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Precoloring extension of Vizing's Theorem for multigraphs

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ABSTRACT

Let G be a graph with maximum degree $\Delta(G)$ and maximum multiplicity $\mu(G)$. Vizing and Gupta, independently, proved in the 1960s that the chromatic index of G is at most $\Delta(G) + \mu(G)$. The distance between two edges e and f in G is the length of a shortest path connecting an endvertex of e and an endvertex of f. A distance-t matching is a set of edges having pairwise distance at least t. Albertson and Moore conjectured that if G is a simple graph, using the palette $\{1, \dots, \Delta(G) + 1\}$, any precoloring on a distance-3 matching can be extended to a proper edge coloring of G. Edwards et al. proposed the following stronger conjecture: For any graph G, using the palette $\{1, \dots, \Delta(G) + \mu(G)\}$, any precoloring on a distance-2 matching can be extended to a proper edge coloring of G. Girão and Kang verified the conjecture of Edwards et al. for distance-9 matchings. In this paper, we improve the required distance from 9 to 3 for multigraphs G with $\mu(G) \geq 2$.

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1. Introduction

In this paper, we follow the book [1] of Stiebitz et al. for notation and terminologies. Graphs in this paper are finite, undirected, without loops, but may have multiple edges. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ are respectively the vertex set and the edge set of G. Let $\Delta(G)$ and $\mu(G)$ be respectively the maximum degree and the maximum multiplicity of G. Let $[k] := \{1, \dots, k\}$ be a **palette** of k available colors. A **k-edge-coloring** of G is a map that assigns to every edge of G a color from the palette $[k]$ such that no two adjacent edges receive the same color (the edge coloring is also called **proper**). Denote by $\mathcal{C}^k(G)$ the set of all k-edge-colorings of G. The **chromatic index** $\chi'(G)$ is the least integer k such that $\mathcal{C}^k(G) \neq \emptyset$. The distance between two edges e and f in G is the length of a shortest path connecting an endvertex of e and an endvertex of f. A **distance-t matching** is a set of edges having pairwise distance at least t. Following this definition, a matching is a distance-1 matching and an induced matching is a distance-2 matching. For a matching M, we use $V(M)$ to denote the set of vertices saturated by M.

In the 1960s, Vizing [2] and, independently, Gupta [3] proved that $\Delta(G) \le \chi'(G) \le \Delta(G)+\mu(G)$, which is commonly called Vizing's Theorem. Vizing's Theorem plays an important role in graph edge coloring. Using the palette $[\Delta(G) + \mu(G)]$, when can we extend a precoloring on a given edge set $F \subseteq E(G)$ to a proper edge coloring of G? Albertson and Moore [4] conjectured that if G is a simple graph, using the palette $[\Delta(G) + 1]$, any precoloring on a distance-3 matching can be extended to a proper edge coloring of G. Edwards et al. [5] proposed a stronger conjecture: **For any graph G, using the palette $[\Delta(G) + \mu(G)]$, any precoloring on a distance-2 matching can be extended to a proper edge coloring of G.** Girão and Kang [6] verified the conjecture of Edwards et al. for distance-9 matchings. In this paper, we improve the required distance from 9 to 3 for multigraphs with the maximum multiplicity at least 2 as follows.

Theorem 1.1. *Let G be a multigraph with $\mu(G) \ge 2$. Using the palette $[\Delta(G)+\mu(G)]$, any precoloring on a distance-3 matching M in G can be extended to a proper edge coloring of G.*

The **density** of a graph G, denoted $\Gamma(G)$, is defined as

$$\Gamma(G) = max\left\{\frac{2|E(H)|}{|V(H)|-1} : H \subseteq G,\ |V(H)| \ge 3 \text{ and } |V(H)| \text{ is odd}\right\}$$

if $|V(G)| \ge 3$ and $\Gamma(G) = 0$ otherwise. Note that for any $X \subseteq V(G)$ with odd $|X| \ge 3$, we have $\chi'(G[X]) \ge \frac{2|E(G[X])|}{|X|-1}$, where $G[X]$ is the subgraph of G induced by X. Therefore, $\chi'(G) \ge \lceil \Gamma(G) \rceil$. So, besides the maximum degree, the density provides another lower bound on the chromatic index of a graph. In the 1970s, Goldberg [7] and Seymour [8] independently conjectured that actually $\chi'(G) = \lceil \Gamma(G) \rceil$ provided $\chi'(G) \ge \Delta(G)+2$. The conjecture was commonly referred to as one of the most challenging problems in graph chromatic theory [1]. In joint work with Zang, two authors of this paper, Chen and Jing gave a proof of the Goldberg–Seymour Conjecture recently [9]. We assume that the Goldberg–Seymour Conjecture is true in this paper.

We will prove Theorem 1.1 in Section 4. In Section 2 we introduce some new structural properties of dense subgraphs. In Section 3 we define a general multi-fan and obtain some generalizations of Vizing's Theorem.

2. Dense subgraphs

Throughout the rest of this paper, we reserve the notation Δ and μ for the maximum degree and the maximum multiplicity of the graph G, respectively. For $u \in V(G)$, let $d_G(u)$ denote the **degree** of u in G. For a vertex set $N \subseteq V(G)$, let $G-N$ be the graph obtained from G by deleting all the vertices in N and edges incident with them. For an edge set $F \subseteq E(G)$, let $G - F$ be the graph obtained from G by deleting all the edges in F but keeping their endvertices. If $F = \{e\}$, we simply write $G - e$. Similarly, we let $G + e$ be the graph obtained from G by adding the edge e to $E(G)$. For disjoint $X, Y \subseteq V(G)$, $E_G(X, Y)$ is the set of edges of G with one endvertex in X and the other in Y. If $X = \{x\}$, we simply write $E_G(x, Y)$. For $X \subseteq V(G)$, the edge set $\partial_G(X) := E_G(X, V(G)\backslash X)$ is called the **boundary** of X in G. For a subgraph H of G, we simply write $\partial_G(H)$ for $\partial_G(V(H))$.

Let G be a graph, $v \in V(G)$ and $\varphi \in \mathcal{C}^k(G)$ for some positive integer k. We define $\varphi(v) = \{\varphi(f) : f \in E(G)$ and f is incident with $v\}$ $\overline{\varphi}(v) = [k] \setminus \varphi(v)$. We call $\varphi(v)$ the set of colors **present** at v and $\overline{\varphi}(v)$ the set of colors **missing** at v. For a vertex set $X \subseteq V(G)$, define $\overline{\varphi}(X) = \bigcup_{v \in X} \overline{\varphi}(v)$. A vertex set $X \subseteq V(G)$ is called φ-**elementary** if $\overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset$ for every two distinct vertices $u, v \in X$. The set X is called φ-**closed** if each color on edges from $\partial_G(X)$ is present at each vertex of X. Moreover, the set X is called **strongly** φ-**closed** if X is φ-closed and colors on edges from $\partial_G(X)$ are pairwise distinct. For a subgraph H of G, let φ_H or $(\varphi)_H$ be the edge coloring of G restricted on H. We say a subgraph H of G is φ-elementary, φ-closed and strongly φ-closed, if $V(H)$ is φ-elementary, φ-closed and strongly φ-closed, respectively. Clearly, if H is φ_H-elementary then H is φ-elementary, but the converse is not true as the edges in $\partial_G(H)$ are removed when we consider φ_H.

A subgraph H of G is k-**dense** if $|V(H)|$ is odd and $|E(H)| = (|V(H)| - 1)k/2$. Moreover, H is a **maximal** k-**dense subgraph** if there does not exist a k-dense subgraph H' containing H as a proper subgraph. An edge e of a graph G is called a **k-critical edge** if $k = \chi'(G-e) < \chi'(G) = k+1$. A graph G is called **k-critical** if $\chi'(H) < \chi'(G) = k+1$ for each proper subgraph H of G. It is easy to see that a connected graph G is k-critical if and only if every edge of G is k-critical. For $e \in E(G)$, let $V(e)$ denote the set of the two endvertices of e. The **diameter** of a graph G, denoted diam(G), is the greatest distance between any pair of vertices in $V(G)$. An i-**edge** is an edge colored with the color i.

Lemma 2.1 ([10]). *Given a graph G, if $\chi'(G) = k \geq \Delta(G)+1$, then distinct maximal k-dense subgraphs of G are pairwise vertex-disjoint.*

Lemma 2.2. *Let G be a graph with $\chi'(G) = k$ and H be a k-dense subgraph of G. Then H is an induced subgraph of G with $\chi'(H) = \Gamma(H) = k$. Furthermore, for any coloring $\varphi \in \mathcal{C}^k(G)$ and $\psi \in \mathcal{C}^k(H)$, H is strongly φ-closed and ψ-elementary.*

Proof. Since H is k-dense, by the definition, $|E(H)| = \frac{|V(H)|-1}{2}k$. Thus $k \leq \Gamma(H) \leq \chi'(H) \leq \chi'(G) = k$ implying $\chi'(H) = \Gamma(H) = k$. Thus H is an induced subgraph of G, since otherwise there exists a subgraph H' of G with $V(H') = V(H)$ such that $\chi'(H') \geq \Gamma(H') > k$, a contradiction to $\chi'(H') \leq \chi'(G) = k$. Since H has odd order, a maximum matching in H has size at most $(|V(H)| - 1)/2$. Therefore, under any k-edge-coloring φ of G, each color class in H is a matching of size exactly $(|V(H)| - 1)/2$. Thus every color in $[k]$ is missing at exactly one vertex of H or it appears exactly once in $\partial_G(H)$. Consequently, H is strongly φ-closed. For any $\psi \in \mathcal{C}^k(H)$, the same argument as above shows that H is ψ-elementary. □

The following lemma is a consequence of the Goldberg–Seymour Conjecture.

Lemma 2.3. *Let G be a multigraph and $e \in E(G)$. If e is a k-critical edge of G and $k \geq \Delta(G)+1$, then $G-e$ has a k-dense subgraph H containing $V(e)$ such that e is also a k-critical edge of $H+e$.*

Proof. Clearly, $\chi'(G) = k+1$ and $\chi'(G-e) = k$. By the assumption of the Goldberg–Seymour Conjecture, $\chi'(G) = \lceil \Gamma(G) \rceil = k+1$. As $\lceil \Gamma(G) \rceil = k+1$, by the definition of density, G has a subgraph H^* of odd order such that $|E(H^*)| > (|V(H^*)|-1)k/2$. Thus $\chi'(G) \geq \chi'(H^*) > \frac{2|E(H^*)|}{|V(H^*)|-1} = k$. Since $\chi'(G-e) = k$, it follows that $e \in E(H^*)$. On the other hand, we have $\frac{2|E(H^*-e)|}{|V(H^*-e)|-1} \leq \lceil \Gamma(H^*-e) \rceil \leq \chi'(H^*-e) \leq \chi'(G-e) = k$, which in turn gives $|E(H^*-e)| \leq (|V(H^*)|-1)k/2$. Thus $|E(H^*-e)| = (|V(H^*)|-1)k/2$. Then $k \leq \lceil \Gamma(H^*-e) \rceil \leq \chi'(H^*-e) \leq \chi'(G-e) = k$ and $k+1 \leq \lceil \Gamma(H^*) \rceil \leq \chi'(H^*) \leq \chi'(G) = k+1$, which implies that $k = \chi'(H^*-e) < \chi'(H^*) = k+1$. Thus $H := H^* - e$ is a k-dense subgraph containing $V(e)$, and e is also a k-critical edge of $H+e$. □

Lemma 2.4. *Let G be a multigraph with $\chi'(G) = k+1 \geq \Delta(G)+2$ and e be a k-critical edge of G. We have the following statements.*

(a) $G-e$ has a unique maximal k-dense subgraph H containing $V(e)$, and e is also a k-critical edge of $H+e$.

(b) For any $\varphi \in \mathcal{C}^k(G-e)$, H is φ_H-elementary and strongly φ-closed.

(c) If $\chi'(G) = \Delta(G)+\mu(G)$, then $\Delta(H+e) = \Delta(G)$, $\mu(H+e) = \mu(G)$ and diam$(H+e) \leq$ diam$(H) \leq 2$.

Proof. By Lemma 2.3, $G-e$ contains a k-dense subgraph H containing $V(e)$ and e is also a k-critical edge of $H+e$. We may assume that H is a maximal k-dense subgraph, and the uniqueness of H is a direct consequence of Lemma 2.1. This proves (a). By applying Lemma 2.2 on $G-e$, we immediately have statement (b).

