Introductory physics: Drawing inspiration from the mathematically possible to characterize the observable

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A calculus that characterizes the interaction between quantities, and the mathematical implications of those interactions, will help prepare students who take physics to use mathematics for quantifying the natural world, and uncovering its laws. In this talk I characterize essential features of reasoning with quantity in physics, and some implications for the teaching of calculus.

Introduction

Conceptually understanding what calculus is doing when its most basic functions represent relations between physical quantities is a more valuable learning outcome for students of physics than demonstrating mastery of multiple integration techniques in the contexts of challenging integrals, or knowing cold the tests of convergence for unfamiliar series.

Here is why:

1. A proceptual facility with functions whose variables are scalar or vector quantities is a central feature to expert reasoning in physics. Instructors expect students to have quick facility with this as well, based on their prerequisite math courses.
2. The relationship between physical quantities, their change, their rates of change with respect to time and position, and their accumulation from these rates of change is central to understanding the meaning of the laws of physics.
3. The clear majority of the models in introductory physics involve linear, inverse, sine, cosine and quadratic functions. Students are expected to know the derivatives and antiderivatives of **these** functions, symbolically and graphically, as well as their behavior at physically significant points and extreme cases.

A significant majority of the students who are taking calculus in the US at any given time will subsequently take introductory physics - with the main exception being calculus courses for business and economics majors. I argue that rate and accumulation reasoning are likely important for all calculus students, even those who won't take a physics course.

In calculus and in introductory physics, we are essentially teaching the same students. But do they perceive what we are doing as being the same things? Arguably, students are "culture-shifting" between doing math and doing physics, which limits the quantitative resources they tap into when taking a physics course (Taylor & Loverude, 2023). Bajracharya, Sealey and Thompson interviewed math majors as part of a study to uncover how they made sense of a negative definite integral. They observed that invoking a physics example of a stretched spring helped catalyze sense making. Although the physical context helped math majors conceptualize the accumulation, there was a perceived departure from the pure math world to make meaning, as articulated by one interviewee, "when you think about just, like, the pure math problems, that's all you really think about — just the fact that dx is just telling you … what variable to use (in the integral) … but … here, it represents, it represents something…" (Bajracharya, Sealey, & Thompson, 2023). In physics, every variable represents something physical, we'd like students to imagine the potential of x and y in calculus to represent a whole variety of quantities, even when they're not prompted to do so.

It is challenging to serve all the future needs of the students in a service course as ubiquitous as calculus. Physics is asking for just a bit less breadth in the interest of more depth, such that students can spontaneously decide that taking an integral, or a derivative, or representing a function as a series, is a sensible thing to do in a physics context. Why would you integrate? When is it useful to approximate a function by terms in a series? And can do it as well. The tradeoff is that by considering the interplay between quantities: fundamental quantities, their rates of change, and the accumulation of the product quantities they form, can perhaps help a more diverse group of students conceptualize calculus as well.

There is a natural tension between the learning objectives of a calculus course and what students really need for a physics course. It is true that our worldviews differ. Physics is about modeling the physical world by inventing quantities and their relationships to each other. The ultimate test of models is if they predict what happens in nature. Validated models represent the corpus of knowledge in physics. Mathematics has different constraints, and its validity test is logical proof. Developing reliable capacity to solve problems is an added utilitarian emphasis in both disciplines, to make sure that students can "do" math/physics after having taken a course. While becoming efficient at problem solving is an important learning objective, an excessive focus on sharpening this skill comes at a price. Much is missing in the quantitative reasoning behind why we do what we do, rendering most students unaware as to how they can use their quantitative insight to think creatively in physics.

