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Author(s): Joel Hass, Abigail Thompson and Anastasiia Tsvietkova

Source: *Indiana University Mathematics Journal*, 2021, Vol. 70, No. 2 (2021), pp. 525-534

Published by: Indiana University Mathematics Department

Stable URL: <https://www.jstor.org/stable/10.2307/27114633>

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Alternating Links Have at Most Polynomially Many Seifert Surfaces of Fixed Genus

JOEL HASS, ABIGAIL THOMPSON & ANASTASIIA TSVIETKOVA

ABSTRACT. Let L be a non-split prime alternating link with $n > 0$ crossings. We show that for each fixed g , the number of genus- g Seifert surfaces for L is bounded by an explicitly given polynomial in n . The result also holds for all spanning surfaces of fixed Euler characteristic. Previously known bounds were exponential.

1. INTRODUCTION

A *Seifert surface* for a link in S^3 is a connected orientable surface, embedded in $S^3 \setminus L$, whose boundary is isotopic to the link. If the orientability condition is omitted, the surface is called a *spanning surface*. Let L be a non-split prime alternating link with an n -crossing diagram, where $n > 0$. In this paper, we give an upper bound on the number of isotopy classes of spanning surfaces of L that have a fixed Euler characteristic. The bound is given by an explicit polynomial in n . Our methods apply to Seifert surfaces, spanning surfaces, and to more general essential surfaces with non-meridional boundary. However, in this paper we focus our attention on Seifert surfaces and state our results mainly for that class of surface. We show that the number of genus- g Seifert surfaces in the complement of an n -crossing alternating diagram is at most $(4n)^{64g^2-48g}$.

In general, a knot complement can contain many, and in some cases infinitely many, non-isotopic Seifert surfaces of a given genus (see, e.g., [2]). But in a hyperbolic manifold the number of spanning surfaces is always finite. This can be seen by homotoping each surface to a least area representative and applying the Gauss-Bonnet theorem and Schoen's curvature estimates [6, 19]. Alternatively, the surfaces can be homotoped to pleated surfaces as in [20]. This type of argument

applies also to π_1 -injective immersions, but is not constructive and gives no explicit bound on the number of surfaces of a given genus. Normal surface theory can also be used to bound the number of spanning surfaces of genus g . Each surface can be isotoped to be normal, and can then be expressed as a sum of finitely many fundamental normal surfaces. However, this process leads to an exponential bound on the number of spanning surfaces of genus g as a function of n . This is due to the exponential growth of the number of fundamental surfaces of a given genus, and even of the number of vertex fundamental surfaces, as a function of the number of tetrahedra t [7]. An additional difficulty is that an incompressible surface may not be fundamental, so that one must also count combinations of lower genus fundamental surfaces that combine to form a given genus [5].

The surfaces we consider are embedded and incompressible, but not necessarily disjoint. The number of disjoint incompressible surfaces in a manifold is easier to bound. This was first observed by Kneser [13] for closed incompressible surfaces. Kneser showed that the number of such surfaces is bounded by a linear function of the number of tetrahedra t required to triangulate the manifold. For a link complement, t is a linear multiple of the number of crossings in the link diagram. Hence, the number of disjoint spanning surfaces realizable for a link L grows linearly with n [9]. Kneser's arguments also apply to surfaces with boundary in link complements.

If we fix a link complement, our results additionally yield an exponential upper bound in terms of genus (rather than crossing number). This can be compared with the results of Masters [14] and Kahn-Markovic [12] for essential immersed closed surfaces in a closed manifold, where the manifold is fixed and it is shown that the number of surfaces grows exponentially with the genus.

