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On Littlewood's Estimate for the Modulus of the Zeta Function on the Critical Line

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Inspired by a result of Soundararajan, assuming the Riemann hypothesis (RH), we prove a new inequality for the logarithm of the modulus of the Riemann zeta function on the critical line in terms of a Dirichlet polynomial over primes and prime powers. Our proof uses the Guinand-Weil explicit formula in conjunction with extremal one-sided bandlimited approximations for the Poisson kernel. As an application, by carefully estimating the Dirichlet polynomial, we revisit a 100-year-old estimate of Littlewood and give a slight refinement of the sharpest known upper bound (due to Chandee and Soundararajan) for the modulus of the zeta function on the critical line assuming RH, by providing explicit lower-order terms.

Keywords and Phrases: Riemann zeta-function, Poisson kernel, band-limited approximations.

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1. Introduction

Let $\zeta(s)$ denote the Riemann zeta function. Assuming the Riemann hypothesis (RH), a classical estimate of Littlewood [10] from 1924 states that, for sufficiently large t, there is a constant $C > 0$ such that

$$|\zeta(\tfrac{1}{2} + it)| \ll \exp\Big(C \frac{\log t}{\log\log t}\Big).$$

In the past 100 years, no improvement on this estimate has been made apart from reducing the permissible values of C. With Littlewood's estimate in mind, in this paper we prove a variation of an inequality for $\log|\zeta(\frac{1}{2} + it)|$ due to Soundararajan in terms of a Dirichlet polynomial over the primes and prime powers. Assuming RH, for sufficiently large t, Soundararajan [12, Main Proposition] proved that*

$$\log|\zeta(\tfrac{1}{2}+it)| \leq \mathrm{Re}\sum_{n\leq x}\frac{\Lambda(n)}{n^{1/2+\lambda/\log x+it}\log n}\frac{\log(x/n)}{\log x}+\frac{(1+\lambda)}{2}\frac{\log t}{\log x}+O\Big(\frac{1}{\log x}\Big), \tag{1.1}$$

for $2 \leq x \leq t^2$ and $\lambda \geq \lambda_0 = 0.4912\ldots$, where λ_0 denotes the unique positive real number satisfying $e^{-\lambda_0} = \lambda_0 + \lambda_0^2/2$. Here, as usual, we let $\Lambda(n)$ denote the von Mangoldt function defined to be $\log p$, if $n = p^m$ with p a prime number and $m \geq 1$ an integer, and to be zero otherwise.

The usefulness of Soundararajan's inequality (1.1) is that there is considerable flexibility in choosing the parameter x. For example, assuming RH, choosing $x = (\log t)^{2-\varepsilon}$ for any $\varepsilon > 0$, and estimating the sum over n trivially, Soundararajan [12, Corollary C] deduced that

$$|\zeta(\tfrac{1}{2}+it)| \ll \exp\bigg(\Big(\frac{1+\lambda_0}{4}+o(1)\Big)\frac{\log t}{\log\log t}\bigg) \leq \exp\Big(\frac{3}{8}\frac{\log t}{\log\log t}\Big)$$

for sufficiently large t. At the time, this was the sharpest known version of Littlewood's result, improving on earlier work of Ramachandra and Sankaranarayanan [11]. The flexibility in the parameter x in (1.1) also allowed Soundararajan [12, Main Theorem] to study the frequency of large values of $|\zeta(\frac{1}{2}+it)|$ and allowed Harper [8, Theorem 1] to give sharp upper bounds for $2k$-th moment of the zeta function on the critical line; see also [12, Corollary A]. An overview of these (and other related) ideas concerning the distribution of values of zeta and L-functions can be found in the recent survey article of Soundararajan [13].

In [12], Soundararajan asked for an upper bound for $|\zeta(\frac{1}{2}+it)|$ on RH that attained the limit of existing methods. Using the Guinand-Weil explicit formula, it was shown in [5] that this problem could be framed in terms of bandlimited minorants of the function $\log((4+x^2)/x^2)$. Assuming RH, drawing upon the work of Carneiro and Vaaler [4], Chandee and Soundararajan [5, Theorem 1.1] use the optimal such minorants to prove that

$$\log|\zeta(\tfrac{1}{2}+it)| \leq \Big(\frac{\log 2}{2}+o(1)\Big)\frac{\log t}{\log\log t}, \tag{1.2}$$

as $t \to \infty$. They initially proved that the term of $o(1)$ is $O(\log\log\log t/\log\log t)$ but it was later observed in [1] and [2] that the term of $o(1)$ can be taken to be $O(1/\log\log t)$.

The goal of this note is to use the Guinand-Weil explicit formula, in conjunction with extremal bandlimited majorants and minorants for the Poisson kernel constructed in [2, 3], to give an analogue of Soundararajan's inequality

*Throughout the paper we use the traditional big-O notation $f = O(g)$ (or Vinogradov's notation $f \ll g$) to mean that $|f(t)| \leq C\,|g(t)|$ for a certain constant $C > 0$. In the subscript we indicate the parameters in which such constant C may depend on.

for $\log|\zeta(\frac{1}{2}+it)|$ in (1.1) and then to use this new inequality to give a slight refinement of Chandee and Soundararajan's bound in (1.2). For example, assuming RH, we show that

$$\log\left|\zeta(\tfrac{1}{2}+it)\right| \le \frac{\log 2}{2}\frac{\log t}{\log\log t} + \left(\frac{\log 2}{2}+\log^2 2\right)\frac{\log t}{(\log\log t)^2} + O\left(\frac{\log t}{(\log\log t)^3}\right),$$

as $t\to\infty$. Moreover, our new analogue of (1.1) has an explicit weight function in the Dirichlet polynomial over primes and prime powers, and it maintains the flexibility in the parameter x. So it potentially remains useful in applications such as studying the frequency of large values of $|\zeta(\frac{1}{2}+it)|$.

