PLAYFUL MATH: WHEN STUDENT AUTHORING GENERATES NOVEL MATHEMATICS

Amy Ellis	Dru Horne	Anna Bloodworth
University of Georgia	University of Georgia	University of Georgia
amyellis@uga.edu	dhorne8@uga.edu	anna.bloodworth@uga.edu

We propose the construct of playful math to support instances of student authoring. Authoring positions students as authors of mathematics in an environment in which students and teachers meet as epistemological equals. By emphasizing student agency and autonomy, playful math encourages students to explore self-selected goals as they design novel problems for one another or for their teachers. We introduce two types of novel mathematics that emerged from student authoring, Unfamiliar Problem and Catalyst, and share one example of each to envision a mathematics education future that celebrates student authoring.

Keywords: Algebra and Algebraic Thinking, Cognition, Problem-Based Learning.

Supporting students to develop and solve their own problems can enhance creativity, understanding, and positive attitudes towards mathematics (Kaur & Rosli, 2021; Kontorovich et al., 2012). Problem posing is seldom considered in relation to creating new mathematics, but some researchers have written about the experience of learning new ideas from their students' activity. For instance, Norton and Flanagan (2022) described how the ideas they developed about nested number sequences and logarithms as maps between multiplicative worlds were informed by their research on children's mathematics, and Ellis (2022) noted that her participants' mathematics "served as a source of novel mathematics for me as a researcher, as it could also do for teachers" (p. 24). We propose that student authoring of mathematics can create opportunities for both students and their instructors to experience new ideas, and that one way to foster authoring is through playful math. By *authoring of mathematics*, we refer to students producing something original (Cheng et al., 2022), using their mathematical voices to "enquire, interrogate, and reflect upon what is being learned" (Povey et al., 1999, p. 243). This use of authoring draws on Povey et al.'s (1999) notion of author/ity, in which teachers and students consider themselves to be members of a knowledge-making community where they "meet as epistemological equals" (p. 234). This perspective positions students as creators, not just doers, of mathematics.

Playful math describes the activities and features of an instructional environment that can facilitate mathematical play (Ellis et al., 2022). This can include task features, instructional moves, and engagement with artifacts. In playful math, students have agency to explore self-selected goals and to author novel problems. We present two examples of student authoring that introduced new mathematics both for the students and for us. We distinguish two types of new mathematics that can emerge from these contexts, Unfamiliar Problem and Catalyst.

Problem Posing and Mathematical Play

Problem-posing tasks are ones that "require teachers or students to generate new problems and questions based either on given situations or on mathematical expressions or diagrams" (Cai et al., 2020, p. 2). Problem posing can counteract the belief that there is one right way to do mathematics, as there is no one "right" question to ask (Palmér & van Bommel, 2020). It can also promote a sense of agency (Brown & Walter, 2004) and can improve motivation and creativity (Kontorovich et al., 2012). We have developed playful math environments as a vehicle for fostering student agency and author/ity. When students experience author/ity, they author problems that raise new mathematics not only for them, but also for us as teacher-researchers.

Defining and Designing for Mathematical Play

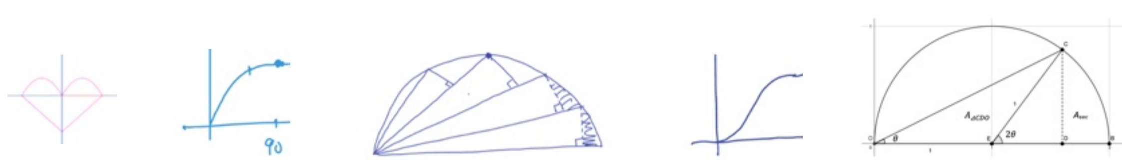
We define mathematical play to include (a) agency in exploration, (b) self-selection of goals, (c) immersion, and (d) enjoyment (Ellis et al., 2022). *Agency in exploration* means that students choose whether and how to participate (Huizinga, 1955) and how to accomplish their goals (Jasien & Horn, 2018). *Self-selection of goals* acknowledges that a learner's agency in determining goals (or sub-goals) is crucial for play (Dewey, 1916/1966). The final two traits are *immersion* and *enjoyment.* Mathematical play is imaginative and creative (Featherstone, 2000), and most accounts of students' mathematical play mention enjoyment (e.g., Sukstrienwong, 2018). Mathematical play can support experimentation, reflection, and persistence (Barab et al., 2010; Gresalfi et al., 2018), and it can provide a productive route for exploring and conjecturing (Mason, 2019; Williams-Pierce, 2019). Given these benefits, we set out to see if we could encourage mathematical play for secondary and undergraduate students.

