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Khovanov homology of strongly invertible knots and their quotients

Robert Lipshitz and Sucharit Sarkar

Abstract. We construct a spectral sequence relating the Khovanov homology of a strongly invertible knot to the annular Khovanov homologies of the two quotient knots. Using this spectral sequence, we re-prove that Khovanov homology distinguishes certain slice disks. We also give an analogous spectral sequence for $\widehat{HF}$ of the branched double cover.

Contents

1. Introduction

A knot $K \subset \mathbb{R}^3$ is called *strongly invertible* if K meets some straight line in exactly two points, and is preserved setwise by rotation by 180° around that line; this straight line is called the *axis*. While strongly invertible knots have been studied for decades (see [**Sak86**] and the references therein), they have recently seen a surge in interest. For example, there is a somewhat mysterious, noncommutative concordance group of strongly invertible knots [**Sak86, DP23**], as well as natural equivariant analogues of the slice genus [**BI22**]. Related to this, many of the pairs of non-isotopic slice disks or more general slice surfaces which have appeared in the literature recently come from strongly invertible knots [**Hay, SS22, HS**], a phenomenon which has led to connections with Heegaard Floer-theoretic invariants [**DMS**]. In addition to Heegaard Floer homology, Donaldson's diagonalization theorem [**BI22**], the G-signature theorem and related equivariant signature techniques [**AB, DP**], Kojima-Yamasaki's η-function [**KY79, Sak86, BI22, DPF**],

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and Khovanov homology [**Cou09, Wat17, Sna18, LW21**] have also been applied recently to study strongly invertible knots.

The quotient of K by its strong inversion is naturally an embedded arc with boundary on the axis. By gluing this arc to part of the axis, we obtain a quotient knot; see Figures 1.1 and 5.2. (In S^3, instead of $\mathbb{R}^3$, there are two equally natural choices of quotient knot, corresponding to the two choices of half-axis; cf. [**Sak86, BI22**].) The main goal of this paper is to construct a spectral sequence relating the Khovanov homology of a strongly invertible knot K and a variant of the Khovanov homology of its quotient.

Similar results have been proved before, for other symmetries. Stoffregen-Zhang [**SZ**] and Borodzik-Politarczyk-Silvero [**BPS21**] showed that there is a spectral sequence relating the Khovanov homology of a periodic knot (a knot preserved by rotation around an axis disjoint from it) and the annular Khovanov homology of its quotient (see also [**Cor, Zha18**]). An analogous result relating the symplectic Khovanov homology of a 2-periodic knot and of its quotient was proved earlier by Seidel-Smith [**SS10**]. Using the same technical tool, Hendricks proved a similar relationship for knot Floer homology [**Hen15**], and analogous results have been given for other symmetries in Heegaard and monopole Floer homology [**Hen12, HLS16, LT16, LM18, Lar, HLL22**].

Like Stoffregen-Zhang's and Borodzik-Politarczyk-Silvero's spectral sequence for periodic knots, the spectral sequence we construct for strongly invertible knots relates the Khovanov homology of K to the annular Khovanov homology of its quotient $\overline{K}$. If τ denotes the 180° rotation around the axis then by a slight τ-equivariant perturbation of the knot K we may assume that the projection of K to the plane perpendicular to the axis is a knot diagram. Following the literature [**Boy21, BI22**], we call such a diagram *intravergent*. The quotient knot $\overline{K}$ may be viewed as an annular knot in two natural ways, $\overline{K}_0$ and $\overline{K}_1$, corresponding to taking the quotients of the 0-resolution or the 1-resolution of the fixed crossing of the intravergent diagram K; see Figure 1.1 and 5.2. (These quotients depend on the diagram K; see Remark 4.5 and Proposition 4.6.) The annular Khovanov chain complexes of these knots are related by an *axis-moving map* $f^+\colon \Sigma^{0,0,1}ACKh(\overline{K}_1) \to ACKh(\overline{K}_0)$, where $\Sigma^{a,b,c}$ denotes a (homological, quantum, annular) trigrading shift by (a,b,c); we construct the map f^+, which is a special case of Akhmechet-Khovanov's maps associated to anchored cobordisms [**AK**], in Section 3. By a slight abuse of notation, we define the annular Khovanov chain complex of the pair of annular knots $(\overline{K}_1, \overline{K}_0)$ to be the mapping cone of f^+,

$$ACKh(\overline{K}_1, \overline{K}_0) = \operatorname{Cone}\left(\Sigma^{0,0,1}ACKh(\overline{K}_1) \xrightarrow{f^+} ACKh(\overline{K}_0)\right),$$

and the annular Khovanov homology of the pair to be the homology of this complex which, over any field $\mathbb{F}$, is also (unnaturally) isomorphic to the homology of the mapping cone of the induced map on homology,

$$\begin{aligned} AKh(\overline{K}_1, \overline{K}_0; \mathbb{F}) &= H_* \operatorname{Cone}\left(\Sigma^{0,0,1}ACKh(\overline{K}_1; \mathbb{F}) \xrightarrow{f^+} ACKh(\overline{K}_0; \mathbb{F})\right) \\ &\cong H_* \operatorname{Cone}\left(\Sigma^{0,0,1}AKh(\overline{K}_1; \mathbb{F}) \xrightarrow{f^+} AKh(\overline{K}_0; \mathbb{F})\right). \end{aligned}$$

THEOREM 1.1. *Given a strongly invertible knot K with annular quotients $\overline{K}_0$, $\overline{K}_1$ there is a spectral sequence with the following properties:*

(Θ-1) The E^1-page is $Kh(K;\mathbb{F}_2)\otimes\mathbb{F}_2[\theta^{-1},\theta]$ with d^1-differential the map $\theta(\mathrm{Id}+\tau_*)$,

$$\cdots \longrightarrow \theta^{-1}Kh(K;\mathbb{F}_2) \xrightarrow{\theta(\mathrm{Id}+\tau_*)} \theta^0 Kh(K;\mathbb{F}_2) \xrightarrow{\theta(\mathrm{Id}+\tau_*)} \theta^1 Kh(K;\mathbb{F}_2) \longrightarrow \cdots,$$

where τ_* is induced by the strong inversion.

(Θ-2) The d^r-differential preserves the quantum grading and increases the θ-power grading by r.

(Θ-3) The spectral sequence converges to $AKh(\overline{K}_1,\overline{K}_0;\mathbb{F}_2)\otimes\mathbb{F}_2[\theta^{-1},\theta]$. Keeping track of quantum gradings, the summand of the spectral sequence in quantum grading j converges to

$$\bigoplus_{\substack{\bar{\imath},\bar{\jmath},k\in\mathbb{Z}\\ 2\bar{\jmath}+k=j-1+3N_- -6\overline{N}_-}} H_*\,\mathrm{Cone}\left(AKh_{\bar{\imath},\bar{\jmath},k-1}(\overline{K}_1;\mathbb{F}_2) \xrightarrow{f^+} AKh_{\bar{\imath},\bar{\jmath},k}(\overline{K}_0;\mathbb{F}_2)\right)\otimes\mathbb{F}_2[\theta^{-1},\theta] \tag{1.1}$$

where N_- (respectively $\overline{N}_-$) is the number of negative crossings of K (respectively $\overline{K}$).

Since the quantity $3N_- - 6\overline{N}_-$ comes up frequently, let

$$\Delta = N_- - 2\overline{N}_-. \tag{1.2}$$

See Remark 4.5 and Proposition 4.6 for a little further discussion of the grading shift, and Section 2 for our grading conventions for Khovanov homology. Like the periodic knot case, the proof of Theorem 1.1 uses the Khovanov stable homotopy type and Smith theory.

Corollary 1.2. *For any quantum grading j, we have*

$$\sum_i \dim Kh_{i,j}(K;\mathbb{F}_2) \geq \sum_{\substack{\bar{\imath},\bar{\jmath},k\in\mathbb{Z}\\ 2\bar{\jmath}+k=j-1+3\Delta}} \dim AKh_{\bar{\imath},\bar{\jmath},k}(\overline{K}_1,\overline{K}_0;\mathbb{F}_2).$$

Proof. This follows from Theorem 1.1 by comparing the ranks of the E^1-page and E^∞-page. □

Corollary 1.3. *Assume that in some quantum grading j, $Kh_{*,j}(K;\mathbb{F}_2)$ is supported in a single homological grading i. Then,*

$$\dim Kh_{i,j}(K;\mathbb{F}_2) - 2\,\mathrm{rank}((\mathrm{Id}+\tau_*)_{i,j}) = \sum_{\substack{\bar{\imath},\bar{\jmath},k\in\mathbb{Z}\\ 2\bar{\jmath}+k=j-1+3\Delta}} \dim AKh_{\bar{\imath},\bar{\jmath},k}(\overline{K}_1,\overline{K}_0;\mathbb{F}_2).$$

where $(\mathrm{Id}+\tau_*)_{i,j}$ is the induced endomorphism on $Kh_{i,j}(K;\mathbb{F}_2)$.

Proof. This also follows from Theorem 1.1 by equating the ranks of the E^2-page and E^∞-page: since θ has homological, quantum bigrading $(-1,0)$, statement (Θ-2) in the theorem implies that the d^r-differentials vanish for $r>1$. □

Corollary 1.4. *Assume that in some quantum grading j, $Kh_{*,j}(K;\mathbb{F}_2)$ is supported in a single homological grading i. Suppose also that*

$$\bigoplus_{\bar{\imath},\bar{\jmath},k\,|\,2\bar{\jmath}+k=j-1+3\Delta} AKh_{\bar{\imath},\bar{\jmath},k}(\overline{K}_1,\overline{K}_0;\mathbb{F}_2) = 0.$$

Then, the endomorphism $(\mathrm{Id}+\tau_*)_{i,j}$ on $Kh_{i,j}(K;\mathbb{F}_2)$ has rank $\frac{1}{2}\dim Kh_{i,j}(K;\mathbb{F}_2)$. In particular, if $Kh_{i,j}(K;\mathbb{F}_2)$ is 2-dimensional then, up to a change of basis, τ_* is given by the matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$.

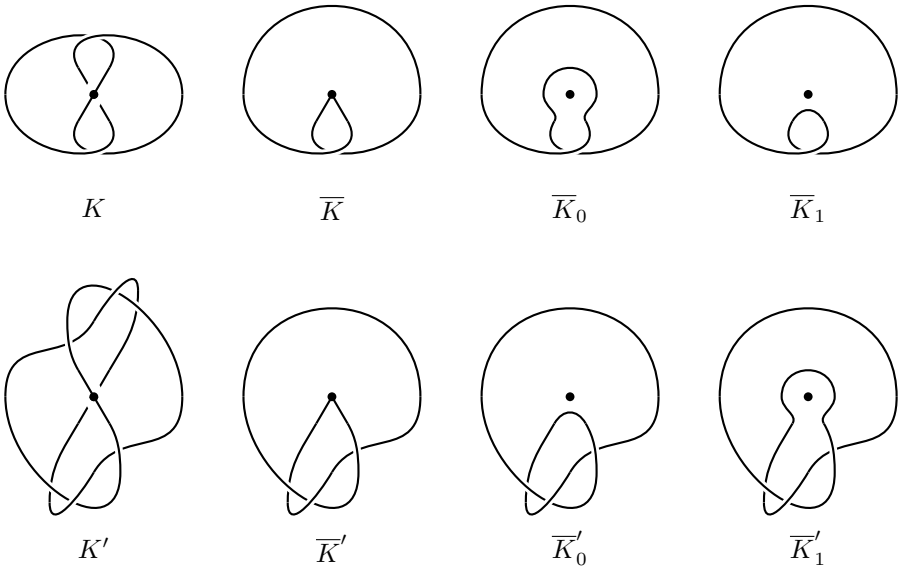

FIGURE 1.1. **Strong inversions of the trefoil.** Each row shows an intravergent diagram for a strong inversion of the trefoil, the quotient knot, and the two annular quotients. The two strong inversions are isotopic in S^3 but not $\mathbb{R}^3$; in S^3, the two different quotients correspond to the two choices of half-axis.

