Moving Beyond Show and Tell in Proof-Based Courses

Kathleen Melhuish, Cody L. Patterson, Paul C. Dawkins, and Kristen Lew

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As instructors, we have likely all encountered the situation where a certain few students dominate the mathematical activity in our classrooms. We may find this especially to be the case in classes where students are encountering proofs for the first few times. Not only do these classes raise the level of abstraction, they deeply change the game students are asked to play. Students need to argue from a particular logical-deductive approach and use symbol-dense language that follows a whole new set of rules. Too often, these classes narrow the mathematics game from one with exploration, discovery, and testing of unfinished ideas to a precise and prescriptive exercise in setting up hypotheses, unpacking definitions, and arriving at a predetermined conclusion. As a result, it can be easy for proof classrooms to be a place where the only students perceived as competent or "good at math" are those who already can produce a symbolic proof, and quickly.

As mathematics instructors and education researchers we have repeatedly challenged ourselves to involve more students in proof-based classes. Just as important is that we involve students in such a way that we do not regularly put a fixed set of students in a position to guide the work of the other students. We want all students to develop a sense of their own mathematical capability and be seen as capable by their peers.

So how do we do this in ways that do not unnecessarily constrain mathematical activity, that preserve the intentions of our proof-based courses, and at the same time involve more students (and perhaps more mathematics)?

In this article, we share some of our research-based attempts to change the nature of students' mathematical activity in proof-based classes that might address these challenges. We draw on several heuristics:

- (authenticity principle) The undergraduate proof-based classroom should support students in apprenticeship into the mathematical discipline and thus should reflect the tools of research mathematics. [MVLE22].
- (access principle) The undergraduate proof-based classroom should start with current student backgrounds in order to provide opportunities for all students to engage in the activities of research mathematicians. [MDLS22].
- (participation principle) The undergraduate proof-based classroom should support all students in contributing to the mathematical activity in such a way that participation is relatively equal. [MDLS22].
- (expansion principle) The undergraduate proof-based classroom should expand students' tools and knowledge rather than try to replace their current ways of reasoning [WM22].

While these principles exist in theory, we have found it a worthwhile challenge to consider how we, as instructors,

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Communicated by Notices *Associate Editor Yvonne Lai.*

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DOI: https://doi.org/10.1090/noti3072

can bring them to life in proof-based courses. We have adopted two overarching innovations that have successfully supported this work. First, we develop collaborative tasks that treat theorems and proofs not only as objects for students to construct, but also as objects of study, expanding students' access to different forms of mathematical activity. Second, we adapt and incorporate a number of participation-broadening teaching practices more thoroughly studied at the K–12 level into the proof-based setting.

In this article, we will discuss the three tasks we developed specifically for abstract algebra. We note that these tasks were developed and refined over six implementations which included testing with small groups of students (two times), then in the classroom (four times). The classroom testing began with a member of the project team (Melhuish) implementing both in-person and online versions, and then an instructor outside of the team (Patterson). Current instructor guides for these three tasks are available by request; please visit `https://rume.txstate.edu/curricular-materials.html`.

We organize the remainder of the paper as follows. First, we share an overview of the three abstract algebra tasks, which embody three different ways we encourage students to explore theorems and their proofs. Subsequently, Patterson reflects on his implementation of the tasks as someone outside of the team, and shares some insights into the learning opportunities the tasks can create for students. Finally, we discuss the next steps in this project, including developing parallel tasks for other proof-based courses to be made available for instructors.

Orchestrating Discussions Around Proof (ODAP)

One way we have been framing changes in student classroom activity is moving beyond just proof construction. But what exactly does that mean? First, we want to reiterate that developing and polishing proofs is an essential activity, and we do not intend to diminish this. However, take a moment and think about the types of mathematical work you've done as a researcher. We bet you could easily list many kinds of essential work other than writing the formal proof product. Perhaps you can recall exploring examples, tweaking statements to test new assumptions, reading an article to better understand a technique, or assessing the validity of an argument in a manuscript you are reviewing. In fact, many mathematicians have lamented the way that the final formal proof obscures the mathematical work that comes before it. In our recent work, we have documented the ways that mathematicians go about their work and compiled a list of 10 different tools used towards 9 aims found in the work of research mathematicians [MVLE22]. For instance, a mathematician may generate *examples* of a certain type of object in order to *test* a conjecture about that class of object. Or they may *analyze or refine* parts of a statement that describes that class of object in order to *construct* a more elegant definition of that class. We used this analysis as a launching point for designing tasks that engage students in an expanded set of mathematical activities beyond just proof production.