A related problem for closed surfaces was studied by the authors in [8], where it was proved that the number of closed incompressible genus- g surfaces in a prime alternating link complement is bounded by $C_g n^{40g^2}$, where C_g is an explicit constant depending only on the genus g , and n is the number of crossings. The proof for Seifert surfaces and spanning surfaces presented here needs to consider cases that cause difficulties not encountered with closed surfaces. In [8] a surface is put in standard position with respect to the projection plane and the link diagram, and then decomposed into disks bounded by polygons and lying above or below the plane. By summing the contributions to the Euler characteristic of the surface of each region, a bound on the number and complexity of the region is obtained. Each region makes a negative contribution to the Euler characteristic, bounding their number, and the possible ways for polygons to appear on a link diagram can then be analyzed, providing an upper bound for the surface count. The standard position and Euler characteristic arguments can be extended to Seifert surfaces, but the resulting regions for surfaces with boundary include cases that contribute zero to the Euler characteristic computation. This leads to an exponential explosion in the number of possible intersection configurations relative to the number of crossings. However, many of these configurations give rise to isotopic surfaces, and

we can show that the number of surfaces up to isotopy still grows polynomially with the number of crossings. This is carried out in Sections 4 and 5.

2. STANDARD POSITION FOR SEIFERT SURFACES

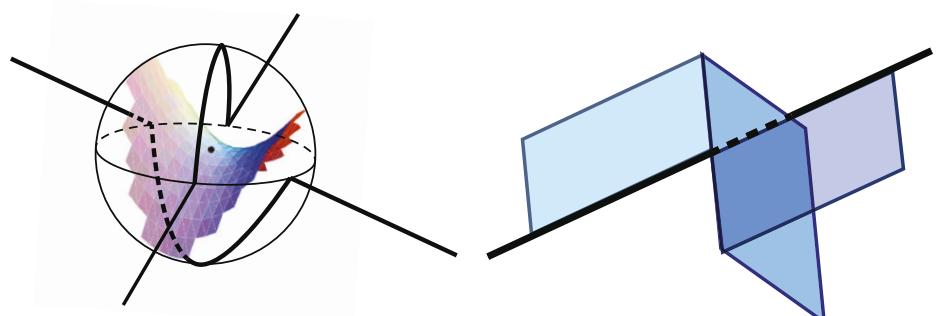
A standard position for surfaces in an alternating knot complement was introduced by Menasco for closed surfaces and for surfaces with meridional boundary [15], and extended by Menasco and Thistlethwaite to general surfaces with boundary [17]. We briefly review these techniques, with some minor modifications to the arguments.

A reduced alternating diagram D of a prime alternating link L can be placed in a projection plane Q except for two small arcs near each crossing, one of which drops below Q , and one of which rises above it. Then, L lies on a union of two overlapping 2-spheres in S^3 , S_+ , and S_- , which agree with Q except along small balls around each crossing, called *bubbles*. The spheres S_+ and S_- go over the top and bottom hemispheres of each bubble, respectively. We denote by B_+ the ball in S^3 lying above the projection plane and bounded by S_+ , and by B_- the ball lying below the projection plane and bounded by S_- .

Suppose F is a spanning surface for L . It is shown in Proposition 2.1 of [17] that F can be isotoped rel boundary so that it intersects Q transversally except in two situations:

- (1) F meets a bubble in a saddle near a crossing, as in Figure 1(a).
- (2) At finitely many points, F twists around a strand of L as in Figure 1(b).

F is then said to be in the *standard position*. In (2), we call the arc of intersection of $F \cap (Q \setminus D)$, together with its endpoints on D , an *interior arc*. An interior arc can run along Q to connect two points on D , or between a point on D and a saddle. An arc of $F \cap Q$ that coincides with an arc of D and lies between two interior arcs is called a *boundary arc*.



(a) F is tangent to the projection plane at a saddle point contained in a bubble. (b) An interior arc of $F \cap Q$ ends at a point where F twists from being above to below Q .

FIGURE 2.1.

Here, F is divided into regions in B_+ and B_- by closed curves in $F \cap S_+$ or $F \cap S_-$. Parts of these curves lie in the boundary of F (i.e., run along the link). We assign to each closed curve C in $F \cap S_+$ or $F \cap S_-$ a word in the letters B and S , defined up to cyclic order, as follows. Orient C and, starting from an arbitrary non-crossing point, follow C until returning to that point. For every saddle passed on the way, add an S to the word, and for every point that passes between interior and boundary arcs, add a B . Figure 2.2 gives an example of a link and a curve of $F \cap S_+$ that gives the word $BSSSS$. The link is depicted in black, the curves of $F \cap S_+$ with dashes, and the curves of $F \cap S_-$ with dots.

