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On a free Schrödinger solution studied by Barceló–Bennett–Carbery–Ruiz–Vilela

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Abstract. We present a free Schrödinger solution studied by Barceló–Bennett–Carbery–Ruiz–Vilela and show why it can be viewed as a sharp example for the recently discovered refined decoupling theorem.

1. Introduction

Let $d \geq 2$. Consider the free Schrödinger equation:

$$\begin{cases} \mathrm{i}u_t - \Delta_x(u) = 0, \\ u(x,0) = f(x) \end{cases} \tag{1.1}$$

for $(x,t) \in \mathbb{R}^{d-1} \times \mathbb{R}$. By taking the Fourier transform of both sides, we know $\operatorname{supp} \hat{u} \subset \widetilde{P^{d-1}}$. Here $\widetilde{P^{d-1}}$ is the paraboloid:

$$\widetilde{P^{d-1}} = \{\xi_d = |\xi'|^2\}, \quad \xi = (\xi', \xi_d) := (\xi_1, \dots, \xi_{d-1}, \xi_d) \in \mathbb{R}^d.$$

Because of the above property, such functions u are closely related to the *Fourier restriction theory* and have been extensively studied by Fourier analysts. In light of the Littlewood-Paley decomposition, people are often interested in functions g on $\mathbb{R}^n$ such that $\operatorname{supp} \hat{g}$ is in the *truncated paraboloid*

$$P^{d-1} = \{\xi_d = |\xi'|^2, \, |\xi_j| \leq 1, \forall 1 \leq j \leq d-1\}.$$

Fourier analysts are then interested in $L^p \to L^q$ estimates of such g on certain subsets of $\mathbb{R}^d$, and a great amount of related recent progress has been made.

In this note, we first review an example of such a function g studied by Barceló–Bennett–Carbery–Ruiz–Vilela in [**1**]. Next, we present a recent result known as "refined decoupling" (proved independently by Guth–Iosevich–Ou–Wang [**7**] and Du–Zhang). Refined decoupling has seen powerful applications in recent years such as in the Falconer distance problem [**5, 7**] and small cap decouplings [**4**]. It would thus be interesting to know various sharp examples for this estimate. In this note, we show that Barceló-Bennett–Carbery–Ruiz–Vilela's free Schrödinger solutions are

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always (almost) sharp examples for refined decoupling. Moreover, we show in the end that this example is also sharp for an L^2 estimate used as a key step in many recent arguments studying the Falconer distance problem.

REMARK 1.1. Historically, [**1**] introduced this example and generalizations to provide useful test cases for L^2-average decay estimates of Fourier transforms of fractal measures. Therefore, it is perhaps not surprising that the example may be relevant for testing against other related results such as refined decoupling.

REMARK 1.2. In [**3**], inequality (4) is essentially a restatement of refined decoupling and it was remarked (Remark 1.3) that Knapp examples make refined decoupling (almost) sharp too. To put [**3**] into historical context, Guth [**6**] provided another different example capturing limits of decoupling. Guth's example was worked out carefully in [**3**] to show sharpness of their study of Mizohata-Takeuchi conjecture using refined decoupling estimate.

2. Barceló–Bennett–Carbery–Ruiz–Vilela's free Schrödinger solution

Let $0 < \sigma < 1/2$ and $R > 1$ be fixed parameters. Let $\mathrm{d}\omega$ be the hypersurface measure on P^{d-1}. Barceló–Bennett–Carbery–Ruiz–Vilela's free Schrödinger solution is a function g such that

$$\hat{g}(\xi) = h(\xi)\,\mathrm{d}\omega$$

where

$$h(\xi) = \sum_{\substack{l_1,\dots,l_{d-1}\in\mathbb{Z},\\ 1\le l_1,\dots,l_{d-1}<R^\sigma}} 1_{(l_1R^{-\sigma}-R^{-1},l_1R^{-\sigma}+R^{-1})\times\cdots\times(l_{d-1}R^{-\sigma}-R^{-1},l_{d-1}R^{-\sigma}+R^{-1})}.$$

We now state the most relevant properties of the above function g here. By elementary computations, one can check that $|g| \sim R^{(d-1)(\sigma-1)}$ at all points of the form $(n_1R^\sigma, n_2R^\sigma, \dots, n_{d-1}R^\sigma, n_dR^{2\sigma})$ inside the ball B_{c_dR} of radius c_dR, where $n_j \in \mathbb{Z}$ and $c_d > 0$ is a small constant only depending on the ambient dimension. Note that $R^{(d-1)(\sigma-1)}$ is comparable to $\|g\|_\infty$ by triangle inequality. Moreover, $|g| \sim R^{(d-1)(\sigma-1)}$ inside a ball of radius $\sim_d 1$ around every point above by a similar computation. We refer the reader to [**1**] for more detailed justification of these facts.