In order to state our results, with the range $0\le u<1$ in mind, we define the following special function:

$$F(u) := \int_0^\infty \frac{\sinh(2uy)}{\cosh^2 y}\,\mathrm{d}y = \frac{\pi u}{\sin(\pi u)} - u\left(\frac{\Gamma'}{\Gamma}\left(\frac{u+1}{2}\right) - \frac{\Gamma'}{\Gamma}\left(\frac{u}{2}\right)\right) + 1. \quad (1.3)$$

The identity in (1.3) follows from [7, Eq. 3.512.1 and 3.541.8]. Note that $F(0)=0$ and that F is in fact analytic in the open ball of radius 1 centered at $u=0$, with simple poles at $u=\pm1$. Using standard facts about the gamma function and the cosecant function (e.g., [7, Eq. 1.411.11 and 8.374]), one has the series expansion

$$F(u) = 2\log 2\cdot u + 2\sum_{k=1}^{\infty}\left(1-\frac{1}{2^{2k}}\right)\zeta(2k+1)\,u^{2k+1}, \quad \text{for } |u|<1. \quad (1.4)$$

Our main result is the following inequality.

Theorem 1.1. *Assume RH. For $t\ge 10$ and $x\ge 2$, we have*

$$\begin{aligned}\log\left|\zeta(\tfrac{1}{2}+it)\right| \le \mathrm{Re}\sum_{n\le x}\frac{\Lambda(n)}{n^{1/2+it}\log x}\,F\left(\frac{\log(x/n)}{\log x}\right)&\\ +\log 2\cdot\frac{\log t}{\log x} + O\left(\frac{\sqrt{x}\,\log x}{t}+1\right).&\end{aligned} \quad (1.5)$$

Remark 1.1. For $2\le n\le x$, we have the bounds

$$\begin{aligned}0 \le \frac{1}{\log x}\cdot F\left(\frac{\log(x/n)}{\log x}\right) &\le \frac{1}{\log n} - \frac{1}{\log(x^2/n)}\\ &\le \min\left\{\frac{1}{\log n}, \frac{2\,\log(x/n)}{\log^2 n}\right\}.\end{aligned} \quad (1.6)$$

To see this, note that by using the elementary inequality $\cosh x\ge e^x/2$ for $x\ge 0$ in the integral formulation of F, one arrives at the bound

$$F(u) \le 2\int_0^\infty \left(e^{-2(1-u)x} + e^{-2(1+u)x}\right)\mathrm{d}x = \frac{2u}{1-u^2}. \quad (1.7)$$

Then, (1.6) plainly follows from (1.7).

Remark 1.2. A similar inequality appears in the work of Chandee and Soundararajan [5, Equation (5)]. Letting $x = e^{2\pi\Delta}$ in their work, for $t \geq 10$ and $x \geq 2$, they show that

$$\log\left|\zeta(\tfrac{1}{2}+it)\right| \leq \mathrm{Re}\sum_{n\leq x}\frac{\Lambda(n)}{n^{1/2+it}}\,W(n;x)+\frac{\log t}{\log x}\log\frac{2}{1+x^{-2}}+O\left(\frac{\sqrt{x}\log x}{t}+1\right),$$

where the weight function $W(n;x)$ is not given explicitly but is shown to satisfy the bound $|W(n;x)| \ll 1$ for $2 \leq n \leq x$. This suffices for the pointwise bound for $\left|\zeta(\frac{1}{2}+it)\right|$ in (1.2) but is not amenable to applications such as studying the distribution of large values of $\zeta(s)$ on the critical line.

Using Theorem 1.1, we obtain a slight refinement of (1.2) with lower-order terms.

Theorem 1.2. *Assume RH. For $t \geq 10$ and any integer $K \geq 4$, we have*

$$\begin{aligned}\log\left|\zeta(\tfrac{1}{2}+it)\right| \leq{}& \frac{\log 2}{2}\frac{\log t}{\log\log t} + \left(\frac{\log 2}{2} + \log^2 2\right)\frac{\log t}{(\log\log t)^2}\\ &+ \left(2\log^2 2 + 2\log^3 2\right)\frac{\log t}{(\log\log t)^3}\\ &+ \sum_{k=4}^{K} C_k\cdot\frac{\log t}{(\log\log t)^k} + O_K\left(\frac{\log t}{(\log\log t)^{K+1}}\right),\end{aligned}$$

where the constants C_k are effectively computable as described in § 4.3.

Remark 1.3. Within our setup, there are different ways to arrive at an inequality of the form

$$\log|\zeta(\tfrac{1}{2}+it)| \leq \sum_{k=1}^{K} C_k\cdot\frac{\log t}{(\log\log t)^k} + O_K\left(\frac{\log t}{(\log\log t)^{K+1}}\right),$$

with $C_1 = (\log 2)/2$ and each C_k explicit for $2 \leq k \leq K$; see, for instance, equation (4.5). We go one step further and address the problem of doing this in an optimal way (within our framework). Running our process to obtain the values of C_k, in addition to the values of C_1, C_2, and C_3 stated in Theorem 1.2, we arrive at

$$\begin{aligned}C_4 &= -L + 6L^3 + 4L^4 + \frac{9\,\zeta(3)}{4},\\ C_5 &= -\frac{4L}{3} - 8L^2 + 16L^4 + 8L^5 + \frac{9\,\zeta(3)}{2} + 18\,\zeta(3)L,\\ C_6 &= \frac{4L}{3} - \frac{40L^2}{3} - 40L^3 + 40L^5 + 16L^6\\ &\quad + \left(-9 + 45L + 90L^2 - \frac{81\,\zeta(3)}{16L}\right)\zeta(3) + \frac{225\,\zeta(5)}{4},\end{aligned}$$

where $L := \log 2$.

