Accumulation as a tool towards blending reasoning about quantity and rate of change in physics contexts

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The amount, change, rate and accumulation of physical quantities are essential features of reasoning with calculus and physics. Experts in physics and mathematics use rate of change reasoning throughout their process of developing and making sense of graphical models; distinguishing between rate and quantity is an essential part of that. We suggest that rate vs time graphs offer an opportunity for direct instruction on distinguishing between rate and quantity, as well as blending this reasoning to determine an accumulation. Here we share some pilot-tested graphical reasoning activities that we have developed based on the ways experts and students reason.

Keywords: covariation, graphical reasoning, quantity, physics

INTRODUCTION

Making sense of quantity, rate of change, and accumulation are central features of calculus (Carlson et al., 2002; Samuels, 2022). Research in mathematics education and physics education has demonstrated that distinguishing between quantity, rate of change, and accumulation is difficult for students (Sealey, 2014; Trowbridge & McDermott, 1980; Von Korff & Rebello, 2012; Yu, 2024). Research has also demonstrated that physics experts distinguish between rate and quantity in part by identifying physically meaningful points in graphical representations and reasoning about the rate of change around those points (Zimmerman et al., 2023). One possible way that calculus and physics instructors may be able to help their students learn to think this way is by using direct instruction of these expert-like behaviours.

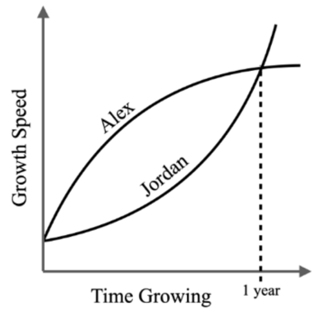
Graphical representations with meaningful accumulated *physical* quantities typically involve a rate of change represented on the vertical axis, and time or position on the horizontal axis. In physical contexts, it is also common that the rate of change is a quantity in its own right (e.g. speed is the time rate of change of position, the accumulated quantity in a graph of speed vs time is a displacement). Reasoning about accumulated quantities using graphs of rate vs. time therefore requires students to be able to identify the physical quantity represented by the vertical axis as a rate, interpret the meaning of its rate of change, and use both pieces of information to determine the accumulation as a distinct quantity. Rate vs. time graphs thus provide a rich representation that blends several ways of reasoning about quantity and rate of change, that are ubiquitous in physics courses.

STUDENT REASONING AROUND ACCULUMATION TASKS

The item shown in Figure 1 is one example, featuring a rate (growth speed) vs time graph and asking students to reason about an accumulated physical quantity (amount of growth in 1 year). This task is derived from a survey that assesses physics quantitative literacy (White Brahmia et al., 2021; Zimmerman et al., 2022). The data we share come from a series of 29 individual student interviews conducted as part of validating the inventory.

Students were solicited for interviews from an algebra-based introductory physics class at a large U.S. university. The course is typically taken by 3rd and 4th year university students studying life science, most of whom have completed at least one semester of calculus instruction that includes basic integration. Interviews involved one student and one member of the research team; student participants were asked to work through the items while talking out loud. Interviews were audio recorded and transcribed. Students were offered $15 gift cards as a small thank you for their time. We do not claim that these ways of reasoning are representative of all physics students; rather, we share this evidence to illustrate the varied mental resources these students activated about physical quantities and rates of change at the beginning of an introductory physics class after having taken calculus.

The graph at right represents how fast two children are growing vs time. The children are named Alex and Jordan, and their growth is measured starting on their 10th birthday. Which of the following choices best describes how much the children have grown in one year?



a. Alex and Jordan have grown the same amount.
b. Alex has grown more than Jordan.
c. Jordan has grown more than Alex.
d. The graph does not provide enough information to compare how much the two children have grown.

Figure 1: An example of an accumulation item.

Most of students that we interviewed chose answer options (a) or (b). Students answers and justifications are shown in Table 1. Students who chose (a) often did so either because they conflated the quantity "growth amount" with the quantity "growth speed", or because they examined the *average* growth speed which is the same for both children. Students who chose (b) did so either by noticing that the accumulated quantity (how much the children grow) can be found by taking an integral, or by reasoning that Alex's growth speed is larger than Jordan's the entire time.

There are multiple interpretations for the students who used an intersection approach. One could reason they were distracted by the intersection and viewed the vertical axis as representing total growth. However, this student previously articulated that they understood the vertical axis represented growth rate. Another interpretation is that this student conflated quantity and its rate of change while trying to use them together to find the total growth. Students who chose (a) and discussed average rates of change represent an opportunity for direct instruction. These students have productive quantitative resources around accumulation with linear functions that can be built upon, but do not yet have facility with non-linear changing rates of change.

We recognize that students who chose (b) and understand the procedure for taking an integral may or may not have strong conceptual reasoning around accumulation. We suggest that students who chose (b) and spontaneously chose to compare the changing rates of change demonstrate strong conceptual reasoning about accumulation for quantities with changing rates of change.

Answer Choice	Approach	Example Quote
a	Intersection	"They intersect right here, despite having two different curves for their growth. So that means despite their different rates of growth at this specific year, they [have] grown the same amount."
a	Average Rate of Change	"Since they both have the same growth speed at the end of the year, they have grown the same amount? Because... they have like the same average speed."
b	Area Under Curve	"So I'm thinking that it's like a physics problem where it's like the area underneath the graph. That would mean Alex grew more than Jordan."
b	Relative Value of Rate of Change	"Alex, their, like, their growth speed is just higher for more of the year. So they're just gonna grow more."

Table 1: Common student approaches to the item shown in Figure 1.