3. Wave packet decomposition

In order to introduce the refined decoupling inequality, we first briefly recall the *wave packet decomposition*, a standard tool for analyzing functions with Fourier support in P^{d-1}. Here, we present (the rescaled version of) the wave packet decomposition used in [**7**].

Fix a parameter $R > 1$. Decompose P^{d-1} into pieces θ such that the projection of each θ onto the hyperplane of the first $d-1$ coordinates is a square of side length $R^{-\frac{1}{2}}$. By elementary differential geometry, each θ is contained in a box of dimensions $R^{-\frac{1}{2}} \times \cdots \times R^{-\frac{1}{2}} \times R^{-1}$. Pick such a box and let T_θ be its dual box centered at the origin. Note that T_θ is roughly a tube of thickness $R^{\frac{1}{2}}$ and length R. Let $B_R \subset \mathbb{R}^d$ be the ball centered at the origin with radius R. Tile B_R by translations of T_θ and call this family $\mathbb{T}_\theta$.

For a function v whose Fourier support is in $P^{d-1} \subset \mathbb{R}^d$, one can decompose

$$v = \sum_{\theta, T\colon T \in \mathbb{T}_\theta} v_{\theta,T}$$

inside B_R such that:

- Each $\widehat{v_{\theta,T}}$ is supported in a box of dimensions $\sim R^{-\frac{1}{2}} \times \cdots \times R^{-\frac{1}{2}} \times R^{-1}$ containing θ.
- Each $v_{\theta,T}$ (known as a wave packet) is morally supported in T and rapidly decays outside of it.
- Each $|v_{\theta,T}|$ is morally a constant on T and we will call this constant the *magnitude* of $v_{\theta,T}$.
- Different $v_{\theta,T}$ are morally L^2-orthogonal on every $R^{\frac{1}{2}}$-ball. This property is known as *local orthogonality* and is very useful in Fourier restriction type problems. We do not need its detailed description here.

For each θ, we also define

$$v_\theta = \sum_{T \in \mathbb{T}_\theta} v_{\theta,T}.$$

4. Refined decoupling and its sharpness

4.1. The refined decoupling theorem. We can now state the refined decoupling theorem:

THEOREM 4.1 (Refined decoupling [**7**]). *Suppose* $\operatorname{supp} \hat{v} \subset P^{d-1}$ *and* $R > 1$. *Suppose that in the wave packet decomposition of* v *in* B_R, *every two wave packets have comparable magnitudes. Let* $X \subset B_R$ *such that each* $x \in X$ *hits the essential support of* $\leq M$ *wave packets, then for* $p = \frac{2(d+1)}{d-1}$,

$$\|v\|_{L^p(X)} \lesssim_\varepsilon R^\varepsilon M^{\frac{1}{2}-\frac{1}{p}} \left(\sum_\theta \|v_\theta\|^p_{L^p(w_{B_R})} \right)^{\frac{1}{p}}. \tag{4.1}$$

Here we have a weight w_{B_R} included on the right hand side for technical reasons. It behaves like 1_{B_R} but has a rapidly decaying tail. Morally one can think of $\|\cdot\|_{L^p(w_{B_R})}$ as $\|\cdot\|_{L^p(B_R)}$. Theorem 4.1 is named refined decoupling because it is a refinement of the celebrated Bourgain–Demeter decoupling theorem for paraboloids [**2**].

We remark that the assumption that all wave packets have comparable magnitudes in the theorem is usually harmless. In applications, one can usually reduce a general situation to this case by dyadic pigeonholing.

In [**7**], Theorem 4.1 is one of the central ingredients the authors use to make progress on the Falconer distance conjecture in $\mathbb{R}^2$. The theorem is also useful in other problems of similar flavors such as Schrödinger maximal function estimates.

4.2. An almost sharp example for Theorem 4.1. The function g we discussed in §2 is an (almost) sharp example for Theorem 4.1, as we explain below.