2. Preliminary Lemmas

We denote a generic non-trivial zero of $\zeta(s)$ by ρ and, assuming RH, we write $\rho = \frac{1}{2} + i\gamma$. Unconditionally, for $s \neq 1$ and s not coinciding with a zero of $\zeta(s)$, the partial fraction decomposition for $\zeta'(s)/\zeta(s)$ (cf. [6, Chapter 12]) states that

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \Big(\frac{1}{s-\rho} + \frac{1}{\rho} \Big) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\big(\tfrac{s}{2}+1\big) + B + \frac{1}{2}\log \pi - \frac{1}{s-1},$$

where $B = -\sum_{\rho} \mathrm{Re}\,(1/\rho)$. Assuming RH, for $\beta > 0$ and $t \geq 1$, using Stirling's formula for the gamma function, one obtains

$$\mathrm{Re}\,\frac{\zeta'}{\zeta}\big(\tfrac{1}{2}+\beta+it\big) = -\frac{1}{2}\log\frac{t}{2\pi} + \sum_{\gamma} h_{\beta}(t-\gamma) + O\Big(\frac{1}{t}\Big), \tag{2.1}$$

where

$$h_{\beta}(x) := \frac{\beta}{\beta^2 + x^2} \tag{2.2}$$

is the Poisson kernel.

2.1. Extremal Bandlimited Approximations

Recall that an entire function $f : \mathbb{C} \to \mathbb{C}$ is said to be of *exponential type* if

$$\tau(f) := \limsup_{|z|\to\infty} |z|^{-1}\log|f(z)| < \infty.$$

In this case, the number $\tau(f)$ is called the *exponential type* of f. An entire function $f : \mathbb{C} \to \mathbb{C}$ is said to be *real entire* if its restriction to $\mathbb{R}$ is real-valued. If $f \in L^1(\mathbb{R})$ our normalization for the Fourier transform is as follows:

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\xi}\,\mathrm{d}x.$$

This extends to a unitary operator in $L^2(\mathbb{R})$ and functions that have compactly supported Fourier transform are called *bandlimited* functions. In this context, the classical Paley-Wiener theorem is a result that serves as a bridge between complex analysis and Fourier analysis. It says that, for $f \in L^2(\mathbb{R})$, the following two conditions are equivalent: (i) $\mathrm{supp}(\widehat{f}) \subset [-\Delta, \Delta]$ and; (ii) f is equal a.e. on $\mathbb{R}$ to the restriction of an entire function of exponential type at most $2\pi\Delta$.

The so-called Beurling-Selberg extremal problem in approximation theory is concerned with finding one-sided bandlimited approximations to a given function $f : \mathbb{R} \to \mathbb{R}$, in a way that $L^1(\mathbb{R})$-error is minimized. Our first lemma is a reproduction of [2, Lemma 9] due to Carneiro, Chirre, and Milinovich. It presents the extremal Beurling-Selberg majorants and minorants for the Poisson kernel. This construction is derived from the general Gaussian subordination framework of Carneiro, Littmann, and Vaaler [3].

Lemma 2.1 (Extremal functions for the Poisson kernel). *Let $\beta > 0$ be a real number and let $\Delta > 0$ be a real parameter. Let $h_\beta : \mathbb{R} \to \mathbb{R}$ be defined as in* (2.2). *Then there is a unique pair of real entire functions $m^{\pm}_{\beta,\Delta} : \mathbb{C} \to \mathbb{C}$ satisfying the following properties:*

(i) The real entire functions $m^{\pm}_{\beta,\Delta}$ have exponential type at most $2\pi\Delta$.

(ii) The inequalities

$$m^{-}_{\beta,\Delta}(x) \leq h_\beta(x) \leq m^{+}_{\beta,\Delta}(x)$$

hold pointwise for all $x \in \mathbb{R}$.

(iii) Subject to conditions (i) *and* (ii), *the value of the integral*

$$\int_{-\infty}^{\infty} \left\{ m^{+}_{\beta,\Delta}(x) - m^{-}_{\beta,\Delta}(x) \right\} \mathrm{d}x$$

is minimized.

The functions $m^{\pm}_{\beta,\Delta}$ are even and verify the following additional properties:

(iv) The L^1-distances of $m^{\pm}_{\beta,\Delta}$ to h_β are explicitly given by

$$\int_{-\infty}^{\infty} \left\{ m^{+}_{\beta,\Delta}(x) - h_\beta(x) \right\} \mathrm{d}x = \frac{2\pi e^{-2\pi\beta\Delta}}{1 - e^{-2\pi\beta\Delta}}$$

and

$$\int_{-\infty}^{\infty} \left\{ h_\beta(x) - m^{-}_{\beta,\Delta}(x) \right\} \mathrm{d}x = \frac{2\pi e^{-2\pi\beta\Delta}}{1 + e^{-2\pi\beta\Delta}}.$$

(v) The Fourier transforms of $m^{\pm}_{\beta,\Delta}$ are even continuous functions supported on the interval $[-\Delta, \Delta]$ and given by (for $|\xi| \leq \Delta$)

$$\widehat{m}^{\pm}_{\beta,\Delta}(\xi) = \pi \, \frac{e^{2\pi\beta(\Delta - |\xi|)} - e^{-2\pi\beta(\Delta - |\xi|)}}{\left(e^{\pi\beta\Delta} \mp e^{-\pi\beta\Delta}\right)^2}. \tag{2.3}$$

(vi) The functions $m^{\pm}_{\beta,\Delta}$ are explicitly given by

$$m^{\pm}_{\beta,\Delta}(z) = \frac{\beta}{\beta^2 + z^2} \cdot \frac{e^{2\pi\beta\Delta} + e^{-2\pi\beta\Delta} - 2\cos(2\pi\Delta z)}{\left(e^{\pi\beta\Delta} \mp e^{-\pi\beta\Delta}\right)^2}.$$

In particular, the function $m^{-}_{\beta,\Delta}$ is non-negative on $\mathbb{R}$.

