

SIMPLIFIED WEAK GALERKIN METHODS FOR LINEAR ELASTICITY ON NONCONVEX DOMAINS

CHUNMEI WANG AND SHANGYOU ZHANG

ABSTRACT. This paper presents a weak Galerkin (WG) finite element method for linear elasticity on general polygonal and polyhedral meshes, free from convexity constraints, by leveraging bubble functions as central analytical tools. The proposed method eliminates the need for stabilizers commonly used in traditional WG methods, resulting in a simplified formulation. The method is symmetric, positive definite, and straightforward to implement. Optimal-order error estimates are established for the WG approximations in the discrete H^1 -norm, assuming sufficient smoothness of the exact solution, and in the standard L^2 -norm under regularity assumptions for the dual problem. Numerical experiments confirm the efficiency and accuracy of the proposed stabilizer-free WG method.

1. INTRODUCTION

This paper introduces a novel weak Galerkin finite element method for linear elasticity that eliminates the stabilizers traditionally required in WG methods. The innovation of the proposed approach, compared to existing stabilizer-free WG methods (e.g., [59]), lies in its effectiveness on general polygonal and polyhedral meshes, including those with nonconvex geometries, as it operates without requiring convexity assumptions.

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be an open, bounded, and connected domain with a Lipschitz continuous boundary $\partial\Omega$, representing an elastic body subjected to an exterior force \mathbf{f} and a prescribed displacement boundary condition. The kinematic model of linear elasticity aims to determine a displacement vector field \mathbf{u} satisfying

$$(1.1) \quad \begin{aligned} -\nabla \cdot \sigma(\mathbf{u}) &= \mathbf{f}, & \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g}, & \text{on } \partial\Omega, \end{aligned}$$

where $\sigma(\mathbf{u})$ is the symmetric Cauchy stress tensor. For linear, homogeneous, and isotropic materials, the stress tensor is expressed as

$$\sigma(\mathbf{u}) = 2\mu\epsilon(\mathbf{u}) + \lambda(\nabla \cdot \mathbf{u})\mathbf{I},$$

with $\epsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ denoting the linear strain tensor, and μ and λ representing the Lamé constants. In the case of linear plane strain, the Lamé constants

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are defined as

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)},$$

where E is the elasticity modulus, and ν is Poisson's ratio.

The variational formulation of the model problem (1.1) can be formulated as follows: Find an unknown function $\mathbf{u} \in [H^1(\Omega)]^d$ such that $\mathbf{u} = \mathbf{g}$ on $\partial\Omega$ and

$$(1.2) \quad 2(\mu\epsilon(\mathbf{u}), \epsilon(\mathbf{v})) + (\lambda\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in [H_0^1(\Omega)]^d,$$

where $H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$.

The Finite Element Method (FEM) and its variants are extensively used for solving partial differential equations (PDEs) numerically. In the context of elasticity problems, mixed FEMs are particularly popular; however, enforcing strong symmetry on the stress tensor presents a significant challenge. To address this, several strategies have been developed, including relaxing the symmetry constraint on the stress tensor [31], constructing weakly symmetric mixed finite elements [5], and designing nonconforming mixed FEMs [1, 3, 4, 14, 54, 60, 61]. A breakthrough was achieved with a family of conforming mixed elements featuring reduced degrees of freedom, applicable in any dimension. This was accomplished by identifying a critical structure within the discrete stress spaces of symmetric matrix-valued polynomials on simplicial grids and establishing two fundamental algebraic results [16]. The Discontinuous Galerkin (DG) method has also been widely adopted for elasticity problems [12, 53]. A key advantage of DG methods is their ability to discretize problems on an element-by-element basis, seamlessly connecting elements through numerical traces [2, 15]. For linear elasticity, innovative approaches include a three-field decomposition method [6] and a novel hybridized mixed method [13]. Other noteworthy methods, such as the tangential-displacement normal-stress method, have demonstrated robustness against both shear and volume locking [29, 30].

The weak Galerkin finite element method has revolutionized the numerical landscape for solving partial differential equations (PDEs). By harnessing the power of distributions and piecewise polynomials, WG transcends traditional finite element approaches. Unlike its predecessors, WG relaxes the stringent regularity requirements for function approximations, instead leveraging carefully crafted stabilizers to ensure method stability. Recent studies have exhaustively explored the versatility of WG in tackling diverse model PDEs, thereby casting the method as a robust and reliable tool in computational science [18, 19, 49, 55, 20, 21, 22, 23, 51, 56, 10, 48, 27, 17, 33, 58, 43, 47, 44, 45, 46, 50, 52]. Its capacity to adapt to a wide range of PDEs is underscored by its use of weak derivatives and weak continuities in designing numerical schemes based on the weak forms of underlying PDEs. A key advancement within the WG paradigm is the Primal-Dual Weak Galerkin (PDWG) method [24, 25, 7, 8, 9, 26, 34, 35, 57, 11, 39, 40, 38, 41, 42]. PDWG views numerical solutions as constrained minimizations of functionals, with constraints that mimic the weak formulation of PDEs using weak derivatives. This formulation results in an Euler-Lagrange equation that integrates both the primal variable and the dual variable (Lagrange multiplier), yielding a symmetric scheme.

This paper presents a simplified formulation of the WG finite element method that eliminates the need for stabilizers. Unlike the existing stabilizer-free WG methods [59], our approach is versatile, as it works on polytopal meshes without

convexity constraints, and supports flexible polynomial degrees. The critical innovation facilitating these advancements is the incorporation of bubble functions. While this method requires the use of higher-degree polynomials for calculating the discrete weak derivatives, which may pose challenges for certain practical applications, our focus is on the theoretical advancements, particularly in developing WG methods with inherent stabilizers, specifically designed for non-convex elements in finite element partitions.

Our method retains the size and global sparsity of the stiffness matrix, greatly simplifying programming complexity in comparison to traditional stabilizer-dependent WG methods. Theoretical analysis confirms that our WG approximations yield optimal error estimates in both the discrete H^1 and L^2 norms. By providing a stabilizer-free WG method that sustains high performance while reducing computational complexity, this paper makes a significant contribution to the development of finite element methods on non-convex polytopal meshes.

The structure of this paper is as follows. Section 2 provides a concise review of weak differential operators and their discrete counterparts. In Section 3, we present the weak Galerkin scheme that eliminates the need for stabilizers. Section 4 proves the existence and uniqueness of the solution. In Section 5, we derive the error equation for the proposed scheme, followed by Section 6, which focuses on the error estimate for the numerical approximation in the energy norm. Section 7 extends this analysis to establish the error estimate in the L^2 norm. Finally, Section 8 presents numerical tests to validate the theoretical findings from the previous sections.

Throughout this paper, we adopt standard notations. Let D be any open bounded domain in \mathbb{R}^d with a Lipschitz continuous boundary. The inner product, semi-norm and norm in the Sobolev space $H^s(D)$ for any integer $s \geq 0$ are denoted by $(\cdot, \cdot)_{s,D}$, $|\cdot|_{s,D}$ and $\|\cdot\|_{s,D}$, respectively. For simplicity, when the domain D is chosen as $D = \Omega$, the subscript D is omitted from the notations of the inner product and norm. For the case where $s = 0$, the notations $(\cdot, \cdot)_{0,D}$, $|\cdot|_{0,D}$ and $\|\cdot\|_{0,D}$ are further simplified as $(\cdot, \cdot)_D$, $|\cdot|_D$ and $\|\cdot\|_D$, respectively.

2. DISCRETE WEAK STRAIN TENSOR AND DISCRETE WEAK DIVERGENCE

In this section, we briefly review the definition of weak strain tensor and weak divergence, along with their discrete counterparts introduced in [56, 47].

Let T be a polytopal element with boundary ∂T . A weak function on T is defined as $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\}$, where $\mathbf{v}_0 \in [L^2(T)]^d$, $\mathbf{v}_b \in [L^2(\partial T)]^d$. The first component, \mathbf{v}_0 , represents the value of \mathbf{v} in the interior of T , while the second component, \mathbf{v}_b , corresponds to the value of \mathbf{v} on the boundary of T . In general, \mathbf{v}_b is assumed to be independent of the trace of \mathbf{v}_0 . A special case arises when $\mathbf{v}_b = \mathbf{v}_0|_{\partial T}$, where the function $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\}$ is fully determined by \mathbf{v}_0 and can be simply denoted as $\mathbf{v} = \mathbf{v}_0$.

Denote by $W(T)$ the space of all weak functions on T ; i.e.,

$$W(T) = \{v = \{\mathbf{v}_0, \mathbf{v}_b\} : \mathbf{v}_0 \in [L^2(T)]^d, \mathbf{v}_b \in [L^2(\partial T)]^d\}.$$

The weak gradient, denoted by ∇_w , is a linear operator from $W(T)$ to the dual space of $[H^1(T)]^{d \times d}$. For any $\mathbf{v} \in W(T)$, the weak gradient $\nabla_w \mathbf{v}$ is defined as a

bounded linear functional on $[H^1(T)]^{d \times d}$ such that

$$(\nabla_w \mathbf{v}, \boldsymbol{\varphi})_T = -(\mathbf{v}_0, \nabla \cdot \boldsymbol{\varphi})_T + \langle \mathbf{v}_b, \boldsymbol{\varphi} \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \boldsymbol{\varphi} \in [H^1(T)]^{d \times d},$$

where \mathbf{n} is an unit outward normal direction to ∂T .

For any non-negative integer r , let $P_r(T)$ represent the space of polynomials on T with total degree at most r . A discrete weak gradient on T , denoted by $\nabla_{w,r_1,T}$, is a linear operator from $W(T)$ to $[P_{r_1}(T)]^{d \times d}$. For any $\mathbf{v} \in W(T)$, $\nabla_{w,r_1,T} \mathbf{v}$ is the unique polynomial matrix in $[P_{r_1}(T)]^{d \times d}$ satisfying

$$(\nabla_{w,r_1,T} \mathbf{v}, \boldsymbol{\varphi})_T = -(\mathbf{v}_0, \nabla \cdot \boldsymbol{\varphi})_T + \langle \mathbf{v}_b, \boldsymbol{\varphi} \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \boldsymbol{\varphi} \in [P_{r_1}(T)]^{d \times d}.$$

We define the discrete weak strain tensor as follows:

$$\epsilon_{w,r_1,T}(\mathbf{u}) = \frac{1}{2}(\nabla_{w,r_1,T} \mathbf{u} + \nabla_{w,r_1,T} \mathbf{u}^T).$$

For any $\mathbf{v} \in W(T)$, the discrete weak strain tensor, denoted by $\epsilon_{w,r_1,T}(\mathbf{v})$, is the unique polynomial matrix in $[P_{r_1}(T)]^{d \times d}$ satisfying

$$(2.1) \quad (\epsilon_{w,r_1,T}(\mathbf{v}), \boldsymbol{\varphi})_T = -(\mathbf{v}_0, \nabla \cdot \frac{1}{2}(\boldsymbol{\varphi} + \boldsymbol{\varphi}^T))_T + \langle \mathbf{v}_b, \frac{1}{2}(\boldsymbol{\varphi} + \boldsymbol{\varphi}^T) \cdot \mathbf{n} \rangle_{\partial T},$$

for all $\boldsymbol{\varphi} \in [P_{r_1}(T)]^{d \times d}$.

For a smooth $\mathbf{v}_0 \in [H^1(T)]^d$, applying the usual integration by parts to the first term on the right-hand side of (2.1) gives

$$(2.2) \quad (\epsilon_{w,r_1,T}(\mathbf{v}), \boldsymbol{\varphi})_T = (\epsilon(\mathbf{v}_0), \boldsymbol{\varphi})_T + \langle \mathbf{v}_b - \mathbf{v}_0, \frac{1}{2}(\boldsymbol{\varphi} + \boldsymbol{\varphi}^T) \cdot \mathbf{n} \rangle_{\partial T}.$$

for all $\boldsymbol{\varphi} \in [P_{r_1}(T)]^{d \times d}$.