Note that for the function g, if we take X to be the 1-neighborhood of $\{(n_1 R^\sigma, n_2 R^\sigma, \ldots, n_{d-1} R^\sigma, n_d R^{2\sigma}) : n_1, \ldots, n_d \in \mathbb{Z}\} \bigcap B_R^d$, then since the Fourier support of g only intersects $\sim R^{(d-1)\sigma}$ θ's and that the supports of wave packets from one θ are essentially nonoverlapping, the relevant M is $\lesssim R^{(d-1)\sigma}$. In fact, since g attains almost its maximal possible value at each $(n_1 R^\sigma, n_2 R^\sigma, \ldots, n_{d-1} R^\sigma, n_d R^{2\sigma})$, one can further see that M is indeed $\sim R^{(d-1)\sigma}$, but even without this stronger observation one can still see sharpness from the computation below.

Recall that $p = \frac{2(d+1)}{d-1}$. Because of the property that g is almost the largest possible at each $(n_1 R^\sigma, n_2 R^\sigma, \ldots, n_{d-1} R^\sigma, n_d R^{2\sigma})$, as discussed in §2, we see that

$$\|g\|_{L^p(X)} \sim R^{(d-1)(\sigma-1)} |X|^{1/p} \sim R^{(d-1)(\sigma-1)} R^{\frac{d-(d+1)\sigma}{p}} \sim R^{\frac{d-1}{2}\sigma - \frac{(d-1)(d+2)}{2(d+1)}}. \tag{4.2}$$

Let us look at the right hand side of (4.1). Each g_θ is supported in a ball of radius $\sim R^{-1}$, so one can see that each $|g_\theta|$ is $\sim R^{-(d-1)}$ on a ball of radius $\sim R$ centered at the origin and has the same value on the whole space as an upper bound. Hence each

$$\|g_\theta\|_{L^p(w_{B_R})} \sim R^{-(d-1)+\frac{d}{p}} \sim R^{-\frac{(d-1)(d+2)}{2(d+1)}}$$

and the right hand side of (4.1) for $v = g$ is

$$\sim R^\varepsilon \cdot R^{\frac{(d-1)\sigma}{d+1}} \cdot R^{\frac{(d-1)\sigma}{p}} \cdot R^{-\frac{(d-1)(d+2)}{2(d+1)}} \sim R^{\frac{d-1}{2}\sigma - \frac{(d-1)(d+2)}{2(d+1)} + \varepsilon}. \tag{4.3}$$

We see that the right hand sides of (4.2) and (4.3) match except for the R^ε-loss, showing that g is an almost sharp example for Theorem 4.1.

Since the decoupling theorem for the paraboloid by Bourgain–Demeter is weaker than Theorem 4.1, we see that this function g is also sharp for Bourgain–Demeter's decoupling theorem.

As mentioned above, the refined decoupling Theorem 4.1 has other consequences that are useful in problems in geometric measure theory such as the Falconer distance problem. For example, by dyadic pigeonholing and Hölder's ineuqality, it implies the following corollary:

Corollary 4.2. *Let $R > 1$ and $\alpha > 0$. Suppose a set $Y \subset B_R$ is a union of lattice 1-cubes with the following "fractal structure (between scale 1 and R)":*

$$|B_r \bigcap Y| \lesssim r^\alpha, \quad \forall 1 \le r \le R, \forall B_r \subset B_R.$$

Suppose $\operatorname{supp} \hat{v} \subset P^{d-1}$ *with* $\hat{v} = \varphi\, \mathrm{d}\omega$ *such that in the wave packet decomposition of v in B_R, each wave packet has its essential support hits a $\lesssim R^{-\frac{(d-1)}{2}}$ fraction of unit cubes in Y. Then*

$$\|v\|_{L^2(Y)} \lesssim_\varepsilon R^{\frac{1}{d+1}(\alpha - \frac{d-1}{2}) + \varepsilon} \|\varphi\|_{L^2(\mathrm{d}\omega)}. \tag{4.4}$$

By a similar computation, one can see that if we take $\alpha = d - (d+1)\sigma$ and v to be the function g in §2, one has again an almost sharp example for Corollary 4.2. In this example, both sides of (4.4) are comparable to or close to $R^{\frac{d-3}{2}\sigma - \frac{d-2}{2}}$.

Theorem 4.1 and various versions of Corollary 4.2 were used in [**7**] and later works such as [**5**] to make progress towards the Falconer distance conjecture. The sharpness of these two propositions showed in this section suggests that to make further progress beyond e.g. [**7**], one either has to sharpen other components in these two papers or to design new approaches.

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