For each task we developed, we focused on several instructional elements: (1) how do we best launch the task? (2) how do we support group work? (3) how do we work with what students create during group work and orchestrate whole-class discussion? For this article, we are going to unpack the types of tasks with emphasis on the expanded mathematical activity. For more information on other elements, see [MDLS22]. We refer to these tasks as the Orchestrating Discussions Around Proof (ODAP) tasks.

Task 1: Proof and theorem comparison and analysis. The first type of task we developed relied on "opening" proofs and theorems to be objects of students' analysis. We developed this task around a common structural property theorem found in abstract algebra:

Theorem 1. *Suppose G and H are isomorphic groups. If G is abelian, then H is abelian.*

This theorem was selected for several reasons. First, it contains a statement that can be strengthened: only a surjective homomorphism is needed between G and H for H to be abelian. The majority of theorems provided in introductory courses are already in the strongest form and do not allow for such exploration. Second, students often inadvertently weaken the statement when writing a proof, arguing that the images of elements of G commute rather than arbitrary elements in H, and then do not refer back to the surjectivity of the map to deduce that therefore any two arbitrary elements of H commute (See [MLC19], p. 214). Thus, this context provides an opportunity for students to engage in analyzing proofs for hidden assumptions and to explore what assumptions in a statement are needed. A typical structure for this task is:

- The instructor prepares copies of two purported proofs of the theorem. These proofs are designed by the research team to represent two common student approaches to proving the theorem: a "G-first" approach and an "H-first" approach, using two different sets of assumptions:

  Proof. Let $a, b \in G$. Then $\phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a)$ since $ab = ba$. Therefore $\phi(a)\phi(b) = \phi(b)\phi(a)$. Since these elements are in H, H is abelian. □

Proof. Let $c, d \in H$. Then $c = \phi(a)$ and $d = \phi(b)$ for some $a, b \in G$ since ϕ is onto. So $cd = \phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a) = dc$. Therefore, H is abelian. □

- After a discussion of the key definitions, assumptions, and conclusion of the theorem's statement, students are put into pairs. The two proofs are given to each pair, with each partner responsible for one.
- The partners explain their proofs to each other and identify one thing that makes sense about the proof and one thing they have a question about.
- In whole-class discussion, the two proof approaches are compared to identify similarities and differences. A running list is kept on the board. At this point, students typically notice many things including that onto is used in one proof and that the arbitrary elements start in the domain or codomain group, respectively. This allows for a discussion of how different assumptions are or are not used in the arguments.
- Students are given an opportunity to state conjectures based on the reasoning they have examined. At this point, students often suggest some of the following: the target group is abelian if ϕ is a homomorphism; if ϕ is a surjective homomorphism; if ϕ is an isomorphism. Note that two of these three are strengthened versions of the original statement, though the first is not true.
- Students test the statements by analyzing the proofs and generating examples of homomorphisms to arrive at the strongest valid version of the theorem. Students also may "patch" the first version of the proof.

Notice that the task involves partners having clear roles and responsibilities, and that they are asked to engage in many types of mathematical activities with the proof. For more information on implementing this task see [MLH22].