(vii) Assume that $0 < \beta \leq 1$ and $\Delta \geq 1$. For any real number x we have

$$0 < m^-_{\beta,\Delta}(x) \leq h_\beta(x) \leq m^+_{\beta,\Delta}(x) \ll \frac{1}{\beta(1+x^2)}, \tag{2.4}$$

and, for any complex number $z = x + iy$, we have

$$\left|m^+_{\beta,\Delta}(z)\right| \ll \frac{\Delta^2 e^{2\pi\Delta|y|}}{\beta(1+\Delta|z|)} \tag{2.5}$$

and

$$\left|m^-_{\beta,\Delta}(z)\right| \ll \frac{\beta\Delta^2 e^{2\pi\Delta|y|}}{1+\Delta|z|}, \tag{2.6}$$

where the constants implied by the $\ll$ notation are universal.

Remark 2.1. Part (vii) of the the previous lemma is stated for $0 < \beta \leq \frac{1}{2}$ in [2, Lemma 9], but in fact it works as well for any $0 < \beta \leq \beta_0$, with the implied constants depending on such β_0. Here we simply take $\beta_0 = 1$.

2.2. Explicit Formula

Our next lemma is an (unconditional) explicit formula that connects the non-trivial zeros of $\zeta(s)$ and the prime numbers.

Lemma 2.2 (Guinand-Weil explicit formula). *Let $h(s)$ be analytic in the strip $|\mathrm{Im}\, s| \leq \frac{1}{2}+\varepsilon$ for some $\varepsilon > 0$, and assume that $|h(s)| \ll (1+|s|)^{-(1+\delta)}$ for some $\delta > 0$ when $|\mathrm{Re}\, s| \to \infty$. Then*

$$\begin{aligned}\sum_\rho h\left(\frac{\rho - \frac{1}{2}}{i}\right) = h\Big(\frac{1}{2i}\Big) + h\Big(-\frac{1}{2i}\Big) - \frac{1}{2\pi}\,\widehat{h}(0)\,\log\pi \\ + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{iu}{2}\Big)\,\mathrm{d}u \\ - \frac{1}{2\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\left(\widehat{h}\Big(\frac{\log n}{2\pi}\Big) + \widehat{h}\Big(\frac{-\log n}{2\pi}\Big)\right).\end{aligned}$$

Proof. The proof follows from [9, Theorem 5.12].

2.3. Estimates for Zeta on the Critical Strip

Our next lemma provides bounds for the real part of $\zeta'(s)/\zeta(s)$ on the critical strip, under RH. Since these inequalities might be of independent interest, we include both upper and lower bounds for completeness.

Lemma 2.3. *Assume RH. For $t \geq 10$, $x \geq 2$, and $0 < \beta \leq 1$, we have*

$$
\begin{aligned}
&\frac{-\log t}{x^\beta - 1} + \frac{2\,x^\beta}{(x^\beta - 1)^2}\ \mathrm{Re}\sum_{n \leq x} \frac{\Lambda(n)}{n^{1/2+it}} \sinh\Big(\beta \log \frac{x}{n}\Big) + O\bigg(\frac{\sqrt{x}\,\log x}{\beta\, t} + \frac{1}{\beta}\bigg) \\
&\leq -\mathrm{Re}\frac{\zeta'}{\zeta}\big(\tfrac{1}{2} + \beta + it\big) \qquad (2.7)\\
&\leq \frac{\log t}{x^\beta + 1} + \frac{2\,x^\beta}{(x^\beta + 1)^2}\ \mathrm{Re}\sum_{n \leq x} \frac{\Lambda(n)}{n^{1/2+it}} \sinh\Big(\beta \log \frac{x}{n}\Big) + O\bigg(\frac{\beta\sqrt{x}\,\log x}{t} + 1\bigg).
\end{aligned}
$$

Proof. To simplify notation, let $m_\Delta^{\pm} = m_{\beta,\Delta}^{\pm}$ and let $x = e^{2\pi\Delta}$. From (2.1) and property (ii) of Lemma 2.1, we have

$$
\begin{aligned}
-\frac{1}{2}\,\log\frac{t}{2\pi} + \sum_\gamma m_\Delta^-(t-\gamma) + O\Big(\frac{1}{t}\Big) &\leq \mathrm{Re}\,\frac{\zeta'}{\zeta}\big(\tfrac{1}{2} + \beta + it\big) \\
&\leq -\frac{1}{2}\,\log\frac{t}{2\pi} + \sum_\gamma m_\Delta^+(t-\gamma) + O\Big(\frac{1}{t}\Big).
\end{aligned} \tag{2.8}
$$

For a fixed $t > 0$, let $\ell_\Delta^{\pm}(z) := m_\Delta^{\pm}(t-z)$ so that $\widehat{\ell}_\Delta^{\pm}(\xi) = \widehat{m}_\Delta^{\pm}(-\xi)e^{-2\pi i\xi t}$ and the condition $|\ell_\Delta^{\pm}(s)| \ll (1+|s|)^{-2}$ when $|\mathrm{Re}\,s| \to \infty$ in the strip $|\mathrm{Im}\,s| \leq 1$ follows from (2.4), (2.5), (2.6), and an application of the Phragmén-Lindelöf principle. Recalling that $\widehat{m}_\Delta^{\pm}$ are even functions, we apply the Guinand-Weil explicit formula (Lemma 2.2) and find that

$$
\begin{aligned}
\sum_\gamma m_\Delta^{\pm}(t-\gamma) = \Big\{m_\Delta^{\pm}\big(t - \tfrac{1}{2i}\big) + m_\Delta^{\pm}\big(t + \tfrac{1}{2i}\big)\Big\} &- \frac{1}{2\pi}\,\widehat{m}_\Delta^{\pm}(0)\log\pi \\
&+ \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^{\pm}(t-y)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{iy}{2}\Big)\,\mathrm{d}y \\
&- \frac{1}{\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\,\widehat{m}_\Delta^{\pm}\Big(\frac{\log n}{2\pi}\Big)\cos(t\log n).
\end{aligned} \tag{2.9}
$$

We now proceed with an asymptotic analysis of each term on the right-hand side of (2.9).