There is mounting evidence that students struggle with conceptualizing arithmetic and algebra as used in introductory physics (Kuo, Hull, Gupta, & Elby, 2011; White Brahmia et al., 2021). These difficulties carry over into subsequent course taking. In a summary of studies on mathematical reasoning in upper-division physics, the authors found the following common student difficulties, despite having taken many math courses beyond the calculus level:

- activating appropriate mathematical tool without prompting (e.g., delta function, Taylor series)
- recognizing meaning of mathematical expressions
- spontaneous reflection on results (e.g., limiting cases, dimensional analysis)
- generating mathematical expressions from physical description

The students had no problems with executing the mathematics when asked, but they expressed a strong desire to understand what they were doing, and why (Caballero, Wilcox, Doughty, & Pollock, 2015).

This paper explores current educational research focusing on the salient aspects of how some important calculus concepts appear in introductory physics teaching, with recommendations of materials that can help foster a conception of calculus that promotes physics reasoning.

Calculus in introductory physics

Expert physics modeling involves significant overlap of the mathematical and physical worlds

Consider current a priori cognitive models of modeling in physics and in math contexts, on which classroom mathematical modeling activities are framed. The concept of a *cycle* is ubiquitous, exemplified by the Modeling Cycle shown in Figure 1 (Blum & Leiß, 2007; Czocher, 2016). Note the complete separation of the math world and the rest of the world in the mental process. The model implies that mathematizing is done largely in a separate mental place from the context in which it is being done.

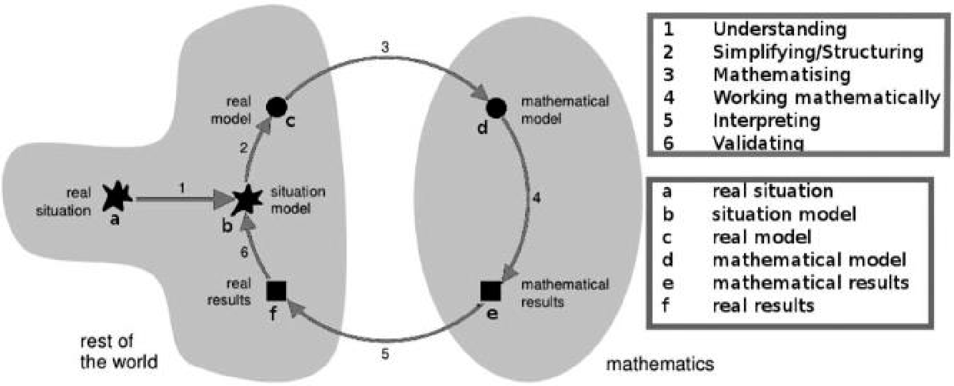

Figure 1: Czocher's redraft of Blum and Leiß's modeling cycle (Blum & Leiß, 2007; Czocher, 2016)

In contrast to the apriori cyclic models, researchers in mathematics education have found little evidence that students' reasoning while modeling is cyclical (Ärlebäck 2009; Borromeo Ferri, 2007). In a recent study, Czocher (2016) conducted interviews throughout an academic term of four engineering majors enrolled in a differential equations course. In each interview, the students were observed solving problems in everyday contexts that required generating mathematical descriptions from a variety of branches of mathematics, including differential equations. The author describes a much finer-grained blending of mathematical reasoning and physical sense-making than is represented in apriori cyclic models of modeling, specifically that ``there are transitions that appear out-of-order. This was largely because three of the modeling transitions (understanding, simplifying/structuring, and validating) appeared early and often throughout the students' modeling processes." The importance of continuous validation to the progress of their mathematization is not predicted by the apriori models. Czocher (2016) presents a fine-grain description of the interpreting and validating that was observed, a portion is reproduced in Table 1.

The students who were less successful spent little time validating, while students who were more successful spent much more time on validation. The subset of skills listed in Table 1 involved in interpreting and validating are precisely the skills physics counts on its students mastering to be successful at modeling in physics -- they are central to mathematization in physics.