Task 2: Proof and theorem comprehension. The second task was developed for the fundamental homomorphism theorem (or first isomorphism theorem):

Theorem 2. *If $\phi : G \to H$ is a group homomorphism, then*

$$G/(\ker \phi) \cong \phi(G).$$

The crux of this task is comprehension of the theorem's statement and proof. That is, we are not asking students to construct the statement or proof, but rather make sense of a conceptually and symbolically dense theorem. In particular, the theorem connects many fundamental ideas from group theory (e.g., isomorphism, homomorphism, quotient groups; see [MGD+23]) and has a proof with multiple parts referencing abstract objects. A typical structure of this task looks like:

- After initial discussion of the theorem statement and needed definitions, small groups are each given a different group homomorphism. The examples use groups that are familiar to students (for example, a homomorphism from $\mathbb{Z}_{12}$ to $\mathbb{Z}_4$, or from a dihedral group to $\mathbb{Z}_2$) and are designed to allow students to quickly build a mapping diagram showing the correspondence between domain elements and codomain elements. The small group works on the board with each member having a different color chalk. One person is responsible for writing the elements in the domain and codomain groups. One person is responsible for drawing a mapping diagram for the homomorphism. One person is responsible for identifying the kernel and its cosets. The final person is responsible for adding the isomorphism map to the picture.
- In a whole-group discussion, students look at the structural similarity across the examples on the board and develop a symbolic representation of the isomorphism that would capture each of the instances.
- Students then anticipate what will need to be proven in the theorem. This list typically includes showing there is an isomorphism (one-to-one, onto, homomorphism), but not that the function is well-defined.
- Students are then given the proof and asked to chunk and label the pieces according to the goals. At this point the class discusses the need to show that the function is well-defined.
- Each small group is then responsible for one chunk of the proof. Within the group, each member is given a card that contains a discussion question that can help the group make sense of their particular section of the proof. (Figure 1 provides some examples of the discussion questions provided.)
- The small groups each present their section to the class.

Notice this task involves substantial mathematical activity despite students being given a theorem and proof. This can allow for emphasis of structural elements and proof approaches. Finally, we note the fact that the examples on the board are joint products (as each student contributes a piece), and small-group time is structured by distributed discussion prompts.

Task 3: Theorem conjecturing and proving from diagrammatic structure. The last of our three abstract algebra tasks was created for Lagrange's theorem:

Theorem 3. *For any finite group G, the order of every subgroup H of G divides the order of G. Alternatively, the quotient of the order $|G|/|H|$ is equal to the index $[G : H]$.*

We selected this theorem not just for its centrality to the course, but because important insights can be gained via reasoning about diagrams and recognizing the multiplicative structure of the cosets. For this task, students explore cosets and then use those insights for Lagrange's theorem. A typical structure of this task looks like:

- Small groups of students are each tasked with finding all the subgroups of the group they are given (or to save time, provided the list). The groups given are relatively small and familiar to students at this point: cyclic groups $\mathbb{Z}_n$ for different values of $n \leq 12$, and possibly a dihedral group of order less than or equal to 10. Students record subgroups on the board and then are asked to conjecture relationships between subgroups and their parent groups by examining examples developed by other students. This often leads to a lengthy list that includes something akin to Lagrange's theorem.
- The statement of Lagrange's theorem is discussed, and students anticipate what needs to be proven: a statement of the form $|G| = k|H|$ for some $k \in \mathbb{Z}$. At this point, it can be helpful to prime students to think of a definition of multiplication that has worked since elementary school.
- Cosets are introduced as a useful tool, and each group of students returns to their parent group and creates cosets for a designated subgroup. Students again make conjectures based on what they notice from the class's example work. This list should include things that can be formalized as lemmas: all cosets are the same size, the union of all cosets is the group, distinct cosets are disjoint. Students usually notice most of these facts as they work, though they may state them differently (for example, "the sets make up the whole group").
- Students are then asked to return to their example and identify what the numbers $|G|$, k, and $|H|$ in the multiplication statement $|G| = k|H|$ represent in terms of the coset list they have developed.
- In small groups, students then work on creating an outline of a proof that draws on the three lemmas. Depending on time, students can work to prove the three lemmas in class or at home.

Much like the prior tasks, this task subdivides the mathematical activity. Students are positioned to make a number of conjectures, and then use tools (multiplicative structure and lemmas) to develop a proof of Lagrange's theorem. We note that earlier versions of this task did not have clear individual responsibilities in the group. In later versions, particular group roles were developed, each with specific guidelines and roles in activity management (e.g., Example Guardian, Meaning Manager, Lemma Liaison, Conjecture Curator).[1] For example, the Conjecture Curator is responsible for recording and presenting what the group notices or conjectures about subgroups in the first part of the task, and about cosets during the coset-building exploration. These roles were designed to support interdependence among team members and individual accountability.

