Calculus quantities and conservation laws in physics

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A focus on quantity in calculus instruction can enrich students' understanding of what they are doing, and why they are doing it. To help prepare calculus students for future learning in a physics course, we present an evidentiary argument that a Riemann sum representation of integration with quantities is important for quantifying in physics. This paper presents two conservation laws encountered in a physics course through the lens of the Fundamental Theorem of Calculus as a potential bridge between the calculus that students learn, and important physical contexts in which it is used.

Keywords: calculus, physics, fundamental theorem, quantity, infinitesimal.

INTRODUCTION

Physics is the science of change, and calculus is its language. Most physics and engineering majors are required to complete calculus and calculus-based introductory physics courses in their first year of study – ideally preparing them to use calculus in physically significant contexts. Calculus helps guide modelling in physics; it is essential to describing how physical quantities are related to each other, and for creating a structure for new ones to emerge. While many students master procedures in their calculus courses, research shows that it is not unusual for them to view the mathematics in mathematics courses as distinct from physics (Bajracharya, Sealey and Thompson, 2023). This paper argues for an agreed-upon objective for calculus learning that students understand *why* they do what they do in a calculus course, as well as how to do it. The physical world creates a *need* for the tools that calculus provides. This need is an opportunity for learning, as seen through the lens of Harel's necessity principle -- students must have an intellectual need for a topic to be able to learn it.

In addition to quantities playing an important role in physics, the quantities of calculus mean more in *calculus learning* than simply being the objects of procedures. Researchers argue that reasoning with an explicit focus on mathematical quantities facilitates students' learning of calculus. For example, the differential dx in an indefinite integral is seen by many as a cue to the variable of integration. Operationally, there is nothing wrong with that interpretation, it helps you efficiently get an answer, but it reveals essentially nothing about why you would want to perform the integral in the first place. Many authors argue for an infinitesimal interpretation of dx as a quantity, because it facilitates visualizing a tiny amount of something, which is valuable in making meaning of the ratio and products involving dx (Thompson, 2011, Oehrtman and Simmons, 2023, Ely and Jones, 2023).

Ratio quantities, product quantities, rates, intervals, accumulation and change are mathematical quantities around which the ideas of calculus are formed. Student conceptualization of these quantities, and how derivatives and integrals emerge from their combination, is at the heart of understanding why one does calculus in the sciences, and not just how to do it. Conceptualizing the unit as part of quantity has been shown to be important for students in mathematics courses. Thompson (2011) emphasizes the importance of the unit as *part* of the quantity itself, e.g. a speed v= 10 m/s. In one study in determining the areas and volumes of shapes, Dorko and Speer (2015) observed that calculus students who wrote correct units could explain dimensions of planar figures and solids, and connect this knowledge to the shapes' units. In contrast, students who struggled with units also struggled with dimensionality.

This brief paper narrows the calculus focus to the evaluation theorem of the fundamental theorem of calculus (FTC) $F(a) - F(b) = \int_a^b f(x)dx$, and the mathematical quantities it combines. It highlights the generative richness of the FTC in the context of two foundational laws of physics – the laws of conservation of energy, and the conservation of momentum. The paper concludes, that through coordination, the two disciplines can help students' conceptual gap narrow.

BACKROUND

Quantification and symbolizing in physics

Quantification involves generating physical quantities as useful and productive objects for making sense of a situation. Consider a sailboat moving across water. What measurable quantities can help describe the motion? What arithmetic makes sense in constructing rates? What are their units? White Brahmia (2019) argues that quantification is at the root of modelling on physics, emphasizing the importance of quantity and its rate of change. Many physics quantities are vector quantities, and signed scalars, presenting an additional challenge for new learners. Many quantities emerge from multiplying and dividing other quantities (e.g., momentum = mass x velocity, density=mass/volume). Procedurally, the arithmetic involved in creating new quantities is not a challenge for most students, however understanding why the arithmetic makes sense can pose a significant challenge (Thompson, 2011).

Physics students struggle with symbolizing quantities and operations (Von Korff and Rebello, 2012). Given the challenges of quantification and symbolizing in introductory physics, students must have a solid understanding of mathematical quantities and their meanings before they blend them with the many new symbols they will encounter in a physics course. For example, Gauss's law combines mathematical quantities, symbolizing and both vector and scalar physical quantities: $\oiint_S \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q}{\varepsilon_o}$

Reliable resources and difficulties applying calculus reasoning in physics

Conceptualizing the summing up of quantities, as exemplified in the Riemann sum, is productive for many students (Meredith and Margonelle, 2008, Von Korff and Rebello, 2012, Sealey, 2014, Ely and Jones, 2023). However, students often struggle with understanding physical quantities that approach zero. Visualizing what happens to the physical quantity represented by the infinitesimal *dx* in a definite integral in the limit that it goes to zero is difficult. Where does it go? What are you summing up if you're multiplying by zero? Research suggests that physics students are more successful when reasoning about summing finite infinitesimals, as this approach helps make the abstract concept of limits more accessible and intuitive. (Meredith and Margonelle, 2008, Von Korff and Rebello, 2012, Oehrtman and Simmons, 2023)

Interpretation of the FTC through mathematics quantities

Mathematics education researchers present a framing of the FTC as a relationship between key mathematical quantities of change, rate and accumulation (Samuels, 2022), see Table 1(a).