Interpreting	Re-contextualizing the mathematical result	• Referring to units • Answering contextual question, not just mathematical one • Interpreting meaning from an equation or its elements, or from the mathematical representation • Referring to conditions/variables/parameters from "simplifying/structuring"
Validating	Verifying results against constraints	• Statements about reasonableness of answer/model • Checking extreme cases and special cases (of variable, parameter, relationship) • Comparing answer to a known result • Estimating an appropriate result • Adding limitations to the model • Talking about ideal results • Comparing merits of different models • Dimensional analysis

Table 1: Adapted from Czocher (2016)

In Zimmerman, Olsho, Loverude and White Brahmia's study of expert modelers in physics (graduate students and faculty), interviewees were asked to create graphical solutions for novel physics tasks (Zimmerman, Olsho, Loverude & White Brahmia, under review). The tasks were isomorphic versions of the kinematics tasks used in the study by Hobson and Moore (Hobson & Moore, 2017), but rendered more challenging for expert physicists by invoking abstracts contexts and quantities. For example, "Going around Gainesville", which asks the interviewee to generate a graph of the distance of a car from Gainesville as a function of the distance it has travelled along the road, became a charged probe moving around a small charged sphere. The task prompts participants to create a graph that relates the electric potential and the total distance traveled, as it moves at constant speed from start to finish.

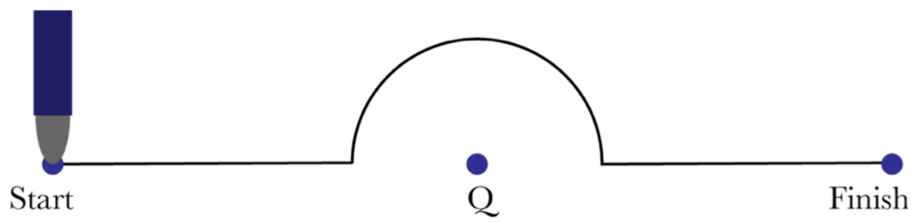

Figure 2: Still from the animation associated with example task (Zimmerman et al., under review)

Zimmerman et al. report many of the mental actions included in Czochers's description of validation are precisely the features that characterize aspects of the study participants' covariational reasoning - specifically their simplification techniques and their tools for covariation when modeling novel physics tasks. A subset of the expert physicists reasoning methods uncovered in this study are represented in the behaviors in Table 1. We note that reasoning with units, dimensional analysis, checking extreme cases, simplifying/structuring and interpreting meaning from an equation and its elements are all essential ingredients in physics modeling.

Many physics students struggle to naturally take up these behaviors in a physics course if they never encountered them in a math course before. In a study conducted by Rowland in the context of a differential equations course, the author found that despite having completed introductory physics, over half of the engineering students were not confident about linking the mathematical expressions they were creating to the physics phenomena they represent, and the clear majority failed to incorporate the notion that the units of each term in the model should be the same (Rowland, 2006). The author argues ``a consideration of units, how they combine, and how they can be used to analyze systems in modelling contexts needs to be an explicit part of instruction." The disconnect between amount and its unit is as much of a problem with physics instruction as it is with mathematics, and it is one we can solve collectively by expanding the overlap of our worlds, such that they aren't perceived by our students as separate mental places.

Quantities are central to the laws of physics

Quantities in physics are either scalars or vectors, and are commonly the result of multiplying and dividing other quantities (e.g., momentum, density). Procedurally, the arithmetic involved in creating new quantities is not a challenge for most students, however deciding when and why the arithmetic makes sense can pose a significant challenge (Thompson, 2011). Vergnaud (1998) argues that multiplication, division, fraction, ratios, proportions, linear functions, dimensional analysis and vector spaces are not mathematically independent, and should be included in a domain he names multiplicative structures. Tuminaro (2007) reports on student difficulties conceptualizing the simplest multiplicative structures in physics contexts.