What Might Tasks Such as ODAP Make Possible?

In this section, we switch point-of-view to share reflections from Cody Patterson, who implemented versions of the ODAP tasks after they were tested and developed by the project team. Patterson has taught proof-based courses in mathematics for ten years, including introduction to proof, real analysis, and abstract algebra. We hope the first-person reflection helps to illustrate the potential of these types of tasks in proof-based courses.

When possible, I (Patterson) teach using an inquiry-oriented pedagogy [LR19], devoting a significant portion of class time to having students explore problems and develop proofs on their own. In some cases, I have had students develop proofs of most of the results of the course with minimal lecture or direct guidance from me. Historically, I have facilitated this by producing a list of theorems for students to prove, with some examples and other exercises interspersed.

When I was invited to try using the ODAP tasks in my algebra class, I saw an opportunity to engage students in ways of thinking about proof that had not been fostered in my teaching to date. I also appreciated the guidance that the task documentation provided on the use of instructional routines such as the task launch and facilitating productive class discussions using students' invented solution approaches [SESH08]. While the ODAP tasks are rooted firmly in disciplinary considerations of what it means to engage in mathematical activity, their incorporation of routines from the K–12 mathematics teaching literature helps structure student-centered activity so that

[1] *We wish to acknowledge Brittney (Bea) Ellis's contribution of these group roles and names in the task redesign process.*

lessons reach their intended endpoints in a timely manner, and leverage the thinking of a wider range of students, not only those most often encouraged to share their thinking publicly during class.

In this section I will share a few anecdotes illustrating how the ODAP tasks engaged my students in forms of mathematical thinking that I did not typically see with traditional tasks that defined participation narrowly as proof construction, while also revealing the mathematical competence of students who often went unheard during typical lessons.

Broadening students' mathematical thinking. The process of developing a proof of a relatively routine proposition builds some skills that are indispensable in the learning of advanced mathematics: using definitions, unpacking assumptions, and using the structure of the desired conclusion to formulate a game plan for the argument. These operations, which rely primarily on logical structure, fall within *syntactic modes* of thinking about proof [Alc10], [WA04]; it is expected that students develop these skills in an introduction to proof course and continue to hone them through other advanced mathematics courses. However, as instructors, we also wish to see students develop *semantic modes* of thinking about proof: ways of thinking that attend specifically to the mathematical objects described by a proposition or argument.

In the first few weeks of my algebra course, I implemented the ODAP task on Lagrange's theorem. As described above, a centerpiece of the lesson is a small-group exploration in which students are given a group and a subgroup and are asked to work at the board to generate all left cosets of the given subgroup. Transcripts of the small-group work from my class gave us insight into how different teams of students approached the coset generation process. One team was tasked with generating left cosets of the alternating group A_3 in the symmetric group S_3:

> Richard: Basically take every element and apply it to the left side. ... So we just start with the first one – with (1), right? So it would be (1)(1), (1)(1 2 3), (1)(1 3 2), and that would give you a coset. Which is going to be itself, right?
> Terry: Which is going to be that. So if $a = (1)$, if a equals the identity, we're just going to get the same group.
> Richard: The same subgroup, yeah.
> Terry: Yeah, but if it equals (1 2) –
> Richard: Are you understanding what we're doing here?
> Will: Yeah. And a can be any of these, right?
> Richard: Yeah, a is all of them. We do all of them. Each one will generate a different coset.

Later, students were asked to discuss and record any relationships they noticed among the left cosets:

> Will: Did y'all notice any relationships?
> Terry: Well, the order of the coset is the order of the subgroup, which is pretty obvious.
> Richard: Interesting that some of them don't have the identity in there.

These transcript segments illustrate how the team's example generation (building a left coset by picking an element a and composing a with each element of the subgroup) set students up to make a key observation about the size of each coset. They also illustrate that while the tasks elicit valuable mathematical thinking from students, they are not a panacea for issues of participation that sometimes emerge in small-group work. Claims that a mathematical relationship is "obvious" can function as a form of mathematical microaggression [CARW23], [Su1510] and may discourage participation from students to whom the relationship might not seem obvious.

