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Background

Metamaterials are architected materials with tailored microstructures that have revolutionized control over wave propagation, enabling new phenomena in wave guidance, steering, and attenuation. Significant among these are phononic crystals (PnCs) which represent a class of composites with a periodic arrangement of layers that repeat themselves in space [1]. In conventional (bi-layered) PnCs, these self-repeating blocks, also known as unit cells, consist of two distinct materials with contrasting inertial and elastic properties. These arrangements give rise to bandgaps which are wide frequency ranges within which wave propagation is forbidden [2]. A lesser-known configuration of PnCs is one where the unit cell is formed via a continuous variation of material properties as opposed to distinct layers of contrasting materials. In here, we investigate novel wave phenomena that arise from such class of PnCs when the elastic modulus follows a harmonic profile. We refer to such structures as harmonically-graded PnCs. While bi-layered PnCs are known to exhibit an infinite number of bandgaps along the frequency axis, a harmonically-graded PnC only reveals a single, well-defined bandgap that can be tuned by varying the parameters of the harmonic stiffness profile.

Project Objectives and Goals

- Study elastic wave propagation in a 1D harmonically-graded PnC and derive a comprehensive mathematical framework which captures the dispersion behavior for any discretized configuration of such PnC.
- Derive a semi-analytical solution that captures the effect of the different parameters that govern the stiffness profile on the emergence/closure of bandgaps.

Analytical Setup

The Plane Wave Expansion (PWE) method is applied to a 1D slender rod exhibiting longitudinal vibrations with a governing equation of motion:

$$\rho(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left[E(x) \frac{\partial u}{\partial x} \right] \quad (1)$$

where u is the axial displacement, the density ρ is assumed constant, and the elastic modulus $E(x)$ is defined as a spatially varying function which can be expanded using a Fourier series. The bi-layered and harmonically-graded PnCs represent two opposite ends of the design spectrum, where the former represents a discretized version of the latter when three discretizing segments are utilized (see Fig. 1). Examining both cases along with the intermediate configurations provides valuable insight into how the band structure evolves as the spatial grading resolution is increased.