For (c), by (a) and Vizing's Theorem, $\Delta(G)+\mu(G)=\chi'(G)=\chi'(H+e)\leq \Delta(H+e)+\mu(H+e)\leq \Delta(G)+\mu(G)$ implying that $\Delta(H+e)=\Delta(G)=\Delta$ and $\mu(H+e)=\mu(G)=\mu$. For any $\varphi\in\mathcal{C}^k(G-e)$, H is φ_H-elementary by (b). For any $x\in V(H)$, with respect to φ_H, all the colors missing at other vertices of H present at x. Note that $k=\Delta+\mu-1$. For each vertex $v\in V(H)$, we have that $|\overline{\varphi}_H(v)|=k-d_H(v)\geq k-\Delta=\mu-1$ if $v\notin V(e)$, and $|\overline{\varphi}_H(v)|=k-d_H(v)+1\geq k-\Delta+1\geq(\mu-1)+1$ if $v\in V(e)$. Denote $|V(H)|$ by n. We then have $d_H(x)\geq|\bigcup_{v\in V(H),v\neq x}\overline{\varphi}_H(v)|\geq(k-\Delta)(n-1)+1=(\mu-1)(n-1)+1$.

Since $\mu(H)\leq\mu(G)=\mu$, we get $|N_H(x)|\geq\frac{d_H(x)}{\mu}\geq\frac{(\mu-1)(n-1)+1}{\mu}$, where $N_H(x)$ is the neighbor set of x in H. Since $\mu\geq 2$, we have $\frac{(\mu-1)(n-1)+1}{\mu}\geq\frac{n}{2}$. Hence, every vertex in H is adjacent to at least half vertices in H. Consequently, every two vertices of H share a common neighbor, which in turn gives $\mathrm{diam}(H)\leq 2$. This proves (c). □

The following technical lemma will be used several times in our proof.

Lemma 2.5. *Let G be a graph with $\chi'(G)=k$ and H be a k-dense subgraph of G. Let ψ and φ respectively be k-edge-colorings of H and $G-E(H)$ such that colors on edges in $\partial_G(H)$ are pairwise distinct under φ. Then the following two statements hold.*

(a) If $k\geq\Delta(G)$, then by renaming color classes of ψ on $E(H)$, we can obtain a (proper) k-edge-coloring of G by combining φ and the modified coloring based on ψ.

(b) For any fixed color $i\in[k]$, if $k\geq\Delta(G)+1$, then by renaming other color classes of ψ on $E(H)$ we can obtain a coloring of G such that all color classes are matchings except the i-edges. The only exception is as follows: exactly one i-edge from $E(H)$ and exactly one i-edge from $\partial_G(H)$ share an endvertex.

Proof. Since $\chi'(G)=k$ and H is k-dense, $\chi'(H)=k$ and H is ψ-elementary by Lemma 2.2. We first show that statement (b) is a consequence of statement (a). Let M_i be the set of edges of G colored by i. Then we know that $|M_i\cap E(H)|=\frac{1}{2}(|V(H)|-1)$ by H being ψ-elementary. Thus $H-M_i$ is $(k-1)$-dense. Now the first part of statement (b) is a consequence of statement (a) by having $G-M_i$ in the place of G. The second part of statement (b) follows easily by the assumption that edges in $\partial_G(H)$ are pairwise distinct under φ. Thus we only show statement (a) below.

We permute some color names of ψ step by step to get a k-edge-coloring ψ^* of H such that $\varphi(v)\subseteq\overline{\psi^*}(v)$ for any $v\in V(H)$. Then the combination of ψ^* and ψ gives a desired k-edge-coloring of G. Let $w\in V(H)$ and $i\in\overline{\psi}(w)\cap\varphi(w)$. By the assumptions of statement (a) and H being ψ-elementary, we have the following properties:

$$|\overline{\psi}(w)|=k-d_H(w)\geq\Delta(G)-d_H(w)\geq d_{G-E(H)}(w)=|\varphi(w)|, \tag{1}$$

$$i\notin\overline{\psi}(u)\cup\varphi(u)\quad\text{for any } u\in V(H)\setminus\{w\}. \tag{2}$$

Let $v\in V(H)$ such that $\varphi(v)\setminus\overline{\psi}(v)\neq\emptyset$. Let $s=|\varphi(v)\setminus\overline{\psi}(v)|$, and $\varphi(v)\setminus\overline{\psi}(v)=\{i_1,\ldots,i_s\}$. By (1), $\overline{\psi}(v)\setminus\varphi(v)$ has a subset $\{j_1,\ldots,j_t\}$ of t distinct elements with $t\geq s$. We now modify ψ as ψ_1 by exchanging the color names i_p and j_p for each $p\in[s]$. The graph H is still ψ_1-elementary by Lemma 2.2 and now we have $\varphi(v)\subseteq\overline{\psi_1}(v)$. By (2), we know that $|\overline{\psi_1}(u)\cap\varphi(u)|\geq|\overline{\psi}(u)\cap\varphi(u)|$ for any $u\in V(H)\setminus\{v\}$. Repeating this process at most another $|V(H)|-1$ times gives us a desired coloring ψ^* of H. □

3. Refinements of multi-fans and some consequences

We first recall Kempe-chains and related terminologies. Let φ be a k-edge-coloring of G using the palette $[k]$. Given two distinct colors α,β, an (α,β)**-chain** is a component of the subgraph induced by edges assigned color α or β in G, which is either an even cycle or a path. We call the operation that swaps the colors α and β on an (α,β)-chain the **Kempe change**. Clearly, the resulting

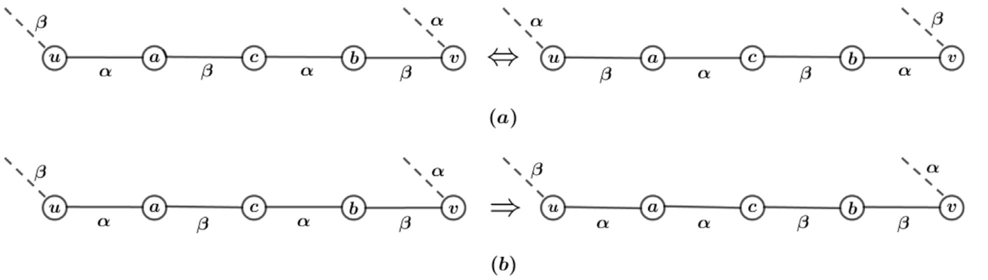

Fig. 1. (a) The Kempe change on one (α,β)-chain $P_u(\alpha,\beta)$ or $P_v(\alpha,\beta)$; (b) The Kempe change on one subchain $P_{[a,b]}(\alpha,\beta)$. (The dashed lines represent missing colors at vertices).

coloring after a Kempe change is still a (proper) k-edge-coloring. Furthermore, we say that a chain has **endvertices** u and v if the chain is a path connecting vertices u and v. For a vertex $v \in V(G)$, we denote by $P_v(\alpha,\beta)$ the unique (α,β)-chain containing the vertex v. For two vertices $u, v \in V(G)$, the two chains $P_u(\alpha,\beta)$ and $P_v(\alpha,\beta)$ are either identical or disjoint. (See Fig. 1(a).) More generally, for an (α,β)-chain, if it is a path and it contains two vertices a and b, we let $P_{[a,b]}(\alpha,\beta)$ be its subchain with endvertices a and b. The operation of swapping colors α and β on the subchain $P_{[a,b]}(\alpha,\beta)$ is still called a Kempe change, but the resulting coloring may no longer be a proper edge coloring. (See Fig. 1(b).)

Let G be a graph with an edge $e \in E_G(x,y)$, and φ be a proper edge coloring of G or $G-e$. A sequence $F = (x, e_0, y_0, e_1, y_1, \ldots, e_p, y_p)$ with integer $p \ge 0$ consisting of vertices and distinct edges is called a (general) **multi-fan** at x with respect to e and φ if $e_0 = e$, $y_0 = y$, for each $i \in [p]$, $e_i \in E_G(x, y_i)$ and there is a vertex y_j with $0 \le j \le i-1$ such that $\varphi(e_i) \in \overline{\varphi}(y_j)$. Note that $y_i = y_j$ can happen for distinct i and j in F, and that the definition of a multi-fan in this paper is slightly general than the one in [1] since the edge e may be colored in G. We say a multi-fan F is **maximal** if there is no multi-fan containing F as a proper subsequence. Similarly, we say a multi-fan F is **maximal without any** i**-edge** if F does not contain any i-edge and there is no multi-fan without any i-edge containing F as a proper subsequence. The set of vertices and edges contained in F are denoted by $V(F)$ and $E(F)$, respectively. Let $e_G(x,y) = |E_G(x,y)|$ for $x, y \in V(G)$. Note that a multi-fan may have repeated vertices. By $e_F(x, y_i)$ for some $y_i \in V(F)$ we mean the number of edges joining x and y_i in F.

Let $s \ge 0$ be an integer. A **linear sequence** $S = (y_0, e_1, y_1, \ldots, e_s, y_s)$ at x from y_0 to y_s in G is a sequence consisting of distinct vertices and distinct edges such that $e_i \in E_G(x, y_i)$ for $i \in [s]$ and $\varphi(e_i) \in \overline{\varphi}(y_{i-1})$ for $i \in [s]$. Clearly for any $y_j \in V(F)$, the multi-fan F contains a linear sequence at x from y_0 to y_j (take a shortest sequence $(y_0, e_1, y_1, \ldots, e_j, y_j)$ of vertices and edges with the property that $e_i \in E_G(x, y_i) \cap E(F)$ for $i \in [j]$ and $\varphi(e_i) \in \overline{\varphi}(y_{i-1})$ for $i \in [j]$). The following local edge recoloring operation will be used in our proof. A **shifting** from y_i to y_j in the linear sequence S is an operation that replaces the current color of e_t by the color of e_{t+1} for each $i \le t \le j-1$ with $1 \le i < j \le s$. Note that the shifting does not change the color of e_j, where e_j joins x and y_j, so the resulting coloring after a shifting is not a proper coloring. In our proof we will uncolor or recolor the edge e_j to make the resulting coloring proper. We also denote by $V(S)$ and $E(S)$ the set of vertices and the set of edges contained in the linear sequence S, respectively. A Δ-**vertex** in G is a vertex with degree exactly Δ in G. A Δ-**neighbor** of a vertex v in G is a neighbor of v that is a Δ-vertex in G.

Lemma 3.1 ([1,11]). *Let G be a graph, $e \in E_G(x,y)$ be a k-critical edge and $\varphi \in \mathcal{C}^k(G-e)$ with $k \ge \Delta(G)$. Let $F = (x, e, y_0, e_1, y_1, \ldots, e_p, y_p)$ be a multi-fan at x with respect to e and φ, where $y_0 = y$. Then the following statements hold.*

(a) $V(F)$ is φ-elementary, and each edge in $E(F)$ is a k-critical edge of G.

(b) If $\alpha \in \overline{\varphi}(x)$ and $\beta \in \overline{\varphi}(y_i)$ for $0 \le i \le p$, then $P_x(\alpha,\beta) = P_{y_i}(\alpha,\beta)$.

(c) If F is a maximal multi-fan at x with respect to e and φ, then x is adjacent in G to at least $\chi'(G) - d_G(y) - e_G(x, y) + 1$ vertices z in $V(F)\backslash\{x, y\}$ such that $d_G(z) + e_G(x, z) = \chi'(G)$.

Lemma 3.2. *Let G be a multigraph with maximum degree Δ and maximum multiplicity $\mu \geq 1$. Let $e \in E_G(x, y)$ and $k = \Delta + \mu - 1$.*

Assume that $\chi'(G) = k + 1$, e is k-critical and $\varphi \in \mathcal{C}^k(G - e)$. Let $F = (x, e, y_0, e_1, y_1, \ldots, e_p, y_p)$ be a multi-fan at x with respect to e and φ, where $y_0 = y$. Then the following statements hold.

(a) If F is maximal, then x is adjacent in G to at least $\Delta + \mu - d_G(y) - e_G(x, y) + 1$ vertices z in $V(F)\backslash\{x, y\}$ such that $d_G(z) = \Delta$ and $e_G(x, z) = \mu$.

(b) If F is maximal, $d_G(y) = \Delta$ and x has only one Δ-neighbor z' in G from $V(F)\backslash\{x, y\}$, then $e_F(x, z) = e_G(x, z) = \mu$ for all $z \in V(F)\backslash\{x\}$ and $d_G(z) = \Delta - 1$ for all $z \in V(F)\backslash\{x, y, z'\}$.

(c) For $i \in [k]$ and $i \notin \overline{\varphi}(y)$, if F is maximal without any i-edge, then F not containing any Δ-vertex of G from $V(F)\backslash\{x, y\}$ implies that $d_G(y) = \Delta$, and there exists a vertex $z^* \in V(F)\backslash\{x, y\}$ with $i \in \overline{\varphi}(z^*)$ such that $d_G(z^*) = \Delta - 1$.