PROOF. The first statement is immediate from Corollary 1.3. The second follows from the first and the fact that any involution of $\mathbb{F}_2^2$ is one of $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, or $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$, and the last two are conjugate to the second one. □

EXAMPLE 1.5. Figure 1.1 shows two intravergent diagrams K and K' for strong inversions of the positive trefoil. (See also [**BI22**, Appendix A].) The first has quotient $\overline{K}$ the unknot; the quotient $\overline{K}'$ of the second is the negative trefoil.

With our grading conventions, the E^1-page of the spectral sequences from Theorem 1.1 for both K and K' are given by the Khovanov homology of the trefoil tensored with $\mathbb{F}_2[\theta^{-1},\theta]$, i.e.,

	0	1	2	3
9				F
7			D	E
5			C	
3	B			
1	A			

,

where each letter is a copy of $\mathbb{F}_2[\theta^{-1},\theta]$. The homological grading is horizontal and quantum grading is vertical. The two involutions are isotopic in S^3; hence, the

induced maps τ_* on Khovanov homology are the same [**MWW22**]. In fact, since τ_* is a grading-preserving involution, it must be the identity map, so the d^1-differential on the spectral sequence from Theorem 1.1 vanishes. So, the table above also gives the E^2-pages.

By Point (Θ-2), the only possibly nontrivial differential is the d^2-differential, for which we could have $d^2(D) = \theta^2 E$. It is easy to check directly that for K this differential is trivial and the spectral sequence collapses at E^2 with total rank 6 over $\mathbb{F}_2[\theta^{-1}, \theta]$. It follows from Theorem 1.1 and the computation of the E^∞-page below that this holds for K' as well.

Direct computation shows that the cones of f^+ for $\overline{K}$ and $\overline{K}'$ are given by

	−1	0
1		$H_{(-1)}$
−1		$G_{(1)}$
−3		

$\xrightarrow{0}$

−1	0	
	$L_{(-2)}$	1
	$K_{(0)}$	−1
$I_{(0)}$	$J_{(2)}$	−3

and

	−3	−2	−1	0
−1				$T_{(-2)}$
−3				$S_{(0)}$
−5		$P_{(0)}$	$Q_{(0)}$	$R_{(2)}$
−7	$N_{(0)}$	$O_{(0)}$		
−9	$M_{(0)}$			

−3	−2	−1	0	
			$Z_{(-1)}$	−1
			$Y_{(1)}$	−3
	$X_{(-1)}$			−5
$V_{(-1)}$	$W_{(1)}$			−7
$U_{(1)}$				−9

respectively. Here, the subscripts indicate the k-grading. In the first case, the map f^+ vanishes; in the second, it is given by the dashed arrows. In particular, in both cases the homology of the mapping cone has rank 6, consistent with Theorem 1.1. One can further use the grading shift to match up the gradings. For example, for K, $\Delta = -2$, while for K', $\Delta = -5$, and so the quantum gradings on K or K' corresponding to the generators of $\overline{K}$ and $\overline{K}'$ are:

G	H	I	J	K	L
7	9	1	3	5	7

N	P	Q	R	V	X
3	7	7	9	1	5

.

Again, this is consistent with Theorem 1.1. Alternatively, one could use Theorem 1.1 to deduce that f^+ is the map given above.

One reason strongly invertible knots have appeared recently is that they have furnished examples of non-isotopic pairs of slice disks. It turns out that Corollary 1.4 and properties of the maps on Khovanov homology can be used to prove that certain pairs of slice disks are distinguished by Khovanov homology, without explicitly computing the maps associated to the slice disks. We illustrate this phenomenon for the knot 9_{46} in Section 5.

The reduced Khovanov homology of K is closely related to the Heegaard Floer homology of the branched double cover $\Sigma(S^3, K)$ of K [**OSz05, Rob13, GW10**], and Theorem 1.1 has an analogue for Heegaard Floer homology:

THEOREM 1.6. *Given a strongly invertible knot K with quotient knot $\overline{K}$ there is an ungraded spectral sequence with E^1-page given by $\widehat{HF}(\Sigma(S^3,K))\otimes\mathbb{F}_2[\theta^{-1},\theta]]$ converging to $\widehat{HF}(\Sigma(S^3,\overline{K}))\otimes\mathbb{F}_2[\theta^{-1},\theta]]$.*

As we will see, this follows from a localization result of Hendricks, Lidman, and the first author for the Heegaard Floer homology of double branched covers [**HLL22**], which in turn follows from a general localization theorem in Lagrangian intersection Floer homology of Large [**Lar**]. Note that in Theorem 1.6, there are two choices for the quotient knot $\overline{K}$, depending on which half of the axis one chooses. The statement holds for either choice. There is also an analogue for the knot Floer homology relative to a preimage of the axis, Theorem 6.3.

Another spectral sequence associated to a strongly invertible knot was constructed by Lobb-Watson [**LW21**], although they used a different kind of diagrams, transvergent rather than intravergent. (They also mention intravergent diagrams briefly, in the discussion around their Figure 7.) It might be interesting to compare their $\mathcal{F}$ spectral sequence with the one constructed here; in particular, this might give an approach to proving that their other, $\mathcal{G}$, spectral sequence collapses [**LW21**, Question 6.5].

We expect that the spectral sequence in Theorem 1.1 is an invariant of the strong inversion on K, but do not pursue this here. (In fact, we expect a somewhat stronger statement to hold: that the complex $CKh(K)$, viewed as a complex of $\mathbb{F}_2[\mathbb{Z}/2]$-modules, is invariant up to quasi-isomorphism. Invariance of the spectral sequence from Theorem 1.6 follows from [**HLL22**, Remark 4.13].) One could also ask about whether the spectral sequence is an invariant of the strongly invertible knot in S^3, i.e., does not depend on the choice of half-axis, as was the case in Example 1.5; this seems hard to prove using an intravergent diagram, but easier if one could relate it to a transvergent one. Theorem 1.6 suggests there might be an interesting reduced version of Theorem 1.1, but we do not pursue that either. One could also consider links in Theorems 1.1 and 1.6, meeting the axis in more than two points, but we also do not pursue that generalization.

This paper is organized as follows. Since there are many conventions for Khovanov homology, we review ours in Section 2. We then introduce the axis-moving maps on annular Khovanov homology and some of their basic properties in Section 3. Section 4 proves the main localization result, Theorem 1.1. We give the application to slice disks in Section 5. We end with the proof of the analogue for Heegaard Floer homology, Theorem 1.6, in Section 6.

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2. Conventions for Khovanov homology

We start by describing our grading conventions for Khovanov homology (which also serves as a terse review of Khovanov homology itself). Given a knot diagram with N crossings numbered 1 through N, consider the Kauffman cube of resolutions, where the complete resolution of the diagram at the vertex $v=(v_1,\dots,v_N)\in\{0,1\}^N$ is obtained by resolving the i^{th} crossing $\times$ by the 0-resolution $\asymp$ if $v_i=0$ or by the 1-resolution $)\,($ if $v_i=1$, for each $i\in\{1,\dots,N\}$. We will view the cube

as $(1 \to 0)^N$, so each edge runs from a vector with a 1 in some coordinate to the corresponding vector with a 0 in that coordinate. Associated to each edge is an elementary saddle cobordism between the corresponding resolutions.

The Frobenius algebra $\mathbb{Z}[X]/(X^2)$ (with comultiplication given by $1 \mapsto 1 \otimes X + X \otimes 1, X \mapsto X \otimes X$) corresponds to a 2-dimensional TQFT. The Khovanov chain complex CKh is obtained by applying this TQFT to the cube of resolutions, and then taking the total complex. More concretely, CKh is freely generated by the Khovanov generators x, which consist of a choice of a vertex $v \in \{0,1\}^N$ and a labeling of the circles in the complete resolution at v by the labels $\{1, X\}$. The homological grading of x is $|v| - N_-$, and the quantum grading is $N - 3N_- + |v| + \#\{\text{circles labeled } X\} - \#\{\text{circles labeled } 1\}$. (Here N_- is the number of negative crossings in the diagram and $|v| = \sum_i v_i$ is the L^1-norm of v.) So, the quantum grading of X is two more than the quantum grading of 1.

The differential on the Khovanov chain complex is a sum of maps along the edges of the cube; it preserves the quantum grading and decreases the homological grading by 1. The component of the differential along the edge from the vertex $v = (v_1, \dots, v_{n-1}, 1, v_{n+1}, \dots, v_N)$ to the vertex $w = (v_1, \dots, v_{n-1}, 0, v_{n+1}, \dots, v_N)$ is $(-1)^{v_1 + \dots + v_{n-1}}$ times the map associated by the TQFT to the saddle cobordism. That is, if the saddle cobordism merges two circles into one, then the map is induced by the multiplication map in $\mathbb{Z}[X]/(X^2)$, and if the saddle cobordism splits a circle into two, then the map is induced by the comultiplication map in $\mathbb{Z}[X]/(X^2)$.

This convention differs from Khovanov's original [**Kho00**] in a couple of ways: the differential in Khovanov's original paper increased the homological grading; and the quantum grading of X was lower than the quantum grading of 1. However, in order to ensure that arc algebras are supported in non-negative quantum gradings, Khovanov switched the latter convention in [**Kho02**], and his subsequent papers follow the switched convention (where the quantum grading of X is higher than the quantum grading of 1). However, Khovanov's original quantum grading convention had a desirable feature that the positive knots (except the unknot) had Khovanov homologies supported in positive quantum gradings; unfortunately, with the switched convention, their Khovanov homologies were supported in negative quantum gradings. With our grading conventions—additionally making the differential decrease the homological grading—we try to tread a middle ground: arc algebras and Khovanov homologies of (non-trivial) positive knots are both supported in non-negative quantum gradings.

The Khovanov chain complex with our convention is the dual of the Khovanov chain complex from Khovanov's original convention, preserving the bigrading (this follows easily from the duality statement [**Kho00**, Proposition 32]). So, over any field, Khovanov homology with our convention is (unnaturally) bigraded isomorphic to the original Khovanov homology; over $\mathbb{Z}$, the free parts are bigraded isomorphic, and the torsion subgroup with our convention is isomorphic to the original torsion subgroup, but with its homological grading shifted down by 1 (and quantum grading unchanged).

