Towards Scalable Voltage Control in Smart Grid: A Submodular Optimization Approach

Zhipeng Liu*, Andrew Clark†, Phillip Lee*, Linda Bushnell*,
Daniel Kirschen*, and Radha Poovendran*
* Dept. of Electrical Engineering, University of Washington, Seattle, WA, 98195-2500
† Dept. of Electrical and Computer Engineering, Worcester Polytechnic Institute, Worcester, MA, 01609
Email: {zhipliu,leep3,lb2,kirschen,rp3}@uw.edu, aclark@wpi.edu *

ABSTRACT

Voltage instability occurs when a power system is unable to meet reactive power demand at one or more buses. Voltage instability events have caused several major outages and promise to become more frequent due to increasing energy demand. The future smart grid may help to ensure voltage stability by enabling rapid detection of possible voltage instability and implementation of corrective action. These corrective actions will only be effective in restoring stability if they are chosen in a timely, scalable manner. Current techniques for selecting control actions, however, rely on exhaustive search, and hence may choose an inefficient control strategy. In this paper, we propose a submodular optimization approach to designing a control strategy to prevent voltage instability at one or more buses. Our key insight is that the deviation from the desired voltage is a supermodular function of the set of reactive power injections that are employed, leading to computationally efficient control algorithms with provable optimality guarantees. Furthermore, we show that the optimality bound of our approach can be improved from 1/3 to 1/2 when the power system operates under heavy loading conditions. We demonstrate our framework through extensive simulation study on the IEEE 30 bus test case.

1. INTRODUCTION

Power systems are expected to operate closer to their stability limits in the coming decades, due to a combination of increased demand for electricity and unpredictable supply from renewable energy sources. A significant challenge in power system stability will be maintaining steady voltages at system buses in the presence of disturbances, such as equipment failures and changes in generation or load [9]. Inability to maintain system voltages (voltage instability) has been responsible for blackouts in the United States, Sweden, Japan, Belgium and France [8].

Voltage instability is caused by an inability of the power system to meet reactive power demand, and is typically corrected by injecting reactive power at load buses through mechanisms such as capacitor switching or transformer tap changing [9]. These corrective actions are traditionally performed locally, in order to limit the coordination required between geographic regions and utility operators [15]. A purely local approach, however, may be insufficient to meet reactive power demand and does not take the interactions between neighboring buses into account, limiting its effectiveness against major disturbances that affect multiple buses simultaneously.

The enhanced monitoring and communication capabilities of the future smart grid could potentially enable centralized or hierarchical voltage control. One such architecture was proposed in [14], and is currently being implemented in Southern California. Under this approach, the current system state (bus voltage magnitudes and phases) is measured at each time instant using Phasor Measurement Units, and is used to evaluate the effectiveness of distributed voltage control and, when necessary, inform the design of a coordinated, centralized response.

A crucial step in any coordinated response is selecting a set of control actions at each bus based on the gathered state information. Since the set of actions at each bus is inherently discrete (e.g., deciding whether to switch on a capacitor bank), designing a control action at each time step is a discrete subset selection problem.

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At present, such actions are chosen via enumeration and evaluation of all possible combinations of control actions at multiple buses [14]. This approach does not scale to large power systems, and may result in a costly, suboptimal response or fail to identify a set of control actions in time to prevent voltage collapse. A computationally efficient optimization approach that exploits underlying structure of the voltage regulation problem to select provably optimal control actions would enable a timely and effective coordinated response to voltage instability. At present, however, there is no such optimization approach available in the existing literature.

This paper presents a submodular optimization approach to voltage control in power systems that is developed based on a standard model of the voltage-reactive power dynamics obtained from the Jacobian of the power flow equation. Submodularity is a diminishing returns property of set functions that enables the development of efficient approximation algorithms. Our fundamental insight is that the metrics typically used to evaluate the effectiveness of a voltage control strategy, such as the deviation from the desired voltage and the switching cost, have an inherent submodular structure. We make the following specific contributions:

