$A_n(t) \to t$$

weakly under the M_1-topology.

References

[BB13] Francois Baccelli and Pierre Brémaud. *Elements of queueing theory: Palm Martingale calculus and stochastic recurrences*. Vol. 26. Springer Science & Business Media, 2013 (cit. on p. 2).

[Bec+24] Logan A Becker, Baowang Li, Nicholas J Priebe, Eyal Seidemann, and Thibaud Taillefumier. "Exact analysis of the subthreshold variability for conductance-based neuronal models with synchronous synaptic inputs". In: *Physical Review X* 14.1 (2024), p. 011021 (cit. on pp. 1–4).

[Bét23] Loïc Béthencourt. "Stable limit theorems for additive functionals of one-dimensional diffusion processes". In: *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* 59.4 (2023), pp. 1882–1906. DOI: 10.1214/22-AIHP1295. URL: https://doi.org/10.1214/22-A (cit. on p. 4).

[Bil99] Patrick Billingsley. *Convergence of probability measures*. Second. Wiley Series in Probability and Statistics: Probability and Statistics. A Wiley-Interscience Publication. New York: John Wiley & Sons Inc., 1999, pp. x+277 (cit. on p. 9).

[BJP12] Tamara Broderick, Michael I. Jordan, and Jim Pitman. "Beta Processes, Stick-Breaking and Power Laws". In: *Bayesian Analysis* 7.2 (2012), pp. 439–476. DOI: 10.1214/12-BA715. URL: https://doi.org/10.1214/12-BA715 (cit. on p. 1).

[BS02] Andrei N. Borodin and Paavo Salminen. *Handbook of Brownian motion—facts and formulae*. Second. Probability and its Applications. Birkhäuser Verlag, Basel, 2002, pp. xvi+672 (cit. on pp. 10–12, 15, 19, 24).

[CCG21] Patrick Cattiaux, Djalil Chafai, and Arnaud Guillin. "Central limit theorems for additive functionals of ergodic Markov diffusions processes". In: *Latin American Journal of Probability and Mathematical Statistics* 9.2 (2021), pp. 337–382 (cit. on p. 4).

[DLMF] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.1.8 of 2022-12-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds. URL: http://dlmf.nist.gov/ (cit. on pp. 21–26).

[GG11] Thomas L Griffiths and Zoubin Ghahramani. "The Indian Buffet Process: An Introduction and Review." In: *Journal of Machine Learning Research* 12.4 (2011) (cit. on p. 1).

[GK02] Wulfram Gerstner and Werner M Kistler. *Spiking neuron models: Single neurons, populations, plasticity*. Cambridge university press, 2002 (cit. on p. 1).

[GN09] Wulfram Gerstner and Richard Naud. "How good are neuron models?" In: *Science* 326.5951 (2009), pp. 379–380 (cit. on p. 1).

[Har88] J Michael Harrison. "Brownian models of queueing networks with heterogeneous customer populations". In: *Stochastic differential systems, stochastic control theory and applications*. Springer, 1988, pp. 147–186 (cit. on p. 1).

[IK79] Mourad E. H. Ismail and Douglas H. Kelker. "Special functions, Stieltjes transforms and infinite divisibility". In: *SIAM J. Math. Anal.* 10.5 (1979), pp. 884–901 (cit. on p. 21).

[IM74] Kiyosi Itô and Henry P. McKean, Jr. *Diffusion processes and their sample paths*. Second printing, corrected, Die Grundlehren der mathematischen Wissenschaften, Band 125. Berlin: Springer-Verlag, 1974, pp. xv+321 (cit. on p. 10).

[Izh07] Eugene M Izhikevich. *Dynamical systems in neuroscience*. MIT press, 2007 (cit. on p. 1).

[JKO09] Milton Jara, Tomasz Komorowski, and Stefano Olla. "Limit theorems for additive functionals of a Markov chain". In: *The Annals of Applied Probability* 19.6 (2009), pp. 2270–2300. DOI: 10.1214/09-AAP610. URL: https://doi.org/10.1214/09-AAP610 (cit. on p. 4).

[JT81] William B Jones and Wolfgang J Thron. *Continued fractions*. en. Encyclopedia of mathematics and its applications. Harlow, England: Longman Higher Education, Feb. 1981 (cit. on p. 23).

[Kal21] Olav Kallenberg. *Foundations of modern probability*. New York: Springer-Verlag, 2021 (cit. on p. 9).

[Kin61] J. F. C. Kingman. "The single server queue in heavy traffic". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 57.4 (1961), pp. 902–904 (cit. on p. 1).

[Kis63] Nand Kishore. "The Rayleigh function". In: *Proc. Amer. Math. Soc.* 14 (1963), pp. 527–533. ISSN: 0002-9939,1088-6826 (cit. on p. 22).

[Kru91] Paul R Krugman. "Target zones and exchange rate dynamics". In: *The Quarterly Journal of Economics* 106.3 (1991), pp. 669–682 (cit. on p. 1).

[KT81] S. Karlin and H.M. Taylor. *A Second Course in Stochastic Processes*. v. 2. Elsevier Science, 1981 (cit. on p. 16).

[Lin05] Vadim Linetsky. "On the Transition Densities for Reflected Diffusions". In: *Advances in Applied Probability* 37.2 (2005), pp. 435–460 (cit. on p. 1).

[LS07] Kristian Stegenborg Larsen and Michael Sørensen. "Diffusion models for exchange rates in a target zone". In: *Mathematical Finance* 17.2 (2007), pp. 285–306 (cit. on p. 1).

[PY03] Jim Pitman and Marc Yor. "Hitting, occupation and inverse local times of one-dimensional diffusions: martingale and excursion approaches". In: *Bernoulli* 9.1 (2003), pp. 1–24 (cit. on pp. 5, 20).

[RR08] Kavita Ramanan and Martin I. Reiman. "The heavy traffic limit of an unbalanced generalized processor sharing model". In: *Ann. Appl. Probab.* 18.1 (2008), pp. 22–58. ISSN: 1050-5164,2168-8737 (cit. on p. 1).

[RS87] L. M. Ricciardi and L. Sacerdote. "On the probability densities of an Ornstein–Uhlenbeck process with a reflecting boundary". In: *Journal of Applied Probability* 24.2 (1987), pp. 355–369. DOI: 10.2307/3214260 (cit. on p. 1).

[Sko56] A. V. Skorokhod. "Limit Theorems for Stochastic Processes". In: *Theory of Probability & Its Applications* 1.3 (1956), pp. 261–290. DOI: 10.1137/1101022 (cit. on p. 4).

[TJ07] Romain Thibaux and Michael I Jordan. "Hierarchical beta processes and the Indian buffet process". In: *Artificial intelligence and statistics*. PMLR. 2007, pp. 564–571 (cit. on p. 1).

[Whi02] Ward Whitt. *Stochastic-process limits*. Springer Series in Operations Research. An introduction to stochastic-process limits and their application to queues. New York: Springer-Verlag, 2002, pp. xxiv+602. ISBN: 0-387-95358-2 (cit. on pp. 6, 7, 9).

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