The Khovanov complex of a link in the annulus inherits an extra *annular* or *winding number* filtration; the homology of the associated graded complex is annular Khovanov homology. We follow the usual conventions in the literature for the annular filtration. Specifically, given a labeled resolution, orient circles labeled 1 counter-clockwise (positively) and circles labeled X clockwise (negatively). Then,

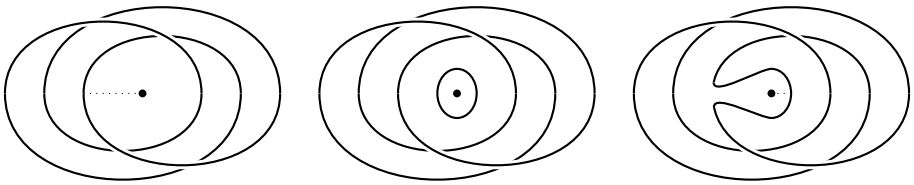

FIGURE 3.1. **The axis moving maps.** Left: the link L and the arc connecting a point p on L to the axis (dotted). Center: the link $L \amalg U$. Right: the link L' and the corresponding arc connecting p' to the axis.

the annular filtration of a labeled resolution is the winding number around the axis. Terms in the differential either preserve the annular filtration or decrease it by 2. (The latter occurs when merging a nullhomotopic circle labeled X with an essential circle labeled 1, merging two essential circles labeled 1, splitting an essential circle labeled 1 into a nullhomotopic circle labeled 1 and an essential circle labeled X, or splitting a nullhomotopic circle labeled X into two essential circles labeled X.)

3. A map on annular Khovanov homology

Let L be an annular link diagram. Fix a point p on L adjacent to the axis of the annulus. Isotoping p across the axis of the annulus gives a new link L'. (See Figure 3.1.) In this section we define and spell out basic properties of the axis moving maps

$$f^+,\ f^-\colon AKh(L) \to AKh(L').$$

While the rest of the results in this paper use $\mathbb{F}_2$-coefficients, in this section, we will work with $\mathbb{Z}$-coefficients.

Let $L \amalg U$ be the result of adding an essential circle U around the axis, adjacent to the axis and disjoint from L. (See Figure 3.1.) The annular Khovanov complex of $L \amalg U$ is $ACKh(L) \otimes ACKh(U) = ACKh(L) \otimes \mathbb{Z}\langle 1, X\rangle$. So, there are two inclusions $\iota_1, \iota_X\colon ACKh(L) \hookrightarrow ACKh(L \amalg U)$, defined by $\iota_1(y) = y \otimes 1$ and $\iota_X(y) = y \otimes X$; these have (homological, quantum, annular) trigradings $(0,-1,1)$ and $(0,1,-1)$, respectively. Merging L and U at the point p gives a map $m\colon ACKh(L \amalg U) \to L$; this map has trigrading $(0,1,0)$. (This is the annular merge map. So, for instance, merging two essential circles labeled 1 is the zero map.) By composing, we get trigrading-preserving maps

$$f^+ = m \circ \iota_1\colon \Sigma^{0,0,1} ACKh(L) \to ACKh(L')$$
$$f^- = m \circ \iota_X\colon \Sigma^{0,2,-1} ACKh(L) \to ACKh(L'),$$

where $\Sigma^{a,b,c}$ denotes a trigrading shift by (a,b,c). The maps $f^\pm$ are compositions of chain maps, hence are chain maps. Abusing notation, we will also use $f^\pm$ to denote the induced map on annular homology.

When we want to indicate the dependence of $f^\pm$ on the point p we will write them as $f^\pm_p$. Also, the point p on L becomes a point p' on L' after the isotopy across the axis.

We give some basic properties of the maps $f^{\pm}$; though we will not use them in the rest of this paper, perhaps they will be useful in related applications.

PROPOSITION 3.1. *Up to homotopy and sign, the maps $f^{\pm}$ are isotopy invariants of the pair of the annular link L and the arc connecting p to the axis. Further,*

$$f^+_{p'} \circ f^+_p = f^-_{p'} \circ f^-_p = 0 \tag{3.1}$$

$$f^+_{p'} \circ f^-_p = f^-_{p'} \circ f^+_p = X\cdot_p \tag{3.2}$$

Here, $X\cdot_p$ is the basepoint action on annular Khovanov homology, the result of merging in a nullhomotopic circle labeled X at the point p, via the annular merge map. Also, the first statement uses naturality of annular Khovanov homology (see, e.g., [**GLW18**]).

PROOF. For the first statement, call an arc from an annular link to the axis *short* if its projection to the annulus is a smooth embedding and disjoint from the projection of the rest of the link to the annulus (i.e., the rest of the link diagram). Given a pair of an annular link L and a (not short) arc γ connecting that link to the axis, there is a canonical isotopy from $L \cup \gamma$ to a pair $L' \cup \gamma'$ where γ' is short: just shrink γ and pull L along with it.

Now, fix link diagrams L_i, $i = 0, 1$, and points p_i on L_i adjacent to the axis, and let γ_i be the short arc from p_i to the axis. Assume that $L_0 \cup \gamma_0$ is isotopic to $L_1 \cup \gamma_1$, via an isotopy $L_t \cup \gamma_t$. Applying the canonical isotopy $L_{t,s} \cup \gamma_{t,s}$ from the previous paragraph to each L_t gives a new isotopy $L_{t,1} \cup \gamma_{t,1}$ from $L_0 \cup \gamma_0$ to $L_1 \cup \gamma_1$ so that the arc $\gamma_{t,1}$ is short for all t. Perturb this isotopy so that the projection to the annulus is generic. This gives a sequence of Reidemeister moves (in the annulus) connecting L_0 to L_1 and disjoint from the arc. Each Reidemeister move induces a map on the annular Khovanov complex, and it is immediate from the definitions that these maps commute with the maps $f^{\pm}$.

Equations (3.1) and (3.2) are clear from the definitions. □

REMARK 3.2. These maps are a special case of maps associated to *anchored cobordisms* by Akhmechet-Khovanov [**AK**]. Specifically, the trace of an isotopy moving p across the axis is an anchored cobordism. To define a map, we must also label the intersection point between this cobordism and the axis (the point where p crosses the axis) by 1 or 2. If the winding number of K around the axis is odd then f^+ corresponds to labeling the intersection point 1 and f^- corresponds to labeling it 2; if the winding number is even then f^+ corresponds to labeling the intersection point 2 and f^- corresponds to labeling it 1 (compare [**AK**, Proof of Theorem 2.19]).

Equations (3.1) and (3.2) follow from Akhmechet-Khovanov's curtain relation [**AK**, Corollary 2.7].

REMARK 3.3. Given (L, p) as above, let L'' be the result of performing a Reidemeister I move at p across the axis, changing the winding number by ± 1 (i.e., a Markov 2 move). Choose the Reidemeister move so that the 1-resolution of the new crossing is the disjoint union of L and a new (essential) circle. Then, up to some overall grading shift, the mapping cone of $f^+ \colon \Sigma^{0,0,1} ACKh(L) \to ACKh(L')$ (respectively $f^- \colon \Sigma^{0,2,-1} ACKh(L) \to ACKh(L')$) is a subcomplex of $ACKh(L'')$: it is the subcomplex where either the new crossing is 0-resolved or the new crossing is 1-resolved and the new essential circle is labeled 1 (respectively X).

4. The localization theorem for strongly invertible knots

In this section, we prove Theorem 1.1. The spectral sequence is constructed as follows. Given an intravergent diagram for K, rotation τ by 180° induces a $\mathbb{Z}/2$ action, which we still call τ, on $CKh(K;\mathbb{F}_2)$. Consider the Tate complex for this $\mathbb{Z}/2$-action, which is given by $CKh(K;\mathbb{F}_2)\otimes\mathbb{F}_2[\theta^{-1},\theta]$ with differential $d+\theta(\mathrm{Id}+\tau)$, where d is the Khovanov differential:

$$\cdots \xrightarrow{\theta(\mathrm{Id}+\tau)} \theta^0 CKh(K;\mathbb{F}_2) \xrightarrow{\theta(\mathrm{Id}+\tau)} \theta^1 CKh(K;\mathbb{F}_2) \xrightarrow{\theta(\mathrm{Id}+\tau)} \cdots. \tag{4.1}$$

If we give θ the (homological, quantum) bigrading $(-1,0)$ then this complex decomposes according to quantum gradings and the differential decreases the homological grading by 1. The complex has a filtration by the θ-power. This filtration induces the spectral sequence in Theorem 1.1.

The main work is to compute the E^∞-page of the spectral sequence (Item (Θ-3)); the other properties are immediate. The strategy is similar to Stoffregen-Zhang's [**SZ**] and Borodzik-Politarczyk-Silvero's [**BPS21**]: we prove that the fixed points of the $\mathbb{Z}/2$-action induced by the strong inversion on a CW complex representing the Khovanov stable homotopy type of K is related to the annular Khovanov stable homotopy type of $(\overline{K}_1,\overline{K}_0)$, and then apply classical Smith theory.

Before embarking on the proof, we briefly summarize the relevant aspects of the Khovanov stable homotopy type [**LS14**], following the more recent box map construction [**LLS20, LLS17**] (see also [**HKK16**]). As sketched in Section 2, in a fixed quantum grading j, the Khovanov chain complex $CKh_{*,j}(K)$ is obtained as follows:

(Kh-1) First construct a cube-shaped diagram of abelian groups, $F_j^{Kh}\colon (1\to 0)^N \to \mathsf{Ab}$, which associates to each vertex v the free abelian group generated by all Khovanov generators x at v with quantum grading j, and associates to each edge $v\to w$ the saddle cobordism map from the TQFT corresponding to the Frobenius algebra $\mathbb{Z}[X]/(X^2)$.

(Kh-2) Viewing F_j^{Kh} as a cube-shaped diagram of chain complexes, take the mapping cone N times, in the N directions of the cube, to obtain a single chain complex. (The signs, like $(-1)^{v_1+\cdots+v_{n-1}}$ from Section 2, appear during this iterated mapping cone construction; the precise signs depend on the order in which the mapping cones are done.)

(Kh-3) Finally, shift the homological grading of the chain complex down by N_- to get the Khovanov chain complex $CKh_{*,j}(K)$. (The total Khovanov complex over all quantum gradings is given by $\bigoplus_j CKh_{*,j}(K)$.)

In order to remove the choice about the order in which to take the mapping cones (which amounts to choosing an ordering of the N crossings of the knot diagram), one can replace Step (Kh-2) by the following:

(Kh-2′) Extend F_j^{Kh} trivially to a diagram $(F_j^{Kh})_+$ from a slightly larger category $(1\to 0)_+^N$ which has an additional object and a unique morphism from every $v\neq 0$ to it, and then take the homotopy colimit to obtain a single chain complex. (The functor $(F_j^{Kh})_+$ sends the new object to the trivial group.)

The lift to a Khovanov stable homotopy type follows the same outline, replacing the cube of abelian groups by a cube of topological spaces. More concretely:

($\mathscr{X}$-0) Fix an integer $D \geq N$.

($\mathscr{X}$-1) Construct a cube-shaped diagram $F_j^{\mathscr{X}}\colon (1 \to 0)^N \to \mathsf{Top}$ of based CW complexes which associates to each vertex a wedge sum of D-dimensional spheres, so that its composition with the reduced homology functor, $\widetilde{H}_D \circ F_j^{\mathscr{X}}$, equals the cube-shaped diagram F_j^{Kh} from Step (Kh-1). (It suffices to construct a homotopy coherent diagram instead of a strictly commuting one.)

($\mathscr{X}$-2) Extend (trivially) to a diagram $(F_j^{\mathscr{X}})_+$ from the larger category $(1 \to 0)_+^N$ mapping the new object to a 1-point space, and then take homotopy colimit to obtain a single topological space.

($\mathscr{X}$-3) Finally, formally desuspend $(D + N_-)$ times to get the Khovanov spectrum $\mathscr{X}_j(K)$ in quantum grading j. (The total Khovanov spectrum over all quantum gradings is given by $\bigvee_j \mathscr{X}_j(K)$.)