- We formulate the problem of selecting a set of VAR devices, such as transformer tap changes and capacitor banks, in order to minimize the voltage control cost, defined as the deviation of the voltage from its desired value plus the total switching cost.
- We prove that the VAR device selection problem is equivalent to submodular maximization with a matroid basis constraint. Our approach exploits the physical intuition that, as additional reactive power is injected into the system, the incremental change in the deviation from the desired voltage is decreasing, and becomes negative as the level of reactive power exceeds the required amount.
- We propose polynomial-time algorithms for computing a voltage control strategy. The algorithms achieve an optimality bound of 1/3 under basic assumptions motivated by the physical properties of power systems. In the special case of power systems under heavy loading, we prove that a randomized greedy algorithm achieves an improved optimality bound of 1/2.
- We evaluate our approach through a numerical study on the IEEE 30 bus test case. We find that our approach results in the same improvement in voltage stability and switching cost as exhaustive search over the possible control actions, but with significantly less computational overhead.

The paper is organized as follows. Section 2 reviews the related work. Section 3 presents the power system model and background on submodularity. Section 4 presents the problem formulation and approximation algorithms based on submodular optimization. Section 5 presents simulation results. Section 6 concludes the paper.

2. RELATED WORK

Voltage stability in power grid has been studied extensively in the existing literature [5, 7, 11, 14, 15]. A theoretical model for voltage collapse was developed in [5] in which the operating point of voltage collapse was identified as the point of bifurcation in power system dynamics. Traditional approach for mitigating voltage instability has mainly been preventive actions based on contingency ranking [7]. However, with the recent advances in real-time monitoring capability using PMUs, development of corrective actions for mitigating voltage instability is becoming an active area of research.

Control-theoretic approaches for voltage regulation have been proposed in [15, 11]. In [15], a sufficient condition for regulating voltage in a distributed manner with limited communication between busses has been derived. Similarly, a distributed control law that guarantees voltage stability has been derived in [11] by varying the reactive power injection at each bus using distributed energy resources (DER). These works, however, assume every bus is capable of dynamically varying reactive power injection and did not take into account the cost associated with reactive power injection.

An optimization problem for regulating voltage while minimizing the economic and operation costs of control in large power systems was studied in [14] where the reactive power control was exerted through switching of VAR devices. This current approach, however, requires enumerating all possible configurations of VAR devices in every bus, limiting its scalability for large power grids.

Submodular optimization techniques for control of networked systems have been proposed in recent years [4, 13, 2, 3]. These techniques, however, focus on optimizing performance parameters, such as robustness to noise, convergence error, and controllability, of generic linear systems. At present, submodularity has not been explored in the context of power system stability and control.

3. SYSTEM MODEL AND PRELIMINARIES

In this section, we present the power system model and introduce notations that will be used throughout the paper. We also give background on submodularity.

3.1 Power System Model

We consider a power grid with n buses. We say that bus k is a neighbor of bus i if there is a transmission line between buses i and k. The set of neighboring buses of bus i is denoted as $N(i)$. We denote the admittance of the transmission line between bus i and k as $y_{ik} = g_{ik} - jb_{ik}$ where g_{ik} and b_{ik} denote the conductance and susceptance respectively. We assume that active power loss over the transmission lines are negligible, i.e., $g_{ik} \approx 0$ for all transmission lines. We also assume that a subset of buses, Ω, are capable of switching in capacitor and reactor banks. For bus $i \in \Omega$, the change in reactive power injection induced by switching in capacitor or reactor back is denoted as ΔQ_i.

We denote the magnitude and the phase angle of complex voltage at bus i as V_i and θ_i. From power flow analysis [9], the reactive power flow between bus i and k is given as

$$Q_{ik} = V_i^2 b_{ik} - V_i V_k b_{ik} \cos\theta_{ik} \tag{1}$$

where $\theta_{ik} = \theta_i - \theta_k$. Then, the reactive power injection at bus i, denoted as Q_i is given as

$$Q_i = \sum_{k \in N(i)} Q_{ij} = \left(\sum_k V_i^2 b_{ik} - V_i V_k b_{ik} \cos\theta_{ik} \right) \tag{2}$$

by the conservation of power. We assume that the change in reactive power flow is mainly induced by the changes in terminal voltage magnitudes and the change induced by the phase angle is negligible. This assumption is consistent with the existing literature [14]. Under this assumption, by linearizing Q_{ik} around the current operating point, we obtain