Assume that $\chi'(G) = k$, $\varphi \in \mathcal{C}^k(G)$ and $V(G)$ is φ-elementary. Then the following statement holds.

(d) If a multi-fan F' is maximal at x with respect to e and φ in G, then x having no Δ-neighbor in G from $V(F')$ implies that $d_G(z) = \Delta - 1$ for all $z \in V(F')\backslash\{x\}$ and every edge in F' is colored by a missing color at some vertex in $V(F')$. Furthermore, for $i \in [k]$ and $\varphi(e) \notin \overline{\varphi}(V(F'))$, if F' is maximal without any i-edge, then F' not containing any Δ-vertex in G from $V(F')\backslash\{x\}$ implies that there exists a vertex $z^* \in V(F')\backslash\{x\}$ with $i \in \overline{\varphi}(z^*)$ such that $d_G(z^*) = \Delta - 1$.

Proof. For statements (a), (b) and (c), $V(F)$ is φ-elementary by Lemma 3.1(a). As F is maximal, for any $\alpha \in \overline{\varphi}(V(F))$, we know that there exists $z \in V(F)$ such that $\varphi(xz) = \alpha$. As a consequence, we know that $\sum_{z\in V(F)\backslash\{x\}} e_F(x, z) = 1 + \sum_{z\in V(F)\backslash\{x\}} |\overline{\varphi}(z)|$, where the term 1 counts the uncolored edge e. Statement (a) holds easily by Lemma 3.1(c). Assume that there are q distinct vertices in $V(F)\backslash\{x\}$.

For (b), we have

$$\begin{aligned} q\mu &\geq \sum_{z\in V(F)\backslash\{x\}} e_G(x, z) \geq \sum_{z\in V(F)\backslash\{x\}} e_F(x, z) = 1 + \sum_{z\in V(F)\backslash\{x\}} |\overline{\varphi}(z)| \\ &\geq 1 + (k - \Delta + 1) + (k - \Delta) + (q - 2)(k - \Delta + 1) = q(k - \Delta + 1) = q\mu, \end{aligned}$$

as $|\overline{\varphi}(y)| = k - \Delta + 1$, $|\overline{\varphi}(z')| = k - \Delta$ and $|\overline{\varphi}(z)| \geq k - \Delta + 1$ for $z \in V(F)\backslash\{x, y, z'\}$. Therefore, $e_F(x, z) = e_G(x, z) = \mu$ for each $z \in V(F)\backslash\{x\}$ and $d_G(z) = \Delta - 1$ for each $z \in V(F)\backslash\{x, y, z'\}$. This proves (b).

Next for (c), suppose first that $i \notin \overline{\varphi}(z^*)$ for any $z^* \in V(F)\backslash\{x\}$. Then F is maximal without any i-edge implies that F is maximal. By (a), x has at least one Δ-neighbor in F from $V(F)\backslash\{x, y\}$. This gives a contradiction to the assumption that F does not contain any Δ-vertex of G from $V(F)\backslash\{x, y\}$. Thus we have $i \in \overline{\varphi}(z^*)$ for some $z^* \in V(F)\backslash\{x\}$. As $i \notin \overline{\varphi}(y)$ by the assumption in the statement, we know that $z^* \neq y$. Since $V(F)$ is φ-elementary, x must be incident with an i-edge. Since now there is no i-edge in F and $i \in \overline{\varphi}(z^*)$, we have

$$\begin{aligned} q\mu &\geq \sum_{z\in V(F)\backslash\{x\}} e_G(x, z) \geq \sum_{z\in V(F)\backslash\{x\}} e_F(x, z) = 1 + (|\overline{\varphi}(z^*)| - 1) + \sum_{z\in V(F)\backslash\{x, z^*\}} |\overline{\varphi}(z)| \\ &= \sum_{z\in V(F)\backslash\{x\}} |\overline{\varphi}(z)| \geq 1 + k - \Delta + (q - 1)(k - \Delta + 1) = q(k - \Delta + 1) = q\mu. \end{aligned}$$

Therefore, $d_G(y) = \Delta$ and $d_G(z) = \Delta - 1$ for each $z \in V(F)\backslash\{x, y\}$. This proves (c).

Now for the first part of (d), as $\varphi(e)$ may be contained in $\overline{\varphi}(V(F'))$, we have

$$\begin{aligned} q\mu &\geq \sum_{z\in V(F')\backslash\{x\}} e_G(x, z) \geq \sum_{z\in V(F')\backslash\{x\}} e_{F'}(x, z) \geq \sum_{z\in V(F')\backslash\{x\}} |\overline{\varphi}(z)| \\ &\geq q(k - \Delta + 1) = q\mu, \end{aligned}$$

as $|\overline{\varphi}(z)| \geq k - \Delta + 1$ for $z \in V(F')\backslash\{x\}$. Therefore, $e_{F'}(x, z) = e_G(x, z) = \mu$ and $d_G(z) = \Delta - 1$ for all $z \in V(F')\backslash\{x\}$, and every edge in F' is colored by a missing color at some vertex in $V(F')$. For the furthermore part of (d), we also have that there exists a vertex $z^* \in V(F')\backslash\{x\}$ with $i \in \overline{\varphi}(z^*)$, since otherwise, x has at least one Δ-neighbor in F' from $V(F')\backslash\{x,y\}$, a contradiction. Since now $\varphi(e) \notin \overline{\varphi}(V(F'))$ and there is no i-edge in F' with $i \in \overline{\varphi}(z^*)$, we have

$$q\mu \geq \sum_{z\in V(F')\backslash\{x\}} e_G(x,z) \geq \sum_{z\in V(F')\backslash\{x\}} e_{F'}(x,z) = 1 + (|\overline{\varphi}(z^*)| - 1) + \sum_{z\in V(F')\backslash\{x,z^*\}} |\overline{\varphi}(z)|$$
$$= \sum_{z\in V(F')\backslash\{x\}} |\overline{\varphi}(z)| \geq q(k - \Delta + 1) = q\mu.$$

Therefore, $d_G(z) = \Delta - 1$ for each $z \in V(F')\backslash\{x\}$. This proves (d). □

Let G be a graph with maximum degree Δ and maximum multiplicity μ. Berge and Fournier [12] strengthened the classical Vizing's Theorem by showing that if M^* is a maximal matching of G, then $\chi'(G - M^*) \leq \Delta + \mu - 1$. An edge $e \in E_G(x,y)$ is **fully** G**-saturated** if $d_G(x) = d_G(y) = \Delta$ and $e_G(x,y) = \mu$. For every graph G with $\chi'(G) = \Delta + \mu$, observe that G contains a $(\Delta + \mu - 1)$-critical subgraph H with $\chi'(H) = \Delta + \mu$ and $\Delta(H) = \Delta$ by Lemma 2.4(c), and G contains at least two fully G-saturated edges by Lemma 3.2(a).

Lemma 3.3. *For a fixed matching M of a graph G, if $\mu(G) \geq 2$ and $\chi'(G - M) = \Delta(G) + \mu(G)$, then there exists a matching M^* of $G - V(M)$ such that $\chi'(G - (M \cup M^*)) = \Delta(G) + \mu(G) - 1 =: k$ and every edge $e \in M^*$ is k-critical and fully G-saturated in the graph $H_e + e$, where H_e is the unique maximal k-dense subgraph of $G - (M \cup M^*)$ containing $V(e)$.*

Proof. Let M^* be a matching of $G - V(M)$ consisting of fully G-saturated edges. We further choose M^* such that M^* is maximal. Then $G - (M \cup M^*)$ has no fully G-saturated edge by the maximality of M^*. We claim that $\chi'(G - (M \cup M^*)) = k$. For otherwise, we have $\chi'(G - (M \cup M^*)) = k + 1 = \Delta + \mu$. We let G' be a $(\Delta + \mu - 1)$-critical subgraph of G. Clearly, we have $\Delta(G') = \Delta$. Let $e \in E_{G'}(x,y)$ such that $d_{G'}(x) = \Delta$. By considering a maximal multi-fan at x with respect to a coloring $\varphi \in \mathcal{C}^k(G' - e)$ and e, Lemma 3.2(a) implies that x has a Δ-neighbor z in G' for which $e_{G'}(x,z) = \mu$. Thus any edge in $E_{G'}(x,z)$ is a fully G-saturated edge, a contradiction to the choice of M^*.

Thus $\chi'(G - (M \cup M^*)) = k$. If there exists $e \in M^*$ such that $\chi'(G - (M \cup M^*\backslash\{e\})) = k$, we remove e out of M^*. Thus we may assume that for each $e \in M^*$, $\chi'(G - (M \cup M^*\backslash\{e\})) = k + 1$, i.e., each e is a k-critical edge of $G - (M \cup M^*\backslash\{e\})$. By Lemma 2.4$(a)$, there exists a unique maximal k-dense subgraph H_e of $G - (M \cup M^*)$ such that $V(e) \subseteq V(H_e)$ and e is also a k-critical edge of $H_e + e$. Notice that $\Delta(H_e + e) = \Delta$ and $\mu(H_e + e) = \mu$ by Lemma 2.4(c). It is now only left to show that each $e \in M^*$ is full G-saturated in the graph $H_e + e$. Suppose on the contrary that there exists $e \in M^*$ such that e is not fully G-saturated in $H_e + e$.

Since e is a k-critical edge of $G - (M \cup M^*\backslash\{e\})$, we let $\varphi \in \mathcal{C}^k(G - (M \cup M^*))$. By Lemma 2.2, H_e is φ_{H_e}-elementary and strongly φ-closed. Let $V(e) = \{x,y\}$ and F_x be a maximal multi-fan at x with respect to e and φ_{H_e}. By Lemma 3.2(a), x has a Δ-neighbor, say x_1, in H_e from $V(F_x)\backslash\{x,y\}$. By Lemma 3.1(a), the edge $e_{xx_1} \in E_G(x,x_1)$ in F_x is also a k-critical edge of $H_e + e$. By Lemma 3.2(a) again, in a maximal multi-fan F_{x_1} at x_1 with respect to e_{xx_1} there exists a fully G-saturated edge e'. Let $M' = (M^*\backslash\{e\}) \cup \{e'\}$. Since every vertex of $V(M \cup M^*)$ has degree less than Δ in $G - (M \cup M^*)$, it follows that $M \cup M'$ is a matching of G. Let $H_{e'} = H_e + e - e'$. Clearly, $H_{e'}$ is also k-dense. Applying Lemma 3.1(a) with respect to the multi-fan F_{x_1}, we see that e' is also a k-critical edge of $H_e + e$. Thus $\chi'(H_{e'}) = k$ and $H_{e'}$ is also an induced subgraph of $G - (M \cup M')$ by Lemma 2.2. Moreover, $H_{e'}$ is a maximal k-dense subgraph of $G - (M \cup M')$, since otherwise there exists a k-dense subgraph H' containing $H_{e'}$ as a proper subgraph which implies that the k-dense subgraph $H' + e' - e$ is also a k-dense subgraph containing H_e as a proper subgraph in $G - (M \cup M^*)$, a contradiction to the maximality of H_e. As H_e is strongly φ-closed, colors on edges of $\partial_{G-(M\cup M')}(H_{e'}) = \partial_{G-(M\cup M^*)}(H_e)$ are pairwise distinct. Applying Lemma 2.5(a) on any k-edge-coloring of $H_{e'}$ and the k-edge-coloring of $G - (M \cup M' \cup E(H_{e'}))$, we have $\chi'(G - (M \cup M')) = k$. In order to claim that we can replace e by e' in M^*, and so repeat the same process for every edge f of M^* that is not fully G-saturated in $H_f + f$, where H_f is the maximal k-dense subgraph of $G - (M \cup M^*)$ with $V(f) \subseteq V(H_f)$, we discuss that this replacement will not affect the properties of other edges in M^* as follows.