Of these, the hardest step is Step ($\mathscr{X}$-1), which we undertake by first constructing a lax diagram in a third category—the Burnside 2-category $\mathscr{B}$ of finite sets, finite correspondences, and bijections between correspondences. Specifically, letting $\mathbb{Z}\langle\cdot\rangle\colon \mathscr{B} \to \mathsf{Ab}$ denote the functor that replaces a finite set by the free abelian group generated by it, we do the following:

($\mathscr{B}$-1) Construct a cube-shaped 2-functor $F_j^{\mathscr{B}}\colon (1 \to 0)^N \to \mathscr{B}$ which associates to each vertex v the set of Khovanov generators x at v with quantum grading j, to each edge $v \to w$ a correspondence $F_j^{\mathscr{B}}(v) \xleftarrow{s} F_j^{\mathscr{B}}(v \to w) \xrightarrow{t} F_j^{\mathscr{B}}(w)$, so that $\mathbb{Z}\langle\cdot\rangle \circ F_j^{\mathscr{B}} = F_j^{Kh}$, and to each 2-dimensional face $\begin{smallmatrix} & & v & & \\ u & \swarrow & & \searrow & w \\ & \searrow & v' & \swarrow & \end{smallmatrix}$ a 2-morphism (which is an isomorphism of correspondences) $F_j^{\mathscr{B}}(v \to w) \circ F_j^{\mathscr{B}}(u \to v) \xrightarrow{\cong} F_j^{\mathscr{B}}(v' \to w) \circ F_j^{\mathscr{B}}(u \to v')$ satisfying a coherence relation for every 3-dimensional face. Note that this data specifies, for any $v > w$ in the poset $(1 > 0)^N$, a correspondence $F_j^{\mathscr{B}}(v \to w)\colon F_j^{\mathscr{B}}(v) \to F_j^{\mathscr{B}}(w)$, by composing the correspondences along a sequence of oriented edges connecting v to w; any two such sequences specify canonically isomorphic correspondences. (We can also define the total Burnside functor over all quantum gradings as $\coprod_j F_j^{\mathscr{B}}$ where $\amalg$ is defined by taking disjoints unions of sets and correspondences at vertices and edges, respectively.)

Most of the construction of $F_j^{\mathscr{B}}$ is forced. The only choices are the isomorphisms of correspondences for certain 2-dimensional faces (which we call ladybugs), and we explicitly choose the isomorphisms for those faces (which we call ladybug matchings) [**LS14**, Section 5.4], [**LLS20**, Section 8.1]. Once we have the 2-functor $F_j^{\mathscr{B}}\colon (1 \to 0)^N \to \mathscr{B}$, we can carry out Step ($\mathscr{X}$-1) by the *box map construction*, as follows:

($\mathscr{X}$-1′) For each vertex v, define $F_j^{\mathscr{X}}(v) = \big(\coprod_{x \in F_j^{\mathscr{B}}(v)} B_x\big)/\partial$, where B_x is a D-dimensional rectangular prism (*box*) associated to the Khovanov generator x. For each edge $v \to w$, choose disjoint D-dimensional sub-boxes $\{B_b\}_{b \in F_j^{\mathscr{B}}(v \to w)}$ inside $\coprod_{x \in F_j^{\mathscr{B}}(v)} B_x$ so that each B_b lies in $B_{s(b)}$; define the map $F_j^{\mathscr{X}}(v \to w)\colon F_j^{\mathscr{X}}(v) \to F_j^{\mathscr{X}}(w)$ by sending each sub-box $B_b \subset B_{s(b)}$ to $B_{t(b)}$ by scaling and translation, and the complement of all

these sub-boxes to the basepoint. Call maps as in the previous sentence *box maps*. To extend this to a homotopy coherent diagram $F_j^{\mathscr{X}}$ on the entire cube, we specify, for every chain $v_\ell > \cdots > v_0$ in the poset $(1>0)^N$, a $[0,1]^{\ell-1}$-parameter family of box maps $F_j^{\mathscr{X}}(v_\ell) \to F_j^{\mathscr{X}}(v_0)$ satisfying certain coherence conditions on its boundary, and refining the correspondence $F_j^{\mathscr{B}}(v_\ell \to v_0)\colon F_j^{\mathscr{B}}(v_\ell) \to F_j^{\mathscr{B}}(v_0)$. By induction on ℓ, and using the coherence conditions and the 2-morphisms in the Burnside category, such a family of maps is already defined on the boundary $\partial[0,1]^{\ell-1}$. Extend it to the entire cube $[0,1]^{\ell-1}$ using $(D-2)$-connectedness of the space of labeled sub-boxes and the assumption that $D \geq N$.

The constructions for annular Khovanov complexes and annular Khovanov homotopy types mirror these definitions. There is an extra annular grading, and we only consider maps that preserve that grading. Therefore, in each (quantum, annular) bigrading (j,k), we get diagrams $AF_{j,k}^{Kh}\colon (1\to 0)^N \to \mathsf{Ab}$, $AF_{j,k}^{\mathscr{B}}\colon (1\to 0)^N \to \mathscr{B}$, and $AF_{j,k}^{\mathscr{X}}\colon (1\to 0)^N \to \mathsf{Top}$; their extensions $(AF_{j,k}^{Kh})_+\colon (1\to 0)_+^N \to \mathsf{Ab}$ and $(AF_{j,k}^{\mathscr{X}})_+\colon (1\to 0)_+^N \to \mathsf{Top}$; and the chain complex $ACKh_{*,j,k}$ and the spectrum $A\mathscr{X}_{j,k}$. For the pair $(\overline{K}_1, \overline{K}_0)$ of annular knots whose annular Khovanov chain complex $ACKh(\overline{K}_1, \overline{K}_0)$ is defined as the mapping cone of $\Sigma^{0,0,1}ACKh(\overline{K}_1) \xrightarrow{f^+} ACKh(\overline{K}_0)$, it still may be viewed as a subcomplex of another annular Khovanov chain complex—see Remark 3.3—and therefore, all these constructions work for the pair $(\overline{K}_1, \overline{K}_0)$ as well.

Given an intravergent diagram of K, the strong inversion induces $\mathbb{Z}/2$-actions on these various objects as follows.

LEMMA 4.1. *The* 180° *rotation on the intravergent diagram of K induces a $\mathbb{Z}/2$-action on the cube $(1\to 0)^N$ and an* external $\mathbb{Z}/2$-action *on the 2-functor $F_j^{\mathscr{B}}(K)\colon (1\to 0)^N \to \mathscr{B}$ in each quantum grading j, in the sense of Stoffregen-Zhang* [**SZ**, Definition 3.4].

PROOF. The proof is similar to Stoffregen-Zhang's corresponding result for 2-periodic knots [**SZ**, Proposition 6.4]. The 180° rotation τ around the axis induces a $\mathbb{Z}/2$-action on the N crossings of K, which in turn induces a $\mathbb{Z}/2$-action (also denoted τ) on the cube category $(1\to 0)^N$ after identifying it with $(1\to 0)^{\{\text{crossings of } K\}}$ by ordering the crossings. It also induces a $\mathbb{Z}/2$-action (still denoted τ) on the set of all Khovanov generators in quantum grading j, sending $F_j^{\mathscr{B}}(v)$ to $F_j^{\mathscr{B}}(\tau v)$. Moreover, for each edge $v\to w$, it induces an isomorphism of correspondences

$$\begin{array}{ccccc} F_j^{\mathscr{B}}(v) & \xleftarrow{s} & F_j^{\mathscr{B}}(v\to w) & \xrightarrow{t} & F_j^{\mathscr{B}}(w) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\ F_j^{\mathscr{B}}(\tau v) & \xleftarrow{s} & F_j^{\mathscr{B}}(\tau v\to \tau w) & \xrightarrow{t} & F_j^{\mathscr{B}}(\tau w) \end{array}$$

since for any Khovanov generators $x\in F_j^{\mathscr{B}}(v), y\in F_j^{\mathscr{B}}(w)$, the set $s^{-1}(x)\cap t^{-1}(y)$ has 0 or 1 elements. So the only thing to check is that τ respects the ladybug matchings across 2-dimensional faces, which holds since the ladybug matching is invariant under planar isotopy, and in particular, the 180° rotation τ. □

This $\mathbb{Z}/2$-action on $F_j^{\mathscr{B}}(K)\colon (1\to 0)^N \to \mathscr{B}$ has a *fixed point functor* $(F_j^{\mathscr{B}}(K))^\tau$ [**SZ**, Definition 3.11]. The functor $(F_j^{\mathscr{B}}(K))^\tau$ is defined on the fixed subcategory of the cube category $(1\to 0)^N$, which is itself isomorphic to the cube category $(1\to 0)^{\overline{N}+1}$ where $\overline{N}=(N-1)/2$ is the number of crossings of the quotient diagram $\overline{K}$; $(F_j^{\mathscr{B}}(K))^\tau$ assigns to vertices and edges the fixed subset of the τ-action on the sets and correspondences, respectively. It turns out that these fixed point functors are related to the Burnside 2-functors associated to the pair of annular knots $(\overline{K}_1,\overline{K}_0)$:

LEMMA 4.2. *For any quantum grading j, the fixed point functor $(F_j^{\mathscr{B}}(K))^\tau$ is isomorphic to $\coprod_{\bar{\jmath},k|2\bar{\jmath}+k=j-1+3\Delta} AF^{\mathscr{B}}_{\bar{\jmath},k}(\overline{K}_1,\overline{K}_0)$, where Δ is as in Equation (1.2).*

PROOF. The proof is similar to the 2-periodic case [**SZ**, Theorem 6.7]. First, order the $\overline{N}$ crossings of $\overline{K}$ arbitrarily. Then, at any vertex $v=(v_1,\dots,v_{\overline{N}+1})\in(1\to 0)^{\overline{N}+1}$, the set $AF^{\mathscr{B}}_{\bar{\jmath},k}(\overline{K}_1,\overline{K}_0)(v)$ is defined to be the set of Khovanov generators of $\overline{K}_{v_{\overline{N}+1}}$ over the vertex $(v_1,\dots,v_{\overline{N}})$ in (quantum, annular) bigrading $(\bar{\jmath},k-v_{\overline{N}+1})$.

Order the N crossings of K such that the crossing on the axis is ordered last and the quotient map $\{\text{other crossings of } K\}\to\{\text{crossings of } \overline{K}\}$ is order-preserving. Then, there is an inclusion of cube categories $\iota\colon (1\to 0)^{\overline{N}+1}\to(1\to 0)^N$ which sends the vertex $v=(v_1,\dots,v_{\overline{N}},v_{\overline{N}+1})$ to the vertex $\iota(v)=(v_1,v_1,\dots,v_{\overline{N}},v_{\overline{N}},v_{\overline{N}+1})\in(1\to 0)^N$; the image is precisely the fixed subcategory of $(1\to 0)^N$.