The weak divergence of $\mathbf{v} \in W(T)$, denoted by $\nabla_w \cdot \mathbf{v}$, is a bounded linear functional in the Sobolev space $H^1(T)$, and its action on any $\phi \in H^1(T)$ is given by

$$(\nabla_w \cdot \mathbf{v}, \phi)_T = -(\mathbf{v}_0, \nabla \phi)_T + \langle \mathbf{v}_b \cdot \mathbf{n}, \phi \rangle_{\partial T}.$$

The discrete weak divergence of $\mathbf{v} \in W(T)$, denoted by $\nabla_{w,r_2,T} \cdot \mathbf{v}$, is the unique polynomial in $P_{r_2}(T)$ satisfying

$$(2.3) \quad (\nabla_{w,r_2,T} \cdot \mathbf{v}, \phi)_T = -(\mathbf{v}_0, \nabla \phi)_T + \langle \mathbf{v}_b \cdot \mathbf{n}, \phi \rangle_{\partial T},$$

for any $\phi \in P_{r_2}(T)$.

For a smooth $\mathbf{v}_0 \in [H^1(T)]^d$, applying the usual integration by parts to the first term on the right-hand side of (2.3) gives

$$(2.4) \quad (\nabla_{w,r_2,T} \cdot \mathbf{v}, \phi)_T = (\nabla \cdot \mathbf{v}_0, \phi)_T + \langle (\mathbf{v}_b - \mathbf{v}_0) \cdot \mathbf{n}, \phi \rangle_{\partial T},$$

for any $\phi \in P_{r_2}(T)$.

3. WEAK GALERKIN ALGORITHMS WITHOUT STABILIZERS

Let \mathcal{T}_h be a finite element partition of the domain $\Omega \subset \mathbb{R}^d$ into polytopal elements, where \mathcal{T}_h is assumed to be shape-regular as defined in [52]. Denote by \mathcal{E}_h the set of all edges/faces in \mathcal{T}_h , and let $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ be the set of interior edges/faces. The diameter of an element $T \in \mathcal{T}_h$ is denoted by h_T , and the mesh size of the partition is given by $h = \max_{T \in \mathcal{T}_h} h_T$.

For each element $T \in \mathcal{T}_h$, we define the local weak finite element space as:

$$(3.1) \quad V(k, T) = \{ \{ \mathbf{v}_0, \mathbf{v}_b \} : \mathbf{v}_0 \in [P_k(T)]^d, \mathbf{v}_b \in [P_k(e)]^d \}.$$

By assembling $V(k, T)$ over all the elements $T \in \mathcal{T}_h$ and enforcing continuity on the interior interfaces \mathcal{E}_h^0 , we define the global weak finite element space:

$$(3.2) \quad V_h = \{ \{ \mathbf{v}_0, \mathbf{v}_b \} : \{ \mathbf{v}_0, \mathbf{v}_b \}|_T \in V(k, T), \forall T \in \mathcal{T}_h \}.$$

Additionally, we denote by V_h^0 the subspace of V_h with vanishing boundary value on $\partial\Omega$:

$$(3.3) \quad V_h^0 = \{ \{ \mathbf{v}_0, \mathbf{v}_b \} \in V_h : \mathbf{v}_b = 0 \text{ on } \partial\Omega \}.$$

For simplicity, the discrete weak strain tensor $\epsilon_{w,r_1,T}\mathbf{v}$ and the discrete weak divergence $\nabla_{w,r_2,T} \cdot \mathbf{v}$ are denoted by $\epsilon_w \mathbf{v}$ and $\nabla_w \cdot \mathbf{v}$, respectively. These quantities are computed locally on each element T using definitions (2.1) and (2.3):

$$\begin{aligned} (\epsilon_w \mathbf{v})|_T &= \epsilon_{w,r_1,T}(\mathbf{v}|_T), & \forall T \in \mathcal{T}_h, \\ (\nabla_w \cdot \mathbf{v})|_T &= \nabla_{w,r_2,T} \cdot (\mathbf{v}|_T), & \forall T \in \mathcal{T}_h. \end{aligned}$$

The WG numerical scheme without stabilizers, based on the weak formulation (1.2) for the elasticity problem (1.1), is as follows:

Simplified Weak Galerkin Algorithm 3.1. Find $\mathbf{u}_h = \{ \mathbf{u}_0, \mathbf{u}_b \} \in V_h$ such that $\mathbf{u}_b = Q_b \mathbf{g}$ on $\partial\Omega$ and

$$(3.4) \quad \sum_{T \in \mathcal{T}_h} 2(\mu \epsilon_w(\mathbf{u}_h), \epsilon_w(\mathbf{v}))_T + (\lambda \nabla_w \cdot \mathbf{u}_h, \nabla_w \cdot \mathbf{v})_T = \sum_{T \in \mathcal{T}_h} (\mathbf{f}, \mathbf{v}_0)_T,$$

for all $\mathbf{v} = \{ \mathbf{v}_0, \mathbf{v}_b \} \in V_h^0$. Here Q_b denotes the L^2 projection operator onto the space $P_k(e)$.

This scheme directly solves the elasticity problem without introducing any stabilization terms.

4. SOLUTION EXISTENCE AND UNIQUENESS

To begin, we recall the essential trace inequalities. Given that \mathcal{T}_h is a shape-regular finite element partition of the domain Ω , the following trace inequality holds for any element $T \in \mathcal{T}_h$ and function $\phi \in H^1(T)$ [52]:

$$(4.1) \quad \|\phi\|_{\partial T}^2 \leq C(h_T^{-1} \|\phi\|_T^2 + h_T \|\nabla \phi\|_T^2).$$

For polynomials ϕ , a simplified trace inequality is used [52]:

$$(4.2) \quad \|\phi\|_{\partial T}^2 \leq C h_T^{-1} \|\phi\|_T^2.$$

Next, we define two norms that are crucial in the error analysis. For any $\mathbf{v} = \{ \mathbf{v}_0, \mathbf{v}_b \} \in V_h$, we define the following discrete energy norm

$$(4.3) \quad \|\mathbf{v}\| = \left(\sum_{T \in \mathcal{T}_h} (2\mu \epsilon_w(\mathbf{v}), \epsilon_w(\mathbf{v}))_T + (\lambda \nabla_w \cdot \mathbf{v}, \nabla_w \cdot \mathbf{v})_T \right)^{\frac{1}{2}},$$

and the following discrete H^1 semi-norm

$$(4.4) \quad \|\mathbf{v}\|_{1,h} = \left(\sum_{T \in \mathcal{T}_h} (2\mu\epsilon(\mathbf{v}_0), \epsilon(\mathbf{v}_0))_T + (\lambda \nabla \cdot \mathbf{v}_0, \nabla \cdot \mathbf{v}_0)_T + h_T^{-1} \|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T}^2 \right)^{\frac{1}{2}}.$$

Lemma 4.1. [36] *For $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h$, there exists a constant C such that*

$$\|\epsilon(\mathbf{v}_0)\|_T \leq C \|\epsilon_w(\mathbf{v})\|_T.$$

Proof. Let $T \in \mathcal{T}_h$ be a polytopal element with N edges/faces denoted by e_1, \dots, e_N . It is important to emphasize that the polytopal element T can be non-convex. For each edge/face e_i , we construct a linear equation $l_i(x)$ such that $l_i(x) = 0$ on e_i , defined as follows:

$$l_i(x) = \frac{1}{h_T} \overrightarrow{AX} \cdot \mathbf{n}_i,$$

where $A = (A_1, \dots, A_{d-1})$ is a given point on the edge/face e_i , $X = (x_1, \dots, x_{d-1})$ is an arbitrary point on the edge/face e_i , \mathbf{n}_i is the normal direction to the edge/face e_i , and h_T represents the size of the element T .

The bubble function of the element T can be defined as

$$\Phi_B = l_1^2(x) l_2^2(x) \cdots l_N^2(x) \in P_{2N}(T).$$

It is straightforward to verify that $\Phi_B = 0$ on the boundary ∂T . The function Φ_B can be scaled so that $\Phi_B(M) = 1$ where M represents the barycenter of the element T . Additionally, there exists a sub-domain $\hat{T} \subset T$ such that $\Phi_B \geq \rho_0$ for some constant $\rho_0 > 0$.

For $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h$, let $r_1 = 2N + k - 1$ and set $\boldsymbol{\varphi} = \Phi_B \epsilon(\mathbf{v}_0) \in [P_{r_1}(T)]^{d \times d}$ in (2.2). We then obtain:

$$(4.5) \quad \begin{aligned} & (\epsilon_w(\mathbf{v}), \Phi_B \epsilon(\mathbf{v}_0))_T \\ &= (\epsilon(\mathbf{v}_0), \Phi_B \epsilon(\mathbf{v}_0))_T + \langle \mathbf{v}_b - \mathbf{v}_0, \frac{1}{2} (\Phi_B \epsilon(\mathbf{v}_0) + \Phi_B \epsilon(\mathbf{v}_0)^T) \cdot \mathbf{n} \rangle_{\partial T} \\ &= (\epsilon(\mathbf{v}_0), \Phi_B \epsilon(\mathbf{v}_0))_T, \end{aligned}$$

where we used $\Phi_B = 0$ on ∂T .

From the domain inverse inequality [52], there exists a constant C such that

$$(4.6) \quad (\epsilon(\mathbf{v}_0), \Phi_B \epsilon(\mathbf{v}_0))_T \geq C (\epsilon(\mathbf{v}_0), \epsilon(\mathbf{v}_0))_T.$$

Using the Cauchy-Schwarz inequality along with (4.5)-(4.6), we have

$$(\epsilon(\mathbf{v}_0), \epsilon(\mathbf{v}_0))_T \leq C (\epsilon_w(\mathbf{v}), \Phi_B \epsilon(\mathbf{v}_0))_T \leq C \|\epsilon_w(\mathbf{v})\|_T \|\Phi_B \epsilon(\mathbf{v}_0)\|_T \leq C \|\epsilon_w(\mathbf{v})\|_T \|\epsilon(\mathbf{v}_0)\|_T,$$

which implies

$$\|\epsilon(\mathbf{v}_0)\|_T \leq C \|\epsilon_w(\mathbf{v})\|_T.$$

This completes the proof of the lemma. \square

Lemma 4.2. *For $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h$, there exists a constant C such that*

$$\|\nabla \cdot \mathbf{v}_0\|_T \leq C \|\nabla_w \cdot \mathbf{v}\|_T.$$

Proof. The proof follows the same approach as in Lemma 4.1 \square

Remark 4.1. If the polytopal element T is convex, the bubble function in Lemma 4.1 can be simplified to

$$\Phi_B = l_1(x)l_2(x) \cdots l_N(x).$$

It can be verified that there exists a sub-domain $\hat{T} \subset T$, such that $\Phi_B \geq \rho_0$ for some constant $\rho_0 > 0$, and $\Phi_B = 0$ on the boundary ∂T . Lemmas 4.1, 4.2 can be proved in the same manner using this simplified construction. In this case, we take $r_1 = N + k - 1$ and $r_2 = N + k - 1$.

Recall that T is a d -dimensional polytopal element and e_i is a $(d-1)$ -dimensional edge/face of T . We construct an edge/face-based bubble function

$$\varphi_{e_i} = \prod_{k=1, \dots, N, k \neq i} l_k^2(x).$$

It can be verified that (1) $\varphi_{e_i} = 0$ on the edge/face e_k for $k \neq i$, (2) there exists a subdomain $\hat{e}_i \subset e_i$ such that $\varphi_{e_i} \geq \rho_1$ for some constant $\rho_1 > 0$.