Quantification produces the physical quantities that are used in physics modeling, and it relies on blending physics meaning with a conceptualization of multiplicative structures. For experts, the blending of the mathematical concepts with physics quantities happens unconsciously and seamlessly (Kustusch, Roundy, Dray, & Manogue, 2014; Zimmerman et al., under review). Expert-like math-physics blending is a desired learning outcome of an introductory physics course, yet it needs to be nurtured as part of instruction for students to understand and develop creativity as they learn to interpret physics models. We suggest that the foundation for this blending can be part of a calculus course. For that to happen, we should agree on what we mean by representing quantity.

Sherin developed a symbolic form framework that explains how successful students understand and construct equations in physics. The symbolic form framework posits that students have conceptual schema associated with specific symbolic patterns (e.g., the *ratio* form) commonly invoked to compare two quantities $\left[\frac{x}{y}\right]$ (Sherin, 2001). Dorko and Spear (2015) developed the *Measurement* symbolic form in the context of area and volume in mathematics, which always includes a unit as well as a value. The authors argue that the units are an important part of students' conception of measurement. I make the argument that in physics, where use the term *quantity* instead of measurement, this form should also include a sign, as most quantities students work with in an introductory physics course are vector components and other signed quantities (Olsho, White Brahmia, Smith, & Boudreaux, 2021; White Brahmia, 2019; White Brahmia, Olsho, Smith, Boudreaux, 2020; White Brahmia et al., 2021). The units and the sign carry important meaning, and I suggest that students can be better primed for this onslaught in physics if they encounter quantity in this way in a calculus course.

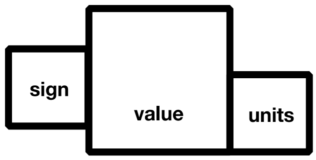

Figure 3: The Quantity symbolic form relevant to physics builds on the Measurement symbolic form by including sign (Dorko & Spear, 2015; White Brahmia, 2019)

Both Czocher's and our (Zimmerman et al., under review) studies provide evidence that successful students, and experts, derive physical meaning from "an equation or its elements" (see Table 1), which are measured or derived quantities in physics models. Calculus provides a mental framework for thinking about the relationships between quantities in physics, and for imagining new ones. The clear majority of quantities in physics have an amount/change/rate/accumulation relationship.

Figure 4 shows a plot of how some fundamental quantities in physics (examples shown are from mechanics) are mathematically processed to create new quantities that eventually play a central role in the fundamental laws of mechanics – Newton's laws and the conservation laws of momentum, energy and angular momentum. The fundamental quantities are directly measurable. All the rest are derived from these measurable quantities. While each of these quantities is sometimes combined with the same type of quantity through arithmetic operations (lengths combine for area, displacement, etc.) many of the quantities that are involved in the laws of physics are related to each other as rates and accumulations (i.e. ``area under the curve"). We adopt "accumulation" as has been put forward by Thompson and others, as it holds much more potential for student comprehension in a physics context than area-under-the-curve does. None of these important quantities are actual areas. The notion of the derivative/antiderivative/accumulation/change relationships are so important in physics, that frequently they are created as new quantities and given their own name - connected through the Fundamental Theorem of Calculus (FTC).

Samuels' Amount Change Rate Accumulation (ACRA) framework of the FTC shows promise for supporting students of physics to conceptualize these relationships (Samuels, 2022, 2023) in the context of a calculus course (see shaded region of Table 2). I've applied the ACRA framework in the unshaded region of Table 2 to demonstrate the essential role the FTC plays understanding the generation of physics quantities, and the physical laws that relate them to each other (e.g., Newton's laws, Conservation laws of energy, momentum, etc.).