By Lemmas 2.1 and 2.2, maximal k-dense subgraphs of $G-(M\cup M^*)$ are induced and vertex-disjoint. Thus for any $f\in M^*\backslash\{e\}$, either $V(H_f)\cap V(H_e)=\emptyset$ or $H_f=H_e$. If $V(H_f)\cap V(H_e)=\emptyset$, then H_f is still the induced maximal k-dense subgraph of $G-(M\cup M')$ containing $V(f)$ and f is k-critical in H_f+f. If $H_f=H_e$, then as $H_{e'}$ is an induced maximal k-dense subgraph of $G-(M\cup M')$ with $V(H_e)=V(H_{e'})$, it follows that $H_f+e-e'=H_{e'}$ is the maximal k-dense subgraph of $G-(M\cup M')$ containing $V(f)$ and f is k-critical in $H_f+e-e'+f$ by Lemma 2.4(a). As $V(f)\cap V(e)=\emptyset$ and $V(f)\cap V(e')=\emptyset$, the property that whether or not f is fully G-saturated in H_f+f is not changed after replacing e by e' in M^*. Therefore, by repeating the replacement process as for the edge e above for every edge f of M^* that is not fully G-saturated in H_f+f, we may assume that each edge $e\in M^*$ is fully G-saturated in H_e+e. The proof is completed. □

4. Proof of Theorem 1.1

Proof. Let $k=\Delta+\mu-1$ and $\Phi: M\to[\Delta+\mu]$ be a given precoloring on M. Note that $\chi'(G-M)\le k+1$ by Vizing's Theorem. The conclusion of Theorem 1.1 holds easily if $\chi'(G-M)\le k$ with the reason as follows. For any k-edge-coloring ψ of $G-M$, if there exists $e\in E(G-M)$ such that e is adjacent in G to an edge $f\in M$ (maybe $V(e)=V(f)$) and $\psi(e)=\Phi(f)$, we recolor each such e with the color $\Delta+\mu$ and get a new coloring ψ' of $G-M$. Under ψ', the edges colored by $\Delta+\mu$ form a matching in G since M is a distance-3 matching. Thus the combination of Φ and ψ' is a $(k+1)$-edge-coloring of G. Therefore, in the remainder of the proof, we assume $\chi'(G-M)=k+1$.

Let $M_{\Delta+\mu}$ be the set of edges precolored with $\Delta+\mu$ in M under Φ. For any uncolored matching $M^*\subseteq G-V(M)$ and any $(k+1)$-edge-coloring or k-edge-coloring φ of $G-(M\cup M^*)$, denote the $\Delta+\mu$ color class of φ by $E^{\varphi}_{M^*}$. In particular, $E^{\varphi}_{M^*}=\emptyset$ if φ is a k-edge-coloring. We introduce the following notation. For $f\in E_G(u,v)\cap M$, if there exists $f_1\in E(G-(M\cup M^*))$ such that $\varphi(f_1)=\Phi(f)$ and $V(f_1)\cap V(f)=\{u\}$ ($V(f_1)=V(f)=\{u,v\}$, respectively), we call f **T1-improper** (Type 1 improper) at u (at u and v, respectively) if $V(f_1)\cap V(M^*)=\emptyset$, and **T2-improper** (Type 2 improper) at u if $V(f_1)\cap V(M^*)\neq\emptyset$. If f is T1-improper or T2-improper at u, we say that f is **improper** at u. Define

$$E_1(M^*,\varphi)=\{f_1\in E(G-(M\cup M^*)): f_1 \text{ is adjacent in } G \text{ to a T1-improper edge}\},$$
$$E_2(M^*,\varphi)=\{f_1\in E(G-(M\cup M^*)): f_1 \text{ is adjacent in } G \text{ to a T2-improper edge}\}.$$

Observe that $E_1(M^*,\varphi)\cup E_2(M^*,\varphi)$ is a matching since M is a distance-3 matching in G. We call the triple $(M^*,E^{\varphi}_{M^*},\varphi)$ **prefeasible** if the following conditions are satisfied:

(a) $M_{\Delta+\mu}\cup M^*\cup E^{\varphi}_{M^*}$ is a matching;
(b) for each $e\in M^*$ such that e is adjacent in G to an edge of $E_2(M^*,\varphi)$, e is k-critical and fully G-saturated in the graph H_e+e, where H_e is the unique maximal k-dense subgraph of $G-(M\cup M^*)$ containing $V(e)$;
(c) the colors on edges of $\partial_{G-(M\cup M^*)}(H_e)$ are all distinct under φ.

Let $(M^*,E^{\varphi}_{M^*},\varphi)$ be a prefeasible triple. Since $M\cup M^*$ is a matching in G, if $(M^*,E^{\varphi}_{M^*},\varphi)$ also satisfies **Condition** (d): $|E_1(M^*,\varphi)|=|E_2(M^*,\varphi)|=0$, then by assigning the color $\Delta+\mu$ to all edges of M^*, we obtain a (proper) $(k+1)$-edge-coloring of G, where the $(k+1)$-edge-coloring is the combination of the precoloring Φ on M, the coloring using the color $\Delta+\mu$ on M^*, and the coloring φ of $G-(M\cup M^*)$. Thus we define a **feasible** triple $(M^*,E^{\varphi}_{M^*},\varphi)$ as one that satisfies Conditions (a)-(d).

The rest of the proof is devoted to showing the existence of a feasible triple $(M^*,E^{\varphi}_{M^*},\varphi)$ of G. Our main strategy is to first fix a particular prefeasible triple $(M^*_0,E^{\varphi_0}_{M^*_0},\varphi_0)$, then modify it step by step into a feasible triple $(M^*,E^{\varphi}_{M^*},\varphi)$. In particular, we will choose M^*_0 and φ_0 such that $E^{\varphi_0}_{M^*_0}=\emptyset$. At the end, when we modify φ_0 into φ, we will ensure that the $\Delta+\mu$ color class of G is $M_{\Delta+\mu}\cup M^*\cup E_1(M^*_0,\varphi_0)\cup E_2(M^*_0,\varphi_0)$. The process is first to modify M^*_0 and φ_0 at the same time to deduce the number of T2-improper edges.

By Lemma 3.3, there exists a matching M^*_0 of $G-V(M)$ such that $\chi'(G-(M\cup M^*_0))=k$ and each edge $e\in M^*_0$ is k-critical and fully G-saturated in H_e+e, where H_e is the unique maximal k-dense subgraph of $G-(M\cup M^*_0)$ containing $V(e)$. By Lemmas 2.1 and 2.2, H_e is induced in $G-(M\cup M^*_0)$ with

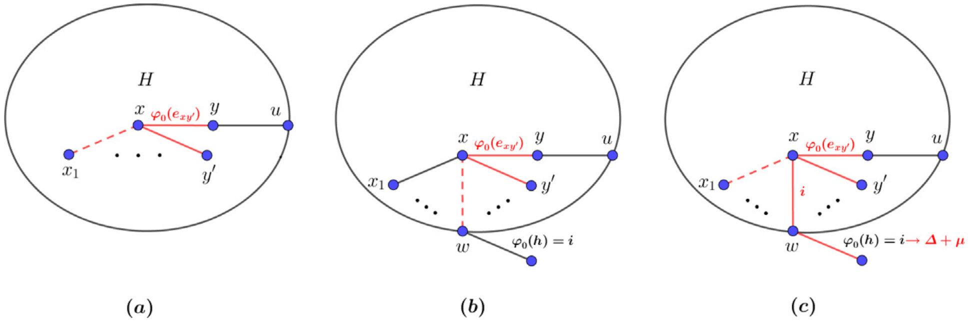

Fig. 2. Operations I, II and III in Case 1. (The edges of the dashed line represent uncolored edges).

$\chi'(H_e) = k$, and H_e and $H_{e'}$ are either identical or vertex-disjoint for any $e' \in M_0^* \setminus \{e\}$. Moreover, by Lemma 2.4, $\mathrm{diam}(H_e + e) \le \mathrm{diam}(H_e) \le 2$, and H_e is $(\varphi_0)_{H_e}$-elementary and strongly φ_0-closed in $G - (M \cup M_0^*)$. As $\chi'(G - M) = k + 1$, we have $|M_0^*| \ge 1$. Let φ_0 be a k-edge-coloring of $G - (M \cup M_0^*)$. Thus $E_{M_0^*}^{\varphi_0} = \emptyset$. Obviously, the triple $(M_0^*, \emptyset, \varphi_0)$ is prefeasible, which we take as our initial triple.

For $(M_0^*, \emptyset, \varphi_0)$, if $|E_1(M_0^*, \varphi_0)| = |E_2(M_0^*, \varphi_0)| = 0$, then we are done. If $|E_1(M_0^*, \varphi_0)| \ge 1$ and $|E_2(M_0^*, \varphi_0)| = 0$, then we recolor each edge in $E_1(M_0^*, \varphi_0)$ with the color $\Delta + \mu$ to produce a $(k + 1)$-edge-coloring φ_1 of $G - (M \cup M_0^*)$, since $E_{M_0^*}^{\varphi_0} = \emptyset$ and $E_1(M_0^*, \varphi_0)$ is a matching. Then as $|E_1(M_0^*, \varphi_1)| = |E_2(M_0^*, \varphi_1)| = 0$ and $M_{\Delta+\mu} \cup M_0^* \cup E_{M_0^*}^{\varphi_1}$ is a matching, it follows that the new triple $(M_0^*, E_1(M_0^*, \varphi_0), \varphi_1)$ is feasible. Then we are also done.

Therefore, we assume that $|E_1(M_0^*, \varphi_0)| \ge 0$ and $|E_2(M_0^*, \varphi_0)| \ge 1$. Recall that for each $e \in M_0^*$, e is fully G-saturated in $H_e + e$. Thus we have the following observation: for an edge $f_{uv} \in M$ with $V(f_{uv}) = \{u, v\}$, if $\{u, v\} \cap V(H_e) = \emptyset$ for any $e \in M_0^*$, then f_{uv} cannot be a T2-improper edge.

Since $|E_2(M_0^*, \varphi_0)| \ge 1$, we consider one T2-improper edge in M, say f_{uv} with $V(f_{uv}) = \{u, v\}$. Suppose that f_{uv} is T2-improper at u and $\Phi(f_{uv}) = i \in [k]$ (as φ_0 is a k-edge-coloring, $i \ne k + 1 = \Delta + \mu$). Then there exist $e_{xy} \in E_G(x, y) \cap M_0^*$ and a maximal k-dense subgraph H of $G - (M \cup M_0^*)$ such that $V(e_{xy}) \subseteq V(H)$ and f_{uv} and e_{xy} are both adjacent in G to an i-edge $e_{yu} \in E_H(y, u)$. Since M is a distance-3 matching and $\mathrm{diam}(H) \le 2$, we have $V(H) \cap V(M \setminus \{f_{uv}\}) = \emptyset$. We will modify φ_0 into a new coloring such that f_{uv} is not T2-improper at u under this new coloring and that no other edge of M_0^* is changed into a new T2-improper edge. We consider the three cases below regarding the location of f_{uv} with respect to H.

Case 1: f_{uv} is not improper at v, or f_{uv} is T1-improper at v but $v \notin V(H)$.

Let F_x be a maximal multi-fan at x with respect to e_{xy} and $(\varphi_0)_H$ in $H + e_{xy}$. There exist at least one Δ-vertex in $V(F_x) \setminus \{x, y\}$ by Lemma 3.2(a) and a linear sequence at x from y to this Δ-vertex in F_x. We consider two subcases as follows.

Subcase 1.1: $V(F_x) \setminus \{x, y\}$ has a Δ-vertex x_1 and there is a linear sequence S at x from y to x_1 such that S contains no i-edge or S contains no vertex w such that w is incident with an i-edge of $\partial_{G-(M\cup M_0^*)}(H)$.

Let $S = (y, e_{xy'}, y', \dots, e_{xx_1}, x_1)$ be the linear sequence (where $y' = x_1$ is possible). We apply Operation I as follows: apply a shifting in S from y to x_1, color e_{xy} with $\varphi_0(e_{xy'})$, uncolor e_{xx_1}, and replace e_{xy} by e_{xx_1} in M_0^*. See Fig. 2(a). Since x_1 is not incident with any edge in $M \cup M_0^*$, $M_1^* := (M_0^* \setminus \{e_{xy}\}) \cup \{e_{xx_1}\}$ is a matching. Denote $H_1 := H + e_{xy} - e_{xx_1}$. Let ψ be the k-edge coloring of H_1 after Operation I. Note that for any vertex $z \in V(H_1)$ that is incident with an edge of $\partial_{G-(M\cup M_1^*)}(H_1)$, if $\overline{\psi}(z) \ne \overline{(\varphi_0)_H}(z)$, then $z \in V(S)$. By the condition of Subcase 1.1 and Operation I, there is no such vertex w such that w is incident with both an i-edge of $E(S)$ and an i-edge of $\partial_{G-(M\cup M_1^*)}(H_1)$. Thus we can rename some color classes of ψ but keep the color i unchanged to match all colors on edges of $\partial_{G-(M\cup M_1^*)}(H_1)$. In this way we obtain a (proper) k-edge-coloring φ_1 of $G - (M \cup M_1^*)$ by Lemma 2.5(b).