Lemma 4.3. For $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h$, let $\boldsymbol{\varphi} = (\mathbf{v}_b - \mathbf{v}_0)\mathbf{n}^T \varphi_{e_i}$, where \mathbf{n} is the unit outward normal direction to the edge/face e_i . The following inequality holds:

$$(4.7) \quad \|\boldsymbol{\varphi}\|_T^2 \leq Ch_T \int_{e_i} ((\mathbf{v}_b - \mathbf{v}_0)\mathbf{n}^T)^2 ds.$$

Proof. We first extend \mathbf{v}_b , defined on the $(d-1)$ -dimensional edge/face e_i , to the entire d -dimensional polytopal element T and claim that \mathbf{v}_b remains a polynomial vector on T after extension. Next, let \mathbf{v}_{trace} denote the trace of \mathbf{v}_0 on e_i and extend \mathbf{v}_{trace} to the entire element T . Like \mathbf{v}_b , \mathbf{v}_{trace} remains a polynomial after the extension. For details on these extensions, see [36, 37].

Now, define $\boldsymbol{\varphi} = (\mathbf{v}_b - \mathbf{v}_0)\mathbf{n}^T \varphi_{e_i}$. We have

$$\begin{aligned} \|\boldsymbol{\varphi}\|_T^2 &= \int_T \boldsymbol{\varphi}^2 dT = \int_T ((\mathbf{v}_b - \mathbf{v}_{trace})\mathbf{n}^T \varphi_{e_i})^2 dT \\ &\leq Ch_T \int_{e_i} ((\mathbf{v}_b - \mathbf{v}_{trace})\mathbf{n}^T \varphi_{e_i})^2 ds \\ &\leq Ch_T \int_{e_i} ((\mathbf{v}_b - \mathbf{v}_0)\mathbf{n}^T)^2 ds, \end{aligned}$$

where we used (1) $\varphi_{e_i} = 0$ on the edge/face e_k for $k \neq i$, (2) there exists a subdomain $\hat{e}_i \subset e_i$ such that $\varphi_{e_i} \geq \rho_1$ for some constant $\rho_1 > 0$, and applied the properties of the projection.

This completes the proof of the lemma. □

Lemma 4.4. For $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h$, let $\phi = (\mathbf{v}_b - \mathbf{v}_0) \cdot \mathbf{n} \varphi_{e_i}$, where \mathbf{n} is the unit outward normal direction to the edge/face e_i . The following inequality holds:

$$(4.8) \quad \|\phi\|_T^2 \leq Ch_T \int_{e_i} (\mathbf{v}_b - \mathbf{v}_0)^2 ds.$$

Proof. This proof follows similarly to Lemma 4.3. □

Lemma 4.5. *There exists positive constants C_1 and C_2 such that for any $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h$, we have*

$$(4.9) \quad C_1 \|\mathbf{v}\|_{1,h} \leq \|\mathbf{v}\| \leq C_2 \|\mathbf{v}\|_{1,h}.$$

Proof. Note that the polytopal element T can be non-convex. Recall that an edge/face-based bubble function is defined as

$$\varphi_{e_i} = \prod_{k=1, \dots, N, k \neq i} l_k^2(x).$$

We extend \mathbf{v}_b from e_i to T , and similarly extend \mathbf{v}_{trace} (trace of \mathbf{v}_0 on e_i) to T . We denote these extensions as \mathbf{v}_b and \mathbf{v}_0 for simplicity. Details of the extensions can be found in Lemma 4.3 and 36, 37.

By choosing $\boldsymbol{\varphi} = (\mathbf{v}_b - \mathbf{v}_0) \mathbf{n}^T \varphi_{e_i}$ in (2.2), we have:

$$(4.10) \quad \begin{aligned} (\epsilon_w(\mathbf{v}), \boldsymbol{\varphi})_T &= (\epsilon(\mathbf{v}_0), \boldsymbol{\varphi})_T + \langle \mathbf{v}_b - \mathbf{v}_0, \frac{1}{2}(\boldsymbol{\varphi} + \boldsymbol{\varphi}^T) \cdot \mathbf{n} \rangle_{\partial T} \\ &= (\epsilon(\mathbf{v}_0), \boldsymbol{\varphi})_T + \int_{e_i} |\mathbf{v}_b - \mathbf{v}_0|^2 \varphi_{e_i} ds, \end{aligned}$$

where we used $\varphi_{e_i} = 0$ on e_k for $k \neq i$ and the fact that $\varphi_{e_i} \geq \rho_1$ for some constant $\rho_1 > 0$ in a subdomain $\hat{e}_i \subset e_i$.

Using Cauchy-Schwarz inequality, (4.10), the domain inverse inequality 52, and Lemma 4.3 gives

$$\begin{aligned} \int_{e_i} |\mathbf{v}_b - \mathbf{v}_0|^2 ds &\leq C \int_{e_i} |\mathbf{v}_b - \mathbf{v}_0|^2 \varphi_{e_i} ds \\ &\leq C(\|\epsilon_w(\mathbf{v})\|_T + \|\epsilon(\mathbf{v}_0)\|_T) \|\boldsymbol{\varphi}\|_T \\ &\leq Ch_T^{\frac{1}{2}}(\|\epsilon_w(\mathbf{v})\|_T + \|\epsilon(\mathbf{v}_0)\|_T) \left(\int_{e_i} ((\mathbf{v}_0 - \mathbf{v}_b))^2 ds \right)^{\frac{1}{2}}, \end{aligned}$$

which, from Lemma 4.1, gives

$$(4.11) \quad h_T^{-1} \int_{e_i} |\mathbf{v}_b - \mathbf{v}_0|^2 ds \leq C(\|\epsilon_w(\mathbf{v})\|_T^2 + \|\epsilon(\mathbf{v}_0)\|_T^2) \leq C\|\epsilon_w(\mathbf{v})\|_T^2.$$

Choosing $\phi = (\mathbf{v}_b - \mathbf{v}_0) \cdot \mathbf{n} \varphi_{e_i}$ in (2.4), gives

$$(4.12) \quad \begin{aligned} (\nabla_w \cdot \mathbf{v}, \phi)_T &= (\nabla \cdot \mathbf{v}_0, \phi)_T + \langle (\mathbf{v}_b - \mathbf{v}_0) \cdot \mathbf{n}, \phi \rangle_{\partial T} \\ &= (\nabla \cdot \mathbf{v}_0, \phi)_T + \int_{e_i} |\mathbf{v}_b - \mathbf{v}_0|^2 \varphi_{e_i} ds, \end{aligned}$$

where we used (1) $\varphi_{e_i} = 0$ on the edge/face e_k for $k \neq i$, (2) there exists a subdomain $\hat{e}_i \subset e_i$ such that $\varphi_{e_i} \geq \rho_1$ for some constant $\rho_1 > 0$.

Using Cauchy-Schwarz inequality, (4.12), the domain inverse inequality 52, and Lemma 4.4 gives

$$\begin{aligned} \int_{e_i} |\mathbf{v}_b - \mathbf{v}_0|^2 ds &\leq C \int_{e_i} |\mathbf{v}_b - \mathbf{v}_0|^2 \varphi_{e_i} ds \\ &\leq C(\|\nabla_w \cdot \mathbf{v}\|_T + \|\nabla \cdot \mathbf{v}_0\|_T) \|\phi\|_T \\ &\leq Ch_T^{\frac{1}{2}}(\|\nabla_w \cdot \mathbf{v}\|_T + \|\nabla \cdot \mathbf{v}_0\|_T) \left(\int_{e_i} (\mathbf{v}_0 - \mathbf{v}_b)^2 ds \right)^{\frac{1}{2}}, \end{aligned}$$

which, from Lemma 4.2, gives

$$(4.13) \quad h_T^{-1} \int_{e_i} |\mathbf{v}_b - \mathbf{v}_0|^2 ds \leq C(\|\nabla_w \cdot \mathbf{v}\|_T^2 + \|\nabla \cdot \mathbf{v}_0\|_T^2) \leq C\|\nabla_w \cdot \mathbf{v}\|_T^2.$$

This, together with (4.11), (4.13), Lemmas 4.1, 4.2, (4.3) and (4.4), gives

$$C_1 \|\mathbf{v}\|_{1,h} \leq \|\mathbf{v}\|.$$

Next, from (2.2), Cauchy-Schwarz inequality and the trace inequality (4.2), we conclude:

$$\left| (\epsilon_w(\mathbf{v}), \boldsymbol{\varphi})_T \right| \leq \|\epsilon(\mathbf{v}_0)\|_T \|\boldsymbol{\varphi}\|_T + Ch_T^{-\frac{1}{2}} \|\mathbf{v}_b - \mathbf{v}_0\|_{\partial T} \|(\boldsymbol{\varphi} + \boldsymbol{\varphi}^T) \cdot \mathbf{n}\|_T,$$

which yields

$$(4.14) \quad \|\epsilon_w(\mathbf{v})\|_T^2 \leq C(\|\epsilon(\mathbf{v}_0)\|_T^2 + h_T^{-1} \|\mathbf{v}_b - \mathbf{v}_0\|_{\partial T}^2).$$

From (2.4), Cauchy-Schwarz inequality and the trace inequality (4.2), we have

$$\left| (\nabla_w \cdot \mathbf{v}, \phi)_T \right| \leq \|\nabla \cdot \mathbf{v}_0\|_T \|\phi\|_T + Ch_T^{-\frac{1}{2}} \|(\mathbf{v}_b - \mathbf{v}_0) \cdot \mathbf{n}\|_{\partial T} \|\phi\|_T,$$

which yields

$$(4.15) \quad \|\nabla_w \cdot \mathbf{v}\|_T^2 \leq C(\|\nabla \cdot \mathbf{v}_0\|_T^2 + h_T^{-1} \|\mathbf{v}_b - \mathbf{v}_0\|_{\partial T}^2),$$

Using (4.14)-(4.15), (4.3) and (4.4), gives

$$\|\mathbf{v}\| \leq C_2 \|\mathbf{v}\|_{1,h}.$$

This completes the proof of the lemma. \square

Remark 4.2. If the polytopal element T is convex, the edge/face-based bubble function in Lemmas 4.3, 4.5 simplifies to

$$\varphi_{e_i} = \Pi_{k=1, \dots, N, k \neq i} l_k(x).$$

It can be verified that (1) $\varphi_{e_i} = 0$ on the edge/face e_k for $k \neq i$, (2) there exists a subdomain $\hat{e}_i \subset e_i$ such that $\varphi_{e_i} \geq \rho_1$ for some constant $\rho_1 > 0$. Lemmas 4.3, 4.5 can be derived similarly using this simplified construction.

Remark 4.3. For any d -dimensional polytopal element T , there exists a hyperplane $H \subset R^d$ such that a finite number l of distinct $(d-1)$ -dimensional edges/faces containing e_i are contained within H . In such cases, Lemmas 4.3, 4.5 can be proved with additional techniques. For more details, see [36, 37], which can be generalized to Lemmas 4.3, 4.5.

Lemma 4.6. [56, 47] (Second Korn's Inequality) Let Ω be a connected, open, bounded domain with a Lipschitz continuous boundary. Assume $\Gamma_1 \subset \partial\Omega$ is a nontrivial portion of the boundary $\partial\Omega$, with dimension $d-1$. For any fixed real number $1 \leq p < \infty$, there exists a constant C such that

$$(4.16) \quad \|\mathbf{v}\|_1 \leq C(\|\epsilon(\mathbf{v})\|_0 + \|\mathbf{v}\|_{L^p(\Gamma_1)}),$$

for any $\mathbf{v} \in [H^1(\Omega)]^d$.

Theorem 4.7. The Weak Galerkin Algorithm without Stabilizers [3, 1] has a unique solution.