2.3.1. First term. From (2.5) and (2.6), for $t \geq 10$ and $x \geq 2$, we see that

$$
\big|m_\Delta^{+}\big(t - \tfrac{1}{2i}\big) + m_\Delta^{+}\big(t + \tfrac{1}{2i}\big)\big| \ll \frac{\Delta^2 e^{\pi\Delta}}{\beta(1+\Delta t)} \ll \frac{\sqrt{x}\log x}{\beta\, t} \tag{2.10}
$$

and

$$
\big|m_\Delta^{-}\big(t - \tfrac{1}{2i}\big) + m_\Delta^{-}\big(t + \tfrac{1}{2i}\big)\big| \ll \frac{\beta\Delta^2 e^{\pi\Delta}}{1+\Delta t} \ll \frac{\beta\sqrt{x}\log x}{t}. \tag{2.11}
$$

2.3.2. Second term. From (2.3), it follows that

$$\widehat{m}_{\Delta}^{+}(0) = \pi\,\frac{e^{\pi\beta\Delta}+e^{-\pi\beta\Delta}}{e^{\pi\beta\Delta}-e^{-\pi\beta\Delta}} = \pi\,\frac{x^{\beta/2}+x^{-\beta/2}}{x^{\beta/2}-x^{-\beta/2}} = \pi\,\frac{x^{\beta}+1}{x^{\beta}-1} \tag{2.12}$$

and

$$\widehat{m}_{\Delta}^{-}(0) = \pi\,\frac{e^{\pi\beta\Delta}-e^{-\pi\beta\Delta}}{e^{\pi\beta\Delta}+e^{-\pi\beta\Delta}} = \pi\,\frac{x^{\beta/2}-x^{-\beta/2}}{x^{\beta/2}+x^{-\beta/2}} = \pi\,\frac{x^{\beta}-1}{x^{\beta}+1}. \tag{2.13}$$

2.3.3. Third term. Recall that the Poisson kernel h_β defined in (2.2) satisfies $\int_{-\infty}^{\infty} h_\beta(y)\,\mathrm{d}y = \pi$. Note also that for $0 < \beta \leq 1$ and $|y| \geq 1$ we have

$$h_\beta(y) = \frac{\beta}{\beta^2+y^2} \leq \frac{1}{1+y^2}. \tag{2.14}$$

Hence, from (2.4) and (2.14), we get

$$\begin{aligned} 0 &\leq \int_{-\infty}^{\infty} m_{\Delta}^{-}(y)\,\log(2+|y|)\,\mathrm{d}y \\ &\leq \int_{-\infty}^{\infty} h_\beta(y)\log(2+|y|)\,\mathrm{d}y \\ &= \int_{-1}^{1} h_\beta(y)\log(2+|y|)\,\mathrm{d}y + \int_{|y|\geq 1} h_\beta(y)\log(2+|y|)\,\mathrm{d}y = O(1). \end{aligned} \tag{2.15}$$

From Stirling's formula, (2.13), and (2.15), for $t \geq 10$, it follows that

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} m_{\Delta}^{-}(t-y)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{iy}{2}\Big)\,\mathrm{d}y \\ &\qquad = \frac{1}{2\pi}\int_{-\infty}^{\infty} m_{\Delta}^{-}(y)\Big(\log\frac{t}{2} + O(\log(2+|y|))\Big)\,\mathrm{d}y \\ &\qquad = \frac{1}{2}\,\frac{x^{\beta}-1}{x^{\beta}+1}\,\log\frac{t}{2} + O(1). \end{aligned} \tag{2.16}$$

Similarly, using Stirling's formula, (2.4), and (2.12), for $t \geq 10$, it follows that

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} m_{\Delta}^{+}(t-y)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{iy}{2}\Big)\,\mathrm{d}y \\ &\qquad = \frac{1}{2\pi}\int_{-\infty}^{\infty} m_{\Delta}^{+}(y)\Big(\log\frac{t}{2} + O(\log(2+|y|))\Big)\,\mathrm{d}y \\ &\qquad = \frac{1}{2}\,\frac{x^{\beta}+1}{x^{\beta}-1}\,\log\frac{t}{2} + O\Big(\frac{1}{\beta}\Big). \end{aligned} \tag{2.17}$$

2.3.4. Fourth term. Now, to handle the sum over prime powers, recall that $x = e^{2\pi\Delta}$ and note that the sum in (2.9) only runs over n with $2 \leq n \leq x$.

Using the explicit description for the Fourier transforms $\widehat{m}_\Delta^{\pm}$ given by (2.3), we get

$$\begin{aligned}
&\frac{1}{\pi}\sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\,\widehat{m}_\Delta^{\pm}\Big(\frac{\log n}{2\pi}\Big)\cos(t\log n)\\
&\qquad=\frac{e^{2\pi\beta\Delta}}{\big(e^{2\pi\beta\Delta}\mp 1\big)^2}\ \mathrm{Re}\sum_{n\leq e^{2\pi\Delta}}\frac{\Lambda(n)}{n^{1/2+it}}\left(\frac{e^{2\pi\beta\Delta}}{n^\beta}-\frac{n^\beta}{e^{2\pi\beta\Delta}}\right)\\
&\qquad=\frac{x^\beta}{\big(x^\beta\mp 1\big)^2}\ \mathrm{Re}\sum_{n\leq x}\frac{\Lambda(n)}{n^{1/2+it}}\left\{\Big(\frac{x}{n}\Big)^\beta-\Big(\frac{x}{n}\Big)^{-\beta}\right\}\\
&\qquad=\frac{2\,x^\beta}{\big(x^\beta\mp 1\big)^2}\ \mathrm{Re}\sum_{n\leq x}\frac{\Lambda(n)}{n^{1/2+it}}\ \sinh\Big(\beta\log\frac{x}{n}\Big).
\end{aligned}\tag{2.18}$$