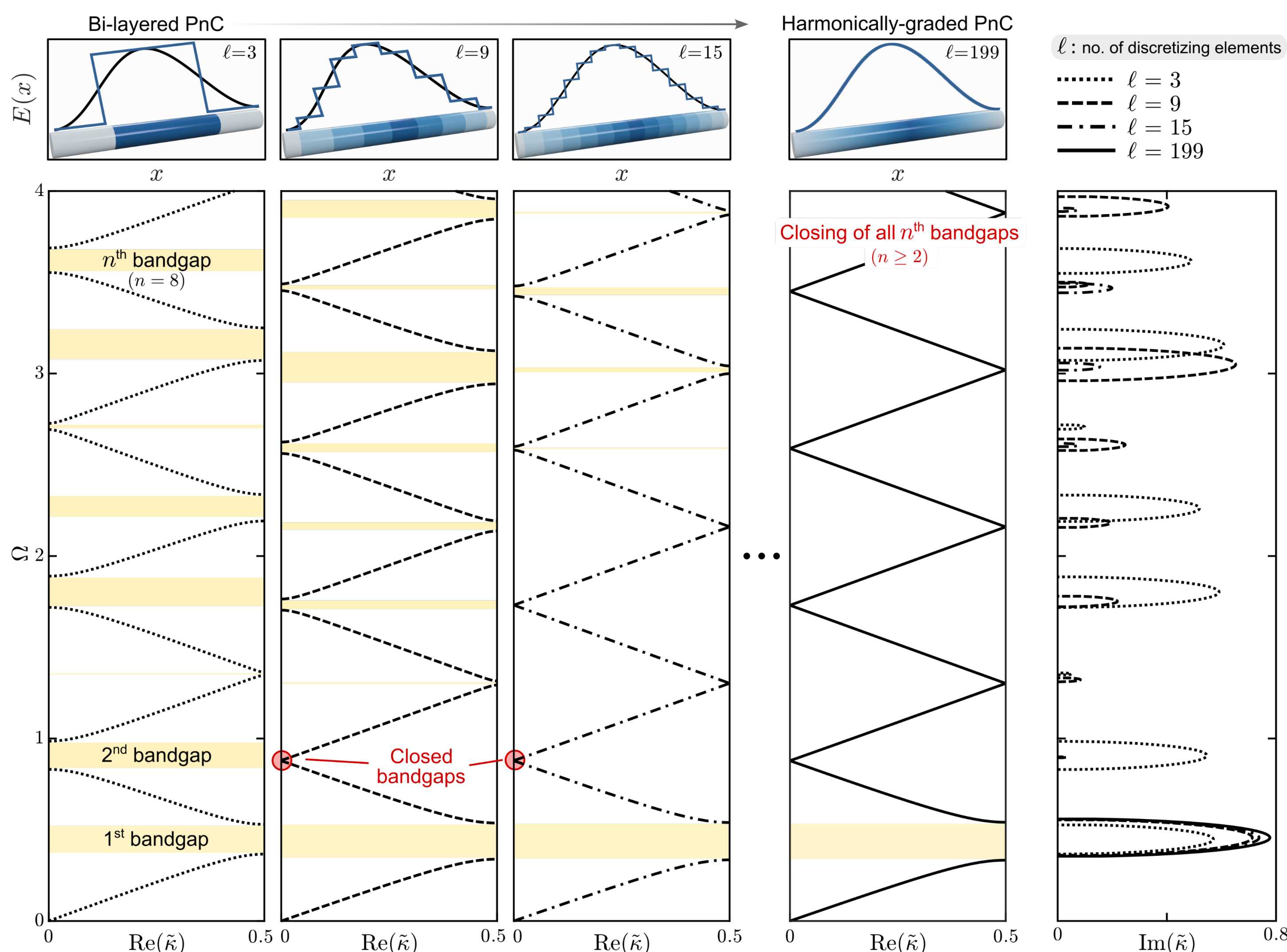


Figure 1: Evolution of the dispersion diagram from a bi-layered to a harmonically-graded PnC

Expressing the displacement in the Bloch form and expanding the elastic modulus $E(x)$ into a Fourier series, then substituting into the 1D wave equation, leads to the eigenvalue problem:

$$\mathbf{A}(\tilde{\kappa}) \mathbf{u} = \Omega^2 \mathbf{u} \quad (2)$$

where the stiffness matrix $\mathbf{A}(\tilde{\kappa})$ is given with the truncation $i, j \in [-d, d]$:

$$A_{(i,j)}(\tilde{\kappa}) = (i - \tilde{\kappa})(j - \tilde{\kappa}) E_{i-j} \quad (3)$$

Defining the Fourier coefficients $E_{i-j} = E_m$, it is vital to identify which Fourier coefficients will be zero. For cosine-based periodic structures, the Fourier coefficients guarantee that $E_m = E_{-m}$ and all coefficients are real, resulting in a symmetric structure and reducing computational cost. The diagonal of the matrix is the main diagonal of the uniform rod's stiffness matrix, which results in pairs of degenerate eigenvalues (no bandgaps for the uniform rod). The degeneracy is lifted if there is direct or indirect coupling between the degenerate eigenvalues.

Data and Results

The elastic modulus of the bi-layered PnC in Fig. 2(a) is a square wave which can be represented by odd cosine harmonics (i.e., $E_0, E_{-1,1}, E_{-3,3}, \dots$). The stiffness matrix $\mathbf{A}(\tilde{\kappa})$ is constructed at the irreducible Brillouin zone edges ($\tilde{\kappa} = 0, 0.5$), showing that:

- Contrary to a uniform (single-material) rod, the stiffness matrix of the bi-layered PnC has several non-zero Fourier coefficients, akin to introducing perturbation to the uniform rod's stiffness matrix and lifting degeneracy for all eigenvalues pairs.
- **At $\tilde{\kappa} = 0.5$** , there are direct couplings between all the degenerate eigenvalues, introducing bandgaps. The degenerate pairs $(0.25E_0)$, $(2.25E_0)$, and $(6.25E_0)$ are coupled through the off-diagonal terms $(-0.25E_1)$, $(-2.25E_3)$, and $(-6.25E_5)$.
- **At $\tilde{\kappa} = 0$** , there are indirect couplings between all the degenerate eigenvalues, introducing bandgaps at that location. For example, the degenerate pair (E_0) is coupled $(4E_0)$ through $(-2E_3)$ and the $(4E_0)$ is coupled with (E_0) through $(2E_1)$. This pattern is demonstrated clearly in Fig. 3(c) as every degenerate pair has a unique color, and the arrow shows the paths connecting the pairs.

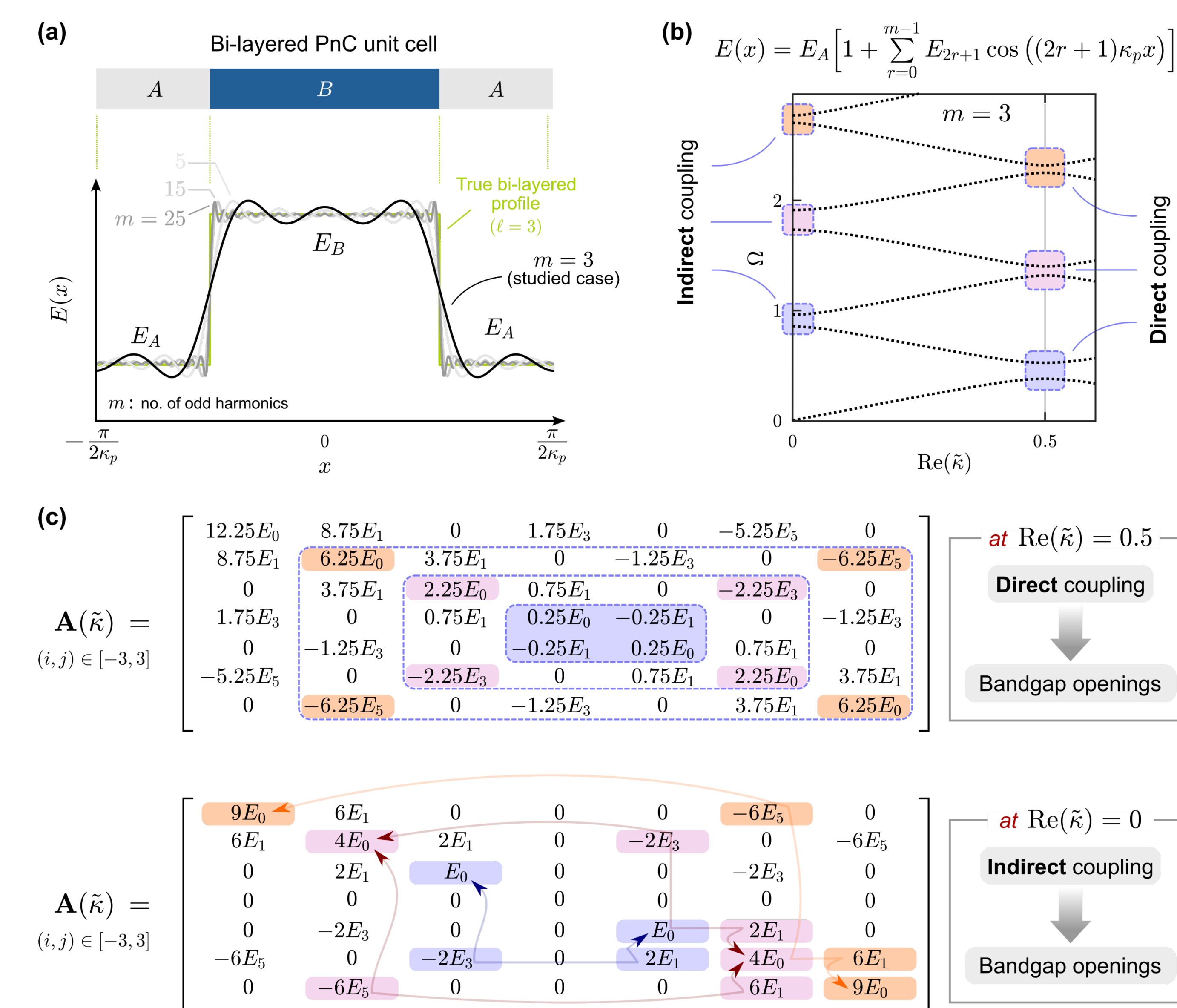


Figure 2: Approximated stiffness profile and structure of the stiffness matrix in a bi-layered PnC

For a **harmonically-graded PnC** where the elastic modulus varies as a cosine function, as shown in Fig. 3(a), the Fourier coefficients include only three non-zero coefficients: E_0, E_1 , and E_{-1} . This simplifies the structure of the stiffness matrix significantly. The equation for a given mode \mathbf{u}_c is now only coupled to its nearest neighbors \mathbf{u}_{c-1} and \mathbf{u}_{c+1} , resulting in a tri-diagonal symmetric structure for the infinite matrix. Upon closely examining the stiffness matrix in Fig. 3(c), we observe the following:

- **At $\tilde{\kappa} = 0.5$** , the degenerate pair $(0.25E_0)$ is directly coupled through the off-diagonal term $(-0.25E_1)$, which lifts the degeneracy, producing two distinct eigenvalues and opening a single bandgap. No further couplings occur, confirming the inability of the harmonically-graded PnC to exhibit more than one bandgap.
- **At $\tilde{\kappa} = 0$** , the upper-right and lower-left blocks of the matrix are zeros, preventing any direct/indirect coupling, and confirming the lack of bandgaps at this location.