Proof. Since the number of equations equals the number of unknowns in (3.4), it suffices to prove the uniqueness of the solution. Assume $\mathbf{u}_h^{(1)} \in V_h$ and $\mathbf{u}_h^{(2)} \in V_h$ are two distinct solutions of the WG Algorithm 3.1. Define $\boldsymbol{\eta}_h = \mathbf{u}_h^{(1)} - \mathbf{u}_h^{(2)} \in V_h^0$. Then, $\boldsymbol{\eta}_h$ satisfies the equation

$$\sum_{T \in \mathcal{T}_h} 2(\mu \epsilon_w(\boldsymbol{\eta}_h), \epsilon_w(\mathbf{v}))_T + (\lambda \nabla_w \cdot \boldsymbol{\eta}_h, \nabla_w \cdot \mathbf{v})_T = 0.$$

By setting $\mathbf{v} = \boldsymbol{\eta}_h$, we obtain $\|\boldsymbol{\eta}_h\| = 0$. Using the fact that $\|\boldsymbol{\eta}_h\| = 0$ and (4.9), we have

$$\|\boldsymbol{\eta}_h\|_{1,h} = 0.$$

This implies that $\epsilon(\boldsymbol{\eta}_0) = 0$ and $\nabla \cdot \boldsymbol{\eta}_0 = 0$ on each element T , and that $\boldsymbol{\eta}_0 = \boldsymbol{\eta}_b$ on each ∂T .

Since $\epsilon(\boldsymbol{\eta}_0) = 0$ on each element T , it follows that $\boldsymbol{\eta}_0 \in RM(T) \subset [P_1(T)]^d$. Consequently, $\boldsymbol{\eta}_0 = \boldsymbol{\eta}_b$ on each ∂T and $\boldsymbol{\eta}_0$ is continuous across the domain Ω . Using the fact that $\boldsymbol{\eta}_b = 0$ on $\partial\Omega$, we conclude that $\boldsymbol{\eta}_0 = 0$ on $\partial\Omega$. Applying the second Korn's inequality (4.16), we conclude that $\boldsymbol{\eta}_0 \equiv 0$ in Ω . Since $\boldsymbol{\eta}_0 = \boldsymbol{\eta}_b$ on each ∂T , we have $\boldsymbol{\eta}_b \equiv 0$ in Ω . Therefore, $\mathbf{u}_h^{(1)} = \mathbf{u}_h^{(2)}$.

Finally, note that this result holds for any $\lambda \geq 0$, completing the proof. \square

5. ERROR EQUATIONS

On each element $T \in \mathcal{T}_h$, let Q_0 denote the L^2 projection onto $P_k(T)$. On each edge/face $e \subset \partial T$, recall that Q_b is the L^2 projection operator onto $P_k(e)$. For any $\mathbf{w} \in [H^1(\Omega)]^d$, we define the L^2 projection into the weak finite element space V_h , denote by $Q_h \mathbf{w}$, such that

$$(Q_h \mathbf{w})|_T := \{Q_0(\mathbf{w}|_T), Q_b(\mathbf{w}|_{\partial T})\}, \quad \forall T \in \mathcal{T}_h.$$

We also denote by Q_{r_1} and Q_{r_2} the L^2 projection operators onto the finite element spaces of piecewise polynomials of degrees r_1 and r_2 respectively.

Lemma 5.1. *The following property holds:*

$$(5.1) \quad \epsilon_w(\mathbf{w}) = Q_{r_1} \epsilon(\mathbf{w}), \quad \forall \mathbf{w} \in [H^1(T)]^d.$$

Proof. For any $\mathbf{w} \in [H^1(T)]^d$, using (2.2), we obtain

$$\begin{aligned} (\epsilon_w(\mathbf{w}), \boldsymbol{\varphi})_T &= (\epsilon(\mathbf{w}), \boldsymbol{\varphi})_T + \langle \mathbf{w}|_{\partial T} - \mathbf{w}|_T, \frac{1}{2}(\boldsymbol{\varphi} + \boldsymbol{\varphi}^T) \cdot \mathbf{n} \rangle_{\partial T} \\ &= (\epsilon(\mathbf{w}), \boldsymbol{\varphi})_T \\ &= (Q_{r_1} \epsilon(\mathbf{w}), \boldsymbol{\varphi})_T, \end{aligned}$$

for any $\boldsymbol{\varphi} \in [P_{r_1}(T)]^{d \times d}$. This completes the proof of this lemma. \square

Lemma 5.2. *The following property holds:*

$$(5.2) \quad \nabla_w \cdot \mathbf{w} = Q_{r_2}(\nabla \cdot \mathbf{w}), \quad \forall \mathbf{w} \in [H^1(T)]^d.$$

Proof. The proof of this lemma follows similarly to that of Lemma 5.1. \square

Let \mathbf{u} and $\mathbf{u}_h \in V_h$ denote the exact solution of the elasticity problem (1.1) and its numerical approximation arising from the WG Algorithm 3.1 respectively. We define the error function \mathbf{e}_h as follows

$$(5.3) \quad \mathbf{e}_h = \mathbf{u} - \mathbf{u}_h \in V_h^0.$$

Lemma 5.3. *The error function \mathbf{e}_h given in (5.3) satisfies the following error equation:*

$$(5.4) \quad \sum_{T \in \mathcal{T}_h} (2\mu \epsilon_w(\mathbf{e}_h), \epsilon_w(\mathbf{v}))_T + (\lambda \nabla_w \cdot \mathbf{e}_h, \nabla_w \cdot \mathbf{v})_T = \ell(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{v} \in V_h^0,$$

where

$$\ell(\mathbf{u}, \mathbf{v}) = \sum_{T \in \mathcal{T}_h} \langle \mathbf{v}_b - \mathbf{v}_0, 2\mu(Q_{r_1} - I)\epsilon(\mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T} + \langle (\mathbf{v}_b - \mathbf{v}_0) \cdot \mathbf{n}, \lambda(Q_{r_2} - I)(\nabla \cdot \mathbf{u}) \rangle_{\partial T}.$$

Proof. Using (5.1), (5.2), and letting $\boldsymbol{\varphi} = Q_{r_1}\epsilon(\mathbf{u})$ in (2.2) and $\phi = Q_{r_2}(\nabla \cdot \mathbf{u})$ in (2.4), gives

$$\begin{aligned} & \sum_{T \in \mathcal{T}_h} (2\mu \epsilon_w(\mathbf{u}), \epsilon_w(\mathbf{v}))_T + (\lambda \nabla_w \cdot \mathbf{u}, \nabla_w \cdot \mathbf{v})_T \\ &= \sum_{T \in \mathcal{T}_h} (2\mu Q_{r_1}\epsilon(\mathbf{u}), \epsilon_w \mathbf{v})_T + (\lambda Q_{r_2}(\nabla \cdot \mathbf{u}), \nabla_w \cdot \mathbf{v})_T \\ &= \sum_{T \in \mathcal{T}_h} (2\mu \epsilon(\mathbf{v}_0), Q_{r_1}\epsilon(\mathbf{u}))_T + \langle 2\mu(\mathbf{v}_b - \mathbf{v}_0), \frac{1}{2}(Q_{r_1}\epsilon(\mathbf{u}) + Q_{r_1}\epsilon(\mathbf{u})^T) \cdot \mathbf{n} \rangle_{\partial T} \\ & \quad + (\lambda \nabla \cdot \mathbf{v}_0, Q_{r_2}(\nabla \cdot \mathbf{u}))_T + \langle \lambda(\mathbf{v}_b - \mathbf{v}_0) \cdot \mathbf{n}, Q_{r_2}(\nabla \cdot \mathbf{u}) \rangle_{\partial T} \\ (5.5) \quad &= \sum_{T \in \mathcal{T}_h} (2\mu \epsilon(\mathbf{v}_0), \epsilon(\mathbf{u}))_T + \langle 2\mu(\mathbf{v}_b - \mathbf{v}_0), Q_{r_1}\epsilon(\mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T} \\ & \quad + (\lambda \nabla \cdot \mathbf{v}_0, \nabla \cdot \mathbf{u})_T + \langle \lambda(\mathbf{v}_b - \mathbf{v}_0) \cdot \mathbf{n}, Q_{r_2}(\nabla \cdot \mathbf{u}) \rangle_{\partial T} \\ &= \sum_{T \in \mathcal{T}_h} (\mathbf{f}, \mathbf{v}_0) + \langle 2\mu \epsilon(\mathbf{u}) \cdot \mathbf{n}, \mathbf{v}_0 \rangle_{\partial T} + \langle \lambda \nabla \cdot \mathbf{u}, \mathbf{v}_0 \cdot \mathbf{n} \rangle_{\partial T} \\ & \quad + \langle \mathbf{v}_b - \mathbf{v}_0, 2\mu Q_{r_1}\epsilon(\mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T} + \langle (\mathbf{v}_b - \mathbf{v}_0) \cdot \mathbf{n}, \lambda Q_{r_2}(\nabla \cdot \mathbf{u}) \rangle_{\partial T} \\ &= \sum_{T \in \mathcal{T}_h} (\mathbf{f}, \mathbf{v}_0) + \langle \mathbf{v}_b - \mathbf{v}_0, 2\mu(Q_{r_1} - I)\epsilon(\mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T} \\ & \quad + \langle (\mathbf{v}_b - \mathbf{v}_0) \cdot \mathbf{n}, \lambda(Q_{r_2} - I)(\nabla \cdot \mathbf{u}) \rangle_{\partial T}, \end{aligned}$$

where we used (1.1), the usual integration by parts, and the fact that $\sum_{T \in \mathcal{T}_h} \langle 2\mu \epsilon(\mathbf{u}) \cdot \mathbf{n}, \mathbf{v}_b \rangle_{\partial T} = \langle 2\mu \epsilon(\mathbf{u}) \cdot \mathbf{n}, \mathbf{v}_b \rangle_{\partial \Omega} = 0$ and $\sum_{T \in \mathcal{T}_h} \langle \lambda \nabla \cdot \mathbf{u}, \mathbf{v}_b \cdot \mathbf{n} \rangle_{\partial T} = \langle \lambda \nabla \cdot \mathbf{u}, \mathbf{v}_b \cdot \mathbf{n} \rangle_{\partial \Omega} = 0$ since $\mathbf{v}_b = 0$ on $\partial \Omega$.

Finally, subtracting (3.4) from (5.5) completes the proof of the lemma. \square

6. ERROR ESTIMATES

Lemma 6.1. [52] *Let \mathcal{T}_h be a finite element partition of the domain Ω that satisfies the shape regular assumption outlined in [52]. For any $0 \leq s \leq 1$, $0 \leq m \leq k + 1$,*

and $0 \leq n \leq k$, there holds

$$(6.1) \quad \sum_{T \in \mathcal{T}_h} h_T^{2s} \|\epsilon(\mathbf{u}) - Q_{r_1} \epsilon(\mathbf{u})\|_{s,T}^2 \leq Ch^{2m-2} \|\mathbf{u}\|_m^2,$$

$$(6.2) \quad \sum_{T \in \mathcal{T}_h} h_T^{2s} \|\nabla \cdot \mathbf{u} - Q_{r_2} \nabla \cdot \mathbf{u}\|_{s,T}^2 \leq Ch^{2m-2} \|\mathbf{u}\|_m^2,$$

$$(6.3) \quad \sum_{T \in \mathcal{T}_h} h_T^{2s} \|\mathbf{u} - Q_0 \mathbf{u}\|_{s,T}^2 \leq Ch^{2n+2} \|\mathbf{u}\|_{n+1}^2.$$

Lemma 6.2. Assume the exact solution \mathbf{u} of the elasticity problem (1.1) is sufficiently regular, such that $\mathbf{u} \in [H^{k+1}(\Omega)]^d$. Then, there exists a constant C such that the following estimate holds:

$$(6.4) \quad \|\mathbf{u} - Q_h \mathbf{u}\| \leq Ch^k \|\mathbf{u}\|_{k+1}.$$

Proof. Recall that μ and λ are the Lamé constants. Using (2.2), the Cauchy-Schwarz inequality, the trace inequalities (4.1)-(4.2), and the estimate (6.3) with $n = k$ and $s = 0, 1$, we obtain:

$$\begin{aligned} & \sum_{T \in \mathcal{T}_h} (2\mu \epsilon_w(\mathbf{u} - Q_h \mathbf{u}), \boldsymbol{\varphi})_T \\ &= \sum_{T \in \mathcal{T}_h} (2\mu \epsilon(\mathbf{u} - Q_0 \mathbf{u}), \boldsymbol{\varphi})_T + \langle 2\mu(Q_0 \mathbf{u} - Q_b \mathbf{u}), \frac{1}{2}(\boldsymbol{\varphi} + \boldsymbol{\varphi}^T) \cdot \mathbf{n} \rangle_{\partial T} \\ &\leq \left(\sum_{T \in \mathcal{T}_h} \|2\mu \epsilon(\mathbf{u} - Q_0 \mathbf{u})\|_T^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \|\boldsymbol{\varphi}\|_T^2 \right)^{\frac{1}{2}} \\ &\quad + \left(\sum_{T \in \mathcal{T}_h} \|2\mu(Q_0 \mathbf{u} - Q_b \mathbf{u})\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \|\boldsymbol{\varphi}\|_{\partial T}^2 \right)^{\frac{1}{2}} \\ &\leq C \left(\sum_{T \in \mathcal{T}_h} \|\epsilon(\mathbf{u} - Q_0 \mathbf{u})\|_T^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \|\boldsymbol{\varphi}\|_T^2 \right)^{\frac{1}{2}} \\ &\quad + C \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_0 \mathbf{u} - \mathbf{u}\|_T^2 + h_T \|Q_0 \mathbf{u} - \mathbf{u}\|_{1,T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|\boldsymbol{\varphi}\|_T^2 \right)^{\frac{1}{2}} \\ &\leq Ch^k \|\mathbf{u}\|_{k+1} \left(\sum_{T \in \mathcal{T}_h} \|\boldsymbol{\varphi}\|_T^2 \right)^{\frac{1}{2}}, \end{aligned}$$

for any $\boldsymbol{\varphi} \in [P_{r_1}(T)]^{d \times d}$.