Define the complexity of a surface F in standard position to be a pair (s, c) , where s is the sum of the number of S associated with all curves in $F \cap S_+$ and $F \cap S_-$, and c is the number of such curves. If F minimizes this complexity in lexicographic order among all standard position surfaces in its isotopy class, then F is said to have *minimal complexity*.

We refer to a segment of the link diagram D that travels between two successive crossings of L and lies in the projection plane Q as an *edge* of D , and use *arc* to refer to subcurves of $F \cap S_+$ and $F \cap S_-$ running between the S and B .

The following lemma summarizes the properties of spanning surfaces in standard position that follow from the work of Menasco and Thistlethwaite [17].

Lemma 2.1. *Suppose a spanning surface F has minimal complexity. Then, the curves of $F \cap S_+$ and $F \cap S_-$ and the associated words in the letters B , S have the following properties:*

- (1) *The curves of $F \cap S_+$ and $F \cap S_-$ subdivide F into disks, each disk lying either in $F \cap B_+$ or $F \cap B_-$.*
- (2) *No curve passes through the same bubble twice.*
- (3) *Every edge of the diagram D meets an interior arc of $F \cap Q$.*
- (4) *No curve contains two interior arcs with endpoints on an edge A and lying on the same side of A .*
- (5) *An equal number of curves of $F \cap S_+$ pass through each side of a saddle. The same holds for $F \cap S_-$.*
- (6) *No curve passes through a saddle and then meets an edge of D adjacent to the saddle.*
- (7) *No word is empty.*
- (8) *Letters B in a word appear in consecutive pairs.*
- (9) *There is no word consisting entirely of instances of S .*
- (10) *Each word has length at least four.*

Proof. We indicate the proofs for $F \cap S_+$ below. The same arguments apply to $F \cap S_-$.

For (1), see Proposition 2.2 (i) in [17]. For (2) and (4), see Proposition 2.2 (ii), and for (6), see Proposition 2.3.

The proof of (3) follows from the fact that L is alternating and is the boundary of F . Hence, every edge is adjacent to an overpass and underpass, and a part of the surface near an edge changes from being in B_+ to being in B_- somewhere

between the two adjacent crossings. This gives rise to an interior arc meeting the edge. Claim (5) follows from the fact that each saddle of a surface results in one intersection curve on each side of a crossing. For (7), if there is a component of $F \cap S_+$ with no saddles or punctures, take an innermost such component. It bounds a disk in B_+ , so we can isotop F to eliminate the curve of intersection with S_+ and reduce the complexity. For (8), note that each boundary arc in a curve contributes two successive instances of B . Claim (9) is proved in Lemma 2 of [15] for closed surfaces. The proof for spanning surfaces is exactly the same.

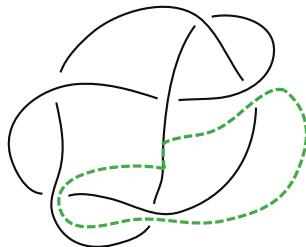


FIGURE 2.2. The link L and a $BBSSS$ curve from $F \cap S_+$

For (10), if the word for a curve has just two letters, it is one of BB , SB , or SS . The word SS contradicts (2), SB contradicts (8), and BB contradicts (4). Among 3-letter words, we have ruled out SSS . A curve of type BBS can be perturbed so that it intersects the link L at most three times. Thus, the perturbed curve intersects twice, implying either that D is not prime or that Case (6) is violated. Hence, the length of the word is at least four. \square

3. DECOMPOSING A SURFACE INTO REGIONS

We henceforth consider a surface F in standard position with minimal complexity. In this section, we decompose the surface into polygonal faces, determined by the intersections of F with Q , and analyze the contributions of the faces to the Euler characteristic of the surface. A similar technique was used in [8] for closed surfaces, and in [17] for a different purpose.