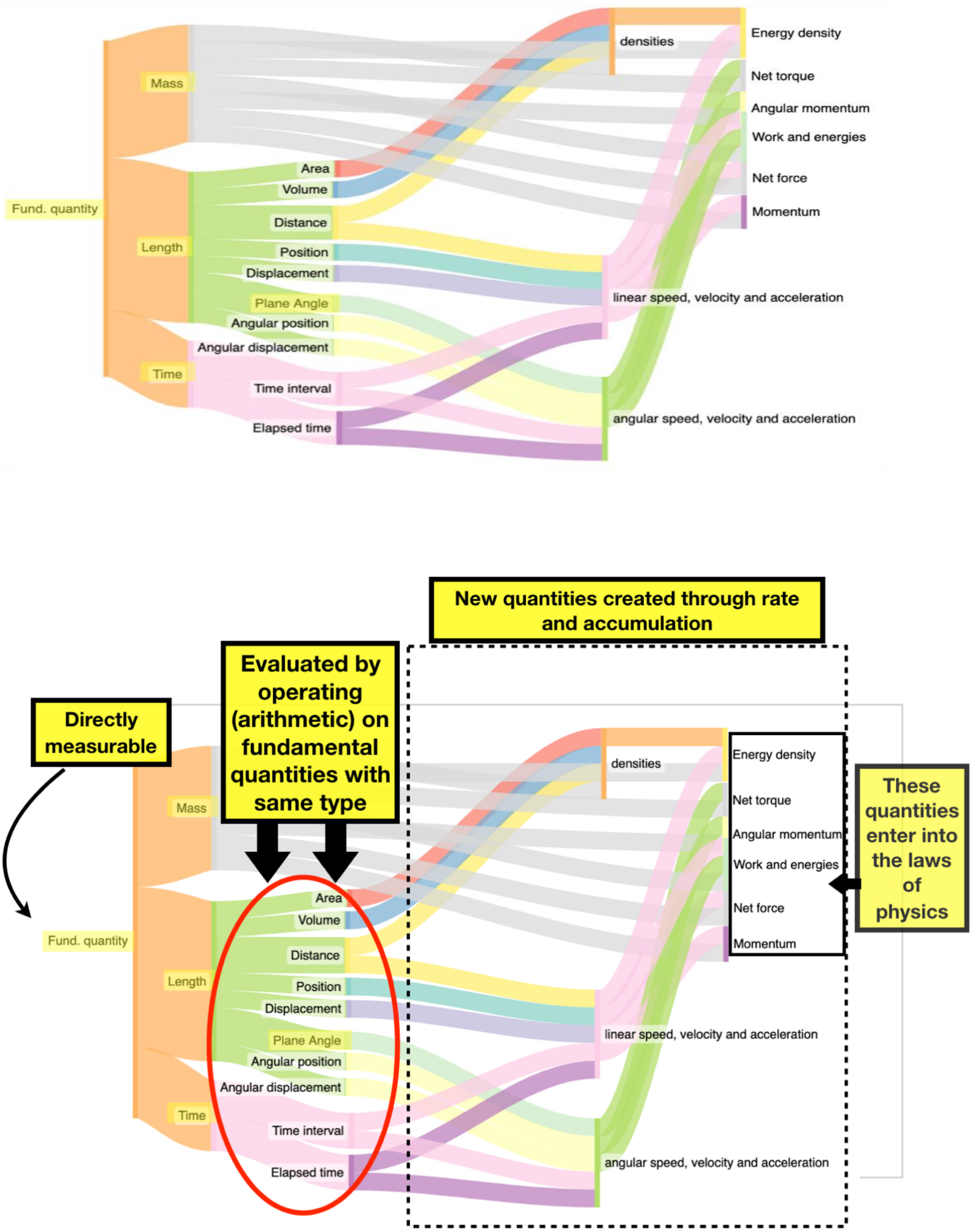

Figure 4: Quantities encountered in introductory mechanics

I argue that quantification is the neglected first step in modeling in physics (White Brahmia, 2019), a neglect that increases the likelihood that students' beliefs about doing physics is that their job is to find the right equation (Hammer, 1989; Kuo, Hull, Gupta, & Elby, 2011). The notion that they can participate in the mathematical creativity of quantification, and modeling, is largely lost on them. Physics has a long way to go such that all students feel confident in their capacity to engage in creative mathematization. Given the preponderance of calculus concepts involved, deepening students' conceptual understanding of what they are doing and why they are doing it in calculus can help students' feel more confident modeling in physics.