2.3.5. Conclusion. To derive the bound on the right-hand side of (2.7), we combine (2.8), (2.9), (2.11), (2.13), (2.16), and (2.18), to find that

$$\begin{aligned}
\mathrm{Re}\frac{\zeta'}{\zeta}\big(\tfrac12+\beta+it\big)\geq \frac{-1}{x^\beta+1}\ \log\frac{t}{2\pi}&\\
-\frac{2\,x^\beta}{\big(x^\beta+1\big)^2}\ \mathrm{Re}\sum_{n\leq x}\frac{\Lambda(n)}{n^{1/2+it}}\ &\sinh\Big(\beta\log\frac{x}{n}\Big)\\
+O\bigg(\frac{\beta\sqrt{x}\log x}{t}+1\bigg)&.
\end{aligned}$$

To derive the bound on the left-hand side of (2.7), we combine (2.8), (2.9), (2.10), (2.12), (2.17), and (2.18), to conclude that

$$\begin{aligned}
\mathrm{Re}\frac{\zeta'}{\zeta}\big(\tfrac12+\beta+it\big)\leq \frac{1}{x^\beta-1}\ \log\frac{t}{2\pi}&\\
-\frac{2\,x^\beta}{\big(x^\beta-1\big)^2}\ \mathrm{Re}\sum_{n\leq x}\frac{\Lambda(n)}{n^{1/2+it}}\ &\sinh\Big(\beta\log\frac{x}{n}\Big)\\
+O\bigg(\frac{\sqrt{x}\,\log x}{\beta\,t}+\frac{1}{\beta}\bigg)&.
\end{aligned}$$

This completes the proof of Lemma 2.3. □

3. Proof of Theorem 1.1

For $t\geq 10$, observe that

$$\begin{aligned}
\log|\zeta(\tfrac12+it)|&=-\int_0^1\mathrm{Re}\frac{\zeta'}{\zeta}\big(\tfrac12+u+it\big)\,\mathrm{d}u+\log|\zeta(\tfrac32+it)|\\
&=-\int_0^1\mathrm{Re}\frac{\zeta'}{\zeta}\big(\tfrac12+u+it\big)\,\mathrm{d}u+O(1).
\end{aligned}\tag{3.1}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{\log\big(1+x^{-u}\big)}{\log x}\right] = -\frac{1}{x^u+1},$$

we note that

$$\log t\int_0^1 \frac{\mathrm{d}u}{x^u+1} \leq \log t\int_0^\infty \frac{\mathrm{d}u}{x^u+1} = \log 2\cdot\frac{\log t}{\log x}. \tag{3.2}$$

Applying the upper bound from Lemma 2.3 in (3.1), and using (3.2), we get

$$\begin{aligned}\log|\zeta(\tfrac12+it)| &\leq \log t\int_0^1 \frac{\mathrm{d}u}{x^u+1} + \mathrm{Re}\sum_{n\leq x}\frac{\Lambda(n)}{n^{1/2+it}}W_0(n;x) + O\bigg(\frac{\sqrt{x}\,\log x}{t}+1\bigg)\\ &\leq \log 2\cdot\frac{\log t}{\log x} + \mathrm{Re}\sum_{n\leq x}\frac{\Lambda(n)}{n^{1/2+it}}W_0(n;x) + O\bigg(\frac{\sqrt{x}\,\log x}{t}+1\bigg),\end{aligned} \tag{3.3}$$

where

$$W_0(n;x) = \int_0^1 \frac{2\,x^u}{\big(x^u+1\big)^2}\sinh\Big(u\log\frac{x}{n}\Big)\,\mathrm{d}u.$$

Now observe that

$$\begin{aligned}W_0(n,x) &= \frac12\int_0^\infty \frac{\sinh\big(u\log\frac{x}{n}\big)}{\cosh^2\big(\frac{u}{2}\log x\big)}\,\mathrm{d}u + O\bigg(\int_1^\infty\frac{\mathrm{d}u}{n^u}\bigg)\\ &= \frac{1}{\log x}\int_0^\infty \frac{\sinh\big(2y\frac{\log(x/n)}{\log x}\big)}{\cosh^2 y}\,\mathrm{d}y + O\bigg(\frac{1}{n\log n}\bigg)\\ &= \frac{1}{\log x}\,F\Big(\frac{\log(x/n)}{\log x}\Big) + O\bigg(\frac{1}{n\log n}\bigg),\end{aligned}$$

with F as in (1.3). Therefore

$$\mathrm{Re}\sum_{n\leq x}\frac{\Lambda(n)}{n^{1/2+it}}W_0(n;x) = \mathrm{Re}\sum_{n\leq x}\frac{\Lambda(n)}{n^{1/2+it}\log x}\,F\Big(\frac{\log(x/n)}{\log x}\Big) + O(1). \tag{3.4}$$

Theorem 1.1 now follows directly from (3.3) and (3.4).

4. Proof of Theorem 1.2

We now deduce Theorem 1.2 from Theorem 1.1.