We have established playful math five design principles to encourage (but not guarantee) mathematical play. They are (1) enable free exploration within constraints; (2) engender anticipation within the task; (3) provide a method for intrinsic feedback; (4) offer meaningful challenge while still being feasible; and (5) allow the student to act as both designer and player. As an example, we draw on an activity in which students investigate growing shapes, graphing a shape's area compared to its length as it sweeps left to right. To playify the task, we created the Guess My Shape game, in which students create secret shapes of their choice (design principles 1 and 5), construct graphs comparing length and area (principles 2, 4, and 5), and challenge each other to determine the shape based on the graph alone (or vice versa; principles 2, 3, and 4). Our principles are consistent with several features of problem-posing tasks, but the open nature of the Guess My Shape game offers greater agency than typical tasks to support author/ity and enable students to author problems reflecting their own mathematical interests.

Data Examples: Sector Areas and Vertical Line Segments

Unfamiliar Problem: Determining Areas in a Semicircle

In the following example, Phyllis and Ryan (secondary pre-service teachers) decided to create a heart shape (Figure 2a). They imagined a line segment on the *x*-axis that swept counterclockwise, rotating 360° to sweep out the shape. The task for the other students was to graph the area swept as a function of the angle swept by the line segment. Phyllis and Ryan reasoned that the initial part of the graph, from 0° to 90°, would increase at a decreasing rate (Figure 2b). However, when the other students encountered the challenge, they thought that the area would first increase at an increasing rate from 0° to the peak of the semi-circle, and then increase at a decreasing rate from the peak to 90°. To determine this, they created equiangular partitions and reasoned perceptually about the rates of change (Figure 2c), deciding that the area should be "bigger and then smaller" (Figure 2d).



(a) (b) (c) (d) (e)

Figure 2: Heart shape (a), Phyllis and Ryan's graph (b), partitioning the semi-circle (c),

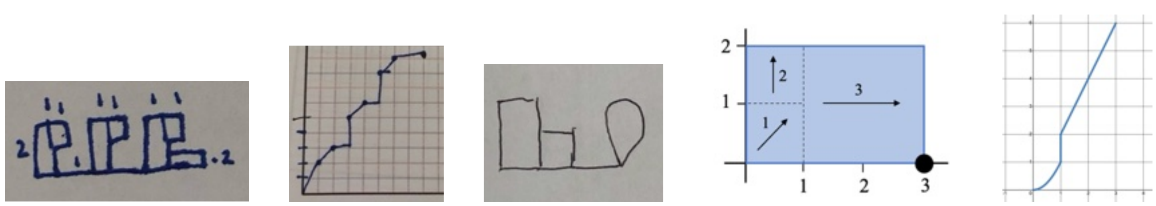
Meredith and Toby's graph (d), and a semicircle with radius 1 (e)

When the groups compared their solutions, they resolved the discrepancy by redrawing a more precise version of Figure 2c with smaller partitions. They concluded that the area indeed increased at a decreasing rate throughout the semi-circle, but they acknowledged that this decision was based on a perceptual judgement. The students' disagreement led us to realize that we did not know how to directly compute the area of these equiangular portions. We wondered how to find the area between two non-radii chords without computing a double integral in polar coordinates. Thus, the students' authoring led to a novel Unfamiliar Problem for us. An Unfamiliar Problem is a problem addressing a new mathematical idea or challenge for the problem-solver, in this case, us as the teacher-researchers. Certainly, the mathematical ideas in the problem are not novel, but we experienced them as unfamiliar in that we were not aware of a solution method. We solved the problem by drawing a semicircle whose radius is 1 (Figure 2e). Denote by A the area covered by $\angle COB$, which can be decomposed as the sum of the area of triangle $\triangle COE$, denoted by $A_{\triangle COE}$, and the area of the sector corresponding to $\angle CEB$, denoted by A_{sec}. If we take OE as the base of $\triangle COE$, the length of the height is $\overline{CD}$, which is $\sin(2\theta)$. So, we find $A_{\triangle COE} = \frac{1}{2}(1)\sin(2\theta)$ and $A_{sec} = \frac{1}{2}r^2\alpha = \frac{1}{2}(1)^2(2\theta) = \theta$, hence $A = \frac{1}{2}\sin(2\theta) + \theta$. This function does indeed increase at a decreasing rate from 0 to $\frac{\pi}{2}$.