We construct a natural isomorphism

$$\eta\colon \Big(\coprod_{\substack{\bar{\jmath},k\\ 2\bar{\jmath}+k=j-1+3\Delta}} AF^{\mathscr{B}}_{\bar{\jmath},k}(\overline{K}_1,\overline{K}_0)\Big)\longrightarrow\Big((F_j^{\mathscr{B}}(K))^\tau\circ\iota\Big)$$

between the two Burnside functors. For any vertex $v\in(1\to 0)^{\overline{N}+1}$ and any Khovanov generator $x\in AF^{\mathscr{B}}_{\bar{\jmath},k}(\overline{K}_1,\overline{K}_0)(v)$, let $\eta(x)$ be the Khovanov generator of K over the vertex $\iota(v)$ which labels each circle in the $\iota(v)$-resolution of K by the same label that x labels its quotient circle in the $(v_1,\dots,v_{\overline{N}})$-resolution of $\overline{K}_{v_{\overline{N}+1}}$. It is a straightforward calculation that $\eta(x)$ has quantum grading j if and only if $(\bar{\jmath},k)$ satisfies $2\bar{\jmath}+k=j-1+3\Delta$. For the reader's convenience, we summarize the calculation below.

In the $(v_1,\dots,v_{\overline{N}})$-resolution of the annular knot $\overline{K}_{v_{\overline{N}+1}}$, let a_1 and a_X be the numbers of essential circles that x labels by 1 and X, respectively, and let b_1 and b_X be the numbers of non-essential circles that x labels by 1 and X, respectively. Then,

$$\begin{aligned}\bar{\jmath}&=\overline{N}-3\overline{N}_-+(|v|-v_{\overline{N}+1})+(a_X+b_X)-(a_1+b_1)\\ k-v_{\overline{N}+1}&=a_1-a_X\\ 2\bar{\jmath}+k+6\overline{N}_-+1&=(2\overline{N}+1)+(2|v|-v_{\overline{N}+1})+(a_X+2b_X)-(a_1+2b_1).\end{aligned}$$

In the $\iota(v)=(v_1,v_1,\dots,v_{\overline{N}},v_{\overline{N}},v_{\overline{N}+1})$-resolution of K, the number of circles labeled $1,X$ by the generator $\eta(x)$ is (a_1+2b_1) and (a_X+2b_X), respectively. Then,

$$\begin{aligned}j+3N_-&=N+|\iota(v)|+(a_X+2b_X)-(a_1+2b_1)\\ &=(2\overline{N}+1)+(2|v|-v_{\overline{N}+1})+(a_X+2b_X)-(a_1+2b_1).\end{aligned}$$

Therefore, $2\bar{\jmath}+k = j-1+3(N_- - 2\overline{N}_-) = j-1+3\Delta$.

To make the notation less cumbersome, let $AF^{\mathscr{B}}_{[\bar{\jmath},k]} = \coprod_{\bar{\jmath},k|2\bar{\jmath}+k=j-1+3\Delta} AF^{\mathscr{B}}_{\bar{\jmath},k}$.

Next, we must specify the natural isomorphism η on the 1-morphisms, that is, for any edge $v \to w$ in $\{1\to 0\}^{\overline{N}+1}$, we must specify an isomorphism between correspondences:

$$AF^{\mathscr{B}}_{[\bar{\jmath},k]}(\overline{K}_1,\overline{K}_0)(v\to w) \xrightarrow{\eta} F^{\mathscr{B}}_j(K)(\iota(v)\to\iota(w)).$$

As in the proof of Lemma 4.1, for any generators $x \in AF^{\mathscr{B}}_{[\bar{\jmath},k]}(\overline{K}_1,\overline{K}_0)(v)$, $y \in AF^{\mathscr{B}}_{[\bar{\jmath},k]}(\overline{K}_1,\overline{K}_0)(w)$, the set $s^{-1}(x)\cap t^{-1}(y) \subset AF^{\mathscr{B}}_{[\bar{\jmath},k]}(\overline{K}_1,\overline{K}_0)(v\to w)$ has either 0 or 1 elements, and the set $s^{-1}(\eta(x))\cap t^{-1}(\eta(y)) \subset F^{\mathscr{B}}_j(K)(\iota(v)\to\iota(w))$ also has either 0 or 1 elements, correspondingly; therefore, the isomorphism η between the correspondences is forced. This is checked by a direct case analysis: When $v_{\overline{N}+1} = w_{\overline{N}+1}$, then this is Stoffregen-Zhang's case analysis for the 2-periodic link $K_{v_{\overline{N}+1}}$ [**SZ**, Theorem 6.8]. When $v_{\overline{N}+1} > w_{\overline{N}+1}$, the proof is the following similar (but shorter) case analysis. Consider the axis-moving isotopy from the $(v_1,\dots,v_{\overline{N}})$-resolution of the annular knot $\overline{K}_1$ to the corresponding resolution of the annular knot $\overline{K}_0$. There are two cases, depending on whether an essential circle becomes inessential or an inessential circle becomes essential. In the first (respectively second) case, $s^{-1}(x)\cap t^{-1}(w) \subset AF^{\mathscr{B}}_{[\bar{\jmath},k]}(\overline{K}_1,\overline{K}_0)(v\to w)$ is non-empty (and has only one element) if and only if x and y label the moving circle by X (respectively 1) and all other circles by the same labels. In the picture for K, we get a corresponding saddle cobordism from the $\iota(v)$-resolution of K to the $\iota(w)$-resolution of K. In the first (respectively second) case, the saddle is a split (respectively merge) and $s^{-1}(\eta(x))\cap t^{-1}(\eta(y)) \subset F^{\mathscr{B}}_j(K)(\iota(v)\to\iota(w))$ is non-empty (and has only one element) if and only if $\eta(x)$ and $\eta(y)$ label all the circles involved in the saddle by X (respectively 1), and all other circles by the same labels.

Finally, we have to check that compatibility of these isomorphisms across 2-dimensional faces. That is, given a 2-dimensional face $u \rightrightarrows \begin{matrix} v \\ v' \end{matrix} \rightrightarrows w$ in $\{1\to 0\}^{\overline{N}+1}$, we have to check that the following diagram commutes:

$$\begin{array}{ccc}
\begin{array}{l} AF^{\mathscr{B}}_{[\bar{\jmath},k]}(\overline{K}_1,\overline{K}_0)(v\to w) \\ \quad \circ AF^{\mathscr{B}}_{[\bar{\jmath},k]}(\overline{K}_1,\overline{K}_0)(u\to v)\end{array} & \longrightarrow & \begin{array}{l} F^{\mathscr{B}}_j(K)(\iota(v)\to\iota(w)) \\ \quad \circ F^{\mathscr{B}}_j(K)(\iota(u)\to\iota(v))\end{array} \\
\downarrow & & \downarrow \\
\begin{array}{l} AF^{\mathscr{B}}_{[\bar{\jmath},k]}(\overline{K}_1,\overline{K}_0)(v'\to w) \\ \quad \circ AF^{\mathscr{B}}_{[\bar{\jmath},k]}(\overline{K}_1,\overline{K}_0)(u\to v')\end{array} & \longrightarrow & \begin{array}{l} F^{\mathscr{B}}_j(K)(\iota(v')\to\iota(w)) \\ \quad \circ F^{\mathscr{B}}_j(K)(\iota(u)\to\iota(v'))\end{array}
\end{array}$$

where the horizontal arrows are induced by the isomorphisms that we just constructed, and the vertical arrows are induced by the isomorphisms that are part of the data for the respective Burnside 2-functors. Unless the 2-dimensional face is a ladybug, for any pair of generators $x \in AF^{\mathscr{B}}_{[\bar{\jmath},k]}(\overline{K}_1,\overline{K}_0)(u), y \in AF^{\mathscr{B}}_{[\bar{\jmath},k]}(\overline{K}_1,\overline{K}_0)(w)$, each of $s^{-1}(x)\cap t^{-1}(w) \subset AF^{\mathscr{B}}_{[\bar{\jmath},k]}(\overline{K}_1,\overline{K}_0)(u\to w)$ and $s^{-1}(\eta(x))\cap t^{-1}(\eta(y)) \subset F^{\mathscr{B}}_j(K)(\iota(u)\to\iota(w))$ has 0 or 1 elements, and so the check is automatic. Thus, the only case remaining is when the 2-dimensional face is a ladybug. However, recall from Remark 3.3 that the functor $AF^{\mathscr{B}}_{[\bar{\jmath},k]}(\overline{K}_1,\overline{K}_0)$ may be viewed as a subfunctor

of the Burnside functor associated to a different annular knot obtained from $\overline{K}_1$ by performing a Reidemeister I move. However, such a crossing (coming from a Reidemeister I move) cannot be involved in a ladybug configuration. Therefore, in order to be a ladybug, we must have $u_{\overline{N}+1} = v_{\overline{N}+1} = v'_{\overline{N}+1} = w_{\overline{N}+1}$; and in that case, commutativity of the above diagram follows from the analogue for the 2-periodic link $K_{u_{\overline{N}+1}}$ [**SZ**, Lemma 6.15]. □

Now, given the $\mathbb{Z}/2$-action on the Burnside functor $F_j^{\mathscr{B}}(K)$, and the above identification of its fixed point functor with those of the quotient annular knots $(\overline{K}_1, \overline{K}_0)$, all that remains is to refine these actions and the fixed points to the category of topology spaces. This is precisely Stoffregen-Zhang's central thesis:

PROPOSITION 4.3. [**SZ**, Proposition 5.10] *Let $F^{\mathscr{B}}\colon (1 \to 0)^N \to \mathscr{B}$ be a Burnside 2-functor with an external $\mathbb{Z}/2$-action τ and $(F^{\mathscr{B}})^\tau$ denote the fixed point functor. Then, the homotopy coherent diagram $F^{\mathscr{X}} : (1 \to 0)^N \to \mathsf{Top}$ refining $F^{\mathscr{B}}$ using the box map construction, as in Step ($\mathscr{X}$-1′), may be chosen $\mathbb{Z}/2$-equivariantly so that the fixed point homotopy coherent diagram $(F^{\mathscr{X}})^\tau$ refines $(F^{\mathscr{B}})^\tau$ using box maps.*

PROOF. For the reader's convenience, we sketch the proof (summarizing the proofs of [**SZ**, Lemma 4.7 and Proposition 5.10]).