Physics quantity	$f(b) - f(a)$	=	$\int_{x=a}^{x=b} df$	=	$\int_a^b \frac{df}{dx} dx$	=	$\int_a^b f'(x)dx$
	Total change (accumulation)		Infinite sum of small change		Infinite sum of dep. variable change for each interval × interval		Infinite sum of rate × interval
impulse	$p(t_2) - p(t_1)$	=	$\int_{t=t_1}^{t=t_2} dp$		$\int_{t_1}^{t_2} \frac{dp}{dt} dt$		$\int_{t_1}^{t_2} F(t)dt$
	Change in momentum		Same as above		Same as above		Infinite sum of force × time interval
work done *on* system	$U(x_2) - U(x_1)$	=	$\int_{x=x_1}^{x=x_2} dU$		$\int_{x_1}^{x_2} \frac{dU}{dx} dx$		$\int_{x_1}^{x_2} F(x)dx$
	Change in potential energy		Same as above		Same as above		Infinite sum of force × displacement

Table 1: (a) Shaded region represents mathematical quantities in the FTC (Samuels, 2022) (b) Unshaded region is an FTC representation of conservation laws.

The term *change* here is used to refer uniquely to the change in the dependent variable. While both Δy and Δx are referred to as "change" in mathematics, in the context of scientific measurement they represent different types of change. One is manipulated, and the other is a response, even though they covary. The right –hand side is a sum of infinitesimally small products. Each product has a specific value of the rate as one factor, and infinitesimally small *interval* of the independent variable as the other. The key mathematical quantities are the *change,* the infinitesimal *products*, and the *accumulation* that is found through summing up these tiny products.

EXAMPLES OF FTC IN PHYSICS: CONSERVATION LAWS

The conservation laws are introduced in students' first physics course, mechanics. The conservation of total mechanical energy and the total momentum of a system form the foundation of mechanics. The Fundamental Theorem of Calculus (FTC) provides a framework for representing these conservation laws, as represented in Table 1 (b).

The total energy of a system changes when an external object exerts a force on the system as its position changes along the direction of the force. The dot product of the two vector quantities (force and displacement) in the integral result in a scalar change in energy that is exactly equal to the cumulative effect of the force acting over a *displacement* (position interval). This accumulation is so significant that it is given a specific name: *work*. Work is the only way to change the mechanical energy a system. In the context in Table 1(b), the force both drives and quantifies the rate at which work is done as the object's position changes. Similarly, the total momentum of a system changes when a force acts over a *time interval*. The vector change in momentum is equal to the cumulative effect of the vector force over that total time interval. This accumulation, too, is so crucial that it is given a name: *impulse*. Here, the force both drives and quantifies the time rate at which momentum changes.

A student who is well-versed from calculus in the mathematical quantities that make up the FTC will be better-positioned to take up the new and challenging ideas that it frames when they encounter them in the context of physics. Energy and momentum are abstract, being able to fall back on a facility with conceptualizing the calculus can make learning them easier. Emphasizing the meaning of the operators, quantities and their symbols in the FTC can help prepare students to more easily frame the applications they will encounter in their subsequent courses.

	operators	**quantities**	**language**	**physics examples**
change, interval	d Δ	dy Δx	dep. var. – change indep. var. – interval	impulse as a change in momentum displacement as a change of position
rate of change	$\frac{\Delta}{\Delta x}$ $\frac{d}{dx}$	$\frac{\Delta y}{\Delta x}$ $\frac{dy}{dx}$	ratio of change to interval	acceleration as the time rate of change of velocity force as the time rate of change of momentum
accumulation	$\sum$ $\int_b^a$	$\sum_i \left(\frac{dy}{dx}\right)_i dx_i$ $\int_a^b f(x)dx$	sum of many small pieces	work impulse

Table 2: FTC symbols and quantities common across calculus and physics.

IMPLICATIONS FOR CALCULUS AND PHYSICS INSTRUCTION

Symbols and quantities carry deep significance, and calculus instruction can convey that to students. Table 2 presents those that recur in the FTC, and merit instructional time in calculus. Physics instructors can help enrich their students' mathematical knowledge by knowing the calculus quantities and correctly using them in physics instruction help their students' calculus knowledge emerge in physics contexts.

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