We claim that $(M_1^*, \emptyset, \varphi_1)$ is a prefeasible triple. As $M_{\Delta+\mu} \cup M_1^*$ is a matching, we verify that M_1^* and φ_1 satisfy the corresponding conditions. Clearly H_1 is k-dense with $V(H_1) = V(H)$ and $\partial_{G-(M\cup M_1^*)}(H_1) = \partial_{G-(M\cup M_0^*)}(H)$ and $\chi'(H_1) = \chi'(H) = k$, and e_{xx_1} is k-critical and fully G-saturated in $H_1 + e_{xx_1}$. Furthermore, as distinct maximal k-dense subgraphs are vertex-disjoint we know that each edge $e \in M_1^* \setminus \{e_{xx_1}\}$ is still contained in a k-dense subgraph of $G - (M \cup M_1^*)$ such that e is k-critical and fully G-saturated in the graph $H_e + e$ if e is adjacent in G to an edge of $E_2(M_1^*, \varphi_1)$, where H_e is the unique maximal k-dense subgraph of $G-(M\cup M_0^*)$ containing $V(e)$ if H_e and H_1 are vertex-disjoint, and $H_e = H_1$ otherwise. Since φ_1 is a k-edge-coloring of $G-(M\cup M_1^*)$, H_e is strongly φ_1-closed for each $e \in M_1^*$. Therefore, $(M_1^*, \emptyset, \varphi_1)$ is a prefeasible triple.

Next, we claim that $|E_2(M_1^*, \varphi_1)| = |E_2(M_0^*, \varphi_0)| - 1$. Note that under φ_1, we still have $\varphi_1(e_{yu}) = i$. Since $e_{xy}, e_{yu} \in E(H_1)$, $e_{xx_1} \in M_1^*$ and e_{xx_1} is not adjacent to e_{yu} in $G - (M \cup M_1^*)$, we see that now f_{uv} is no longer T2-improper at u but T1-improper at u with respect to M_1^* and φ_1. For any edge $f \in M \setminus \{f_{uv}\}$, since both x and x_1 are Δ-vertices of $H + e_{xy}$ and $V(H_1) \cap V(M \setminus \{f_{uv}\}) = \emptyset$, we see that the distance between f and e_{xx_1} in $G - (M \cup M_1^*)$ is at least 2. Thus the property of f being T1-improper or T2-improper is not changed under M_1^* and φ_1. Thus the new triple $(M_1^*, \emptyset, \varphi_1)$ is prefeasible with $|E_1(M_1^*, \varphi_1)| = |E_1(M_0^*, \varphi_0)| + 1$ and $|E_2(M_1^*, \varphi_1)| = |E_2(M_0^*, \varphi_0)| - 1$, and so we can consider $(M_1^*, \emptyset, \varphi_1)$ instead.

Subcase 1.2: For any Δ-vertex in $V(F_x) \setminus \{x, y\}$, any linear sequence from y to this Δ-vertex contains both an i-edge h_i and a vertex w such that w is incident with an i-edge h of $\partial_{G-(M\cup M_0^*)}(H)$.

Let $F \subseteq F_x$ be the maximal multi-fan at x without any i-edge with respect to e_{xy} and $(\varphi_0)_H$. By the condition of Subcase 1.2, F does not contain any Δ-vertex from $V(F)\setminus\{x, y\}$ in H. By Lemma 3.2(c), there exists a vertex $z^* \in V(F)\setminus\{x, y\}$ with $i \in \overline{(\varphi_0)_H}(z^*)$ and $d_H(z^*) = \Delta - 1$. Since $V(F_x)$ is $(\varphi_0)_H$-elementary by Lemma 3.1(a) and every color on edges of $\partial_{G-(M\cup M_0^*)}(H)$ under φ_0 is a missing color at some vertex of H under $(\varphi_0)_H$, it follows that $z^* = w$, i.e., $d_H(w) = \Delta - 1$ and $d_{G-(M\cup M_0^*)}(w) = \Delta$. Thus the i-edge h is the only edge incident with w from $\partial_{G-(M\cup M_0^*)}(H)$, and w is not adjacent in G to any edge from $M \cup M_0^*$. Let $S = (y, e_{xy'}, y', \ldots, e_{xx_1}, x_1)$ be a linear sequence at x from y to x_1, where x_1 is a Δ-vertex. Notice that w is in S by the condition of Subcase 1.2. We consider the following two subcases according whether the boundary i-edge h belongs to $E_1(M_0^*, \varphi_0)$.

Subcase 1.2.1: $h \notin E_1(M_0^*, \varphi_0)$, i.e., h is not adjacent in G to any precolored i-edge in M.

Let $e_{xw} \in E_H(x, w)$ be an edge in S. We apply Operation II as follows: apply a shifting in S from y to w, color e_{xy} with $\varphi_0(e_{xy'})$, uncolor e_{xw}, and replace e_{xy} by e_{xw} in M_0^*. See Fig. 2(b). Since $d_{G-(M\cup M_0^*)}(w) = \Delta$, $M_1^* := (M_0^*\setminus\{e_{xy}\}) \cup \{e_{xw}\}$ is a matching. Denote $H_1 := H + e_{xy} - e_{xw}$. Let ψ be the k-edge coloring of H_1 after Operation II. Note that for any vertex $z \in V(H_1)$ that is incident with an edge of $\partial_{G-(M\cup M_1^*)}(H_1)$, if $\overline{\psi}(z) \neq \overline{(\varphi_0)}_H(z)$, then z is contained in the subsequence of S from y to w. Since h is the only i-edge of $\partial_{G-(M\cup M_1^*)}(H_1)$, there is no such vertex w such that w is incident with both an i-edge contained in the subsequence of S from y to w and an i-edge of $\partial_{G-(M\cup M_1^*)}(H_1)$ after Operation II. Thus we can rename some color classes of ψ but keep the color i unchanged to match all colors on boundary edges of $\partial_{G-(M\cup M_1^*)}(H_1)$. In this way we obtain a (proper) k-edge-coloring φ_1 of $G - (M \cup M_1^*)$ by Lemma 2.5(b).

By the similar argument in the proof of Subcase 1.1, it can be verified that $(M_1^*, \emptyset, \varphi_1)$ is prefeasible, and that f_{uv} is no longer T2-improper at u but T1-improper at u with respect to M_1^* and φ_1. For any edge $f \in M \setminus \{f_{uv}\}$, we see that the distance between f and e_{xw} is at least 2 or just 1 when h is adjacent in G to f with $\Phi(f) \neq i$. Thus the property of f being T1-improper or T2-improper is not changed under M_1^* and φ_1. Thus the new triple $(M_1^*, \emptyset, \varphi_1)$ is prefeasible with $|E_1(M_1^*, \varphi_1)| = |E_1(M_0^*, \varphi_0)| + 1$ and $|E_2(M_1^*, \varphi_1)| = |E_2(M_0^*, \varphi_0)| - 1$, and so we can consider $(M_1^*, \emptyset, \varphi_1)$ instead.

Subcase 1.2.2: $h \in E_1(M_0^*, \varphi_0)$, i.e., h is adjacent in G to some precolored i-edge f_i in M.

We apply Operation III as follows: recolor the i-edge h with the color $\Delta + \mu$, apply a shifting in S from y to x_1, color e_{xy} with $\varphi_0(e_{xy'})$, uncolor e_{xx_1}, and replace e_{xy} by e_{xx_1} in M_0^*. See Fig. 2(c). By the same argument as in the proof of Subcase 1.1, we know that $M_1^* := (M_0^*\setminus\{e_{xy}\}) \cup \{e_{xx_1}\}$ is a matching. Denote $H_1 := H + e_{xy} - e_{xx_1}$. Let ψ be the k-edge coloring of H_1 after Operation III. Note that there is no i-edge in $\partial_{G-(M\cup M_1^*)}(H_1)$ after Operation III. By the similar argument as in the proof of Subcase 1.1, we can rename some color classes of ψ but keep the color i unchanged to match all colors on edges of $\partial_{G-(M\cup M_1^*)}(H_1)$. In this way we obtain a (proper) $(k+1)$-edge-coloring φ_1 of $G-(M\cup M_1^*)$ by Lemma 2.5(b).

We claim that (M_1^*, h, φ_1) is a prefeasible triple. As $M\cup M_1^*$ is a matching and h is adjacent to f_i and $\Phi(f_i)=i\in[k]$, it follows that h is not adjacent to any edge from $M_{\Delta+\mu}\cup M_1^*$, which implies that $M_{\Delta+\mu}\cup M_1^*\cup\{h\}$ is a matching. By the same argument as in the proof of Subcase 1.1, we know that e_{xx_1} is k-critical and fully G-saturated in $H_1+e_{xx_1}$, and each edge $e\in M_1^*\setminus\{e_{xx_1}\}$ is still contained in a k-dense subgraph of $G-(M\cup M_1^*)$ such that e is k-critical and fully G-saturated in the graph H_e+e if e is adjacent in G to an edge of $E_2(M_1^*,\varphi_1)$, where H_e is the unique maximal k-dense subgraph of $G-(M\cup M_0^*)$ containing $V(e)$ if H_e and H_1 are vertex-disjoint, and $H_e=H_1$ otherwise. If the color $\Delta+\mu$ is not used on edges of $\partial_{G-(M\cup M_1^*)}(H_e)$, then colors on edges of $\partial_{G-(M\cup M_1^*)}(H_e)$ are all distinct by the fact that H_e is strongly φ_1-closed. If the color $\Delta+\mu$ is used on edges of $\partial_{G-(M\cup M_1^*)}(H_e)$, then it was used on exactly one edge of $\partial_{G-(M\cup M_1^*)}(H_e)$. This, together with the fact that H_e is $(\varphi_1)_{H_e}$-elementary, implies that colors on edges of $\partial_{G-(M\cup M_1^*)}(H_e)$ are all distinct. Therefore, (M_1^*, h, φ_1) is a prefeasible triple.

By the same argument as in the proof of Subcase 1.1, we know that now f_{uv} is no longer T2-improper at u but T1-improper at u with respect to M_1^* and φ_1, and that for any edge $f\in M\setminus\{f_{uv}\}$, the distance between f and e_{xx_1} in $G-(M\cup M_1^*)$ is at least 2. Except the i-edge f_i of M that is adjacent in G to h, the property of f being T1-improper or T2-improper is not changed under M_1^* and φ_1. The edge f_i is originally T1-improper at w_i, and now is no longer improper at w_i with respect to φ_1, where we assume $h\in E_G(w,w_i)$. Thus $|E_1(M_1^*,\varphi_1)|=|E_1(M_0^*,\varphi_0)|+1-1$ and $|E_2(M_1^*,\varphi_1)|=|E_2(M_0^*,\varphi_0)|-1$, and so we can consider $(M_1^*,\{h\},\varphi_1)$ instead. Note that assigning the color $\Delta+\mu$ to h will not affect the modification of φ_0 into φ and M_0^* into M^*, since $h\in E_1(M_0^*,\varphi_0)$ and we will assign the color $\Delta+\mu$ to all edges in $E_1(M_0^*,\varphi_0)$ in the final process.

Case 2: f_{uv} is T2-improper at v with $v\in V(H')$ for a maximal k-dense subgraph H' other than H.

For this case, we apply the same operations as we did in Case 1 first with respect to the vertex u in H and then with respect to the vertex v in H'. Recall that $V(H)\cap V(H')=\emptyset$ and $E_1(M_0^*,\varphi_0)$ is a matching. By Case 1, the operations applied within $G[V(H)]$ or $G[V(H)]+h_u$ do not affect the operations applied within $G[V(H')]$ or $G[V(H')]+h_v$, where h_u and h_v are the two possible i-edges with $h_u\in\partial_{G-(M\cup M_0^*)}(H)\cap E_1(M_0^*,\varphi_0)$ and $h_v\in\partial_{G-(M\cup M_0^*)}(H')\cap E_1(M_0^*,\varphi_0)$. Furthermore, if h_u and h_v exist at the same time, then $V(h_u)\cap V(h_v)=\emptyset$ and there is no maximal k-dense subgraph H'' other than H and H' such that $V(H'')\cap V(h_u)\neq\emptyset$ and $V(H'')\cap V(h_v)\neq\emptyset$. Denote the matching resulting from M_0^* by M_1^*, and the coloring resulting from φ_0 by φ_1. By Case 1, $E^{\varphi_1}_{M_1^*}\subseteq\{h_u,h_v\}$, $M_{\Delta+\mu}\cup M_1^*\cup\{h_u,h_v\}$ is a matching, and $(M_1^*,E^{\varphi_1}_{M_1^*},\varphi_1)$ also satisfies Conditions (b) and (c). Thus $(M_1^*,E^{\varphi_1}_{M_1^*},\varphi_1)$ is a prefeasible triple. With respect to M_1^* and φ_1, f_{uv} is no longer T2-improper but is T1-improper at both u and v. Furthermore, we have $|E_1(M_1^*,\varphi_1)|\ge|E_1(M_0^*,\varphi_0)|$ and $|E_2(M_1^*,\varphi_1)|=|E_2(M_0^*,\varphi_0)|-2$. Thus we can consider $(M_1^*,E^{\varphi_1}_{M_1^*},\varphi_1)$ instead.