Fix $D_1, D_2 \geq N$. For every $x \in \coprod_v F^{\mathscr{B}}(v)$, associate a $(D_1 + D_2)$-dimensional box $B_x \cong [0,1]^{D_1+D_2}$; endow it with the $\mathbb{Z}/2$-action τ which reflects the first D_1-coordinates and is the identity along the last D_2-coordinates, that is,

$$\tau(x_1, \dots, x_{D_1+D_2}) = (1 - x_1, \dots, 1 - x_{D_1}, x_{D_1+1}, \dots, x_{D_1+D_2}).$$

As in Step ($\mathscr{X}$-1′), for chains $v_\ell > \cdots > v_0$ in the poset $(1 > 0)^N$, we will construct a $[0,1]^{\ell-1}$-parameter family of box maps $F_j^{\mathscr{X}}(v_\ell) \to F_j^{\mathscr{X}}(v_0)$ refining the correspondence $F^{\mathscr{B}}(v_\ell \to v_0)\colon F^{\mathscr{B}}(v_\ell) \to F^{\mathscr{B}}(v_0)$ by induction on ℓ. These maps will already be specified on the boundary $\partial[0,1]^{\ell-1}$ by the compatibility condition. There are two cases:

- If the entire chain $v_\ell > \cdots > v_0$ is not fixed by τ, choose one of the two chains $\mathfrak{c} = (v_\ell > \cdots > v_0)$ or $\tau\mathfrak{c} = (\tau v_\ell > \cdots > \tau v_0)$ arbitrarily; without loss of generality, say we pick $\mathfrak{c}$. Construct the $[0,1]^{\ell-1}$-parameter family of box maps for $\mathfrak{c}$, refining the correspondence $F^{\mathscr{B}}(v_\ell \to v_0)$, arbitrarily using the $(D_1 + D_2 - 2)$-connectedness of the space of labeled sub-boxes. Then define the $[0,1]^{\ell-1}$-parameter family of box maps for the other chain $\tau\mathfrak{c}$, refining the correspondence $F^{\mathscr{B}}(\tau v_\ell \to \tau v_0)$, by pre-composing and post-composing by τ, as well as relabeling the sub-boxes by the map $\tau\colon F^{\mathscr{B}}(v_\ell \to v_0) \to F^{\mathscr{B}}(\tau v_\ell \to \tau v_0)$.
- If the entire chain $v_\ell > \cdots > v_0$ is fixed by τ, construct the $[0,1]^{\ell-1}$-parameter family of box maps refining the correspondence $F^{\mathscr{B}}(v_\ell \to v_0)$ as follows.
  - Let $A \subset F^{\mathscr{B}}(v_\ell \to v_0)$ be the subset not fixed by τ. From every pair $\{a, \tau a\} \subset A$, choose one element arbitrarily. Let $B \subset A$ be the subset of chosen elements. Pick the $[0,1]^{\ell-1}$-parameter family of sub-boxes labeled by B in the complement of the τ-fixed subspace of the boxes using the $(D_1 - 2)$-connectedness of that space. Construct the $[0,1]^{\ell-1}$-parameter of sub-boxes labeled by $A \setminus B$ by applying τ.
  - Let $C \subset F^{\mathscr{B}}(v_\ell \to v_0)$ be the subset fixed by τ. Pick the $[0,1]^{\ell-1}$-parameter family of sub-boxes labeled by C symmetrically with respect to τ. (First choose a $[0,1]^{\ell-1}$-family of D_2-dimensional boxes inside the fixed subset $\{\frac{1}{2}\}^{D_1} \times$

$[0,1]^{D_2}$ using the (D_2-2)-connectedness of that space, and then thicken the boxes τ-equivariantly to get (D_1+D_2)-dimensional boxes, while staying disjoint from the sub-boxes labeled by A.)

This produces a homotopy coherent diagram $F^{\mathscr{X}}$ refining $F^{\mathscr{B}}$ using (D_1+D_2)-dimensional box maps, and the fixed point functor $(F^{\mathscr{X}})^\tau$ is also a homotopy coherent diagram refining $(F^{\mathscr{B}})^\tau$ using D_2-dimensional box maps. □

Combining these ingredients, we get:

PROPOSITION 4.4. *The strong inversion of K induces a $\mathbb{Z}/2$-action on the Khovanov spectrum $\mathscr{X}(K)$ whose geometric fixed point set is $A\mathscr{X}(\overline{K}_1,\overline{K}_0)$ up to some formal (de)suspension. Keeping track of quantum gradings, the geometric fixed point set of the $\mathbb{Z}/2$-action on $\mathscr{X}_j(K)$ is the spectrum $\bigvee_{\bar{\jmath},k|2\bar{\jmath}+k=j-1+3\Delta} A\mathscr{X}_{\bar{\jmath},k}(\overline{K}_1,\overline{K}_0)$, up to some formal (de)suspension.*

PROOF. Choose the homotopy coherent diagram $F_j^{\mathscr{X}}(K)$ $\mathbb{Z}/2$-equivariantly, as in Proposition 4.3. Up to some (de)suspension, the Khovanov spectrum $\mathscr{X}_j(K)$ is the homotopy colimit of the extended diagram $(F_j^{\mathscr{X}}(K))_+$. The geometric fixed point set of this homotopy colimit is the homotopy colimit of the extended fixed point functor $(F_j^{\mathscr{X}}(K))^\tau_+$. But by Lemma 4.2 and Proposition 4.3, the fixed point functor $(F_j^{\mathscr{X}}(K))^\tau$ refines Burnside functor $\coprod_{\bar{\jmath},k|2\bar{\jmath}+k=j-1+3\Delta} AF^{\mathscr{B}}_{\bar{\jmath},k}(\overline{K}_1,\overline{K}_0)$, and therefore, up to some (de)suspension, the homotopy colimit of $(F_j^{\mathscr{X}}(K))^\tau_+$ is simply

$$\bigvee_{\bar{\jmath},k|2\bar{\jmath}+k=j-1+3\Delta} A\mathscr{X}_{\bar{\jmath},k}(\overline{K}_1,\overline{K}_0),$$

as claimed. □

PROOF OF THEOREM 1.1. Since the spectral sequence is induced by the θ-filtration on the Tate complex, Formula (4.1), we only need to prove Item (Θ-3): the other parts are immediate from the definition. However, this is simply the classical Smith inequality applied to the Proposition 4.4, stated in the language of spectral sequences. To wit, the Tate complex from Equation (4.1) computes the localized equivariant homology of $\mathscr{X}(K)$, which by the classical localization theorem, equals the localized equivariant homology of the geometric fixed point set $A\mathscr{X}(\overline{K}_1,\overline{K}_0)$, which simply equals its homology $AKh(\overline{K}_1,\overline{K}_0)$, tensored with $\mathbb{F}_2[\theta^{-1},\theta]$. (It is also easy to keep track of the quantum gradings using Proposition 4.4.) □

REMARK 4.5. The expression $\Delta = N_- - 2\overline{N}_-$ appears as grading shifts in Theorem 1.1, but it is not an invariant of the knot K and its strong inversion. Geometrically, the 2-periodic annular links K_0, K_1 obtained by resolving the crossing of K on the axis, and their quotient annular knots $\overline{K}_0, \overline{K}_1$, are only well-defined up to how many times they wind around the axis, and Δ captures information about this winding number. In more detail, if B (respectively T) denotes the underpass (respectively overpass) of K near its crossing on the axis, then orient the quotient knot $\overline{K}$ by orienting the quotient arc $\overline{B}$ (respectively $\overline{T}$) towards (respectively away from) the axis. This induces orientations of the two annular knots $\overline{K}_0$ and $\overline{K}_1$, as well as their pre-images K_0 and K_1 (but not of the original knot K). Let W be the winding number of K_0 (equivalently, $\overline{K}_0$) around the axis; this is one higher than the winding number of K_1 (equivalently, $\overline{K}_1$) around the axis. Then $W-\Delta$ is an

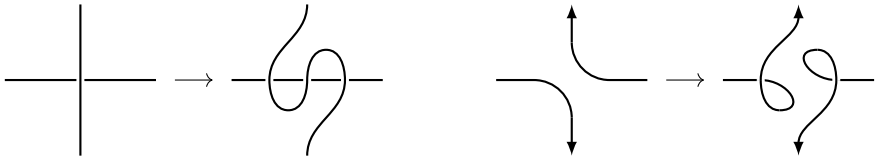

FIGURE 4.1. **An additional move.** The left half shows a new move for intravergent diagrams, corresponding to rotating the overpass T counter-clockwise over the underpass B. The right half shows the corresponding change for the 2-periodic link K_0 (with the orientation from Remark 4.5).

invariant of the knot K and its strong inversion; we prove this as Proposition 4.6 below.

PROPOSITION 4.6. *The quantity $W - \Delta$ is independent of the choice of the intravergent diagram, and in fact equals twice the axis linking number invariant* [**BI22**, Definition 4.6] *of the knot K and its strong inversion.*

PROOF. We first prove that $W - \Delta$ is an invariant. Given two intravergent diagrams for K and its strong inversion, connect them by a generic $\mathbb{Z}/2$-equivariant isotopy in $\mathbb{R}^3$; this produces a generic isotopy (in $\mathbb{R}^3$) connecting the quotient diagrams for $\overline{K}$. As in the proof of Reidemeister's theorem, this implies that the two diagrams for $\overline{K}$ are related by a finite sequence of Reidemeister moves (away from the axis), as well as a new move, corresponding to the situation when during the isotopy of K, the projection of the underpass B becomes tangent to the projection of the overpass T at the axis. The Reidemeister moves of $\overline{K}$ lift to $\mathbb{Z}/2$-equivariant pairs of Reidemeister moves for K, while this new move lifts to the move shown in the left half of Figure 4.1. (Actually, there are two moves, depending on whether the overpass T rotates clockwise or counter-clockwise over B. Figure 4.1 shows the move for the counter-clockwise rotation; the other move can be obtained by performing a $\mathbb{Z}/2$-equivariant pair of Reidemeister II moves of T over B near the axis, and then the above move in reverse.)

The Reidemeister moves for $\overline{K}$—lifting to a $\mathbb{Z}/2$-equivariant pair of Reidemeister moves for K—do not change W, the winding number of K_0. For the Reidemeister I move, depending on the shape of the clasp, either N_- increases by 2 and $\overline{N}_-$ increases by 1, or both N_- and $\overline{N}_-$ are unchanged. For the Reidemeister II move, N_- increases by 2 and $\overline{N}_-$ increases by 1, and for the Reidemeister III move, both N_- and $\overline{N}_-$ are unchanged. So, in each case $\Delta = N_- - 2\overline{N}_-$ does not change. Finally, for the new move from Figure 4.1, N_- increases by 1 and $\overline{N}_-$ also increases by 1, so Δ decreases by 1, but the winding number W decreases by 1, so the quantity $W - \Delta$ is preserved.

Next we will prove that this invariant $W - \Delta$ equals twice Boyle-Issa's axis linking number invariant [**BI22**, Definition 4.6]. Fix an orientation of the knot K. By performing the move from Figure 4.1 once if necessary, we may assume the crossing of K on the axis is a positive crossing. Then K_0 is a 2-component link, and it inherits an orientation from K. To avoid confusion, let o_{can} denote the canonical orientation of K_0 from Remark 4.5, and let o_{ind} denote the induced

orientation from K. These two orientations agree on one of the components of K_0, and disagree on the other.

The number of negative crossings of K_0 with orientation o_{ind} is N_- and the number of the negative crossings of K_0 with orientation o_{can} is $2\overline{N}_-$. Therefore, $\Delta = N_- - 2\overline{N}_-$ is twice the linking number between the two components of K_0 (with orientation o_{can}); in particular, it is an even number.