Letting $\boldsymbol{\varphi} = \epsilon_w(\mathbf{u} - Q_h \mathbf{u})$ yields

$$(6.5) \quad \begin{aligned} & \sum_{T \in \mathcal{T}_h} (2\mu \epsilon_w(\mathbf{u} - Q_h \mathbf{u}), \epsilon_w(\mathbf{u} - Q_h \mathbf{u}))_T \\ & \leq Ch^k \|\mathbf{u}\|_{k+1} \left(\sum_{T \in \mathcal{T}_h} \|\epsilon_w(\mathbf{u} - Q_h \mathbf{u})\|_T^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Using (2.4), the Cauchy-Schwarz inequality, the trace inequalities (4.1)-(4.2), and the estimate (6.3) with $n = k$ and $s = 0, 1$, we have

$$\begin{aligned}
& \sum_{T \in \mathcal{T}_h} (\lambda \nabla_w \cdot (\mathbf{u} - Q_h \mathbf{u}), \phi)_T \\
&= \sum_{T \in \mathcal{T}_h} (\lambda \nabla \cdot (\mathbf{u} - Q_0 \mathbf{u}), \phi)_T + \langle \lambda (Q_0 \mathbf{u} - Q_b \mathbf{u}) \cdot \mathbf{n}, \phi \rangle_{\partial T} \\
&\leq \left(\sum_{T \in \mathcal{T}_h} \|\lambda \nabla \cdot (\mathbf{u} - Q_0 \mathbf{u})\|_T^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \|\phi\|_T^2 \right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{T \in \mathcal{T}_h} \|\lambda (Q_0 \mathbf{u} - Q_b \mathbf{u}) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \|\phi\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\leq C \left(\sum_{T \in \mathcal{T}_h} \|\nabla \cdot (\mathbf{u} - Q_0 \mathbf{u})\|_T^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \|\phi\|_T^2 \right)^{\frac{1}{2}} \\
&\quad + C \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_0 \mathbf{u} - \mathbf{u}\|_T^2 + h_T \|Q_0 \mathbf{u} - \mathbf{u}\|_{1,T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|\phi\|_T^2 \right)^{\frac{1}{2}} \\
&\leq Ch^k \|\mathbf{u}\|_{k+1} \left(\sum_{T \in \mathcal{T}_h} \|\phi\|_T^2 \right)^{\frac{1}{2}},
\end{aligned}$$

for any $\phi \in P_{r_2}(T)$.

Letting $\phi = \nabla_w \cdot (\mathbf{u} - Q_h \mathbf{u})$ gives

$$\begin{aligned}
& \sum_{T \in \mathcal{T}_h} (\lambda \nabla_w \cdot (\mathbf{u} - Q_h \mathbf{u}), \nabla_w \cdot (\mathbf{u} - Q_h \mathbf{u}))_T \\
(6.6) \quad & \leq Ch^k \|\mathbf{u}\|_{k+1} \left(\sum_{T \in \mathcal{T}_h} \|\nabla_w \cdot (\mathbf{u} - Q_h \mathbf{u})\|_T^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Combining (6.5) and (6.6), we conclude that

$$\|\mathbf{u} - Q_h \mathbf{u}\|^2 \leq Ch^k \|\mathbf{u}\|_{k+1} \left(\left(\sum_{T \in \mathcal{T}_h} \|\epsilon_w(\mathbf{u} - Q_h \mathbf{u})\|_T^2 \right)^{\frac{1}{2}} + \left(\sum_{T \in \mathcal{T}_h} \|\nabla_w \cdot (\mathbf{u} - Q_h \mathbf{u})\|_T^2 \right)^{\frac{1}{2}} \right).$$

This completes the proof of the lemma. \square

Theorem 6.3. Assume the exact solution \mathbf{u} of the elasticity problem (1.1) is sufficiently regular such that $\mathbf{u} \in [H^{k+1}(\Omega)]^d$. Then, there exists a constant C , such that the following error estimate holds:

$$(6.7) \quad \|\mathbf{u} - \mathbf{u}_h\| \leq Ch^k \|\mathbf{u}\|_{k+1}.$$

Proof. For the right-hand side of the error equation (5.4), using the Cauchy-Schwarz inequality, the trace inequality (4.1), the estimates (6.1)-(6.2) with $m = k + 1$ and

$s = 0, 1$, and (4.9), we have

$$\begin{aligned}
& \left| \sum_{T \in \mathcal{T}_h} \langle \mathbf{v}_b - \mathbf{v}_0, 2\mu(Q_{r_1} - I)\epsilon(\mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T} \right. \\
& \quad \left. + \langle (\mathbf{v}_b - \mathbf{v}_0) \cdot \mathbf{n}, \lambda(Q_{r_2} - I)(\nabla \cdot \mathbf{u}) \rangle_{\partial T} \right| \\
& \leq C \left(\sum_{T \in \mathcal{T}_h} \|2\mu(Q_{r_1} - I)\epsilon(\mathbf{u}) \cdot \mathbf{n}\|_T^2 + h_T^2 \|2\mu(Q_{r_1} - I)\epsilon(\mathbf{u}) \cdot \mathbf{n}\|_{1,T}^2 \right)^{\frac{1}{2}} \\
& \quad \cdot \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|\mathbf{v}_b - \mathbf{v}_0\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \quad + C \left(\sum_{T \in \mathcal{T}_h} \|\lambda(Q_{r_2} - I)(\nabla \cdot \mathbf{u})\|_T^2 + h_T^2 \|\lambda(Q_{r_2} - I)(\nabla \cdot \mathbf{u})\|_{1,T}^2 \right)^{\frac{1}{2}} \\
& \quad \cdot \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|\mathbf{v}_b - \mathbf{v}_0\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
& \leq Ch^k \|\mathbf{u}\|_{k+1} \|\mathbf{v}\|_{1,h} \\
& \leq Ch^k \|\mathbf{u}\|_{k+1} \|\mathbf{v}\|.
\end{aligned} \tag{6.8}$$

Substituting (6.8) into (5.4) gives

$$\sum_{T \in \mathcal{T}_h} (2\mu\epsilon_w(\mathbf{e}_h), \epsilon_w(\mathbf{v}))_T + (\lambda \nabla_w \cdot \mathbf{e}_h, \nabla_w \cdot \mathbf{v})_T \leq Ch^k \|\mathbf{u}\|_{k+1} \|\mathbf{v}\|. \tag{6.9}$$

Now, applying the Cauchy-Schwarz inequality and letting $\mathbf{v} = Q_h \mathbf{u} - \mathbf{u}_h$ in (6.9), along with the estimate (6.4), we obtain

$$\begin{aligned}
& \|\mathbf{u} - \mathbf{u}_h\|^2 \\
& = \sum_{T \in \mathcal{T}_h} (2\mu\epsilon_w(\mathbf{u} - \mathbf{u}_h), \epsilon_w(\mathbf{u} - Q_h \mathbf{u}))_T + (2\mu\epsilon_w(\mathbf{u} - \mathbf{u}_h), \epsilon_w(Q_h \mathbf{u} - \mathbf{u}_h))_T \\
& \quad + (\lambda \nabla_w \cdot (\mathbf{u} - \mathbf{u}_h), \nabla_w \cdot (\mathbf{u} - Q_h \mathbf{u}))_T + (\lambda \nabla_w \cdot (\mathbf{u} - \mathbf{u}_h), \nabla_w \cdot (Q_h \mathbf{u} - \mathbf{u}_h))_T \\
& \leq \left(\sum_{T \in \mathcal{T}_h} 2\mu \|\epsilon_w(\mathbf{u} - \mathbf{u}_h)\|_T^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} 2\mu \|\epsilon_w(\mathbf{u} - Q_h \mathbf{u})\|_T^2 \right)^{\frac{1}{2}} \\
& \quad + \left(\sum_{T \in \mathcal{T}_h} \lambda \|\nabla_w \cdot (\mathbf{u} - \mathbf{u}_h)\|_T^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \lambda \|\nabla_w \cdot (\mathbf{u} - Q_h \mathbf{u})\|_T^2 \right)^{\frac{1}{2}} \\
& \quad + \sum_{T \in \mathcal{T}_h} (2\mu\epsilon_w(\mathbf{u} - \mathbf{u}_h), \epsilon_w(Q_h \mathbf{u} - \mathbf{u}_h))_T + (\lambda \nabla_w \cdot (\mathbf{u} - \mathbf{u}_h), \nabla_w \cdot (Q_h \mathbf{u} - \mathbf{u}_h))_T \\
& \leq \|\mathbf{u} - \mathbf{u}_h\| \|\mathbf{u} - Q_h \mathbf{u}\| + Ch^k \|\mathbf{u}\|_{k+1} \|Q_h \mathbf{u} - \mathbf{u}_h\| \\
& \leq \|\mathbf{u} - \mathbf{u}_h\| h^k \|\mathbf{u}\|_{k+1} + Ch^k \|\mathbf{u}\|_{k+1} (\|Q_h \mathbf{u} - \mathbf{u}\| + \|\mathbf{u} - \mathbf{u}_h\|) \\
& \leq \|\mathbf{u} - \mathbf{u}_h\| h^k \|\mathbf{u}\|_{k+1} + Ch^k \|\mathbf{u}\|_{k+1} h^k \|\mathbf{u}\|_{k+1} + Ch^k \|\mathbf{u}\|_{k+1} \|\mathbf{u} - \mathbf{u}_h\|.
\end{aligned}$$

This simplifies to

$$\|\mathbf{u} - \mathbf{u}_h\| \leq Ch^k \|\mathbf{u}\|_{k+1}.$$

Thus, the proof of the theorem is complete. \square

7. ERROR ESTIMATES IN L^2 NORM

The standard duality argument is applied to derive the L^2 error estimate. Recall that the error function $\mathbf{e}_h = \mathbf{u} - \mathbf{u}_h = \{\mathbf{e}_0, \mathbf{e}_b\}$ represents the difference between the exact solution \mathbf{u} and the computed solution \mathbf{u}_h . Additionally, we define $\zeta_h = Q_h \mathbf{u} - \mathbf{u}_h = \{\zeta_0, \zeta_b\}$.