Expert Modeling in physics involves a small number of functions

Models in physics typically involve only a small finite number of functions. At the level of introductory mechanics, the laws of physics are dominated by linear and inverse functions, with the more complex combinations of functions that are frequently addressed in a calculus course rarely or never appearing.

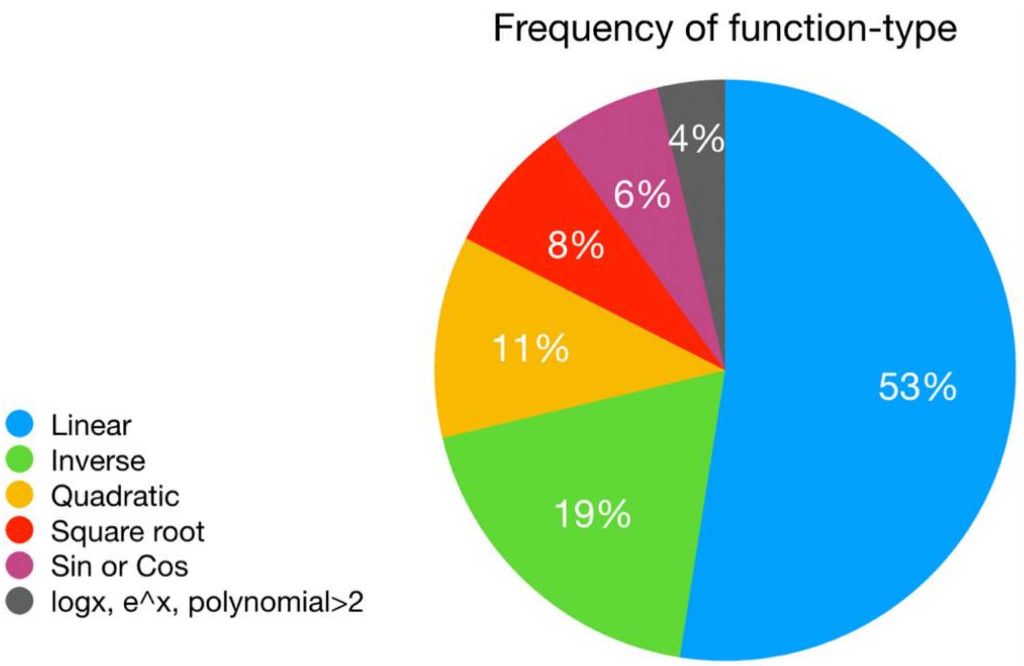

Figure 5: Functions encountered in introductory mechanics

I generated Figure 5 by going through a list of the essential formulas for introductory physics, which is representative of just about any standard college physics textbook (Elert, 2023), and sorting it by function type, noting the frequency of appearance for each function type. The uncertainty on the values shown is likely a few percent. Each of the limited number of functions listed in Figure 5 are central to the covariational reasoning of physics.