Unfamiliar problems can emerge when students have the freedom to explore directions of their own interest. They introduce genuine problem-solving experiences for one another and, in this case, also for us as teacher-researchers. Even though the mathematics was not novel from the perspective of the field, we found the problem to be interesting and worth exploring. Unfamiliar problems create problem-*solving* experiences, rather than problem-*posing* experiences.

Catalyst: Vertical Line Segments

The second example comes from teaching sessions with three middle-school students, Artemis, Apollo, and Francis, who had limited familiarity with graphing or linear functions. In this example, the students decided to create a Guess My Shape challenge for the teacher-researcher (TR), inventing a shape that they called "waves" (Figure 3a). The students graphed the first "wave" correctly, but beginning with the second "wave", they made an iconic translation of the vertical section of their shape directly into the graph, in which the vertical segment represented an increase of 2 square units with no change in the horizontal distance of the graph. They repeated this iconic translation for the final "wave."



(a) (b) (c) (d) (e)

Figure 4: The wave drawing (a), the associated area-length graph (b), TR1's graph (c), up-down square task (d), graph of up-down square task (e)

The vertical line segments created a perturbation for the TR, who wondered how to represent an increase in area without gaining any horizontal length. She proposed a solution in the moment by setting up a new convention of a "bubble", in which there is only one point on the line of horizontal sweeping that nevertheless generates an amount of area (in this case, 2 square units, Figure 3c). This task also resulted in establishing a new convention that any area generated in future tasks should be attached to the line of sweeping, to avoid the difference between the rectangular area in the first "wave" of Figure 3a with that seen in Figure 3c.

The bubble was a spontaneous response to a puzzling situation, but it led us to wonder whether we could create a swept shape that would produce a legitimate vertical segment for its area / length graph. In this case, the students' authoring resulted in an Unfamiliar Problem for themselves, as they tackled the challenge of creating the graph, but it also created a Catalyst for us, in that it provoked a new question: What if an area / length graph *could* have a vertical line segment? What shapes could produce such a segment? A Catalyst is a situation that challenges or reveals an ambiguity about an accepted (or implicit) convention or rule. It can thus engender problem-posing activity, such as the creation of novel sweeping shapes.

We continued to wonder about this question and reasoned that the x-axis quantity would need to stop growing as the area continued to grow. This led to the shape in Figure 3d. In this shape, the square first grows both in length and height, producing area at a constantly changing rate of change (a quadratic graph). Once the square reaches 1 square unit, it then grows up to produce an additional square unit, but without sweeping additional horizontal length, resulting in a vertical line segment. The rectangle then sweeps to the right, producing an additional 4 square units of area at a constant rate (producing a linear graph, Figure 3e). In the graph, the x-axis quantity is the horizontal distance traveled by the dot. We also realized that once the dot stopped moving horizontally, the area could grow up and down multiple times. In creating this problem, we reflected on the fact that the shape of the graph and the trace of the graph are different. This realization led to further problem posing, creating related tasks that incorporated both linear and quadratic growth in the vertical line segments, which can only be distinguished by considering the graph's trace.

Discussion

In both examples, the students experienced Unfamiliar Problems through authoring. However, the novel mathematics that we experienced as teacher-researchers differed. With the semicircle, we experienced an Unfamiliar Problem that required us to devise a solution method we had not previously encountered. The mathematical ideas were not new, but we were challenged to solve a novel (to us) problem. In contrast, the vertical line segment acted as a Catalyst to challenge us to imagine new mathematics. By asking "What if an area / length graph has a vertical line segment?", we introduced new questions for ourselves, such as "Are there sweeping shapes that could produce such a graph, and if so, what would they look like?" This led to a novel set of problems inspired by up-down square, as well as a consideration of the ways in which two graphs can look identical even as their traces differ.

Student authoring can raise unique challenges for teachers, who may be faced with navigating unfamiliar ideas or puzzling situations while interacting with their students. We acknowledge that this can be difficult. However, we envision author/ity environments in which it is allowable for teachers and students to occasionally shift roles, in which teachers experiencing puzzlement or new learning can be normalized and celebrated, and in which we see our students' activity as sources of new learning for us, just as our instruction can be for them.