Now perform the move from Figure 4.1 Δ times. (If $\Delta < 0$, then perform the reverse move $-\Delta$ times.) In the new diagram, the crossing on the axis is still positive, so the above discussion applies. Now $\Delta = 0$, and so the invariant is simply the new winding number W. Also, the linking number between the two components of K_0 in the new diagram is zero, so K_0 is the 2-component butterfly link [**BI22**, Definition 4.1], and by definition its winding number W is twice the axis linking number invariant. □

Remark 4.7. Given a theorem about Khovanov homology, it is natural to wonder if it lifts a result about the Jones polynomial. Let $V_K(q)$ be the unreduced Jones polynomial, that is, the graded Euler characteristic of $Kh(K)$. Let $J_K(q) = V_K(q)/(q+q^{-1})$ denote the reduced Jones polynomial. For an annular knot K, let $AV_K(q,a)$ be the graded Euler characteristic of annular Khovanov homology, which was studied briefly by Roberts [**Rob13**, Section 2]. By Theorem 1.1,

$$V_K(q) \equiv q^{1-3\Delta}(qAV_{\overline{K}_1}(q^2,q) + AV_{\overline{K}_0}(q^2,q)) \pmod 2.$$

It is easy to see from Kauffman's state sum formula that if we quotient by $(q^2+q^{-2})-(q+q^{-1})$ then $AV(q^2,q) \equiv V(q^2)$ and $V(q) \equiv V(q^2)$. Therefore, we have

$$V_K(q) \equiv q^{-3\Delta}(q^2+q)V_{\overline{K}}(q^2) \equiv q^{-3\Delta}(q^2+q)V_{\overline{K}}(q) \pmod{2, q^2-q-q^{-1}+q^{-2}},$$

where $\overline{K}$ denotes either $\overline{K}_0$ or $\overline{K}_1$, viewed as an ordinary, not annular, knot. Since $q^2-q-q^{-1}+q^{-2} = (q+q^{-1})(q-1+q^{-1})$ over $\mathbb{F}_2[q^{-1},q]$, we may divide by $(q+q^{-1})$ to get the equation for reduced Jones polynomial

$$J_K(q) \equiv q^{-3\Delta}(q^2+q)J_{\overline{K}}(q^2) \equiv q^{-3\Delta}(q^2+q)J_{\overline{K}}(q) \equiv J_{\overline{K}}(q) \tag{4.2}$$

modulo $(2, q-1+q^{-1})$. An analogous result can also be obtained for 2-periodic knots using [**SZ**, Theorem 1.3], giving the 2-periodic case of a formula of Murasugi's [**Mur88**, Theorem 1] and, using the fact that $J_K(i) \equiv 1 \pmod 2$, the 2-periodic case of Yokota's refinement [**Yok91**, Theorem 2]. However, Formula (4.2) is actually vacuous, since if we quotient by $(q+q^{-1})-1$, in Kauffman's state sum formula each circle contributes 1, and so for any knot or link diagram K with N crossings, N_- of which are negative, we get

$$J_K(q) \equiv \sum_{v\in\{0,1\}^N} q^{N+|v|-3N_-} = (1+q)^N q^N q^{-3N_-} \equiv 1 \pmod{2, q-1+q^{-1}}.$$

(Murasugi's and Yokota's formulas are also vacuous for 2-periodic knots, though interesting for higher periods. For Murasugi, this is [**Mur88**, Proposition 7]; Yokota only states his results for odd primes, presumably for this reason.)

5. An application to slice disks

Consider the knot $K = 9_{46}$. It bounds two slice disks as illustrated in Figure 5.1; denote them D_1 and D_2, and view them as cobordisms in $[0,1]\times\mathbb{R}^3$ from K to the empty link $\varnothing$. Let $\widehat{D}_i$ denote the image of D_i under the map $(t,x,y,z) \mapsto$

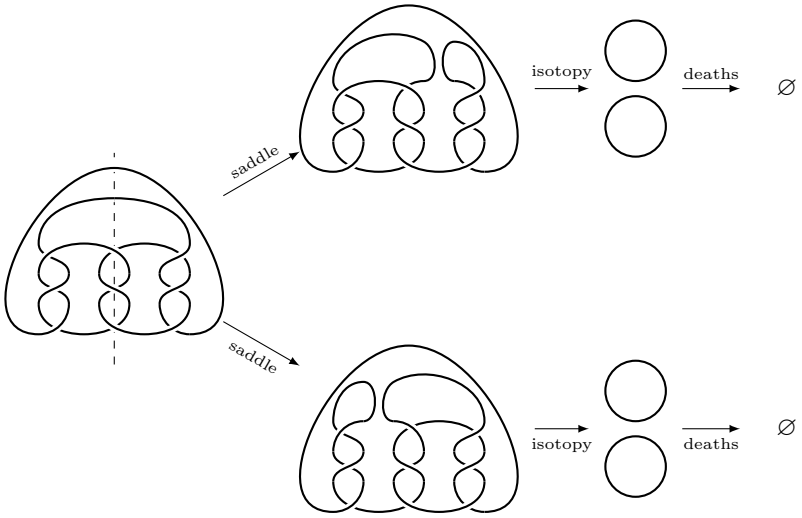

FIGURE 5.1. **The knot $K = 9_{46}$ and a pair of slice disks for it.** The knot is on the left, and the two movies on the two rows represent its two slice disks. Note that the two movies are related by a 180° rotation around the dashed line.

$(1-t, x, y, z)$, so $\widehat{D}_i$ is a cobordism from $\varnothing$ to K. Sundberg-Swann showed that the disks D_1 and D_2 are distinguished by their induced maps on Khovanov homology. We will recover this result using Theorem 1.1. In fact, we get a little more; see Porism 5.2 below. The argument is reminiscent of the recent work of Dai-Mallick-Stoffregen using Heegaard Floer homology [**DMS**].

THEOREM 5.1. [**SS22**] *The slice disks D_1 and D_2 induce different maps on Khovanov homology $Kh(9_{46}; \mathbb{F}_2) \to Kh(\varnothing; \mathbb{F}_2) = \mathbb{F}_2$.*

PROOF. For any cobordism F, let F_* denote the induced map on Khovanov homology. We will find an element $\gamma \in Kh_{0,1}(K; \mathbb{F}_2)$ satisfying $(D_1)_*(\gamma) \neq (D_2)_*(\gamma)$.

The Khovanov homology of K in quantum grading ± 1, retrieved from the Knot Atlas [**BM**] and converted to the conventions of Section 2, is

	-1	0
1		$\mathbb{Z}^2$
-1	$\mathbb{F}_2$	$\mathbb{Z}$

and therefore the Khovanov homology over $\mathbb{F}_2$ of K in these quantum gradings is

	-1	0
1		$\mathbb{F}_2^2$
-1	$\mathbb{F}_2 \longleftarrow$	$\mathbb{F}_2^2$

where the arrow indicates a rank one Bockstein homomorphism associated to the coefficient sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$.

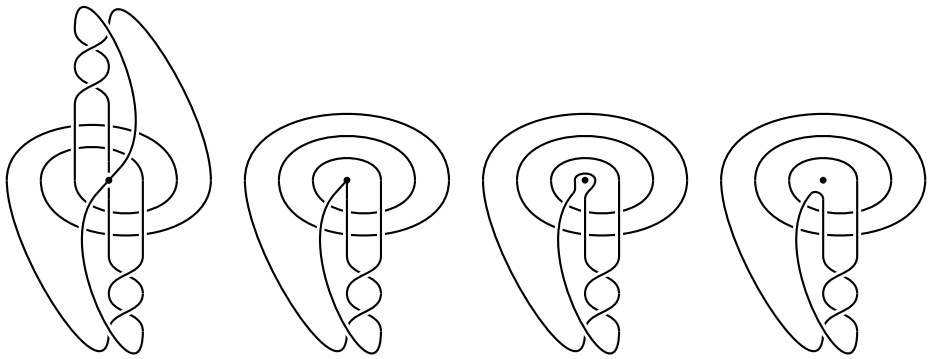

FIGURE 5.2. **The strongly invertible knot $K = 9_{46}$, its quotient knot $\overline{K}$, and the two induced annular trefoil knots $\overline{K}_0$ and $\overline{K}_1$.** Here K has been redrawn as an intravergent diagram with the axis coming straight out of the page through the marked point; it is also the axis for the two annular trefoils.

Let $\alpha \in Kh_{0,-1}(K;\mathbb{F}_2)$ be the generator of the kernel of the Bockstein homomorphism, and let $\beta \in Kh_{0,1}(K;\mathbb{F}_2)$ be the image of α under the X-action (the basepoint map). If $\mathrm{Id}^\bullet$ denotes the identity cobordism from K to itself decorated with a single dot, then this X-action is the map $\mathrm{Id}^\bullet_*$ induced on Khovanov homology by $\mathrm{Id}^\bullet$.

For $i \in \{1,2\}$, consider the cobordism $F_i = D_i \circ \mathrm{Id}^\bullet \circ \widehat{D}_i$ from $\varnothing$ to $\varnothing$. Since F_i is a (knotted) dotted sphere, by a result of Rasmussen and Tanaka [**Ras, Tan06**] (or, more precisely, a trivial extension of it [**LS22**, Lemma 6.16]), $(F_i)_* = (D_i)_* \circ \mathrm{Id}^\bullet_* \circ (\widehat{D}_i)_*$ is the identity map on Khovanov homology $\mathbb{F}_2 = Kh(\varnothing;\mathbb{F}_2) \to Kh(\varnothing;\mathbb{F}_2) = \mathbb{F}_2$. Therefore, both $(\widehat{D}_1)_*$ and $(\widehat{D}_2)_*$ map the generator of $Kh(\varnothing;\mathbb{F}_2)$ (in bigrading $(0,0)$) to α, which is the unique non-zero element of $Kh_{0,-1}(K;\mathbb{F}_2)$ in the kernel of the Bockstein. Therefore, both $\mathrm{Id}^\bullet_* \circ (\widehat{D}_1)_*$ and $\mathrm{Id}^\bullet_* \circ (\widehat{D}_2)_*$ map the generator of $Kh(\varnothing;\mathbb{F}_2)$ to $\beta \in Kh_{0,1}(K;\mathbb{F}_2)$, and both $(D_1)_*$ and $(D_2)_*$ map β to the generator of $Kh(\varnothing;\mathbb{F}_2)$. Choose some $\gamma \in Kh_{0,1}(K;\mathbb{F}_2)$ so that $\{\beta,\gamma\}$ is a basis of $Kh_{0,1}(K;\mathbb{F}_2)$.

The knot K is strongly invertible with respect to the 180° rotation around the dashed vertical line in Figure 5.1; call the involution τ and the induced map on Khovanov homology $\tau_*\colon Kh(K;\mathbb{F}_2) \to Kh(K;\mathbb{F}_2)$. The two slice disks D_1 and D_2 are related by the involution (Id,τ) of $[0,1] \times \mathbb{R}^3$. Therefore, $\tau_* \circ \mathrm{Id}^\bullet_* \circ (\widehat{D}_1)_* = \mathrm{Id}^\bullet_* \circ (\widehat{D}_2)_*$, which implies $\tau_*(\beta) = \beta$.