We decompose F using the arcs of $F \cap Q$. These form a graph on F . At each saddle, four polygonal faces and four arcs of $F \cap Q$ meet at the saddle point, where we add a vertex to the graph. We also add a vertex at every intersection of D and an interior arc of $F \cap Q$. The resulting graph Γ has vertices of valence four at the centers of the saddles and vertices of valence three at endpoints of interior arcs that meet D . The graph Γ cuts F into a collection of disks by Lemma 2.1 (1) that we call “polygonal faces or regions.” The vertices and edges of a region are the vertices and edges of Γ , respectively.

Lemma 3.1. *The Euler characteristic of F can be computed by adding the contribution of each region. A region E contributes $1 - s_0/4 - b_0/4$ to $\chi(F)$, where s_0 is*

the number of instances of S in the word associated with the boundary of E , and b_0 is the number of instances of B .

Proof. Enumerate all curves C_i , $i = 1, \dots, r$, in $F \cap S_+$ and $F \cap S_-$. Suppose C_i is the boundary of a region E_i of F with interior in B_+ or B_- . The Euler characteristic of F can be recovered by summing the contributions of each of these regions E_i , $i = 1, \dots, r$.

Four distinct regions share a vertex at a saddle, and two regions share a vertex at the endpoint of an interior arc that meets D . Two regions share an edge of Γ that coincides with an interior arc, and there is just one region at every edge of Γ that corresponds to a boundary arc. Hence, the Euler characteristic $\chi(F) = v - e + f$ can be distributed among vertices and edges as follows.

For vertices, $+\frac{1}{4}$ is allocated to each vertex of a region on a saddle (i.e., S contributes $+\frac{1}{4}$ as a vertex), $+\frac{1}{2}$ to each vertex at the end of an interior arc that meets D (i.e., B contributes $+\frac{1}{2}$ as a vertex). For edges, the contribution is $-\frac{1}{2}$ for an edge of Γ that corresponds to an interior arc, and -1 to an edge that corresponds to a boundary arc.

Now let us distribute the contributions of the edges of Γ between their vertices. Every B is an endpoint of one interior arc and one boundary arc. Every S is an endpoint of two interior arcs. In an interior arc with the contribution of $-\frac{1}{2}$, we can view the contribution of B as $-\frac{1}{4}$ and of S as $-\frac{1}{4}$. In a boundary arc with the contribution -1 , the contribution of every B is $-\frac{1}{2}$. Thus, the contribution of a region with s_0 vertices at saddles and b_0 vertices at the ends of interior arcs that meet D is

$$s_0 \left(\frac{1}{4} - \frac{1}{4} - \frac{1}{4} \right) + b_0 \left(\frac{1}{2} - \frac{1}{4} - \frac{1}{2} \right) + 1 = 1 - \frac{s_0}{4} - \frac{b_0}{4}. \quad \square$$

Remark 3.2. It follows from Lemma 2.1 (10) that all curves of $F \cap S_+$ or $F \cap S_-$ make a negative contribution to the Euler characteristic of a minimal complexity spanning surface, no greater than $-\frac{1}{4}$, except for *BBBB* and *BBSS* curves, which contribute zero. We analyze *BBBB* curves in the next section.

4. COLLECTIONS OF *BBBB* CURVES

We say that a collection of regions in F is *connected* if the dual graph, formed by taking a vertex for each region and an edge for two regions that share a common arc, is connected. A connected collection of *BBBB* regions is *maximal* if it is not a strict subset of a connected collection of *BBBB* regions.