References

Akay, H., & Boz, N. (2010). The effect of problem posing oriented analyses-II course on the attitudes toward

mathematics and mathematics self-efficacy of elementary prospective mathematics teachers. *Australian journal of teacher education*, *35*(1), 59-75.
Barab, S.A., Gresalfi, M., & Ingram-Goble, A. (2010). Transformational play: Using games to position person, content, and context. *Educational Researcher, 39*(7), 525 – 536.
Bloodworth, A., Horne, D., & Ellis, A.B. (2023). Students' and mathematicians' playful engagement. In Cook, S., Katz, B. & Moore-Russo D. (Eds.), *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 108-117). University of Nebraska.
Brown, S. (2009). *Play: How It Shapes the Brain, Opens the Imagination, and Invigorates the Soul.* Avery.
Brown, S. I., & Walter, I. M. (2004). *The art of problem posing* (3rd edition). Lawrence Erlbaum Associates.
Burghardt, G. M. (2011). Defining and recognizing play. In A. Pelligrini (Ed.), *The Oxford handbook of the development of play* (pp. 9–18). Oxford University Press.
Cai, J. Chen, T., Li, X. Xu, R. Zhang, S. Hu, Y. Zhang, L. & Song, N. (2020). Exploring the impact of a problem-posing workshop on elementary school mathematics teachers' conceptions on problem posing and lesson design. *International Journal of Educational Research*, *102*, 101404.
Cai, J., Moyer, J. C., Wang, N., Hwang, S., Nie, B., & Garber, T. (2013). Mathematical problem posing as a measure of curricular effect on students' learning. *Educational Studies in Mathematics*, *83*(1), 57-69.
Casey, B., Andrews, N., Schindler, H., Kersh, J.E., Samper, A., & Copley, J. (2008). The development of spatial skills through interventions involving block building activities. *Cognition and Instruction, 26*(3), 269-309.
Cheng, W. K., Ng, O. L., & Ni, Y. (2022). Enhancing student authorship and broadening personal latitude in the mathematics classroom with rich dialogic discourse. *ECNU Review of Education*, 20965311221142887.
Crespo, S., & Sinclair, N. (2008). What makes a problem mathematically interesting? Inviting prospective teachers to pose better problems. *Journal of Mathematics Teacher Education, 11*, 395-415.
Dewey, J. (1916 / 1966). *Democracy and Education*. Macmillan.
Edo, M., Planas, N., & Badillo, E. (2009). Mathematical learning in a context of play. *European Early Childhood Education Research Journal*, *17*(3), 325-341.
Ellis, A.B. (2022). Decentering to build asset-based learning trajectories. In Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J. N., Strayer, J., & Drown, S. (Eds.), *Proceedings of the Forty-Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 15 – 29). Middle Tennessee State University.
Ellis, A.B., Horne, D., Bloodworth, A., Nielsen, A., & Ely, R. (2022). Playful math: Modeling students' engagement in play-based algebra activities. In Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J. N., Strayer, J., & Drown, S. (Eds.), *Proceedings of the Forty-Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 771 – 780). Middle Tennessee State University.
Featherstone, H. (2000). "-Pat + Pat = 0": Intellectual play in elementary mathematics. *For the Learning of Mathematics, 20*(2), 14 – 23.
Gresalfi, M., Horn, I., Jasien, L., Wisittanawat, P., Ma, J.Y., Radke, S.C.,…,Sfard, A. (2018). Playful mathematics learning: Beyond early childhood and sugar-coating. In J. Kay & R. Luckin (Eds.), *13th International Conference of the Learning Sciences* (pp. 1335 – 1342). London, UK: University College London.
Huizinga, J. (1955). *Homo Ludens: A Study of the Play Element in Culture*. Boston, MA, Beacon Press.
Jasien, L., & Horn, I.S. (2018). "Ohhh, now I can do it!": School-age children's spontaneous mathematical sensemaking in construction play. In J. Kay & R. Luckin (Eds.), *13th International Conference of the Learning Sciences* (pp. 624 - 631). London, UK: University College London.
Kaur, A., & Rosli, R. (2021). Problem Posing in Mathematics Education Research: A Systematic Review. *International Journal of Academic Research in Progressive Education and Development*, *10*(1), 438-456.
Kontorovich, I., Koichu, B., Leikin, R., & Berman, A. (2012). An exploratory framework for handling the complexity of mathematical problem posing in small groups. *The Journal of Mathematical Behavior, 31*(1), 149–161. https://doi.org/10.1016/j.jmathb.2011.11.002
Levine, S.C., Ratliff, K.R., Huttenlocher, J., & Cannon, J. (2012). Early puzzle play: A predictor of preschoolers' spatial transformation skill. *Developmental Psychology, 48*(2), 530 – 542.
Mason, J. (2019). Pre-parative and post-parative play as key components of mathematical problem solving. In *Problem solving in mathematics instruction and teacher professional development* (pp. 89-110). Springer, Cham.
Norton, A., & Flanagan, K. (2022). The Prime Number Theorem as a Mapping between Two Mathematical Worlds. In Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J. N., Strayer, J., & Drown, S. (Eds.), *Proceedings*