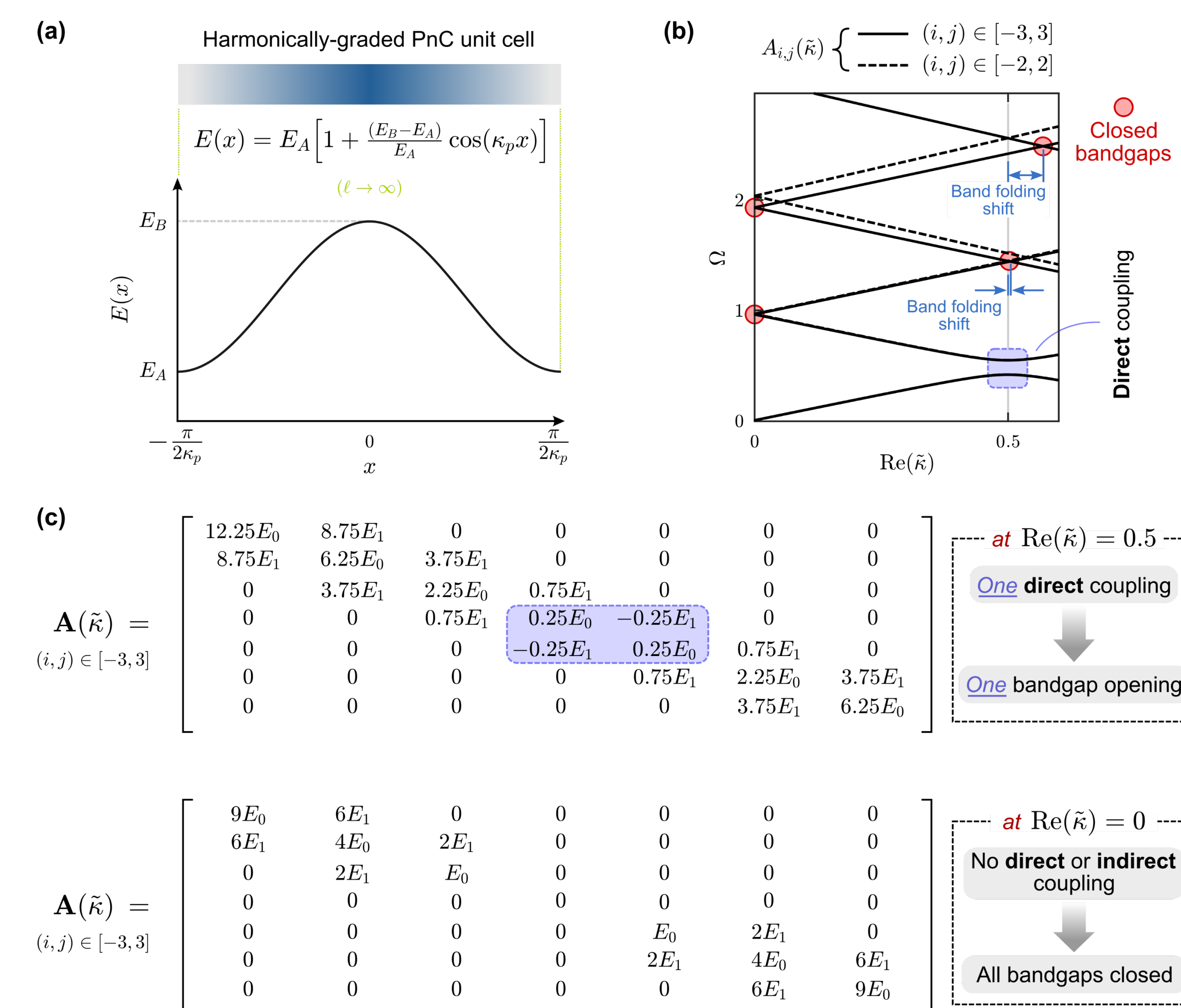


Figure 3: Stiffness profile and structure of the stiffness matrix in a harmonically-graded PnC