$$\Delta Q_{ik} = (2V_i b_{ik} - V_k b_{ik} \cos\theta_{ik})\, \Delta V_i - V_i b_{ik} \cos\theta_{ik} \Delta V_k$$

where ΔV_i is the change in voltage magnitude at bus i. From (2), the total change in injection of reactive power at bus i is given as $\Delta Q_i = \sum_{k \in N(i)} \Delta Q_{ik}$. Let $\Delta \mathbf{q} = [\Delta Q_1, \ldots, \Delta Q_n]^T$ be the vector of changes in reactive power injection, and $\Delta \mathbf{v} = [\Delta V_1, \ldots, \Delta V_n]^T$ be the vector of changes in voltage magnitudes. Then, $\Delta \mathbf{q}$ can be written as

$$\Delta \mathbf{q} = J \Delta \mathbf{v} \tag{3}$$

where the diagonal entries of matrix J are given as

$$J_{ii} = \sum_{k \in N(i)} (2V_i b_{ik} - V_k b_{ik} \cos\theta_{ik}) \tag{4}$$

and the off-diagonal entries are given as

$$J_{ik} = \begin{cases} -V_i b_{ik} \cos\theta_{ik}, & \text{if } k \in N(i) \\ 0, & \text{else} \end{cases}$$

We assume that the matrix J is invertible under normal operating condition. This assumption is consistent with existing literatures [5, 12] where the operating point at which the matrix J becomes singular was identified as the point of voltage collapse since no change in reactive power injection could stabilize the change in voltage magnitudes. Under this assumption, the effect of changes in reactive power injections on the voltage magnitudes can be written as

$$\Delta \mathbf{v} = J^{-1} \Delta \mathbf{q}. \tag{5}$$

3.2 Background on Submodularity

Let V be a finite set, and let 2^V denote the set of all subsets of V. A function $f : 2^V \rightarrow \mathbb{R}$ is *submodular* if, for any sets S and T,

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T). \tag{6}$$

Equivalently [6], a function is submodular if, for any sets S and T with $S \subseteq T$ and any $v \notin T$,

$$f(S \cup \{v\}) - f(S) \geq f(T \cup \{v\}) - f(T). \tag{7}$$

Eq. (7) can be interpreted as a *diminishing returns* property, wherein the incremental benefit of adding an element to a set S decreases as more elements are added to S. This is analogous to concavity of continuous functions. It can be shown that a nonnegative weighted sum of submodular functions is supermodular. A function $f : 2^V \rightarrow \mathbb{R}$ is *supermodular* if $-f$ is submodular.

As a notation, let $|S|$ denote the cardinality of a set S. A matroid is defined as follows.

Definition 1. *Let V be a finite set, and let $\mathcal{I}$ be a collection of subsets of V. The pair $\mathcal{M} = (V, \mathcal{I})$ is a* matroid *if the following three conditions hold: (i) $\emptyset \in \mathcal{I}$, (ii) If $B \in \mathcal{I}$, then $A \in \mathcal{I}$ for all $A \subseteq B$, and (iii) If $A, B \in \mathcal{I}$ and $|A| < |B|$, then there exists $v \in B \setminus A$ such that $(A \cup \{v\}) \in \mathcal{I}$.*

The collection $\mathcal{I}$ is referred to as the set of *independent sets* of the matroid $\mathcal{M}$. A maximal independent set is a *basis*. It can be shown that all bases of a matroid have the same cardinality. The following lemma gives a property of submodular functions over matroid bases.

Lemma 1 ([10]). *Let $\mathcal{M} = (V, \mathcal{I})$ be a matroid, and let $f : 2^V \rightarrow \mathbb{R}$ be a submodular function defined on V. Define S to be a matroid basis such that, for all u and v with $u \in S$, $v \notin S$, and $(S \setminus \{u\} \cup \{v\}) \in \mathcal{I}$), $(1+\epsilon) f(S) \geq f(S \setminus \{u\} \cup \{v\})$. Then for any basis C of $\mathcal{M}$,*

$$2(1+\epsilon) f(S) \geq f(S \cup C) + f(S \cap C). \tag{8}$$

A sub-class of matroids is defined by the following lemma.