Physics quantity	$f(b)-f(a)$	=	$\int_{x=a}^{x=b} df$	=	$\int_a^b \frac{df}{dx}dx$	=	$\int_a^b f'(x)dx$
	Total change (accumulation)		Infinite sum of every infinitesimal change		The integral (infinite sum) of every (infinitesimal change) ÷ (infinitesimal input change) × (infinitesimal input change)		The integral (infinite sum) of infinitesimal rate (as a function) times infinitesimal input change
displacement	$x(t_2)-x(t_1)$	=	$\int_{t=t_1}^{t=t_2} dx$		$\int_{t_1}^{t_2} \frac{dx}{dt}dt$		$\int_{t_1}^{t_2} v(t)dt$
	Change in position		Same as above…in position		Same as above		The integral of the (signed) velocity times tiny time intervals
Δv (velocity change)	$v(t_2)-v(t_1)$	=	$\int_{t=t_1}^{t=t_2} dv$		$\int_{t_1}^{t_2} \frac{dv}{dt}dt$		$\int_{t_1}^{t_2} a(t)dt$
	Change in velocity		Same as above…in velocity		Same as above		Same as above (acceleration)
impulse	$p(t_2)-p(t_1)$	=	$\int_{t=t_1}^{t=t_2} dp$		$\int_{t_1}^{t_2} \frac{dp}{dt}dt$		$\int_{t_1}^{t_2} F(t)dt$
	Change in momentum		Same as above…in momentum		Same as above		Same as above (force)
work done *on* system	$U(x_2) - U(x_1)$	=	$\int_{x=x_1}^{x=x_2} dU$		$\int_{t_1}^{t_2} \frac{dU}{dx}dx$		$\int_{t_1}^{t_2} F(x)dx$
	Change in potential energy		Same as above…in potential energy		Same as above		The integral of the (signed) force times tiny displacements

Table 2: My extension of ACRA (shaded) FTC to include important physical quantities (unshaded)

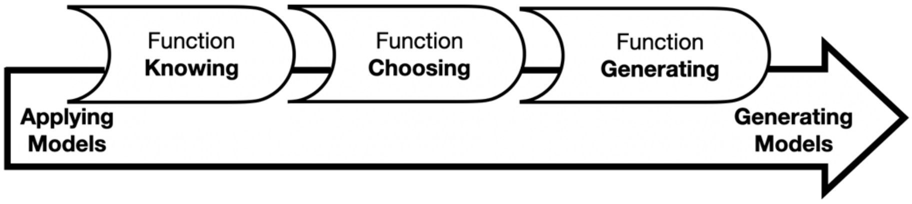

Figure 6: Experts interaction with functions when modeling

Knowing what they look like graphically, how the behave covariationally, how they behave in the limits of very large and very small values of the independent variable, and any other special cases that are specific to the function (e.g. min/max/zeros/special arguments of sine or cosine functions) facilitates modeling for experts (Zimmerman et al., under review). Students who have this deep understanding of these functions before taking a physics course will be at a significant cognitive advantage; it is expected knowledge. In the Zimmerman et al study, we found that when modeling, experts engaged in behaviors of function knowing, function choosing or function generating – which become more cognitively demanding moving from left to right in Figure 6. Experts first look for a function they know based on a similar context (e.g. circular motion invokes sinusoidal functions), and if that fails they tend to choose from the list in Figure 5. If that is unsuccessful, then they try generating a graphical function by invoking covariational reasoning tools (see Table 1 and Zimmerman et al.), designating several physically significant points. They engage in "neighborhood analysis" by considering the first derivative in the neighborhood of these points, and then connecting the points with a line or curve, by considering the 2^{nd} derivative behavior between the points.

An important feature of function choosing and function generating is that they are generally evoked in the context of some sort of data that might (or might not) show a trend consistent with a meaningful function. This modeling scenario is ubiquitous in physics, whether graphically modelling an imaginary situation, or collecting actual data in an experiment and modeling the patterns that emerge from the data. Clean analytical solutions are the exception rather than the norm beyond the introductory course when comparing the real-world patterns to mathematical functions. Making approximations are a standard part of rendering a messy physical system tractable. Einstein famously said, "Everything should be made as simple as possible, but not simpler." Rather than resorting to messy functions, we always hope for one of the functions in Figure 5. Series representations of those functions, especially their first couple of terms, become a standard tool for modeling the physical world beyond the introductory course, and are even invoked in a couple of contexts there as well (e.g. small angle approximation for simple pendulum). Knowing how common approximations are used, and why, would be a wonderful outcome of calculus for physics students.