The dual problem for the elasticity problem (1.1) seeks \mathbf{w} such that

$$(7.1) \quad \begin{aligned} -\nabla \cdot (2\mu\epsilon(\mathbf{w}) + \lambda(\nabla \cdot \mathbf{w})\mathbf{I}) &= \zeta_0, & \text{in } \Omega, \\ \mathbf{w} &= 0, & \text{on } \partial\Omega. \end{aligned}$$

Assume that the dual problem (7.1) satisfies the following regularity estimate:

$$(7.2) \quad \|\mathbf{w}\|_{1+\alpha} \leq C\|\zeta_0\|,$$

where $\frac{1}{2} < \alpha \leq 1$.

Theorem 7.1. *Let \mathbf{u} be the exact solution of the elasticity problem (1.1), assuming sufficient regularity such that $\mathbf{u} \in [H^{k+1}(\Omega)]^d$. Let $\mathbf{u}_h \in V_h$ be the numerical solution of the WG Algorithm 3.1. If the regularity assumption (7.2) holds for the dual problem (7.1), then there exists a constant C such that*

$$\|\mathbf{e}_0\| \leq Ch^{k+\alpha}\|\mathbf{u}\|_{k+1}.$$

Proof. Testing (7.1) with ζ_0 gives

$$(7.3) \quad \begin{aligned} \|\zeta_0\|^2 &= (-\nabla \cdot (2\mu\epsilon(\mathbf{w}) + \lambda(\nabla \cdot \mathbf{w})\mathbf{I}), \zeta_0) \\ &= \sum_{T \in \mathcal{T}_h} (2\mu\epsilon(\mathbf{w}) + \lambda(\nabla \cdot \mathbf{w})\mathbf{I}, \nabla \zeta_0)_T - \langle 2\mu\epsilon(\mathbf{w}) \cdot \mathbf{n}, \zeta_0 \rangle_{\partial T} \\ &\quad - \langle \lambda \nabla \cdot \mathbf{w}, \zeta_0 \cdot \mathbf{n} \rangle_{\partial T} \\ &= \sum_{T \in \mathcal{T}_h} (2\mu\epsilon(\mathbf{w}), \epsilon(\zeta_0))_T + (\lambda \nabla \cdot \mathbf{w}, \nabla \cdot \zeta_0)_T \\ &\quad - \langle 2\mu\epsilon(\mathbf{w}) \cdot \mathbf{n}, \zeta_0 - \zeta_b \rangle_{\partial T} - \langle \lambda \nabla \cdot \mathbf{w}, (\zeta_0 - \zeta_b) \cdot \mathbf{n} \rangle_{\partial T}, \end{aligned}$$

where integration by parts has been used along with the fact that $\sum_{T \in \mathcal{T}_h} \langle 2\mu\epsilon(\mathbf{w}) \cdot \mathbf{n}, \zeta_b \rangle_{\partial T} = \langle 2\mu\epsilon(\mathbf{w}) \cdot \mathbf{n}, \zeta_b \rangle_{\partial\Omega} = 0$ and $\sum_{T \in \mathcal{T}_h} \langle \lambda \nabla \cdot \mathbf{w}, \zeta_b \cdot \mathbf{n} \rangle_{\partial T} = \langle \lambda \nabla \cdot \mathbf{w}, \zeta_b \cdot \mathbf{n} \rangle_{\partial\Omega} = 0$ since $\zeta_b = Q_b \mathbf{u} - \mathbf{u}_b = 0$ on $\partial\Omega$.

Letting $\mathbf{u} = \mathbf{w}$ and $\mathbf{v} = \zeta_h$ in (5.5), we get:

$$\begin{aligned} &\sum_{T \in \mathcal{T}_h} (2\mu\epsilon_w(\mathbf{w}), \epsilon_w(\zeta_h))_T + (\lambda \nabla_w \cdot \mathbf{w}, \nabla_w \cdot \zeta_h)_T \\ &= \sum_{T \in \mathcal{T}_h} (2\mu\epsilon(\zeta_0), \epsilon(\mathbf{w}))_T + \langle 2\mu(\zeta_b - \zeta_0), Q_{r_1} \epsilon(\mathbf{w}) \cdot \mathbf{n} \rangle_{\partial T} \\ &\quad + (\lambda \nabla \cdot \zeta_0, \nabla \cdot \mathbf{w})_T + \langle \lambda(\zeta_b - \zeta_0) \cdot \mathbf{n}, Q_{r_2}(\nabla \cdot \mathbf{w}) \rangle_{\partial T}. \end{aligned}$$

This can be rewritten as:

$$\begin{aligned} &\sum_{T \in \mathcal{T}_h} (2\mu\epsilon(\zeta_0), \epsilon(\mathbf{w}))_T + \lambda(\nabla \cdot \zeta_0, \nabla \cdot \mathbf{w})_T \\ &= \sum_{T \in \mathcal{T}_h} (2\mu\epsilon_w(\mathbf{w}), \epsilon_w(\zeta_h))_T + (\lambda \nabla_w \cdot \mathbf{w}, \nabla_w \cdot \zeta_h)_T \\ &\quad - \langle 2\mu(\zeta_b - \zeta_0), Q_{r_1} \epsilon(\mathbf{w}) \cdot \mathbf{n} \rangle_{\partial T} - \langle \lambda(\zeta_b - \zeta_0) \cdot \mathbf{n}, Q_{r_2}(\nabla \cdot \mathbf{w}) \rangle_{\partial T}. \end{aligned}$$

Substituting this equation into (8.1) and using (5.4), we obtain:

$$\begin{aligned}
& \|\zeta_0\|^2 \\
&= \sum_{T \in \mathcal{T}_h} (2\mu\epsilon_w(\mathbf{w}), \epsilon_w(\zeta_h))_T + (\lambda \nabla_w \cdot \mathbf{w}, \nabla_w \cdot \zeta_h)_T \\
&\quad - \langle 2\mu(\zeta_b - \zeta_0), (Q_{r_1} - I)\epsilon(\mathbf{w}) \cdot \mathbf{n} \rangle_{\partial T} \\
&\quad - \lambda \langle (\zeta_b - \zeta_0) \cdot \mathbf{n}, (Q_{r_2} - I)(\nabla \cdot \mathbf{w}) \rangle_{\partial T} \\
&= \sum_{T \in \mathcal{T}_h} (2\mu\epsilon_w(\mathbf{w}), \epsilon_w(\mathbf{e}_h))_T + (\lambda \nabla_w \cdot \mathbf{w}, \nabla_w \cdot \mathbf{e}_h)_T \\
&\quad + (2\mu\epsilon_w(\mathbf{w}), \epsilon_w(Q_h \mathbf{u} - \mathbf{u}))_T + (\lambda \nabla_w \cdot \mathbf{w}, \nabla_w \cdot (Q_h \mathbf{u} - \mathbf{u}))_T \\
&\quad - \ell(\mathbf{w}, \zeta_h) \\
(7.4) \quad &= \sum_{T \in \mathcal{T}_h} (2\mu\epsilon_w(Q_h \mathbf{w}), \epsilon_w(\mathbf{e}_h))_T + (2\mu\epsilon_w(\mathbf{w} - Q_h \mathbf{w}), \epsilon_w(\mathbf{e}_h))_T \\
&\quad + (\lambda \nabla_w \cdot Q_h \mathbf{w}, \nabla_w \cdot \mathbf{e}_h)_T + (\lambda \nabla_w \cdot (\mathbf{w} - Q_h \mathbf{w}), \nabla_w \cdot \mathbf{e}_h)_T \\
&\quad + (2\mu\epsilon_w(\mathbf{w}), \epsilon_w(Q_h \mathbf{u} - \mathbf{u}))_T + (\lambda \nabla_w \cdot \mathbf{w}, \nabla_w \cdot (Q_h \mathbf{u} - \mathbf{u}))_T \\
&\quad - \ell(\mathbf{w}, \zeta_h) \\
&= \ell(\mathbf{u}, Q_h \mathbf{w}) + \sum_{T \in \mathcal{T}_h} (2\mu\epsilon_w(\mathbf{w} - Q_h \mathbf{w}), \epsilon_w(\mathbf{e}_h))_T \\
&\quad + (\lambda \nabla_w \cdot (\mathbf{w} - Q_h \mathbf{w}), \nabla_w \cdot \mathbf{e}_h)_T + (2\mu\epsilon_w(\mathbf{w}), \epsilon_w(Q_h \mathbf{u} - \mathbf{u}))_T \\
&\quad + (\lambda \nabla_w \cdot \mathbf{w}, \nabla_w \cdot (Q_h \mathbf{u} - \mathbf{u}))_T - \ell(\mathbf{w}, \zeta_h) \\
&= \sum_{i=1}^6 J_i.
\end{aligned}$$

We will estimate the six terms J_i for $i = 1, \dots, 6$ on the last line of (7.4) individually.

Regarding to J_1 , using the Cauchy-Schwarz inequality, the trace inequality (4.1), the estimate (6.1) with $m = k + 1$ and $s = 0, 1$, and the estimate (6.3) with

$n = \alpha$, we have:

$$\begin{aligned}
J_1 &= \ell(\mathbf{u}, Q_h \mathbf{w}) \\
&\leq \left| \sum_{T \in \mathcal{T}_h} \langle Q_b \mathbf{w} - Q_0 \mathbf{w}, 2\mu(Q_{r_1} - I)\epsilon(\mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T} \right| \\
&\quad + \left| \sum_{T \in \mathcal{T}_h} \langle (Q_b \mathbf{w} - Q_0 \mathbf{w}) \cdot \mathbf{n}, \lambda(Q_{r_2} - I)(\nabla \cdot \mathbf{u}) \rangle_{\partial T} \right| \\
&\leq \left(\sum_{T \in \mathcal{T}_h} \|Q_b \mathbf{w} - Q_0 \mathbf{w}\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \|2\mu(Q_{r_1} - I)\epsilon(\mathbf{u}) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{T \in \mathcal{T}_h} \|(Q_b \mathbf{w} - Q_0 \mathbf{w}) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \|\lambda(Q_{r_2} - I)(\nabla \cdot \mathbf{u})\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|\mathbf{w} - Q_0 \mathbf{w}\|_T^2 + h_T \|\mathbf{w} - Q_0 \mathbf{w}\|_{1,T}^2 \right)^{\frac{1}{2}} \\
&\quad \cdot \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|2\mu(Q_{r_1} - I)\epsilon(\mathbf{u}) \cdot \mathbf{n}\|_T^2 + h_T \|2\mu(Q_{r_1} - I)\epsilon(\mathbf{u}) \cdot \mathbf{n}\|_{1,T}^2 \right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|(\mathbf{w} - Q_0 \mathbf{w}) \cdot \mathbf{n}\|_T^2 + h_T \|(\mathbf{w} - Q_0 \mathbf{w}) \cdot \mathbf{n}\|_{1,T}^2 \right)^{\frac{1}{2}} \\
&\quad \cdot \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|\lambda(Q_{r_2} - I)(\nabla \cdot \mathbf{u})\|_T^2 + h_T \|\lambda(Q_{r_2} - I)(\nabla \cdot \mathbf{u})\|_{1,T}^2 \right)^{\frac{1}{2}} \\
&\leq Ch^{-1} h^k \|\mathbf{u}\|_{k+1} h^{1+\alpha} \|\mathbf{w}\|_{1+\alpha} \\
&\leq Ch^{k+\alpha} \|\mathbf{u}\|_{k+1} \|\mathbf{w}\|_{1+\alpha}.
\end{aligned}$$

For J_2 , using the Cauchy-Schwarz inequality, (6.4) with $k = \alpha$ and (6.7), we obtain:

$$J_2 \leq \|\mathbf{w} - Q_h \mathbf{w}\| \|\mathbf{e}_h\| \leq Ch^k \|\mathbf{u}\|_{k+1} h^\alpha \|\mathbf{w}\|_{1+\alpha} \leq Ch^{k+\alpha} \|\mathbf{u}\|_{k+1} \|\mathbf{w}\|_{1+\alpha}.$$