Lemma 2. *Let V denote a finite set, and let $V_1, \ldots, V_m$ be a partition of V, i.e., a collection of sets such that $V_1 \cup \cdots \cup V_m = V$ and $V_i \cap V_j = \emptyset$ for $i \neq j$. Let $d_1, \ldots, d_m$ be a collection of nonnegative integers. Define a set $\mathcal{I}$ by $A \in \mathcal{I}$ iff $|A \cap V_i| \leq d_i$ for all $i = 1, \ldots, m$. Then $\mathcal{M} = (V, \mathcal{I})$ is a matroid.*

A matroid defined as in Lemma 2 is a *partition matroid*.

4. PROPOSED VOLTAGE CONTROL FRAMEWORK

This section formulates the problem of selecting a set of buses to inject reactive power into the power system, in order to minimize the deviation from the desired voltage and switching cost. We first describe the formulation, followed by an equivalent submodular optimization problem. Based on the submodularity property, we present a polynomial-time algorithm with provable optimality guarantees. We then describe algorithms that provide improved optimality bounds under the case of heavy loading.

4.1 Problem Formulation

A centralized voltage controller will monitor and control the voltages continuously for all buses. When a voltage deviation occurs, the controller selects a subset of buses, $S \subseteq \Omega$, to switch their capacitor banks from off to on, or from on to off. Activating a capacitor or reactor bank that is currently turned off incurs a cost c_i, while the cost of deactivating a device that is currently on is denoted b_i. The set of devices that are currently turned on is denoted O, while the set of devices that are turned off is denoted F.

The goal of the controller is to choose the best switching action to minimize the resulting voltage deviation with minimal total switching cost. These costs can be captured by the metric

$$f(S) = \sum_{i \in S \cap F} c_i + \sum_{i \in S \cap O} b_i + \lambda \sum_{k=1}^{n} h(v_k + (\Delta V)_k - (v_{ref})_k)$$

where the first term gives the total switching cost, the second term measures the voltage deviation after switching with a trade-off factor λ, and h is an increasing convex function.

Substituting Δv with equation (5), we get

$$f(S) = \sum_{i \in S \cap F} c_i + \sum_{i \in S \cap O} b_i + \lambda \sum_{k=1}^{n} h\left(\sum_{i \in S} \omega_{ki}(\Delta Q_i) - V_k^*\right) \tag{9}$$

where $\omega_{ki} = (J^{-1})_{ki}$ and V_k^* is the kth entry of $(\mathbf{v}_{ref} - \mathbf{v})$.

The problem of selecting an optimal control action S can then be formulated as $\min\{f(S) : S \subseteq \Omega\}$. Equivalently, the problem can be reformulated as selecting a set of capacitors/reactors, denoted O', that should be active (including devices that are switched from off to on, as well as devices that remain on), so that $O' = (S \cap F) \cup (O \setminus S)$. Define the metric

$$\overline{f}(O') = \sum_{i \in O' \setminus O} c_i + \sum_{i \in O \setminus O'} b_i + \lambda \sum_{k=1}^{n} h\left(\sum_{i \in O'} \omega_{ki}|\Delta Q_i| - \hat{V}_k^*\right), \tag{10}$$

where $\hat{V}_k^* = -\left(v_k - (v_{ref})_k - \sum_{j \in O} \omega_{kj}|\Delta Q_j|\right)$. The equivalence between these metrics is established by the following lemma.

LEMMA 3. *If $O' = (S \cap F) \cup (O \setminus S)$, then $f(S) = \overline{f}(O')$.*

PROOF. The function $f(S)$ is equivalent to

$$\begin{aligned} f(S) &= \sum_{i \in S \cap F} c_i + \sum_{i \in S \cap O} b_i + \lambda \sum_{k=1}^{n} h\left(\sum_{j \in S \cap F} \omega_{kj}(\Delta Q_j)\right. \\ &\qquad \left. + \sum_{j \in S \cap O} \omega_{kj}(\Delta Q_j) - V_k^*\right) \\ &= \sum_{i \in S \cap F} c_i + \sum_{i \in S \cap O} b_i + \lambda \sum_{k=1}^{n} h\left(\sum_{j \in S \cap F} \omega_{kj}(\Delta Q_j)\right. \\ &\qquad \left. - \sum_{j \in O \setminus S} \omega_{kj}|\Delta Q_j| - V_k^*\right). \end{aligned}$$