Case 3: f_{uv} is T1-improper or T2-improper at v with $v\in V(H)$.

Let $e_{bv}\in E_H(b,v)$ with $\varphi_0(e_{bv})=i$. Assume first that $d_H(b)<\Delta$. If f_{uv} is T1-improper at v, then we apply the same operations with respect to u as we did in Case 1. Denote the new matching resulting from M_0^* by M_1^*, and the new coloring resulting from φ_0 by φ_1. Then the vertex b is not incident in G with any edge of M_1^* by Operations I-III in Case 1. Thus f_{uv} is no longer T2-improper at u but T1-improper at u with respect to M_1^* and φ_1. Furthermore, we have $|E_1(M_1^*,\varphi_1)|\ge|E_1(M_0^*,\varphi_0)|$ and $|E_2(M_1^*,\varphi_1)|=|E_2(M_0^*,\varphi_0)|-1$. Thus we can consider $(M_1^*,E^{\varphi_1}_{M_1^*},\varphi_1)$ instead.

If f_{uv} is T2-improper at v, let $e_{ab}\in M_0^*$ with $V(e_{ab})=\{a,b\}$. We apply the same operations with respect to u as we did in Case 1. Denote the resulting matching by M_1^*, and the resulting coloring by φ_1. With respect to M_1^* and φ_1, the edge f_{uv} is still T2-improper at v as $d_H(a)<\Delta$ and $d_H(b)<\Delta$. By Case 1, now f_{uv} is no longer T2-improper at u but T1-improper at u with respect to the prefeasible triple $(M_1^*,E^{\varphi_1}_{M_1^*},\varphi_1)$, where $E^{\varphi_1}_{M_1^*}=\emptyset$ or $\{h\}$ with some vertex w and its incident i-edge $h\in\partial_{G-(M\cup M_0^*)}(H)\cap E_1(M_0^*,\varphi_0)$. Denote by H_1 the new k-dense subgraph after the operations with respect to u in $H+e_{xy}$. In particular, the situation under $(M_1^*,\emptyset,\varphi_1)$ is actually the same as the case $d_H(b)=\Delta$ in the previous paragraph since now $d_{H_1}(y)=\Delta$.

Thus we consider only the case that f_{uv} is T2-improper at v, T1-improper at u and $d_{H_1}(y)=\Delta$. Consider a maximal multi-fan F_a at a with respect to e_{ab} and $(\varphi_1)_{H_1}$ in H_1+e_{ab}. Clearly we can

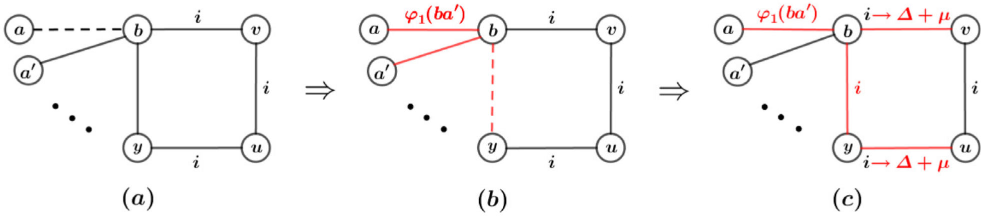

Fig. 3. Operation in Subcase 3.1. (The edges of the dashed line represent uncolored edges).

apply the same operations in Case 1 for v so that f_{uv} is no longer T2-improper at v with respect to the resulting matching M_2^* and coloring φ_2, unless these operations would have to put one edge $e_{ay} \in E_{H_1}(a, y)$ into M_2^*. Then f_{uv} would become T2-improper at u again with respect to M_2^* and φ_2. The only operations that have to uncolor an edge of H_1 incident with y are Operations I and III. Therefore, we make the following two assumptions on F_a in the rest of our proof.

(1) y is the only Δ-vertex in $V(F_a)\backslash\{a, b\}$.
(2) If a linear sequence in F_a at a from b to y contains a vertex w' such that $d_{H_1}(w') = \Delta - 1$ and w' is incident with an i-edge $h' \in \partial_{G-(M\cup M_1^*)}(H_1)$, then $h' \in E_1(M_1^*, \varphi_1)$.

Let F_b be a maximal multi-fan at b with respect to e_{ab} and $(\varphi_1)_{H_1}$ in $H_1 + e_{ab}$. We consider the following three subcases.

Subcase 3.1: F_b contains a linear sequence S at b from a to y such that S does not contain any i-edge.

Let $S = (a, e_{ba'}, a', \ldots, e_{by}, y)$ be the linear sequence (where $a' = y$ is possible). We apply a shifting in S from a to y, color e_{ab} with $\varphi_1(e_{ba'})$, uncolor e_{by}. See Fig. 3(a)–(b). Note that $M_2^* := (M_1^* \setminus \{e_{ab}\})\cup\{e_{by}\}$ is a matching, and $H_2 := H_1 + e_{ab} - e_{by}$ is a k-dense subgraph of $G-(M\cup M_2^*)$. As S does not contain any i-edge, by Lemma 2.5(b), we obtain a k-edge-coloring φ_2 of $G-(M\cup M_2^*)$. Note that f_{uv} is T2-improper at both u and v with respect to M_2^* and φ_2. However, we have $\Phi(f_{uv}) = i$, $\varphi_2(e_{bv}) = \varphi_2(e_{yu}) = i$, and $e_{by} \in M_2^*$ ($bvuyb$ is a cycle with length 4 in G). By assigning the color i to e_{by} and recoloring e_{bv} and e_{yu} with the color $\Delta+\mu$, we obtain a new matching $M_3^* := M_2^*\backslash\{e_{by}\} = M_1^*\backslash\{e_{ab}\}$ of $G-V(M)$ and a new $(k+1)$-edge-coloring φ_3 of $G-(M\cup M_3^*)$. See Fig. 3(c). The edge f_{uv} is now not improper at neither of its endvertices. Note that $E_{M_3^*}^{\varphi_3} = \{e_{bv}, e_{yu}\}$ if $E_{M_1^*}^{\varphi_1} = \emptyset$ and $E_{M_3^*}^{\varphi_3} = \{h, e_{bv}, e_{yu}\}$ if $E_{M_1^*}^{\varphi_1} = \{h\}$. Since $E_{M_3^*}^{\varphi_3} \subseteq (E_1(M_0^*, \varphi_0) \cup E_2(M_0^*, \varphi_0))$ is a matching, and those edges in $E_{M_3^*}^{\varphi_3}$ do not share any endvertex with edges in $M_{\Delta+\mu} \cup M_3^*$, it follows that $M_{\Delta+\mu} \cup M_3^* \cup E_{M_3^*}^{\varphi_3}$ is a matching. Note that $V(H_2)\cap V(M \setminus \{f_{uv}\}) = \emptyset$. For each $e \in M_3^*$ such that e is adjacent in G to an edge of $E_2(M_3^*, \varphi_3)$, e is still k-critical and fully G-saturated in the graph $H_e + e$, where H_e is still the unique maximal k-dense subgraph of $G-(M\cup M_0^*)$ containing $V(e)$ and H_e is also strongly φ_3-closed. Thus the new triple $(M_3^*, E_{M_3^*}^{\varphi_3}, \varphi_3)$ is prefeasible. Furthermore, $|E_1(M_3^*, \varphi_3)| = |E_1(M_1^*, \varphi_1)| - 1 \geq |E_1(M_0^*, \varphi_0)| - 1$ and $|E_2(M_3^*, \varphi_3)| = |E_2(M_1^*, \varphi_1)| - 1 = |E_2(M_0^*, \varphi_0)| - 2$. Thus we can consider $(M_3^*, E_{M_3^*}^{\varphi_3}, \varphi_3)$ instead.

Subcase 3.2: F_b contains a vertex w'' with $d_{H_1}(w'') = \Delta - 1$ and $i \in \overline{(\varphi_1)_{H_1}}(w'')$.

The i-edge e_{bv} is in F_b by the maximality of F_b. Let $S = (a, e_{ba'}, a', \ldots, e_{bw''}, w'', e_{bv}, v)$ be a linear sequence at b from a to v in F_b (where $a = a'$ and $a' = w''$ are possible). Since $i \in \overline{(\varphi_1)_{H_1}}(w'')$, we have that either $i \in \overline{\varphi}_1(w'')$ or w'' is incident with an i-edge $h'' \in \partial_{G-(M\cup M_1^*)}(H_1)$.

Assume first that $i \in \overline{\varphi}_1(w'')$ or w'' is incident with an i-edge $h'' \in \partial_{G-(M\cup M_1^*)}(H_1)$ such that $h'' \in E_1(M_1^*, \varphi_1)$. We apply a shifting in S from a to v, color e_{ab} with $\varphi_1(e_{ba'})$, and uncolor e_{bv}. Note that $e_{bw''}$ was recolored by the color i in the shifting operation. We then recolor the i-edge h'' with the color $\Delta+\mu$ if h'' exists, and rename some color classes of $H_2 := H_1 + e_{ab} - e_{bv}$ but keep the color i unchanged without producing any improper i-edge by Lemma 2.5(b). Finally we assign the color $\Delta+\mu$ to e_{bv}. Note that $h \neq h''$ since $\varphi_1(h) = \Delta+\mu \neq i = \varphi_1(h'')$, and h and h'' cannot both exist in $\partial_{G-(M\cup M_0^*)}(H) = \partial_{G-(M\cup M_1^*)}(H_1)$ since otherwise $\varphi_0(h) = \varphi_0(h'') = i$

contradicting that H is strongly φ_0-closed. Now we obtain a new matching $M_2^* := M_1^*\backslash\{e_{ab}\}$ of $G - V(M)$ and a new (proper) $(k+1)$-edge-coloring φ_2 of $G-(M\cup M_2^*)$ such that f_{uv} is no longer T2-improper at v or even T1-improper at v with respect to a new triple $(M_2^*, E_{M_2^*}^{\varphi_2}, \varphi_2)$, where $E_{M_2^*}^{\varphi_2} = \{e_{bv}\}$ if $E_{M_1^*}^{\varphi_1} = \emptyset$ but h'' does not exist, $E_{M_2^*}^{\varphi_2} = \{e_{bv}, h''\}$ if $E_{M_1^*}^{\varphi_1} = \emptyset$ and h'' exists, and $E_{M_2^*}^{\varphi_2} = \{e_{bv}, h\}$ if $E_{M_1^*}^{\varphi_1} = \{h\}$. Since $E_{M_2^*}^{\varphi_2} \subseteq (E_1(M_0^*, \varphi_0) \cup E_2(M_0^*, \varphi_0))$ is a matching, and those edges in $E_{M_2^*}^{\varphi_2}$ do not share any endvertex with edges in $M_{\Delta+\mu} \cup M_2^*$, it follows that $M_{\Delta+\mu} \cup M_2^* \cup E_{M_2^*}^{\varphi_2}$ is a matching. Note that $V(H_2) \cap V(M \setminus \{f_{uv}\}) = \emptyset$. By the similar argument as in the proof of Subcase 3.1, the new triple $(M_2^*, E_{M_2^*}^{\varphi_2}, \varphi_2)$ is prefeasible. Furthermore, $|E_1(M_2^*, \varphi_2)| \geq |E_1(M_0^*, \varphi_0)|$ and $|E_2(M_2^*, \varphi_2)| = |E_2(M_0^*, \varphi_0)| - 2$. Thus we can consider $(M_2^*, E_{M_2^*}^{\varphi_2}, \varphi_2)$ instead.