Lemma 4.1. *Assume that L is not a $(2, n)$ -torus link. A connected collection of *BBBB* regions forms a subsurface of F with interior homeomorphic to a disk. The set of all maximal connected collections of *BBBB* regions forms a collection of disks with no pair of disks sharing a common arc in their boundaries.*

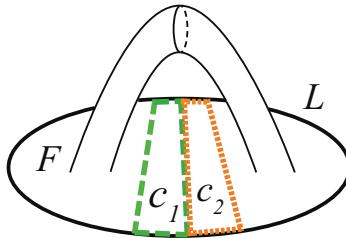


FIGURE 4.1. A connected collection of two regions, each bounded by a *BBBB* curve

Proof. We first use induction to show that a connected collection of *BBBB* regions C has interior homeomorphic to a disk. If C consists of a single *BBBB* region, then it is a disk, since any single region is a disk. Now consider the case when C consists of the union of k distinct *BBBB* regions. The dual graph of C is a finite connected graph, so we can remove some region R to get a collection of $k - 1$ *BBBB* regions C_0 which is also connected. By induction, C_0 is a disk, and C is obtained from C_0 by adding a single *BBBB* region R sharing at least one arc with C_0 . Now, $C_0 \cap R$ consists of either one or two interior *BB* arcs. If one, then since the union of two disks intersecting along a proper arc on their boundary is a disk, it follows that the interior of C is also a disk. If $C_0 \cap R$ has two components, then C is an annulus or Möbius band, properly embedded in F , and ∂C coincides with either one or two components of L . The boundary of a regular neighborhood in S^3 of this annulus or Möbius band is a torus, and since L is alternating, the torus must be compressible in the complement of L [15], and hence unknotted and the boundary of a solid torus in the complement of L . Since each curve of ∂C is isotopic to a curve on this torus that intersects its meridian twice, and L is not split, L must be a $(2, n)$ -torus link, contradicting our hypothesis. We conclude that the connected collection of *BBBB* regions C is homeomorphic to a disk.

A maximal connected collection of *BBBB* regions cannot share an interior arc with another such region, since if it did, neither would be maximal.

Thus, maximal connected collections are disks with no common arc in their boundaries. \square

A consequence of Lemma 4.1 is that the isotopy class of a spanning surface does not depend at all on the location of *BBBB* curves.

Lemma 4.2. *The curves of $F \cap S_+$ and $F \cap S_-$ that are not of type *BBBB*, together with the link L , determine a unique spanning surface F , up to isotopy.*

Proof. By Lemma 4.1, maximal connected collections of *BBBB* regions form disjoint disks when L is not a $(2, n)$ -torus link. The image of the surface in the complement of the disks is determined by the configuration of curves that are not *BBBB* curves. Two Seifert surfaces that agree in the complement of a collection of disks are isotopic, since the link L is not split, and therefore has irreducible

complement. Thus, the isotopy class of the spanning surface F is determined once the curves that are not *BBBB* are specified. Finally, we note that a $(2, n)$ -torus link has a unique incompressible spanning surface. \square

5. THE NUMBER OF SEIFERT SURFACES OF FIXED GENUS

In this section, we bound the number of curves in $F \cap S_+$ and $F \cap S_-$, and the maximum length of the word associated with each curve. This in turn gives an upper bound for the number of Seifert surfaces of a fixed genus, up to isotopy. Here and further, we assume that genus $g > 0$.

Let C_1 denote the set of curves of $F \cap S_+$ and $F \cap S_-$ that are not of type *BBBB* or *BBSS*, and C_2 the set of *BBSS* curves. We have the following.

Lemma 5.1. $|C_1| \leq 8g - 4$.

Proof. By Remark 3.2, every curve in C_1 gives a contribution to the absolute value of the Euler characteristic of F of at least $\frac{1}{4}$. The Euler characteristic of F is $1 - 2g$. Thus, the maximal possible size of $|C_1|$ is $1 - 2g / (-\frac{1}{4}) = 8g - 4$. \square

Lemma 5.2. *The length of the word associated with any curve in $(F \cap S_\pm)$ is at most $8g - 4$.*

Proof. We consider words associated with the curves in C_1 , since all other words are of length 4. It follows from Lemma 3.1 that each word of C_1 gives a negative contribution to the Euler characteristic of F . The Euler characteristic of F is $1 - 2g$, and each word contributes $1 - s_0/4 - b_0/4$ by Lemma 3.1, where b_0 is the number of *B*'s, and s_0 is the number of *S*'s in the word. Thus, the longest possible word has length $b_0 + s_0 = 8g - 4$. \square

Lemma 5.3. $|C_2| \leq 4g - 4$.