of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 920-928). Middle Tennessee State University.
Novak, E., & Tassell, J. (2015). Using video game play to improve education-majors' mathematical performance: An experimental study. *Computers in Human Behavior*, *53*, 124-130.
Palmér, H., & van Bommel, J. (2020). Young students posing problem-solving tasks: what does posing a similar task imply to students? *ZDM*, *52*(4), 743-752.
Parks, A. N. (2015). *Exploring mathematics through play in the early childhood classroom.* National Council of Teachers of Mathematics.
Plass, J.L., O'Keefe, P.A., Homer, B.D., Case, J., Haward, E.O., Stein, M., & Perlin, K. (2013). The impact of individual, competitive, and collaborative mathematics game play on learning, performance, and motivation. *Journal of Educational Psychology, 105*(4), 1050 – 1066.
Povey, H., Burton, L., Angier, C., & Boylan, M (1999). Learners as authors in the mathematics classroom. In L. Burton (Ed.), *Learning Mathematics, from Hierarchies to Networks* (pp. 232 – 245).Falmer.
Sedig, K. (2008). From play to thoughtful learning: A design strategy to engage children with mathematical representations. *Journal of Computers in Mathematics and Science Teaching*, *27*(1), 65-101.
Sengupta-Irving, T., & Enyedy, N. (2015). Why engaging in mathematical practices may explain stronger outcomes in affect and engagement: Comparing student-driven with highly guided inquiry. *Journal of the Learning Sciences, 24*(4), 550-592.
Siegler, R. S. and Ramani, G. B. (2008). Playing linear numerical board games promotes low-income children's numerical development. *Developmental Science, 11*(5): 655–61.
Silver, E. A. (1994). On mathematical problem posing. *For the Learning of Mathematics, 14*(1), 19–28.
Silver, E. A., & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education, 27*(5), 521–539.
Silver, E. A., Mamona-Downs, J., Leung, S. S., & Kenney, P. A. (1996). Posing mathematical problems: An exploratory study. *Journal for Research in Mathematics Education, 27*, 293–309.
Stannard, L.,Wolfgang, C. H., Jones, I., & Phelps, P. (2001). A longitudinal study of the predictive relations among construction play and mathematical achievement. *Early Child Development and Care, 167*(1), 115e125. http://dx.doi.org/10.1080/ 0300443011670110.
Steffe, L., & Wiegel, H. (1994). Cognitive play and mathematical learning in computer microworlds. *Educational Studies in Mathematics, 26*(2/3), 111 – 134.
Stickles, P. (2011). An analysis of secondary and middle school teachers' mathematical problem posing. *Investigations in Mathematics Learning, 3*(2), 1-34.
Su, F. (2017). Mathematics for human flourishing. *The American Mathematical Monthly*, *124*(6), 483-493.
Sukstrienwong, A. (2018). Animo math: The role-playing game in mathematical learning for children. *Tem Journal*, *7*(1), 147.
Wagner, D., & Herbel-Eisenmann, B. (2014). Identifying authority structures in mathematics classroom discourse: A case of a teacher's early experience in a new context. ZDM Mathematics Education, 46(6), 871–882. https://doi.org/10.1007/s11858-014-0587-x
Wang, Z., & Hung, L.M. (2010). Kindergarten children's number sense development through board games. *The International Journal of Learning, 17*(8), 19 – 31.
Widman, Chang-Order, Penuel, & Wortman (2019). Using evaluation tools toward more equitable youth engagement in libraries: Measuring connected learning and beyond. *Young Adult Library Services, 17*(4), 37 – 43.
Williams, G. (2001). Sustained overlap in zones of proximal development? If so, how did that happen? In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th international conference for Psychology of Mathematics Education* (Vol. 1, p. 378). PME.
Williams-Pierce, C. (2019). Designing for mathematical play: failure and feedback. *Information and Learning Sciences.*