Meanwhile, Paolo, whose team was generating left cosets of the subgroup $\{0, 4, 8\}$ of $\mathbb{Z}_{12}$, noticed a relationship between subgroup size, number of cosets, and group order that had been mentioned in passing during a whole-class discussion:

> Paolo: I don't remember if we ever proved it, but I remember something about a correlation between the order of your main group G and divide that by the number of elements you're using from your subgroup, and that's how many unique cosets you're going to get. So like $\{0, 4, 8\}$; $\{1, 5, 9\}$; $\{3, 7, 11\}$. Four, right. We've got order 12, we're using three elements from a subset or a subgroup. Twelve divided by three, four. We've got four unique cosets. But I just don't remember if that's a legitimate rule or if it's happenstance.

As we see from these snapshots of student discussions, this small-group exploration of examples gives students the opportunity to gain two insights that can be hard to obtain through syntactic thinking alone: why all left cosets must be the same size as the subgroup (we generate a left coset by picking a group element and multiplying it by each element of the subgroup), and why the union of the left cosets is the whole group (each group element is in the coset it generates). It also allows students to see that distinct left cosets are disjoint, though additional work (and usually some whole-class discussion) is needed to see why. Part of what makes this segment of the lesson powerful is that it allows students to begin making connections (such as Paolo's dividing 12, the number of elements of $\mathbb{Z}_{12}$, by 3 elements per subset) that infuse meaning into the equation $|G| = |H| \cdot [G : H]$ at the heart of Lagrange's theorem. It is an additional bonus if it sparks students' curiosity about whether this relationship exists for all groups and subgroups or is "happenstance." Ultimately, the goal is for these insights and wonderings to occur consistently for many students as a result of the design of the task, and not only for a few fortunate students as they work on a proof in isolation.

What I found especially intriguing about the implementation of the ODAP tasks was the space they seemed to provide for students to talk about aspects of mathematical practice that are not strictly part of proof writing as traditionally conceived in the undergraduate curriculum. For example, during the coset generation process Paolo talked about the utility of having a standard form for expressing mathematical objects such as sets:

> Paolo: The order that the elements are in doesn't matter as long as they are all there. It's just that standard form is you put them from smallest to largest. The way he explained it to me is that if you've got a bunch of people working on the same thing, but you're not keeping them in a specific order ... it's harder to track if you've got multiples [instances of the same coset] ...

When I implemented the proof analysis task while teaching the structural property theorem lesson described above as Task 1, the contrasts between the two proofs raised questions for some students. Two groups independently asked a key question about the "G-first" proof:

> Blaise: How does it show that H is abelian?
> Jeff: Well, because with—because you originally start off with $\phi(ab)$. You can change it up with having b composed with a since G is abelian ... And so because of that you end up having $\phi(a)\phi(b)$ is the same thing as $\phi(b)\phi(a)$.
> Blaise: See, the only problem though is that I don't think this implies anywhere that this is in H.
> Jeff: Yes. That is true.
> Blaise: 'Cause you never said let a and b be in G, you never said let anything be in H.
>
> Jake: So what makes sense is that this is abelian. $\phi(a)\phi(b)$ is equal to $\phi(b)\phi(a)$. But how are we supposed to know—the question is, how are we supposed to know $\phi(a)\phi(b)$ is in H?
> Joseph: Because ... ϕ is a function G goes to H, but it's not stated. But I guess we can assume it 'cause they are isomorphic.

Both pairs had noticed that the G-first proof does not define ϕ as an isomorphism from G to H, and therefore the link between $\phi(a)$ and $\phi(b)$ and elements of H is not explicit. Another pair noticed that the G-first proof did not use a key property of isomorphisms:

> Diego: How do we know this? We didn't establish like, let this be in G. We didn't establish anything like this.
> Ochn: That's the fundamental problem with this proof. It only proves for some, because the thing I noticed was they never used isomorphic. They only used [homomorphism], so it's like they are proving a different statement almost.