Now we may assume that the i-edge $h'' \notin E_1(M_1^*, \varphi_1)$. Since h and h'' cannot both exist, we have $E_{M_1^*}^{\varphi_1} = \emptyset$. Note that the vertex $w'' \notin V(F_a)$ by Assumption (2) prior to Subcase 3.1. Moreover, w'' is not incident with any edge in $M \cup M_1^*$ and w'' is only incident with the i-edge h'' in $\partial_{G-(M\cup M_1^*)}(H_1)$. Since $d_{G-(M\cup M_1^*)}(w'') = \Delta$ and φ_1 is a k-edge-coloring of $G-(M\cup M_1^*)$ with $k \geq \Delta+1$, there exists a color $\alpha \in \overline{\varphi}_1(w'')$ with $\alpha \neq i$. Since $V(H_1)$ is $(\varphi_1)_{H_1}$-elementary, there exists an α-edge e_1 incident with the vertex a. Thus we can define a maximal multi-fan at a, denoted by F_a', with respect to e_1 and $(\varphi_1)_{H_1}$ in H_1+e_1. (Notice that e_1 is colored by the color α in F_a'.) Moreover, $V(F_a')$ is $(\varphi_1)_{H_1}$-elementary since $V(H_1)$ is $(\varphi_1)_{H_1}$-elementary. By Lemma 3.2(b) and Assumption (1) prior to Subcase 3.1, we have $e_{F_a}(a, b') = e_{H_1+e_{ab}}(a, b') = \mu$ for any vertex b' in $V(F_a)\backslash\{a\}$. Therefore, $V(F_a')\backslash\{a\}$ and $V(F_a)\backslash\{a\}$ are disjoint, since otherwise we have $V(F_a') \subseteq V(F_a)$ and $\alpha \in \overline{(\varphi_1)}_{H_1}(b')$ for some $b' \in V(F_a)$ implying $b' = w'' \in V(F_a)$, a contradiction. Note that if $w'' \notin V(F_a')$, then $V(F_a')\backslash\{a\}$ must contain a Δ-vertex in H_1, since otherwise Lemma 3.2(d) and the fact $(\varphi_1)_{H_1}(e_1) = \alpha \in \overline{\varphi}_1(w'')$ imply that $w'' \in V(F_a')$, a contradiction. Thus F_a' contains a linear sequence $S' = (b_1, e_2, b_2, \ldots, e_t, b_t)$ at a, where $b_1 \in V(e_1)$, b_t (with $t \geq 1$) is a Δ-vertex if $w'' \notin V(F_a')$, and b_t is w'' if $w'' \in V(F_a')$. Notice that b_t is not incident with any edge in $M \cup M_1^*$ by our choice of b_t. Moreover, $b_t \neq y$ since $V(F_a')\backslash\{a\}$ and $V(F_a)\backslash\{a\}$ are disjoint. Let β ($\beta \neq i$) be a color in $\overline{\varphi}_1(b)$. By Lemma 3.1(b), we have $P_b(\beta, \alpha) = P_{w''}(\beta, \alpha)$. We then consider the following two subcases according the set $(V(S')\backslash\{a\}) \cap (V(S)\backslash\{a\})$.

We first assume that $(V(S')\backslash\{a\}) \cap (V(S)\backslash\{a\}) \subseteq \{b_t\}$. If $e_1 \notin P_b(\beta, \alpha)$, then we apply a Kempe change on $P_{[b,w'']}(\beta, \alpha)$, uncolor e_1 and color e_{ab} with α. If $e_1 \in P_b(\beta, \alpha)$ and $P_b(\beta, \alpha)$ meets b_1 before a, then we apply a Kempe change on $P_{[b,b_1]}(\beta, \alpha)$, uncolor e_1 and color e_{ab} with α. If $e_1 \in P_b(\beta, \alpha)$ and $P_{w''}(\beta, \alpha)$ meets b_1 before a, then we uncolor e_1, apply a Kempe change on $P_{[w'',b_1]}(\beta, \alpha)$, apply a shifting in S from a to w'', color e_{ab} with $\varphi_1(e_{ba'})$, and recolor $e_{bw''}$ with β. In all three cases above, e_{ab} is colored with a color in $[k]$ and e_1 is uncolored. Finally we apply a shifting in S' from b_1 to b_t, color e_1 with $\varphi_1(e_2)$, and uncolor e_t. Notice that the above shifting in S' does nothing if $t = 1$. Denote $H_2 := H_1 + e_{ab} - e_t$. Since H_2 is also k-dense and $\chi'(H_2) = k$, we can rename some color classes of $E(H_2)$ but keep the color i unchanged to match all colors on boundary edges without producing any improper i-edge by Lemma 2.5(b). Now we obtain a new matching $M_2^* := (M_1^*\backslash\{e_{ab}\}) \cup \{e_t\}$ and a new (proper) k-edge-coloring φ_2 of $G-(M\cup M_2^*)$ such that f_{uv} is no longer T2-improper at v but T1-improper at v with respect to the new prefeasible triple $(M_2^*, \emptyset, \varphi_2)$. Furthermore, $|E_1(M_2^*, \varphi_2)| = |E_1(M_0^*, \varphi_0)| + 2$ and $|E_2(M_2^*, \varphi_2)| = |E_2(M_0^*, \varphi_0)| - 2$. Thus we can consider $(M_2^*, \emptyset, \varphi_2)$ instead.

Then we assume that there exists $b_j = a^* \in (V(S')\backslash\{a\}) \cap (V(S)\backslash\{a\})$ for some $j \in [t-1]$ and $a^* \in V(S)$. See Fig. 4 for a depiction when $b_1 = b_j = a^* = a'$. In this case we assume a^* is the closest vertex to the vertex a along S. Note that $b_j \neq b$ as $V(F_a')\backslash\{a\}$ and $V(F_a)\backslash\{a\}$ are disjoint. Let $\alpha_j = \varphi_1(e_{j+1}) \in \overline{(\varphi_1)}_{H_1}(b_j)$. By Lemma 3.1($b$), we have $P_b(\beta, \alpha_j) = P_{b_j}(\beta, \alpha_j)$. If $e_{j+1} \notin P_b(\beta, \alpha_j)$, then we apply a Kempe change on $P_{[b,b_j]}(\beta, \alpha_j)$, uncolor e_{j+1} and color e_{ab} with α_j. If $e_{j+1} \in P_b(\beta, \alpha_j)$ and $P_b(\beta, \alpha_j)$ meets b_{j+1} before a, then we apply a Kempe change on $P_{[b,b_{j+1}]}(\beta, \alpha_j)$, uncolor e_{j+1} and color e_{ab} with α_j. If $e_{j+1} \in P_b(\beta, \alpha_j)$ and $P_{b_j}(\beta, \alpha_j)$ meets b_{j+1} before a, then we uncolor e_{j+1}, apply a Kempe change on $P_{[b_j,b_{j+1}]}(\beta, \alpha_j)$, apply a shifting in S from a to b_j (i.e., a^*), color e_{ab} with $\varphi_1(e_{ba'})$, and recolor the edge $e_{bb_j} \in E_{H_1}(b, b_j)$ with β. (See Fig. 4(a)–(c).) In all three cases above, e_{ab} is colored with a color in $[k]$ and e_{j+1} is uncolored. Finally we apply a shifting in S' from b_{j+1} to b_t, color e_{j+1} with $\varphi_1(e_{j+2})$, and uncolor e_t. (See Fig. 4(d).) Notice that the above shifting in S' does nothing if $b_{j+1} = b_t$. Denote $H_2 := H_1 + e_{ab} - e_t$. Since H_2 is also k-dense and $\chi'(H_2) = k$, we can

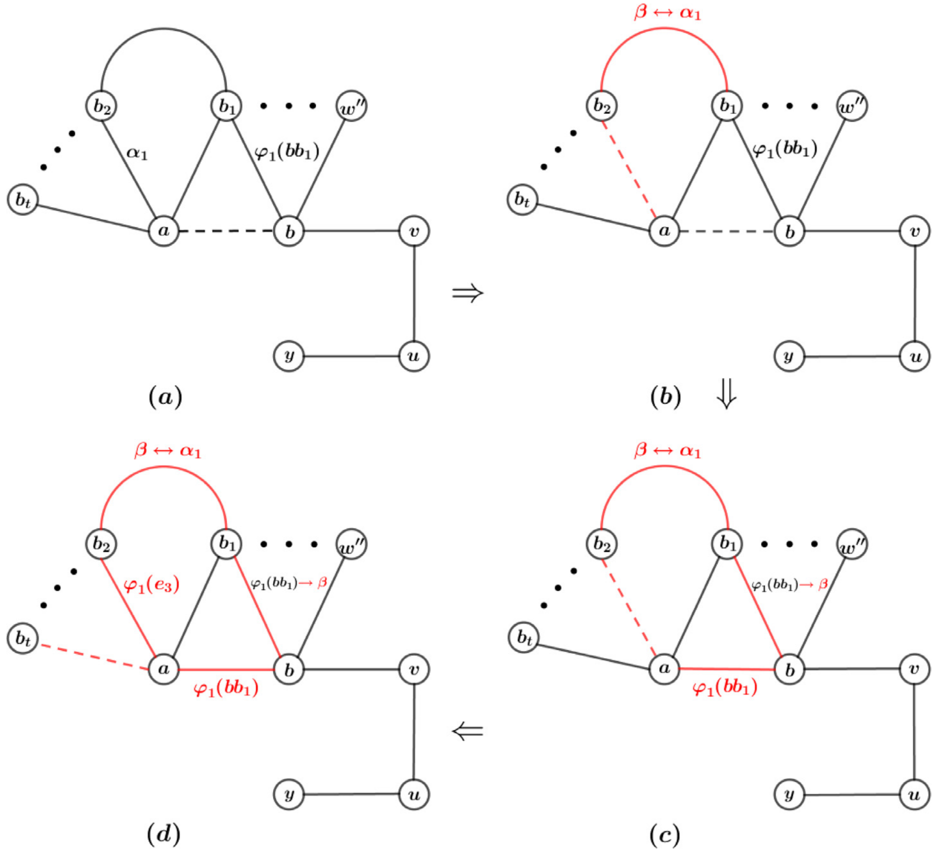

Fig. 4. One possible operation for $b_j = a^* \in (V(S')\backslash\{a\}) \cap (V(S)\backslash\{a\})$ in Subcase 3.2, where $b_1 = b_j = a^* = a'$. (The edges of the dashed line represent uncolored edges).

rename some color classes of $E(H_2)$ but keep the color i unchanged to match all colors on boundary edges without producing any improper i-edge by Lemma 2.5(b). Now we obtain a new matching $M_2^* := (M_1^*\backslash\{e_{ab}\}) \cup \{e_t\}$ of $G - V(M)$ and a new (proper) k-edge-coloring φ_2 of $G - (M \cup M_2^*)$ such that f_{uv} is no longer T2-improper at v but T1-improper at v with respect to the new prefeasible triple $(M_2^*, \emptyset, \varphi_2)$. Furthermore, $|E_1(M_2^*, \varphi_2)| = |E_1(M_0^*, \varphi_0)| + 2$ and $E_2(M_2^*, \varphi_2) = |E_2(M_0^*, \varphi_0)| - 2$. Thus we can consider $(M_2^*, \emptyset, \varphi_2)$ instead.

Subcase 3.3: F_b does not contain a linear sequence at b from a to y without i-edge, and F_b does not contain a vertex w'' with $d_{H_1}(w'') = \Delta - 1$ and $i \in \overline{(\varphi_1)}_{H_1}(w'')$.

We claim that F_b contains a linear sequence S^* at b from a to a Δ-vertex y^* such that $y^* \neq y$ and there is no i-edge in S^*. By Lemma 3.2(a), the multi-fan F_b contains at least one Δ-vertex in H_1. Now if F_b does not contain any linear sequence without i-edges from a to any Δ-vertex in H_1, then by Lemma 3.2(c), the multi-fan F_b contains a vertex w'' with $d_{H_1}(w'') = \Delta - 1$ and $i \in \overline{(\varphi_1)}_{H_1}(w'')$, contradicting the condition of Subcase 3.3. So F_b contains a linear sequence S^* from a to a vertex y^* such that $d_{H_1}(y^*) = \Delta$ and there is no i-edge in S^*. Note that $y^* \neq y$, since otherwise we also have a contradiction to the condition of Subcase 3.3. Thus the claim is proved.

Assume that $S^* = (a, e_{ba'}, a', \ldots, e_{by^*}, y^*)$ at b from a to y^* (where $a' = y^*$ is possible), and S^* contains no i-edge. Let $\theta \in \overline{\varphi}_1(y^*)$.

Subcase 3.3.1: $\theta = i$.