The annular quotient knots $\overline{K}_0$ and $\overline{K}_1$ are shown in Figure 5.2. We consider their annular Khovanov homology in gradings corresponding to the quantum grading $j = 1$ on K. (The grading correction term $\Delta = 4$.) Computer computation, using code by Davis [**Dav**], gives that the annular Khovanov homology of $\overline{K}_1$ in gradings with $2\bar{\jmath} + k + 1 = 12$ is $\mathbb{F}_2^2$, supported in gradings $(2,7,-3)$ and $(3,5,1)$,

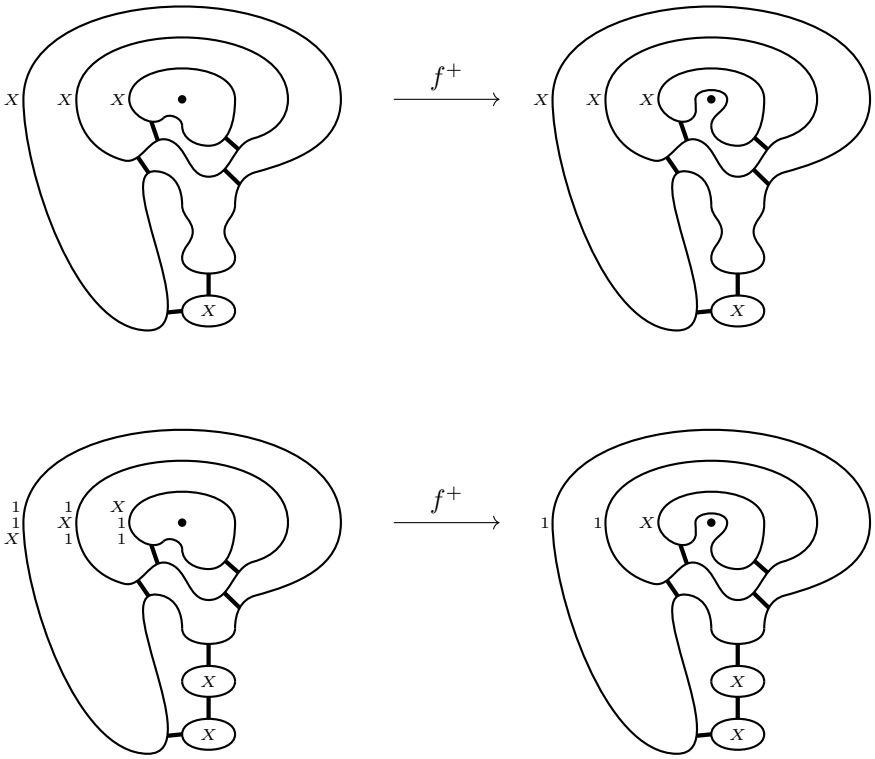

FIGURE 5.3. **Cycles in and their images under f^+.** Top left: a generator for $AKh_{2,7,-3}(\overline{K}_1;\mathbb{F}_2)$. Top right: its image in $AKh_{2,7,-2}(\overline{K}_0;\mathbb{F}_2)$ under f^+. Bottom left: an element of $AKh_{3,5,1}(\overline{K}_1;\mathbb{F}_2)$ which is the sum of three terms, where the two inessential circles and one essential circle are labeled X and the other two essential circles are labeled 1. (We have labeled the inessential circles and listed all possible labels of the three essential circles.) Bottom right: its image in $AKh_{3,5,2}(\overline{K}_0;\mathbb{F}_2)$. Crossings that were 1-resolved, i.e., where there is a Khovanov differential out of this state, are indicated with thick line segments.

while the annular Khovanov homology of $\overline{K}_0$ in gradings with $2\bar{\jmath}+k=12$ is also $\mathbb{F}_2^2$, supported in gradings $(2,7,-2)$ and $(3,5,2)$. It is not hard to find representatives of these cycles in $AKh(\overline{K}_1;\mathbb{F}_2)$ by hand; see Figure 5.3. Their images under f^+ are (distinct) nontrivial elements of $AKh(\overline{K}_0;\mathbb{F}_2)$, so f^+ is an isomorphism. (Verifying that the image is nontrivial by hand is straightforward for the cycle in grading $(3,5,1)$, but is quite tedious for the cycle in grading $(2,7,-3)$, and might be better done by computer.) Thus, $\bigoplus_{\bar{\imath},\bar{\jmath},k|2\bar{\jmath}+k=1-1+3\Delta} AKh_{\bar{\imath},\bar{\jmath},k}(\overline{K}_1,\overline{K}_0;\mathbb{F}_2)=0$ so, by Corollary 1.4 the map $\tau_*\colon Kh_{0,1}(K;\mathbb{F}_2)\to Kh_{0,1}(K;\mathbb{F}_2)$ is given by $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ with respect to an appropriate basis.

Since $\tau_*(\beta) = \beta$, it follows that the basis must be $\{\gamma, \gamma+\beta\}$ and $\tau_*(\gamma) = \gamma+\beta$. So,

$$(D_1)_*(\gamma)+(D_2)_*(\gamma) = (D_1)_*(\gamma)+(D_1)_*(\tau_*(\gamma)) = (D_1)_*(\beta) = 1 \in \mathbb{F}_2 = Kh(\varnothing;\mathbb{F}_2),$$

and hence $(D_1)_*(\gamma) \neq (D_2)_*(\gamma)$, as claimed. □

Porism 5.2. *The knot 9_{46} does not admit an equivariant slice disk, with respect to the $\mathbb{Z}/2$-action on $[0,1]\times\mathbb{R}^3$ by (Id,τ). In fact, for any slice disk D for 9_{46}, D and $(\mathrm{Id},\tau)(D)$ are distinguished by the induced maps on Khovanov homology.*

Proof. The proof is the same as the proof of Theorem 5.1, with D and $(\mathrm{Id},\tau)(D)$ in place of D_1 and D_2. □

Note that the first half of the porism also follows from Sakuma's work [**Sak86**]. The second half implies that D and $(\mathrm{Id},\tau)(D)$ are not even related by taking connected sum with a knotted 2-sphere, since that operation does not change the map on Khovanov homology.

Remark 5.3. For the last portion of the proof of Theorem 5.1, one could instead compute the map τ_* directly; arguably, that involves less work than the argument given. On the other hand, the argument above only requires computing the dimensions of certain annular Khovanov homology groups (including $AKh(\overline{K}_1, \overline{K}_0)$), for which there is a well-known, fast divide-and-conquer algorithm [**BN07**]; computing the action of τ_* efficiently would require further conceptual work.

6. An analogue in Heegaard Floer homology

The main ingredient in the proof of Theorem 1.6 is a recent localization theorem for the Heegaard Floer homology of branched double covers:

Theorem 6.1. [**HLL22**, Theorem 4.1] *Let Y be a closed 3-manifold and $K \subset Y$ an oriented, nullhomologous knot with Seifert surface F. Let $\Sigma(Y,K)$ be the double cover branched along K induced by the Seifert surface F. Then, there is a spectral sequence with E^1-page given by*

$$\widehat{HF}(\Sigma(Y,K)) \otimes \mathbb{F}_2[\theta^{-1},\theta]]$$

converging to

$$\widehat{HF}(Y) \otimes \mathbb{F}_2[\theta^{-1},\theta]].$$

Consider a strongly invertible knot $K \subset \mathbb{R}^3 \subset S^3$ with axis $A \subset S^3$ (a circle). The intersection of K with A decomposes A into two intervals; let A_1 be the interval contained in $\mathbb{R}^3$ and A_2 the other interval. Let $\tau\colon S^3 \to S^3$ be rotation by 180° around A and let $\overline{K} \subset S^3 = S^3/\tau$ be the image of $K \cup A_1$ under the quotient map.

Proof of Theorem 1.6. Let K' be the preimage of A_2 in $\Sigma(S^3,\overline{K})$. We claim that $\Sigma(S^3,K)$ is the double cover of $\Sigma(S^3,\overline{K})$ branched along K'. Theorem 6.1 then gives the result.

The claim follows from [**AB**, Lemma 3.1]; we explain this case of their proof. Let $q\colon S^3 \to S^3/\tau$ be the quotient map. The map q is the double cover branched along $q(A)$. Fix a Seifert surface $\overline{F}$ for $\overline{K}$ meeting $q(A_2)$ transversely, and let $F = q^{-1}(\overline{F})$. Let Y be the result of cutting S^3 along F, so $\partial Y = F_+ \cup_K F_-$. Since F is taken to itself by τ, τ induces an involution of Y. The fixed set of this

involution is a copy of (the preimage of) A_2; the two copies of A_1 (one in F_+ and the other in F_-) are exchanged by the involution.

We can form $\Sigma(S^3, K)$ by gluing two copies of Y together. The involution τ induces an involution $\widetilde{\tau}$ of $\Sigma(S^3, K)$ with fixed set the preimage of A_2. The deck transformation of $\Sigma(S^3, K)$ gives another involution $\widetilde{\sigma}$, exchanging the two copies of Y and commuting with $\widetilde{\tau}$. So, $\widetilde{\sigma}$ descends to an involution σ of $\Sigma(S^3, K)/\widetilde{\tau}$. The quotient $\big(\Sigma(S^3, K)/\widetilde{\tau}\big)/\sigma$ is S^3, and the fixed set of σ is the preimage of $\overline{K}$. Thus, $\Sigma(S^3, K)/\widetilde{\tau} = \Sigma(S^3, \overline{K})$ and

$$\Sigma(S^3, K) = \Sigma(\Sigma(S^3, \overline{K}), K'), \tag{6.1}$$

as claimed. □

Example 6.2. For the knots in Example 1.5, since the double cover of S^3 branched over the trefoil is a lens space, the spectral sequence associated to K has E^1-page $\mathbb{F}_2[\theta^{-1}, \theta]]^3$. The involution acts nontrivially on the set of Spin$^{\mathrm{C}}$-structures on $\Sigma(S^3, K)$, so two of the three generators cancel under the d^1-differential and the spectral sequence collapses at the E^2-page to $\widehat{HF}(S^3) \otimes \mathbb{F}_2[\theta^{-1}, \theta]] = \mathbb{F}_2[\theta^{-1}, \theta]]$. By contrast, for K', the E^1-page and E^∞ page are both given by $\widehat{HF}(\Sigma(S^3, K')) \otimes \mathbb{F}_2[\theta^{-1}, \theta]] = \mathbb{F}_2[\theta^{-1}, \theta]]^3$, and d^i is trivial for all $i \geq 1$. (The action of the involution on the set of Spin$^{\mathrm{C}}$-structures is trivial in this case.) In particular, unlike the spectral sequences for Khovanov homology, the spectral sequences in Heegaard Floer homology distinguish K and K'.

We conclude with a relative version of Theorem 1.6, which follows from a theorem of Large [**Lar**] (see also [**Hen12**]). As in the proof of Theorem 1.6, the preimage of A_2 is a knot K' inside $\Sigma(S^3, \overline{K})$; let $\widetilde{K}'$ be the preimage of K' in $\Sigma(S^3, K)$, which is also the preimage of A_2 in $\Sigma(S^3, K)$.

Theorem 6.3. *With notation as in Theorem 1.6, there is a spectral sequence with E^1-page given by $\widehat{HFK}(\Sigma(S^3, K), \widetilde{K}') \otimes \mathbb{F}_2[\theta^{-1}, \theta]]$ and which converges to $\widehat{HFK}(\Sigma(S^3, \overline{K}), K') \otimes \mathbb{F}_2[\theta^{-1}, \theta]]$.*

Proof. Large proved that given a nullhomologous knot L in a 3-manifold Y and a branched double cover $\Sigma(Y, L)$ of (Y, L) there is a spectral sequence

$$\widehat{HFK}(\Sigma(Y, L), \widetilde{L}) \otimes \mathbb{F}_2[\theta^{-1}, \theta]] \Rightarrow \widehat{HFK}(Y, L) \otimes \mathbb{F}_2[\theta^{-1}, \theta]],$$

where $\widetilde{L}$ is the preimage of L [**Lar**, Theorem 1.5]. (He states the result as a rank inequality.) By Formula (6.1), Large's theorem with $Y = \Sigma(S^3, \overline{K})$ and $L = K'$ gives the result. □

Remark 6.4. The spectral sequence in Theorem 1.6 decomposes along Spin$^{\mathrm{C}}$-structures on $\Sigma(\overline{K})$ (see [**HLL22**, Theorem 1.1]), and the spectral sequence in Theorem 6.3 decomposes along Spin$^{\mathrm{C}}$-structures on $\Sigma(\overline{K})$ and Alexander gradings, i.e., relative Spin$^{\mathrm{C}}$-structures on $(\Sigma(\overline{K}), K')$.

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Email address: lipshitz@uoregon.edu

Department of Mathematics, University of Oregon, Eugene, OR 97403

Email address: sucharit@math.ucla.edu

Department of Mathematics, University of California, Los Angeles, CA 90095