Ochn's comment foreshadows some important mathematical work that an instructor can lead in the subsequent whole-class discussion. While we want students to be attentive to the overall validity of an argument, we would also like for students to be able to dissect the argument, identify parts that are valid, and even consider the possibility that an alternative argument might prove a weaker, stronger, or different claim. Because the elements of G in the G-first argument are arbitrary, the proof can be taken (with perhaps some clarification) as an argument that the image of ϕ is abelian. If ϕ is an isomorphism, this image is H; however, this enables us to prove the more general fact that the image of an abelian group under a homomorphism is abelian. (The H-first argument can also prove this fact with minor modification.)

By setting up encounters with proofs written by hypothetical third parties, tasks like this allow students to engage in collective analysis of the practice of proof-writing, leveraging their own insights about features of proofs that help (or hinder) understanding of an argument. Because the source of the proofs is external, students are free to be candid about parts of proofs that they find unclear or incorrectly reasoned. This helps make students' meta-rules for proof public and available for instructors to treat as objects of inquiry—a practice that can be difficult when class time consists primarily of student presentations of finished proofs.

Broadening who participates in mathematical thinking. When teaching a proof-based course, I often see a few students distinguish themselves as highly confident early in the term; this sometimes discourages other students from participating in whole-class and even small-group discussions. As an instructor, I want every student to see themselves and be seen by peers as capable mathematicians who can contribute to the class's work. Therefore, one aspect of implementation of the ODAP tasks that I particularly enjoyed was that each task incorporated features that set me up to highlight productive mathematical thinking by many students, not just the few who start my course with polished proving skills.

Facilitating students' access to proof. Around the halfway point of my algebra course, I facilitated the ODAP task on the fundamental homomorphism theorem. During the

Well-Defined Person 1

- What is the difference between β and φ? What do elements look like in the domain of β and what do elements look like in the domain of φ? (Consider referring back to a diagram.)

Well-Defined Person 2

- What does it mean for $g_1K = g_2K$? Can you create some generic cosets if K = {e, k1, k2, …}?
  - g1K = { }
  - g2K = { }
- How do we know that $g_1k = g_2$ for some k?

Well-Defined Person 3

- Why do we get to switch between β and φ in lines of this proof?
- Where is homomorphism used in the equations line? Where is kernel used?

Well-Defined Person 4

- What does it mean that β does not depend on the choice of coset representatives?

Figure 1. The comprehension questions given to the group responsible for explaining the proof that β is well-defined. Each question is given to one group member.

proof comprehension portion of the lesson, I divided students into four groups, and tasked each group with developing an explanation of one part of the proof: (1) that the proposed isomorphism $\beta : G/(\ker \phi) \to \phi(G)$ is well-defined, (2) that it is homomorphic, (3) that it is injective, and (4) that it is surjective. To support students' making sense of the mathematics and provide some indicators of elements of each part of the proof that I wanted students to be able to explain, I handed each student a card with a comprehension question; for example, "Why do we get to switch between β and ϕ in lines of this proof?" (Both the textbook proof and the proof comprehension questions are standardized as part of the lesson; see Figure 1 for examples of comprehension questions.) I explained that each student was responsible for leading a small-group discussion of the question on their card, while clearly indicating that understanding the proof and answering the questions was a group responsibility, not an individual one.

When setting up groups for in-class tasks, I usually organized groups strategically based on my knowledge of who worked well together and with the goal of ensuring that each group would have enough ideas (and comfort sharing them) to make progress on the work we were doing. When students organized into their small groups for the proof comprehension task, I noticed that the group in charge of explaining why β is injective consisted of two students who had struggled in the course up to that point, and one who had attended only about 20% of class sessions. I was concerned: what if nobody in the group could make progress toward understanding the section of the proof they were assigned? I was pleased to find my fears were unfounded: the one student in the group who had been in class for the previous part of the lesson presented on behalf of this group, and gave a clear and thorough explanation of the part of the proof that shows β is injective.