Since S^* contains no i-edge, we apply a shifting in S^* from a to y^*, color e_{ab} with $\varphi_1(e_{ba'})$, uncolor e_{by^*}, and rename some color classes of $E(H_1 + e_{ab} - e_{by^*})$ but keep the color i unchanged to match all colors on boundary edges without producing any improper i-edge by Lemma 2.5(b). By coloring e_{by^*} with i and recoloring e_{bv} from i to $\Delta + \mu$, we obtain a new matching $M_2^* := M_1^* \backslash \{e_{ab}\}$ of $G - V(M)$ and a new (proper) $(k+1)$-edge-coloring φ_2 of $G - (M \cup M_2^*)$. Then f_{uv} is no longer T2-improper at v or even T1-improper at v with respect to the new prefeasible triple $(M_2^*, E^{\varphi_2}_{M_2^*}, \varphi_2)$ with $E^{\varphi_2}_{M_2^*} = \{e_{bv}\}$ if $E^{\varphi_1}_{M_1^*} = \emptyset$, and $E^{\varphi_2}_{M_2^*} = \{e_{bv}, h\}$ if $E^{\varphi_1}_{M_1^*} = \{h\}$ (when $y^* \in V(F_x) \cap V(F_b)$). Furthermore, $E^{\varphi_2}_{M_2^*} \subseteq (E_1(M_0^*, \varphi_0) \cup E_2(M_0^*, \varphi_0))$, $|E_1(M_2^*, \varphi_2)| \geq |E_1(M_0^*, \varphi_0)|$ and $|E_2(M_2^*, \varphi_2)| = |E_2(M_0^*, \varphi_0)| - 2$. Thus we can consider $(M_2^*, E^{\varphi_2}_{M_2^*}, \varphi_2)$ instead.

Subcase 3.3.2: $\theta \neq i$.

Since $V(H_1)$ is $(\varphi_1)_{H_1}$-elementary, there exists a θ-edge e_1 incident with the vertex a. Thus by the similar argument as in the proof of Subcase 3.2, we define a maximal multi-fan at a, denoted by F'_a, with respect to e_1 and $(\varphi_1)_{H_1}$ in $H_1 + e_1$, and we have $e_{F_a}(a, b') = e_{H_1+e_{ab}}(a, b') = \mu$ for any vertex b' in $V(F_a)\backslash\{a\}$. Therefore, $V(F'_a)\backslash\{a\}$ and $V(F_a)\backslash\{a\}$ are disjoint, since otherwise we have $V(F'_a) \subseteq V(F_a)$ and $\varphi_1(e_1) = \theta \in \overline{(\varphi_1)_{H_1}}(b')$ for some $b' \in V(F_a)$ implying $y^* = b' \in V(F_a)$, which contradicts Assumption (1). Note that $V(F'_a)\backslash\{a\}$ must contain a Δ-vertex in H_1, since otherwise Lemma 3.2(d) and the fact $(\varphi_1)_{H_1}(e_1) = \theta \in \overline{\varphi}_1(y^*)$ imply that $y^* \in V(F'_a)$, which contradicts $d_{H_1}(y^*) = \Delta$. If F'_a contains a vertex of $V(H_1)$ that is incident with an i-edge of $\partial_{G-(M\cup M_1^*)}(H_1)$ in $G - (M \cup M_1^*)$, then we denote the vertex by w^* and the i-edge by h^*. If F'_a does not contain any linear sequence to a Δ-vertex in H_1 without i-edge and boundary vertex w^*, then by Lemma 3.2(d), the multi-fan F'_a contains a vertex z^* with $i \in \overline{(\varphi_1)}_{H_1}(z^*)$ and $d_H(z^*) = \Delta - 1$. Since H_1 is $(\varphi_1)_{H_1}$-elementary, we have $z^* = w^*$ and $d_{H_1}(w^*) = \Delta - 1$. Thus F'_a contains a linear sequence $S' = (b_1, e_2, b_2, \ldots, e_t, b_t)$ at a, where $b_1 \in V(e_1)$, b_t (with $t \geq 1$) is w^* if there exists w^* with $d_{H_1}(w^*) = \Delta - 1$ such that $h^* \in \partial_{G-(M\cup M_1^*)}(H_1)$ but $h^* \notin E_1(M_0^*, \varphi_0)$, and b_t is a Δ-vertex in H_1 otherwise. Notice that b_t is not incident with any edge in $M \cup M_1^*$ by our choice of b_t. Moreover, if $b_t = w^*$ as defined above, then $b_t = w^*$ is not a vertex in $V(F_b)$ by the condition of Subcase 3.3. And $b_t \neq y$ since $V(F'_a)\backslash\{a\}$ and $V(F_a)\backslash\{a\}$ are disjoint. Let β ($\beta \neq i$) be a color in $\overline{\varphi}_1(b)$. By Lemma 3.1(b), we have $P_b(\beta, \theta) = P_{y^*}(\beta, \theta)$. We then consider the following two subcases according the set $(V(S')\backslash\{a\}) \cap (V(S^*)\backslash\{a\})$.

We first assume that $(V(S')\backslash\{a\}) \cap (V(S^*)\backslash\{a\}) \subseteq \{b_t\}$. If $e_1 \notin P_b(\beta, \theta)$, then we apply a Kempe change on $P_{[b,y^*]}(\beta, \theta)$, uncolor e_1 and color e_{ab} with θ. If $e_1 \in P_b(\beta, \theta)$ and $P_b(\beta, \theta)$ meets b_1 before a, then we apply a Kempe change on $P_{[b,b_1]}(\beta, \theta)$, uncolor e_1 and color e_{ab} with θ. If $e_1 \in P_b(\beta, \theta)$ and $P_{y^*}(\beta, \theta)$ meets b_1 before a, then we uncolor e_1, apply a Kempe change on $P_{[y^*,b_1]}(\beta, \theta)$, apply a shifting in S^* from a to y^*, color e_{ab} with $\varphi_1(e_{ba'})$, and recolor e_{by^*} with β. In all three cases above, e_{ab} is colored with a color in $[k]$ and e_1 is uncolored. Then we apply a shifting in S' from b_1 to b_t, color e_1 with $\varphi_1(e_2)$, and uncolor e_t. Denote $H_2 := H_1 + e_{ab} - e_t$. Since H_2 is also k-dense and $\chi'(H_2) = k$, we can rename some color classes of $E(H_2)$ but keep the color i unchanged to match colors on boundary edges except i-edges by Lemma 2.5(b). Finally recolor h^* with the color $\Delta + \mu$ if $h^* \in \partial_{G-(M\cup M_0^*)}(H) \cap E_1(M_0^*, \varphi_0)$. Now we obtain a new matching $M_2^* := (M_1^*\backslash\{e_{ab}\}) \cup \{e_t\}$ of $G - V(M)$ and a new (proper) $(k+1)$-edge-coloring φ_2 of $G - (M \cup M_2^*)$ such that f_{uv} is no longer T2-improper at v but T1-improper at v with respect to the new prefeasible triple $(M_2^*, E^{\varphi_2}_{M_2^*}, \varphi_2)$, where $\emptyset$ or $\{h\}$ or $\{h^*\} = E^{\varphi_2}_{M_2^*} \subseteq E_1(M_0^*, \varphi_0)$. Furthermore, $|E_1(M_2^*, \varphi_2)| \geq |E_1(M_0^*, \varphi_0)|$ and $|E_2(M_2^*, \varphi_2)| = |E_2(M_0^*, \varphi_0)| - 2$. Thus we can consider $(M_2^*, E^{\varphi_2}_{M_2^*}, \varphi_2)$ instead.

Then we assume that there exists $b_j = a^* \in (V(S')\backslash\{a\}) \cap (V(S^*)\backslash\{a\})$ for some $j \in [t-1]$ and $a^* \in V(S^*)$. See Fig. 5 for a depiction when $b_1 = b_j = a^* = a'$. In this case we assume a^* is the closest vertex to a along S^*. Note that $b_j \neq b$ as $V(F'_a)\backslash\{a\}$ and $V(F_a)\backslash\{a\}$ are disjoint. Let $\theta_j = \varphi_1(e_{j+1}) \in \overline{(\varphi_1)}_{H_1}(b_j)$. By Lemma 3.1($b$), $P_b(\beta, \theta_j) = P_{b_j}(\beta, \theta_j)$. If $e_{j+1} \notin P_b(\beta, \theta_j)$, then we apply a Kempe change on $P_{[b,b_j]}(\beta, \theta_j)$, uncolor e_{j+1} and color e_{ab} with θ_j. If $e_{j+1} \in P_b(\beta, \theta_j)$ and $P_b(\beta, \theta_j)$ meets b_{j+1} before a, then we apply a Kempe change on $P_{[b,b_{j+1}]}(\beta, \theta_j)$, uncolor e_{j+1} and color e_{ab} with θ_j. If $e_{j+1} \in P_b(\beta, \theta_j)$ and $P_{b_j}(\beta, \theta_j)$ meets b_{j+1} before a, then we uncolor e_{j+1}, apply a Kempe change on $P_{[b_j,b_{j+1}]}(\beta, \theta_j)$, apply a shifting in S^* from a to b_j (i.e., a^*), color e_{ab} with $\varphi_1(e_{ba'})$, and recolor the edge $e_{bb_j} \in E_{H_1}(b, b_j)$ with β. (See Fig. 5(a)–(c).) In all three cases above, e_{ab} is colored with a color

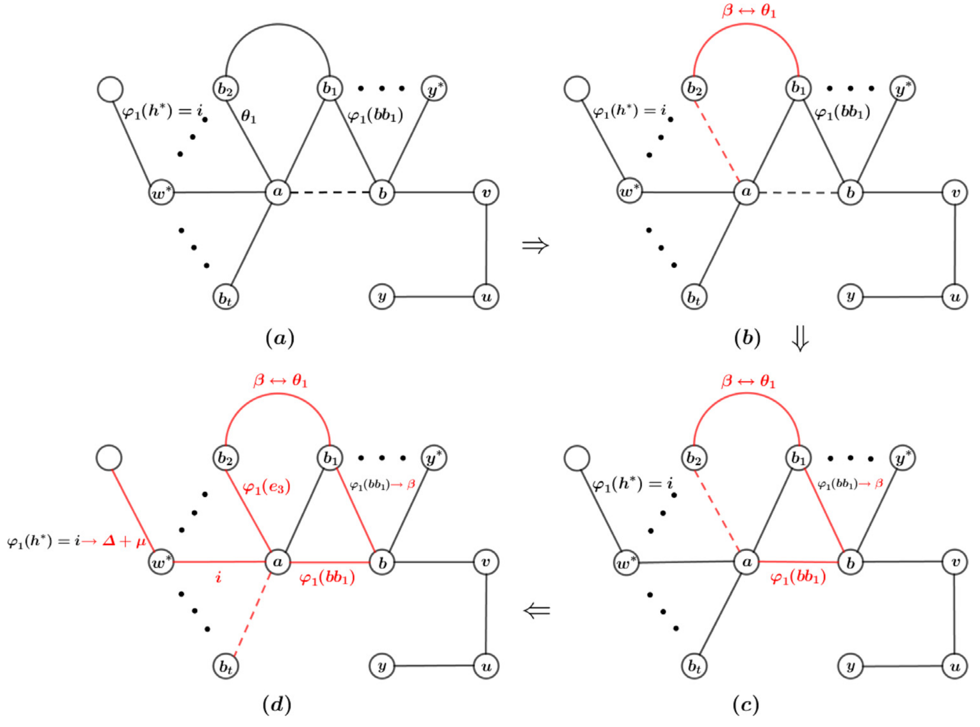

Fig. 5. One possible operation for $b_j = a^* \in (V(S')\backslash\{a\}) \cap (V(S)\backslash\{a\})$ in Subcase 3.3, where $b_1 = b_j = a^* = a'$. (The edges of the dashed line represent uncolored edges).

in $[k]$ and e_{j+1} is uncolored. Denote $H_2 := H_1 + e_{ab} - e_t$. Then we apply a shifting in S' from b_{j+1} to b_t, color e_{j+1} with $\varphi_1(e_{j+2})$, and uncolor the edge e_t, and rename some color classes of $E(H_2)$ but keep the color i unchanged to match all colors on boundary edges except i-edges by Lemma 2.5(b). Finally recolor h^* with $\Delta + \mu$ if $h^* \in \partial_{G-(M\cup M_0^*)}(H) \cap E_1(M_0^*, \varphi_0)$. (See Fig. 5(d).) Now we obtain a new matching $M_2^* = (M_1^*\backslash\{e_{ab}\}) \cup \{e_t\}$ of $G - V(M)$ and a new (proper) $(k+1)$-edge-coloring φ_2 of $G - (M \cup M_2^*)$ such that f_{uv} is no longer T2-improper at v but T1-improper at v with respect to the new prefeasible triple $(M_2^*, E_{M_2^*}^{\varphi_2}, \varphi_2)$, where $\emptyset$ or $\{h\}$ or $\{h^*\} = E_{M_2^*}^{\varphi_2} \subseteq E_1(M_0^*, \varphi_0)$. Furthermore, $|E_1(M_2^*, \varphi_2)| \geq |E_1(M_0^*, \varphi_0)|$ and $|E_2(M_2^*, \varphi_2)| = |E_2(M_0^*, \varphi_0)| - 2$. Thus we can consider $(M_2^*, E_{M_2^*}^{\varphi_2}, \varphi_2)$ instead. The proof is now finished. □

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