4.1. Setup

Assuming RH, we have

$$\sum_{n\le x}\frac{\Lambda(n)}{\sqrt{n}} = 2\sqrt{x} + O\big(\log^3 x\big). \tag{4.1}$$

Since $F(0)=0$ and $F(u)\ge 0$ for $u\in[0,1)$, using (4.1) and summing by parts, it follows that

$$\begin{aligned}
&\left|\operatorname{Re}\sum_{n\le x}\frac{\Lambda(n)}{n^{1/2+it}\log x}\,F\Big(\frac{\log(x/n)}{\log x}\Big)\right| \\
&\le \frac{1}{\log x}\sum_{n\le x}\frac{\Lambda(n)}{\sqrt{n}}F\Big(\frac{\log(x/n)}{\log x}\Big) \\
&= \frac{2}{\log x}\int_{\log 2/\log x}^{1} F'(1-v)\,x^{v/2}\,\mathrm{d}v + O\big(\log^2 x\big) \\
&= \frac{2}{\log x}\int_{1/2}^{1} F'(1-v)\,x^{v/2}\,\mathrm{d}v + O\big(x^{1/4}\log x\big) \\
&= \frac{2\sqrt{x}}{\log x}\int_{0}^{1/2} F'(u)\,x^{-u/2}\,\mathrm{d}u + O\big(x^{1/4}\log x\big).
\end{aligned}$$

Here we have used the fact that $F'(u)$ has a double pole at $u=1$ so that $|F(1-\frac{\log y}{\log x})| \ll \frac{\log^2 x}{\log^2 y}$ for $2\le y\le x$. Differentiating the series expansion for F in (1.4) term-by-term, we see that

$$F'(u) = 2\log 2 + 2\sum_{k=1}^{\infty}\Big(1-\frac{1}{2^{2k}}\Big)(2k+1)\,\zeta(2k+1)\,u^{2k}$$

uniformly for $|u|\le\frac{1}{2}$. Notice that the coefficients of this series are positive. Hence, for any positive integer K, we have

$$\begin{aligned}
\left|\operatorname{Re}\sum_{n\le x}\frac{\Lambda(n)}{n^{1/2+it}\log x}\,F\Big(\frac{\log(x/n)}{\log x}\Big)\right| &\le \frac{(4\log 2)\sqrt{x}}{\log x}\int_0^{1/2} x^{-u/2}\,\mathrm{d}u \\
&+ \frac{4\sqrt{x}}{\log x}\sum_{k=1}^{K}\Big(1-\frac{1}{2^{2k}}\Big)(2k+1)\,\zeta(2k+1)\int_0^{1/2} u^{2k}\,x^{-u/2}\,\mathrm{d}u \\
&+ O_K\bigg(\frac{\sqrt{x}}{\log x}\int_0^{1/2} u^{2K+2}\,x^{-u/2}\,\mathrm{d}u\bigg) + O\big(x^{1/4}\log x\big).
\end{aligned} \tag{4.2}$$

For non-negative integers k, we observe that

$$\int_0^{1/2} u^{2k}\,x^{-u/2}\,\mathrm{d}u \le \int_0^{\infty} u^{2k}\,x^{-u/2}\,\mathrm{d}u = \frac{2^{2k+1}\,\Gamma(2k+1)}{(\log x)^{2k+1}} = \frac{2^{2k+1}(2k)!}{(\log x)^{2k+1}}. \tag{4.3}$$

Therefore, combining (1.5), (4.2), and (4.3), it follows that

$$\begin{aligned}\log\left|\zeta(\tfrac{1}{2}+it)\right| \leq \log 2\cdot\frac{\log t}{\log x} + \frac{(8\log 2)\sqrt{x}}{\log^2 x}&\\ + 8\sqrt{x}\sum_{k=1}^{K}(2^{2k}-1)\,\frac{(2k+1)!\,\zeta(2k+1)}{(\log x)^{2k+2}}&\qquad (4.4)\\ + O_K\bigg(\frac{\sqrt{x}}{(\log x)^{2K+4}}\bigg).&\end{aligned}$$

4.2. Choosing x: Initial Approximations

We roughly want to choose $\sqrt{x}\approx\log t$ in order to minimize the right-hand side of (4.4). If we make the exact choice $\sqrt{x}=\log t$, we obtain

$$\begin{aligned}\log\left|\zeta(\tfrac{1}{2}+it)\right| \leq \frac{\log 2}{2}\frac{\log t}{\log\log t} + \frac{2\log 2\cdot\log t}{(\log\log t)^2}&\\ + 2\log t\sum_{k=1}^{K}\Big(1-\frac{1}{2^{2k}}\Big)\,\frac{(2k+1)!\,\zeta(2k{+}1)}{(\log\log t)^{2k+2}}&\qquad (4.5)\\ + O_K\bigg(\frac{\log t}{(\log\log t)^{2K+4}}\bigg).&\end{aligned}$$

We can make the second term on the right-hand side of this inequality smaller by taking x to be slightly smaller. To this end, we choose

$$\log x = 2\log\log t - 2c \qquad (4.6)$$

in (4.4), and find that

$$\begin{aligned}\log\left|\zeta(\tfrac{1}{2}+it)\right| \leq \frac{\log 2}{2}\frac{\log t}{\log\log t} + \Big(\frac{c\log 2}{2}+2\,e^{-c}\log 2\Big)\frac{\log t}{(\log\log t)^2}&\\ + \Big(\frac{c^2\log 2}{4}+4\,c\,e^{-c}\log 2\Big)\frac{\log t}{(\log\log t)^3}&\qquad (4.7)\\ + O_c\bigg(\frac{\log t}{(\log\log t)^4}\bigg).&\end{aligned}$$

The minimum value of the function $g(c)=\frac{c\log 2}{2}+2\,e^{-c}\log 2$ occurs at $c=2\log 2$ with the minimum value equaling $\frac{\log 2}{2}+\log^2 2$. Choosing $c=2\log 2$ in (4.7), we conclude that

$$\begin{aligned}\log\left|\zeta(\tfrac{1}{2}+it)\right| \leq \frac{\log 2}{2}\frac{\log t}{\log\log t} + \Big(\frac{\log 2}{2}+\log^2 2\Big)\frac{\log t}{(\log\log t)^2}&\\ + \big(2\log^2 2+2\log^3 2\big)\frac{\log t}{(\log\log t)^3}&\\ + O\bigg(\frac{\log t}{(\log\log t)^4}\bigg),&\end{aligned}$$

arriving at the first three terms stated in Theorem 1.2.