I saw several similar occurrences during the lesson. When I chose a student from each group to summarize their part of the proof, I tried to avoid the group member I knew was the most comfortable sharing their thoughts publicly during class. As a result, I (and the class) got to see deep, careful reasoning about the proof from students who had not previously had many public opportunities to be seen as competent. My postmortem lesson notes to the project team highlighted the work of a student I had taught in several courses, including an introduction to proofs course, which he had passed only by a slim margin. Of his presentation of the part of the proof that shows β is well-defined, I wrote, "Today I loved that he spontaneously made the connection between Well-Defined [question] 4 and my question about 'okay but what is this part of the proof doing?' He's been surprising me in a lot of positive ways this semester."

Fostering egalitarian group dynamics. Other ODAP tasks supported my goal of equitable participation in different ways. In past implementations of the ODAP structural property theorem lesson, the instructor had asked students to construct their own proofs of the property for homework; the in-class portion of the task then depended on sorting students into groups so that each group represented multiple different approaches. This often led to some pairs or small groups in which one student had a valid proof that they understood and could explain fluently, and another student had a proof but was not confident in it. This sometimes resulted in one student dominating the small-group discussion and pointing out errors in the other student's argument rather than identifying interesting logical relationships between different arguments, which was our goal. We wanted to give students time to work together to familiarize themselves with multiple arguments, but did not want one student to run away with the group's activity during this time.

The ODAP team addressed this design challenge by adapting two student proofs collected from prior research, pairing students up with each student positioned as the "expert" on a different proof, and providing students with a discussion template in which they take turns sharing what makes sense and where they have questions about each proof. The goal of this discussion template was to mitigate lopsided power dynamics among group members and ensure that each student has something unique to contribute to the discussion. We also wanted to steer students away from evaluation of the proofs; we have found that the impulse to evaluate impedes some students' participation and can bar students from going deeper and uncovering more interesting insights about how different proofs can be related. In my implementation of the task, I found that giving each student a proof to explain, rather than asking them to explain their own proof under some duress, took some of the pressure out of this part of the task for students who might have had difficulty writing a proof for homework, and enabled their participation.

Through my implementation of the ODAP tasks, I found that diversifying the ways in which students interact with proof in the classroom—giving them a "textbook" proof of an important result and providing them with a framework to assist comprehension of the proof, or having them work together to analyze and critique different drafts of a proof—gave me a wider window into the productive ways of thinking that students have about proof. It also created a different threshold for participation in these lessons so that more students could make meaningful connections between proofs and the mathematical objects and relationships they describe.

Where Are We Headed?

After the ODAP project completed, we branched into two follow-up projects. The first project, StEP UP (Structuring Equitable Participation in Undergraduate Proof) has focused on developing group-worthy tasks following the same design principles for other undergraduate proof-based classes. This work has been done in collaboration with mathematicians with a wide range of backgrounds. We have focused on more equitable participation in group work and developing tasks where students can engage in an array of proof activities. Using the three ODAP task structures, we have created tasks for linear algebra, introduction to proof, real analysis, and topology. Our first faculty workshop took place in Summer 2023, and the tasks created are currently being piloted in different courses. We are continuing this work providing workshops for mathematicians and developing a bank of tasks for different courses.

The second project called RAMP (Reading and Appreciating Mathematical Proofs) is developing introduction to proof curriculum materials centered around two kinds of reading activities. The first is reading rich and complex proofs to demonstrate the power and beauty of proofs while supporting students in learning how to read proofs for understanding. We compare this to a literature course in which students learn to read and read to learn. Like ODAP, this invites, develops, and rewards different kinds of mathematical competencies than are commonly featured in proof-based classes. The second reading activity focuses on author stories written by women mathematicians or mathematicians of color who selected and crafted the proofs that students read. The stories cover their backgrounds and how they became mathematicians. These readings seek to humanize mathematics and provide diverse representation from within the mathematical sciences.

As educators who care deeply about abstract mathematics and our students' humanity, we are also striving to improve proof-based courses. We hope this article provided some new ideas to think about in these classes. If you have interest in any of the materials, feel free to reach out to us. You can also learn more about our research group's work at `https://rume.txst.edu`.

ACKNOWLEDGMENT. This work is supported by the National Science Foundation through the Improving Undergraduate STEM Education program (Awards #1836559, #2141925, #2235649). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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