These data suggest that problems that ask students to reason about the accumulated quantity represented in rate vs time (or position) graphs may be a fruitful place to help them learn to differentiate between quantity, rate, and accumulation and to better understand how these three kinds of quantities are related.

EXAMPLES OF ACCUMULATION ACTIVITIES

We designed activities in the context of a large-enrollment (N = 323) algebra-based Introductory Electricity and Magnetism course to support students learning: (1) to distinguish between quantity and rate, (2) to reason about changing-rates-of-change rather than an average, and (3) reasoning that blends procedural and conceptual competency with rates of change, independent of calculus algorithms. We note that deciding whether to treat a physical quantity as a rate, quantity, or accumulation in a particular context is one part of "learning to distinguish" between them. To facilitate variation between instructors' instructional preferences, the activities were designed to be administered as clicker-questions during lecture or as practice exam questions. We also included a small number of these items as exam questions as an early measure of whether student reasoning was improving. These items represent our initial pilot into whether accumulation-based activities may help students learn to reason this way.

An example is shown in Figure 2, in which students compare how much heat is transferred across two rods. They are given a graph of *P*, the rate at which heat is transferred, vs *t*, elapsed time. The rate at which heat is transferred can be thought of as the amount of heat that moves from one end of the rod to the other in each time unit.

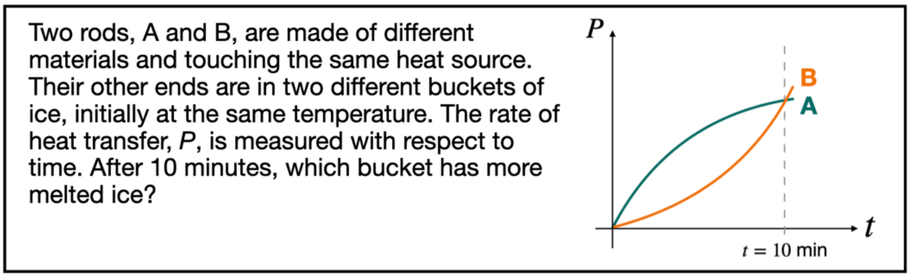


Figure 2: An example of an in-class accumulation activity using a rate vs time graph.

The mid-term exam questions (Fig. 3) provide an early measure of how students' reasoning improved. Although they are mathematically analogous items, they are not rigorous measures of what students learned in the course. Students likely have more facility with some physics contexts (metabolic energy) than others (electric circuits). However, these results paint a picture of how challenging, and context dependent, this kind of reasoning can be for students in a science course–even for students who have completed one or more semesters of calculus. 37% of our students chose the correct answer on the first midterm item, and 57% of our students chose the correct answer on the second. We note that both current and power were directly taught as rate quantities.

We interpret these data as an illustration that students require significantly more opportunities to practice with accumulation than we were able to offer in our preliminary pilot, or than they are getting in their calculus and physics classes alone.

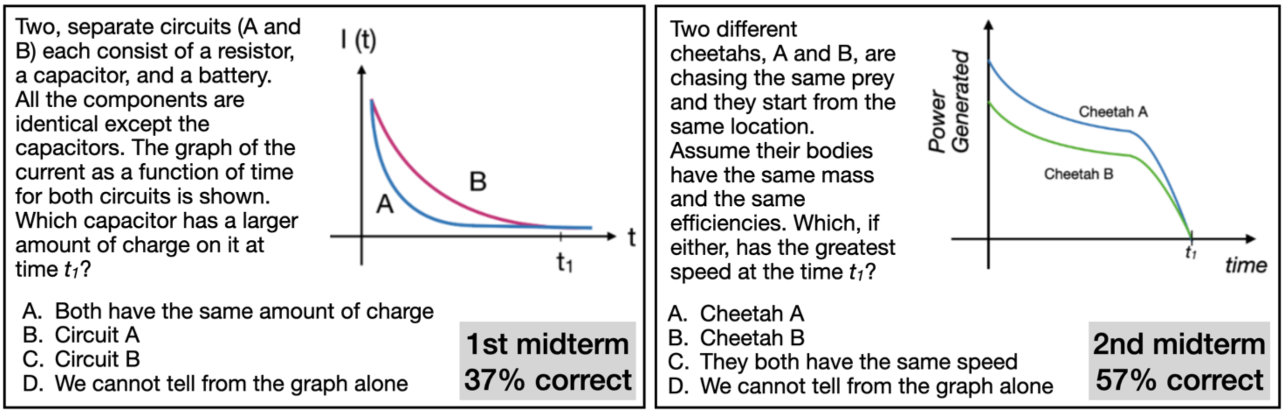


Figure 3: Example exam items from an Introductory to Electricity and Magnetism course (N = 323) The left was given on the first midterm, and the right on the second.

INSTRUCTIONAL IMPLICATIONS

One benefit of incorporating graphical tasks alongside symbolic ones is that there is a high level of conceptual calculus-like reasoning without requiring a high level of procedural proficiency. In introductory physics classes, proficiency with symbolic reasoning is often not consistent across students. It is also typical in physics for graphical questions to act as practice after symbolic ones, despite research that has demonstrated the benefit of a multiple representations approach (Kohl & Finkelstein, 2008). By offering these activities alongside symbolic problems students were grappling with, we leveraged graphical reasoning from the very beginning of the unit.

Our study suggests that university students who have completed calculus and introductory physics are not likely to have strong proficiency with the foundational mathematical ideas of quantity, rates and accumulation. We suggest that these ideas are complex and take time to learn; likely more than any one term university course can manage. Incorporating instruction about accumulation in graphical contexts in calculus courses and across math-based STEM disciplines, that has a common focus and common language, can help students when using calculus to model physical phenomena. We present this work to help foster rich collaboration across disciplines.

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