4.3. The Optimal Choice of x

We now describe the optimal choice of x in terms of t, as hinted by (4.6), but now with a complete power series expansion. The elegant argument presented in this subsection, and the companion Pari script, are due to Don Zagier†.

We make the change of variables $w = \frac{1}{\log\log t}$ and $z = \frac{1}{\log x}$. Define

$$B(w, z) = Lze^{1/w} + e^{1/(2z)} \sum_{m=0}^{\infty} a_m z^{m+1}$$

as in (4.4), where the coefficients L and a_m could be arbitrary numbers but in our case are given by

$$L = \log 2, \quad a_1 = 8L, \quad a_m = \begin{cases} 8(2^{m-1} - 1)\, m!\, \zeta(m), & \text{for } m > 1 \text{ odd}, \\ 0, & \text{for } m \text{ even}. \end{cases}$$

We want an extreme value of B for w fixed, so need w and z related by

$$0 = \frac{\partial B}{\partial z} = Le^{1/w} + e^{1/(2z)} \sum_{m=0}^{\infty} \left((m+1)a_m - \tfrac{1}{2}a_{m+1}\right) z^m .$$

This can be rewritten in turn as

$$e^{1/w} = e^{1/(2z)} \sum_{m=0}^{\infty} b_m z^m, \qquad b_m := \frac{\frac{1}{2}a_{m+1} - (m+1)a_m}{L} \tag{4.8}$$

or, taking logarithms,

$$\frac{1}{w} = \frac{1}{2z} + \log b_0 + \log\left(1 + \sum_{m=1}^{\infty} \frac{b_m}{b_0} z^m\right)$$

(with $b_0 = 4$ and $\log b_0 = 2L$ in our case) or, inverting in the sense of multiplication of power series,

$$w = 2z - 4\log b_0\, z^2 + \left(8\log^2 b_0 - \frac{4b_1}{b_0}\right) z^3 + \cdots$$

or, inverting in the sense of composition of power series,

$$z = \frac{1}{2}w + \frac{\log b_0}{2} w^2 + \left(\frac{1}{2}\log^2 b_0 + \frac{b_1}{4b_0}\right) w^3 + \cdots \tag{4.9}$$

---

†In an earlier version of this paper we had a slightly different line of reasoning for this description, leading to the same coefficients.

or, inverting once more in the sense of multiplication of power series,

$$\frac{1}{z} = \frac{2}{w} - 2\log b_0 - \frac{b_1}{b_0} w + \cdots$$

(this generalizes (4.6)), each time with explicitly computable coefficients that are polynomials in the numbers a_m, $1/L$, $\log b_0$, and $1/b_0$ (so, polynomials with rational coefficients in $(\log 2)^{\pm 1}$ and $\zeta(2k+1)$ in our case). Finally, substituting the value of z as in (4.8) and (4.9) into the function $B(w, z)$, we find

$$\begin{aligned}
B_{\mathrm{opt}}(w) &= e^{1/w}\left(Lz + \sum_{m=0}^{\infty} a_m z^{m+1} \Big/ \sum_{m=0}^{\infty} b_m z^m\right) \\
&= e^{1/w}\left(\left(L + \frac{a_0}{b_0}\right)z + \left(\frac{a_1}{b_0} - \frac{a_0 b_1}{b_0^2}\right)z^2 + \cdots\right) \\
&= e^{1/w}\left(\frac{1}{2}\left(L + \frac{a_0}{b_0}\right)w + \left(\frac{1}{4}\left(\frac{a_1}{b_0} - \frac{a_0 b_1}{b_0^2}\right) + \frac{\log b_0}{2}\left(L + \frac{a_0}{b_0}\right)\right)w^2\right. \\
&\qquad\qquad\left. + C_3 w^3 + \cdots\right),
\end{aligned}$$

which is the desired form proposed in Theorem 1.2.

The following Pari script calculates the first coefficients of this series.

```
a(m) = [8*L,0,144*Z3,0,14400*Z5,0,2540160*Z7,0][m];\\def. of a_m for m=1,...,8; a_0 = 0
b(m) = (a(m+1)/2 - (m+1)*a(m))/L;
w1 = 1/2/z + 2*L + log(1+ sum(m=1,6,b(m)/4*z^m,O(z^7))); Z = serreverse(1/w1);\\ w1 = 1/w
B = L*Z + sum(m=1,8,a(m)*Z^(m+1),O(z^10))/sum(m=1,7,b(m)*Z^m,4+O(z^8));

\\ Time 7 ms.  Output:

w1 == 1/2/z + 2*L - 4*z + (-8 + 18*Z3/L)*z^2 + (-64/3 - 72*Z3/L)*z^3 + O(z^4) \\ true
Z == 1/2*z + L*z^2 + (2*L^2 - 1)*z^3 + (4*L^3 - 6*L - 1 + 9/4*Z3/L)*z^4 + O(z^5) \\ true
coeff(B,1) == L/2
coeff(B,2) == L^2 + L/2
coeff(B,3) == 2*L^3 + 2*L^2
coeff(B,4) == 4*L^4 + 6*L^3 - L + 9/4*Z3
coeff(B,5) == 8*L^5 + 16*L^4 - 8*L^2 - 4/3*L + (18*L + 9/2)*Z3
coeff(B,6) == 16*L^6 + 40*L^5 - 40*L^3 - 40/3*L^2  + 4/3*L+ (90*L^2+45*L-9)*Z3
      - 81/16*Z3^2/L + 225/4*Z5
coeff(B,7) == 32*L^7 + 96*L^6 - 160*L^4 - 80*L^3 + 16*L^2 + 34/5*L
     + 45*(8*L^3 + 6*L^2 - 12/5*L - 1)*Z3 - 81/4*(3-1/L)*Z3^2 + (675*L + 225/2)*Z5
```

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