Proof. Every *BBSS* curve contains two saddles. Theorem 3 of [17] gives an upper bound of $(-\chi(F) - b) = 2g - 2$ for the number of saddles in a minimal complexity diagram, where b is the number of boundary components of F . Each *BBSS* curve meets two saddles, and each saddle meets four curves, so at most $4g - 4$ *BBSS* words are associated with the surface F . Thus, $|C_2| \leq 2(2g - 2) = 4g - 4$. \square

Theorem 5.4. *For a prime non-split alternating link L with n crossings, the number of isotopy classes of genus- g Seifert surfaces is at most $(4n)^{64g^2 - 48g}$.*

Proof. In each isotopy class we choose a surface that is in minimizing standard position. There are at most $8g - 4$ curves in C_1 by Lemma 5.1 and $4g - 4$ curves in C_2 by Lemma 5.3. We count the number of possible configurations these curves can realize, up to isotopy of the curves.

Each of the curves in C_1 has an associated word with length at most $8g - 4$ by Lemma 5.2. Consider a curve of $F \cap S_+$. The link L has n crossings, and each crossing gives rise to two choices for the location of an *S* adjacent to that

crossing. There are $2n$ edges in the link diagram, each edge having two sides, giving $4n$ choices for where an interior arc meets an edge. Choosing successive sides of saddles and edges determines a curve up to isotopy. Therefore, the number of isotopy classes of a curve in C_1 , each such curve having length at most $8g - 4$, is less than $(4n)^{8g-4}$. For the entire collection of up to $8g - 4$ curves, the total number of isotopy classes is bounded by $(4n)^{(8g-4)(8g-4)}$. Similarly, the number of configurations for curves in C_2 is bounded by $(4n)^{(4)(4g-4)}$. Once the curves in C_1 and C_2 are fixed, Lemma 4.2 shows that the spanning surface is determined up to isotopy. This gives an upper bound of $(4n)^{64g^2-48g}$ possible isotopy classes for F . \square

Remark 5.5. We have not used the orientability of F in the arguments above, except when we replace Euler characteristic with genus. Therefore, a similar upper bound holds for the number of spanning surfaces, oriented or not, if genus g is replaced by $(1 - \chi(F))/2$.

Remark 5.6. The bound in Theorem 5.4 is polynomial in n when the genus is fixed. Thus, the number of genus- g Seifert surfaces for a link is bounded by a polynomial function of the number of crossings as L varies. However, for a fixed link L , the number of surfaces can grow exponentially with the genus. This can be compared with the results on immersed closed surfaces in closed hyperbolic 3-manifolds [12], [14], and with similar results for closed surfaces in alternating links [8]. We note that J. Banks constructed explicit families of prime alternating knots for which the number of minimal genus Seifert surfaces grows exponentially with the genus ([1]).

Remark 5.7. The Kakimizu complex of a link L is a simplicial complex whose vertices correspond to isotopy classes of minimal genus Seifert surfaces of L , and edges correspond to disjoint (up to isotopy) pairs of such surfaces ([11]). Theorem 5.4 gives an upper bound on the number of vertices in a connected component of this complex.

Acknowledgements. Each of the authors was partially supported by the National Science Foundation (with respective grant nos. DMS-1758107, DMS-1664587, and DMS-1406588 and DMS-1664425). The third author was also supported by funding from the Okinawa Institute of Science and Technology.

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JOEL HASS & ABIGAIL THOMPSON:
 Department of Mathematics
 University of California, Davis
 One Shields Ave
 Davis, CA 95616, USA
 E-MAIL: hass@math.ucdavis.edu;
thompson@math.ucdavis.edu

ANASTASIIA TSVIETKOVA:
 Department of Mathematics and Computer Science
 Rutgers University-Newark
 101 Warren Street
 Newark, NJ 07102, USA
 E-MAIL: a.tsviet@rutgers.edu

Received: October 6, 2018.