For J_3 , using the Cauchy-Schwarz inequality, (6.4) with $k = \alpha$, and (6.7), we find:

$$J_3 \leq C \|\mathbf{w} - Q_h \mathbf{w}\| \|\mathbf{e}_h\| \leq Ch^k \|\mathbf{u}\|_{k+1} h^\alpha \|\mathbf{w}\|_{1+\alpha} \leq Ch^{k+\alpha} \|\mathbf{u}\|_{k+1} \|\mathbf{w}\|_{1+\alpha}.$$

For J_4 , let Q^0 denote the L^2 -projection onto $[P_0(T)]$. Using (2.1), we get:

$$\begin{aligned}
(7.5) \quad & (2\mu Q^0(\epsilon_w(\mathbf{w})), \epsilon_w(Q_h \mathbf{u} - \mathbf{u}))_T \\
&= - (2\mu(Q_0 \mathbf{u} - \mathbf{u}), \nabla \cdot (Q^0(\epsilon_w(\mathbf{w}))))_T \\
&\quad + \langle 2\mu(Q_b \mathbf{u} - \mathbf{u}), Q^0(\epsilon_w(\mathbf{w})) \cdot \mathbf{n} \rangle_{\partial T} = 0.
\end{aligned}$$

Using (7.5), Cauchy-Schwarz inequality, (5.1) and (6.4), gives

$$\begin{aligned}
J_4 &\leq \left| \sum_{T \in \mathcal{T}_h} (2\mu \epsilon_w(\mathbf{w}), \epsilon_w(Q_h \mathbf{u} - \mathbf{u}))_T \right| \\
&= \left| \sum_{T \in \mathcal{T}_h} (2\mu(\epsilon_w(\mathbf{w}) - Q^0(\epsilon_w(\mathbf{w}))), \epsilon_w(Q_h \mathbf{u} - \mathbf{u}))_T \right| \\
&= \left| \sum_{T \in \mathcal{T}_h} (2\mu(Q_{r_1} \epsilon(\mathbf{w}) - Q^0(Q_{r_1} \epsilon(\mathbf{w}))), \epsilon_w(Q_h \mathbf{u} - \mathbf{u}))_T \right| \\
&\leq \left(\sum_{T \in \mathcal{T}_h} \|Q_{r_1} \epsilon(\mathbf{w}) - Q^0(Q_{r_1} \epsilon(\mathbf{w}))\|_T^2 \right)^{\frac{1}{2}} \|Q_h \mathbf{u} - \mathbf{u}\| \\
&\leq Ch^k \|\mathbf{u}\|_{k+1} h^\alpha \|\mathbf{w}\|_{1+\alpha} \\
&\leq Ch^{k+\alpha} \|\mathbf{u}\|_{k+1} \|\mathbf{w}\|_{1+\alpha}.
\end{aligned}$$

Similar to the estimate for J_4 , we obtain:

$$J_5 \leq Ch^{k+\alpha} \|\mathbf{u}\|_{k+1} \|\mathbf{w}\|_{1+\alpha}.$$

For J_6 , using the Cauchy-Schwarz inequality, the trace inequality (4.1), Lemma 4.5, and the estimates (6.1)-(6.2) with $m = 1 + \alpha$ and $s = 0, 1$, along with (6.4) and (6.7), we obtain:

$$\begin{aligned}
J_6 &= \ell(\mathbf{w}, \zeta_h) \\
&\leq \left| \sum_{T \in \mathcal{T}_h} \langle \zeta_b - \zeta_0, 2\mu(Q_{r_1} - I)\epsilon(\mathbf{w}) \cdot \mathbf{n} \rangle_{\partial T} \right| \\
&\quad + \left| \sum_{T \in \mathcal{T}_h} \langle (\zeta_b - \zeta_0) \cdot \mathbf{n}, \lambda(Q_{r_2} - I)(\nabla \cdot \mathbf{w}) \rangle_{\partial T} \right| \\
&\leq \left(\sum_{T \in \mathcal{T}_h} \|\zeta_b - \zeta_0\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \|2\mu(Q_{r_1} - I)\epsilon(\mathbf{w}) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{T \in \mathcal{T}_h} \|(\zeta_b - \zeta_0) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \|\lambda(Q_{r_2} - I)(\nabla \cdot \mathbf{w})\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|\zeta_b - \zeta_0\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\quad \cdot \left(\sum_{T \in \mathcal{T}_h} \|2\mu(Q_{r_1} - I)\epsilon(\mathbf{w}) \cdot \mathbf{n}\|_T^2 + h_T^2 \|2\mu(Q_{r_1} - I)\epsilon(\mathbf{w}) \cdot \mathbf{n}\|_{1,T}^2 \right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|(\zeta_b - \zeta_0) \cdot \mathbf{n}\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\quad \cdot \left(\sum_{T \in \mathcal{T}_h} \|\lambda(Q_{r_2} - I)(\nabla \cdot \mathbf{w})\|_T^2 + h_T^2 \|\lambda(Q_{r_2} - I)(\nabla \cdot \mathbf{w})\|_{1,T}^2 \right)^{\frac{1}{2}} \\
&\leq Ch^\alpha \|\mathbf{w}\|_{1+\alpha} \|\zeta_h\|_{1,h} \\
&\leq Ch^\alpha \|\mathbf{w}\|_{1+\alpha} \|\zeta_h\| \\
&\leq Ch^\alpha \|\mathbf{w}\|_{1+\alpha} (\|\mathbf{u} - \mathbf{u}_h\| + \|\mathbf{u} - Q_h \mathbf{u}\|) \\
&\leq Ch^\alpha \|\mathbf{w}\|_{1+\alpha} (h^k \|\mathbf{u}\|_{k+1} + h^k \|\mathbf{u}\|_{k+1}) \\
&\leq Ch^{k+\alpha} \|\mathbf{w}\|_{1+\alpha} \|\mathbf{u}\|_{k+1}.
\end{aligned}$$

Substituting the estimates for J_i for $i = 1, \dots, 6$ into (7.4) and using (7.2) gives

$$\|\zeta_0\|^2 \leq Ch^{k+\alpha} \|\mathbf{w}\|_{1+\alpha} \|\mathbf{u}\|_{k+1} \leq Ch^{k+\alpha} \|\mathbf{u}\|_{k+1} \|\zeta_0\|.$$

This gives

$$\|\zeta_0\| \leq Ch^{k+\alpha} \|\mathbf{u}\|_{k+1},$$

which, using triangle inequality and (6.3) with $n = k$, gives

$$\|\mathbf{e}_0\| \leq \|\zeta_0\| + \|\mathbf{u} - Q_0 \mathbf{u}\| \leq Ch^{k+\alpha} \|\mathbf{u}\|_{k+1}.$$

This completes the proof of the theorem. \square

8. NUMERICAL VERIFICATION

In the first test, we solve the linear elasticity equation (3.4) with $\mu = 1$ on the unit square domain $\Omega = (0, 1) \times (0, 1)$. We choose \mathbf{f} and \mathbf{g} in (3.4) such that the exact solution is

$$(8.1) \quad \mathbf{u} = \begin{pmatrix} (x^2 - 2x^3 + x^4)(2y - 6y^2 + 4y^3) \\ -(y^2 - 2y^3 + y^4)(2x - 6x^2 + 4x^3) \end{pmatrix}.$$

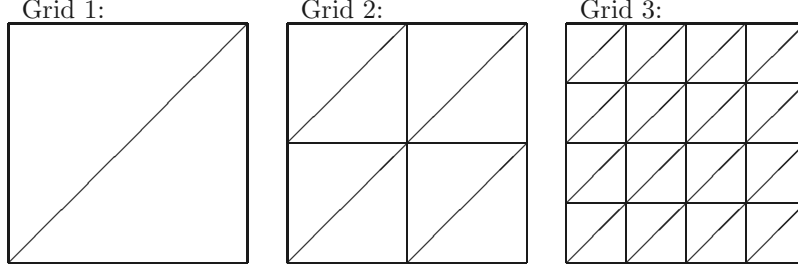


FIGURE 1. The first three grids for the computation in Tables 1-2

We compute the finite element solutions for the solution (8.1) on uniform triangular grids shown in Figure 1 by the P_k WG finite elements, defined in (3.1)–(3.3), for $k = 1, 2, 3$ and 4. The results are listed in Tables 1-2. The optimal order of convergence is achieved for all solutions in all norms. Further, the method is shown pressure robust as the convergence is independent of λ in (3.4).

TABLE 1. The error and the computed order of convergence for the solution (8.1) on Figure 1 meshes: (a) P_1 WG, $\lambda = 1$; (b) P_1 WG, $\lambda = 10^7$; (c) P_2 WG, $\lambda = 1$; (d) P_2 WG, $\lambda = 10^7$.

Method	Grid	$\ \mathbf{u} - \mathbf{u}_0\ _0$	$O(h^r)$	$\ \mathbf{u} - \mathbf{u}_h\ $	$O(h^r)$
(a)	5	0.824E-04	2.0	0.944E-02	1.0
	6	0.207E-04	2.0	0.473E-02	1.0
	7	0.517E-05	2.0	0.237E-02	1.0
(b)	5	0.101E-03	2.0	0.102E-01	1.0
	6	0.254E-04	2.0	0.513E-02	1.0
	7	0.632E-05	2.0	0.256E-02	1.0
(c)	5	0.144E-05	3.0	0.416E-03	2.0
	6	0.180E-06	3.0	0.104E-03	2.0
	7	0.224E-07	3.0	0.261E-04	2.0
(d)	4	0.129E-04	3.0	0.169E-02	1.9
	5	0.160E-05	3.0	0.425E-03	2.0
	6	0.212E-06	2.9	0.107E-03	2.0

TABLE 2. The error and the computed order of convergence for the solution (8.1) on Figure 1 meshes: (a) P_3 WG, $\lambda = 1$; (b) P_3 WG, $\lambda = 10^7$; (c) P_4 WG, $\lambda = 1$; (d) P_4 WG, $\lambda = 10^7$.

Method	Grid	$\ \mathbf{u} - \mathbf{u}_0\ _0$	$O(h^r)$	$\ \mathbf{u} - \mathbf{u}_h\ $	$O(h^r)$
(a)	4	0.427E-06	4.0	0.102E-03	3.0
	5	0.267E-07	4.0	0.128E-04	3.0
	6	0.167E-08	4.0	0.160E-05	3.0
(b)	4	0.457E-06	4.0	0.102E-03	3.0
	5	0.310E-07	3.9	0.127E-04	3.0
	6	0.135E-03	0.0	0.253E-02	0.0
(c)	3	0.549E-06	5.0	0.968E-04	4.0
	4	0.169E-07	5.0	0.601E-05	4.0
	5	0.523E-09	5.0	0.375E-06	4.0
(d)	3	0.568E-06	5.0	0.964E-04	4.0
	4	0.173E-07	5.0	0.599E-05	4.0
	5	0.567E-09	4.9	0.375E-06	4.0

We next compute the finite element solutions for (8.1) on non-convex polygonal grids shown in Figure 2 by the P_k WG finite elements, defined in (3.1)–(3.3), for $k = 1, 2, 3$ and 4. The results are listed in Tables 3–4. The optimal order of convergence is achieved for all solutions in all norms. And the convergence is independent of the incompressible parameter λ in (3.4).

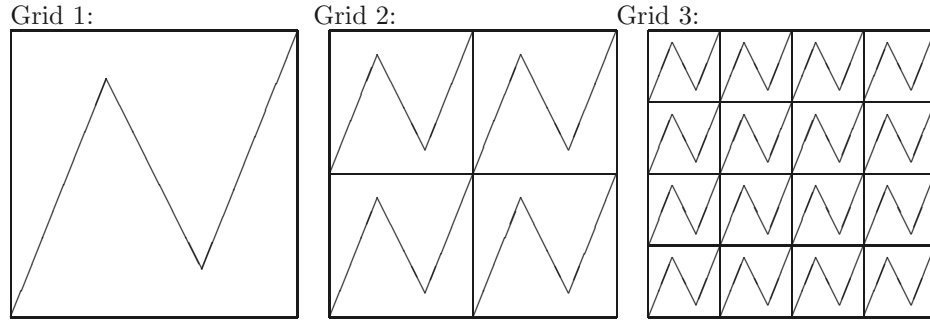


FIGURE 2. The first three non-convex polygonal grids for the computation in Tables 3–4.