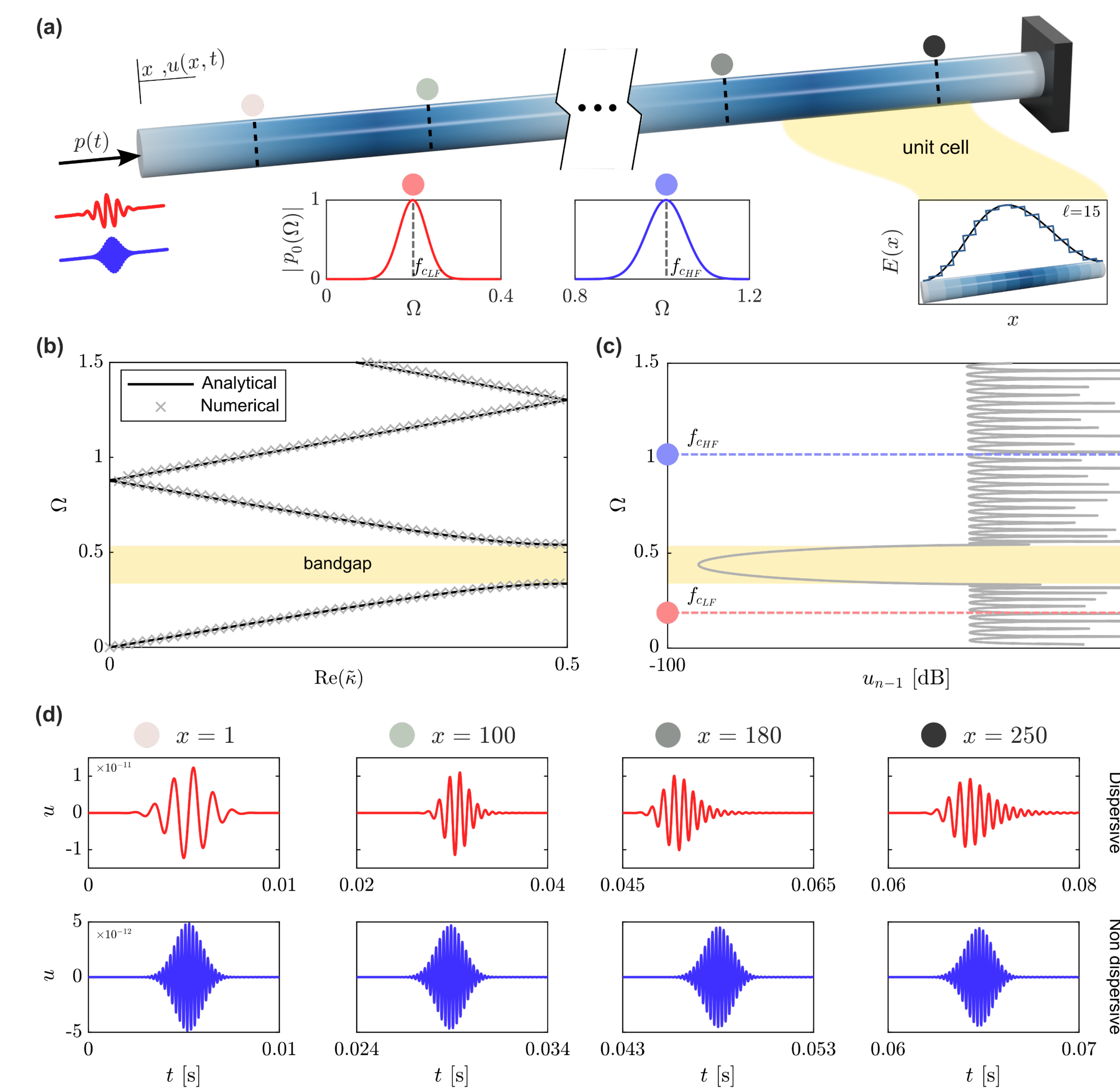


Figure 4: Presence of dispersive and non-dispersive response within a single harmonically-graded PnC consisting of 320 cells ($\ell = 15$ segments per cell) via time-domain FEM simulations

Numerical simulations were conducted to verify the emergence of non-dispersive branches at higher frequencies in a finite harmonically-graded PnC. The response of a finite PnC consisting of 320 unit cells, each discretized using $\ell = 15$ segments, was obtained via finite element modeling (FEM). The numerically-obtained dispersion diagram and frequency response function (FRF) verified analytical predictions, as shown in Figs. 4(b, c). A time-domain analysis was then carried out by subjecting the PnC to two different pressure excitations $p(t)$ in the form of Gaussian wave pulses which are centered around a low frequency f_{CLF} and a high frequency f_{CHF} , as denoted in Fig. 4(a). The displacement field $u(x, t)$ was obtained for both cases at four different locations ($x = 1, 100, 180$, and 250) shown in Fig. 4(d). The results highlight the non-dispersive nature of the PnC at high frequencies, as the incident pulse retains its shape throughout the structure confirming the lack of frequency-dependence in the wave speed, a hallmark feature of wave propagation in elastic non-dispersive media.

Conclusion

The study has demonstrated bandgap closing in PnCs with harmonically-graded, rather than distinctly layered, stiffness, leading to the emergence of non-dispersive wave behavior above a specific cut-off frequency. Our findings established robust correlations between grading parameters and the subsequent wave behavior, enabling control over dispersion patterns for novel metamaterial designs.

REU Experience and Highlights

As part of this REU experience, I worked under the guidance of Prof. Mostafa Nough and PhD candidate Ali Jafari, allowing me to gain experience in fundamental research. The highlights of my research internship can be summarized as follows:

- Developed MATLAB scripts to compute phononic band structures using PWE.
- Applied scientific methods to investigate fundamental questions pertaining to dynamics-related problem (bandgap closing in harmonically graded materials).
- Improved research communication through poster and presentation practice.

Acknowledgment

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Publication

The manuscript "Bandgap closing and non-dispersive wave behavior in phononic crystals with harmonically-graded elastic profiles" is currently under review.

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