Recommendations for the teaching of calculus

While I am not an expert in calculus instruction, I understand that changing the content in courses as institutionalized as tertiary-level calculus courses are in the United States is not straightforward. I suggest here some effective, research-validated materials that help students construct their mathematical knowledge in the contexts of quantity. They were all designed to be used in the context of classroom instruction, ideally in collaborative learning environments.

Developing conceptual foundation

Physics Invention Tasks (White Brahmia, Kanim, Boudreaux): Designed to engage students in authentic quantification, in preparation for subsequent formal learning. Students use data from contrasting cases to invent ratio or product quantities, rules or equations to characterize a variety of physical systems. Students work through sequences of such tasks to ramp up from everyday contexts to more abstract physics contexts. We have field tested sets of invention tasks, called invention sequences, both at the pre-college level, in middle school and high school, and in a variety of introductory physics courses, from pre-service teachers to engineering students. https://depts.washington.edu/pits/Background.html

Precalculus: Pathways to Calculus (Carlson, Oehrtman, Moore, O'Bryan):

Textbook, workbook and supplemental materials that facilitate student construction of calculus ideas that are particularly relevant in physics, especially constant rate of change and linear function, and changing rates of change, using covariation. Includes vector quantities, sequence and series representation as approximation. Focusses on less breadth in the variety of functions in favor of building a deeper understanding of the functions themselves using multiple representations and many relevant applications, while students are constructing their knowledge, not as an afterthought. https://www.greatriverlearning.com/product-details/2212

Calculus course activities

DIRACC Calculus: (Thompson, Ashbrook, Milner) The intention of this work is that students understand a calculus that is about more than lines, areas, and pseudo connections with quantitative situation, with focus on their reasoning about quantities and relationships among quantities. The focus on the FTC as relating rates of change and accumulations such that students must conceptualize rate of change as a relationship between quantities who vary is well-aligned our students' needs. The use of dynamic graphs as a representation is brilliant, and will help prime students for the ubiquitous reference to "goes like" reasoning their instructors use from the very first day (Zimmerman, Olsho, White Brahmia, Boudreaux, Smith, & Eaton, 2020). http://patthompson.net/ThompsonCalc

ACRA framework: The relationships between quantities of single-variable calculus can be described using the ACRA Framework (Samuels, 2022, Samuels 2023). An example of a quantity-focused approach to the FTC is in the shaded region of Table 2. This mode of reasoning entails "conceptualizing a situation in terms of quantities and relationships among quantities" (Thompson & Carlson, 2017), where a quantity is a measurable attribute combined with a way to measure that attribute. (contact Joshua Samuels directly for materials)

CLEAR Calculus: (Oehrtman, Tallman, Reed, Martin) Instructional activities that generalize across contexts to extract common mathematical structure, that are designed to foster quantitative reasoning and modeling skills required for STEM fields. Students both develop useful tools, and engage in activities that reveal the mathematics to be learned, thereby developing productive understandings that can serve as a strong foundation for further study in math and science. The approach to approximation here is well-suited to physics students. https://clearcalculus.okstate.edu/

Conclusion

A calculus course could include many fascinating topics that can unleash quantitative imagination and creativity. I've argued that for those calculus students who intend to pursue majors that also involve taking physics courses, that a calculus that characterizes the interaction between quantities, and the mathematical implications of those interactions, will help prepare those students to use calculus ideas for quantifying the natural world, and uncovering its laws. The students will see the world through a mathematical frame, with all its wonder and potential, and try out their skills predicting what nature will, and will not, reveal through observation. Mathematizing physics is founded in measurable and derived quantities, including its sign and units. The function library of physical laws isn't vast, but conceptualizing those functions that appear is essential. Conceptually understanding what calculus is doing when its most basic functions represent relations between physical quantities opens the door for students to learn physics as Newton did. There is a growing collection of effective activities that can help calculus students learn to quantify, and deepen their facility with the formalism associated with function, changes in quantity, rates of change, accumulation and approximation. This paper was written to help foster discussions and provide impetus for the great work described herein to continue, and to inspire more to come.

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