TABLE 3. The error and the computed order of convergence for the solution (8.1) on Figure 2 non-convex polygonal meshes: (a) P_1 WG, $\lambda = 1$; (b) P_1 WG, $\lambda = 10^5$; (c) P_2 WG, $\lambda = 1$; (d) P_2 WG, $\lambda = 10^5$.

Method	Grid	$\ \mathbf{u} - \mathbf{u}_0\ _0$	$O(h^r)$	$\ \mathbf{u} - \mathbf{u}_h\ $	$O(h^r)$
(a)	5	0.393E-03	1.8	0.332E-01	1.0
	6	0.104E-03	1.9	0.166E-01	1.0
	7	0.265E-04	2.0	0.829E-02	1.0
(b)	5	0.320E-03	1.8	0.339E-01	1.0
	6	0.831E-04	1.9	0.170E-01	1.0
	7	0.210E-04	2.0	0.850E-02	1.0
(c)	4	0.116E-03	2.9	0.241E-01	1.9
	5	0.147E-04	3.0	0.613E-02	2.0
	6	0.184E-05	3.0	0.154E-02	2.0
(d)	4	0.116E-03	2.9	0.240E-01	1.9
	5	0.146E-04	3.0	0.612E-02	2.0
	6	0.183E-05	3.0	0.154E-02	2.0

TABLE 4. The error and the computed order of convergence for the solution (8.1) on Figure 2 non-convex polygonal meshes: (a) P_3 WG, $\lambda = 1$; (b) P_3 WG, $\lambda = 10^5$; (c) P_4 WG, $\lambda = 1$; (d) P_4 WG, $\lambda = 10^5$.

Method	Grid	$\ \mathbf{u} - \mathbf{u}_0\ _0$	$O(h^r)$	$\ \mathbf{u} - \mathbf{u}_h\ $	$O(h^r)$
(a)	4	0.203E-04	3.9	0.684E-02	2.9
	5	0.129E-05	4.0	0.877E-03	3.0
	6	0.811E-07	4.0	0.110E-03	3.0
(b)	3	0.298E-03	3.4	0.494E-01	2.3
	4	0.203E-04	3.9	0.684E-02	2.9
	5	0.130E-05	4.0	0.876E-03	3.0
(c)	3	0.109E-03	4.8	0.254E-01	3.7
	4	0.353E-05	4.9	0.165E-02	3.9
	5	0.111E-06	5.0	0.104E-03	4.0
(d)	3	0.109E-03	4.8	0.254E-01	3.7
	4	0.374E-05	4.9	0.165E-02	3.9
	5	0.427E-05	0.0	0.109E-03	3.9

In the 3D numerical computation, the domain for problem (3.4) is the unit cube $\Omega = (0, 1) \times (0, 1) \times (0, 1)$. We choose an \mathbf{f} and an \mathbf{g} in (3.4) so that the exact solution is

$$(8.2) \quad \mathbf{u} = \begin{pmatrix} e^{y+z} \\ e^{z+x} \\ e^{z+x} \end{pmatrix}.$$

We compute the finite element solutions for (8.2) on the tetrahedral grids shown in Figure 3 by the P_k WG finite elements, defined in (3.1)–(3.3), for $k = 1, 2, 3$ and 4. The results are listed in Tables 5–6. The optimal order of convergence is achieved for all solutions in all norms. And the numerical results indicate the method is pressure robust.

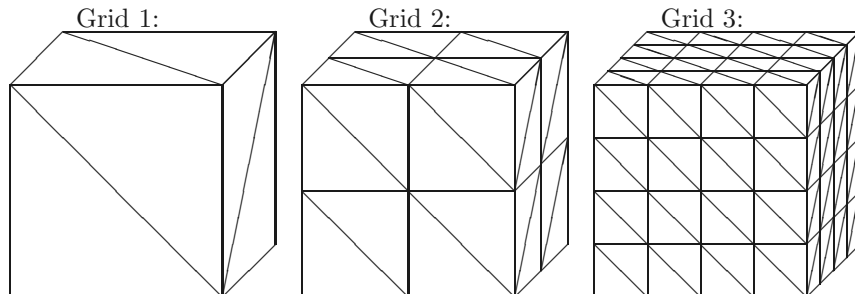


FIGURE 3. The first three grids for the computation in Tables 5–6

TABLE 5. The error and the computed order of convergence for the solution (8.2) on Figure 3 tetrahedral meshes: (a) P_3 WG, $\lambda = 1$; (b) P_3 WG, $\lambda = 10^4$; (c) P_4 WG, $\lambda = 1$; (d) P_4 WG, $\lambda = 10^4$.

Method	Grid	$\ \mathbf{u} - \mathbf{u}_0\ _0$	$O(h^r)$	$\ \mathbf{u} - \mathbf{u}_h\ $	$O(h^r)$
(a)	3	0.107E-01	1.90	0.625E+00	0.90
	4	0.267E-02	2.00	0.322E+00	0.96
	5	0.663E-03	2.01	0.163E+00	0.98
(b)	3	0.157E-01	1.89	0.672E+00	0.97
	4	0.389E-02	2.01	0.338E+00	0.99
	5	0.966E-03	2.01	0.170E+00	0.99
(c)	3	0.516E-03	2.87	0.373E-01	1.90
	4	0.662E-04	2.96	0.956E-02	1.96
	5	0.835E-05	2.99	0.241E-02	1.98
(d)	2	0.493E-02	2.29	0.151E+00	1.77
	3	0.604E-03	3.03	0.391E-01	1.95
	4	0.741E-04	3.03	0.992E-02	1.98

We next compute the finite element solutions for (8.2) on the nonconvex polyhedral grids shown in Figure 4 by the P_k WG finite elements, defined in (3.1)–(3.3), for $k = 1, 2$ and 3. The results are listed in Table 7. The optimal order of convergence is achieved for all solutions in all norms. And the numerical results indicate the method is pressure robust.

TABLE 6. The error and the computed order of convergence for the solution (8.2) on Figure 3 tetrahedral meshes: (a) P_3 WG, $\lambda = 1$; (b) P_3 WG, $\lambda = 10^4$; (c) P_4 WG, $\lambda = 1$; (d) P_4 WG, $\lambda = 10^4$.

Method	Grid	$\ \mathbf{u} - \mathbf{u}_0\ _0$	$O(h^r)$	$\ \mathbf{u} - \mathbf{u}_h\ $	$O(h^r)$
(a)	2	0.298E-03	3.61	0.136E-01	2.74
	3	0.203E-04	3.88	0.179E-02	2.92
	4	0.131E-05	3.96	0.229E-03	2.97
(b)	2	0.328E-03	3.91	0.140E-01	2.80
	3	0.215E-04	3.93	0.181E-02	2.95
	4	0.138E-05	3.96	0.230E-03	2.98
(c)	2	0.187E-04	4.70	0.106E-02	3.80
	3	0.629E-06	4.89	0.692E-04	3.94
	4	0.202E-07	4.96	0.439E-05	3.98
(d)	1	0.620E-03	0.00	0.156E-01	0.00
	2	0.200E-04	4.95	0.108E-02	3.85
	3	0.650E-06	4.95	0.697E-04	3.95

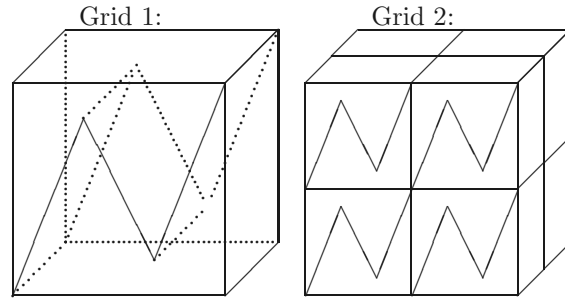


FIGURE 4. The first three grids for the computation in Table 7

TABLE 7. The error and the computed order of convergence for the solution (8.2) on Figure 4, nonconvex polyhedral meshes: (a) P_1 WG, $\lambda = 1$; (b) P_1 WG, $\lambda = 10^4$; (c) P_2 WG, $\lambda = 1$; (d) P_2 WG, $\lambda = 10^4$; (e) P_3 WG, $\lambda = 1$; (f) P_3 WG, $\lambda = 10^4$.

Method	Grid	$\ \mathbf{u} - \mathbf{u}_0\ _0$	$O(h^r)$	$\ \mathbf{u} - \mathbf{u}_h\ $	$O(h^r)$
(a)	4	0.185E-01	2.0	0.217E+01	1.0
	5	0.466E-02	2.0	0.109E+01	1.0
(b)	4	0.190E-01	2.0	0.217E+01	1.0
	5	0.480E-02	2.0	0.109E+01	1.0
(c)	4	0.859E-03	3.0	0.159E+00	2.0
	5	0.109E-03	3.0	0.401E-01	2.0
(d)	3	0.672E-02	2.9	0.620E+00	1.9
	4	0.861E-03	3.0	0.159E+00	2.0
(e)	2	0.719E-02	3.8	0.497E+00	2.8
	3	0.484E-03	3.9	0.656E-01	2.9
(f)	2	0.716E-02	3.8	0.496E+00	2.8
	3	0.481E-03	3.9	0.656E-01	2.9

We finally compute the finite element solutions for (8.2) on the nonconvex polyhedral grids shown in Figure 5 by the P_k WG finite elements, defined in (3.1)–(3.3), for $k = 1$ and 3. The results are listed in Table 8. The optimal order of convergence is achieved for all solutions in all norms. And the numerical results indicate the method is pressure robust.

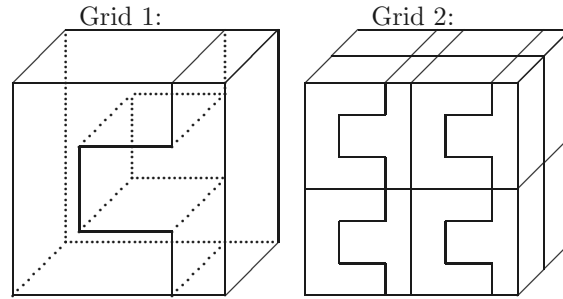


FIGURE 5. The first three grids for the computation in Table 8.

TABLE 8. The error and the computed order of convergence for the solution (8.2) on Figure 5, nonconvex polyhedral meshes: (a) P_1 WG, $\lambda = 1$; (b) P_1 WG, $\lambda = 10^4$; (c) P_2 WG, $\lambda = 1$; (d) P_2 WG, $\lambda = 10^4$.

Method	Grid	$\ \mathbf{u} - \mathbf{u}_0\ _0$	$O(h^r)$	$\ \mathbf{u} - \mathbf{u}_h\ $	$O(h^r)$
(a)	3	0.520E-01	2.0	0.386E+01	1.0
	4	0.127E-01	2.0	0.191E+01	1.0
	5	0.314E-02	2.0	0.952E+00	1.0
(b)	2	0.309E+00	1.9	0.104E+02	1.0
	3	0.808E-01	1.9	0.517E+01	1.0
	4	0.207E-01	2.0	0.252E+01	1.0
(c)	2	0.289E-01	2.9	0.135E+01	1.9
	3	0.341E-02	3.1	0.336E+00	2.0
	4	0.408E-03	3.1	0.835E-01	2.0
(d)	1	0.355E+00	0.0	0.811E+01	0.0
	2	0.421E-01	3.1	0.166E+01	2.3
	3	0.487E-02	3.1	0.389E+00	2.1

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FL 32611, USA.

Email address: `chunmei.wang@ufl.edu`

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF DELAWARE, NEWARK, DE 19716, USA

Email address: `szhang@udel.edu`