The set $S \cap F$ is equal to $O' \setminus O$, while the set $S \cap O$ is equal to $O \setminus O'$. Similarly, $O' = (S \cap F) \cup (O \setminus S)$, and hence

$$\begin{aligned} f(S) &= \sum_{i \in O' \setminus O} c_i + \sum_{i \in O \setminus O'} b_i \\ &\quad + \lambda \sum_{k=1}^{n} h\left(\sum_{j \in O'} \omega_{kj}|\Delta Q_j| - \hat{V}_k^*\right) \\ &= \overline{f}(O'), \end{aligned}$$

completing the proof. □

The problem of selecting a set of buses to inject reactive power can then be formulated as either $\min\{f(S) : S \subseteq \Omega\}$ or $\min\{\overline{f}(O') : O' \subseteq \Omega\}$. These are both combinatorial optimization problems, and hence cannot be efficiently solved or approximated unless the objective function possesses additional structure, such as submodularity. While neither $f(S)$ nor $\overline{f}(O')$ is supermodular (unless under certain conditions on the power system state; see Section 4.4), an equivalent submodular objective function can still be derived, as shown in the following section.

4.2 Submodular Optimization Approach

As a first step, we give an equivalent problem formulation with an expanded ground set. We then demonstrate that this problem has the structure of submodular maximization with a matroid basis constraint.

Define the extended ground set $\overline{\Omega}$ by

$$\overline{\Omega} = \{v_{ij} : i \in \Omega, j = 0, 1\}.$$

Here v_{i0} can be viewed as the event that the capacitor/reactor bank at bus $i \in \Omega$ is switched off, while v_{i1} is the event that the capacitor/reactor bank at bus i is switched on. For each bus i, let $P_i = \{j \in \Omega : \omega_{ij} > 0\}$ and $R_i = \{j \in \Omega : \omega_{ij} < 0\}$, i.e., the set of buses where an injection of reactive power causes an increase (P_i) or decrease (R_i) in voltage at bus i. Finally, for any set $A \subseteq \overline{\Omega}$, define sets A_0 and A_1 by

$$A_0 = \{i \in \Omega : v_{i0} \in A\}, \quad A_1 = \{i \in \Omega : v_{i1} \in A\}.$$

For each bus i, let the function $\hat{f}_i : 2^{\overline{\Omega}} \rightarrow \mathbb{R}$ be defined by

$$\hat{f}_i(A) = h\left(\sum_{j \in P_i \cap A_1} \omega_{ij}|\Delta Q_j| + \sum_{j \in R_i \cap A_0} |\omega_{ij}||\Delta Q_j| - \hat{V}_i\right),$$

where $\hat{V}_i = \hat{V}_i^* - \sum_{j \in R_i} \omega_{ij}|\Delta Q|_j$. Define a system-wide cost function by

$$\hat{f}(A) = \lambda \sum_{i \in \Omega} \hat{f}_i(A) + \sum_{i \in A_0 \cap O} b_i + \sum_{i \in A_1 \cap F} c_i.$$

The following lemma establishes the equivalence between this cost function and the objective function of the previous subsection.

Lemma 4. *Suppose that the set $A \subseteq \overline{\Omega}$ satisfies $|A \cap \{v_{i0}, v_{i1}\}| = 1$ for all $i \in \Omega$. Then $\hat{f}(A) = \overline{f}(A_1)$.*

Proof. First, by definition

$$\sum_{i \in A_1 \setminus O} c_i + \sum_{i \in O \setminus A_1} b_i = \sum_{i \in A_1 \setminus O} c_i + \sum_{i \in A_0 \cap O} b_i.$$

Hence, it suffices to show that, for each $i \in \Omega$,

$$\hat{f}_i(A) = h\left(\sum_{j \in A_1} \omega_{ij}|\Delta Q_j| - \hat{V}_i^*\right)$$

when the condition $|A \cap \{v_{i0}, v_{i1}\}| = 1$ holds for all $i \in \Omega$. We have

$$\sum_{j \in A_1} \omega_{ij}|\Delta Q_j| \tag{11}$$

$$= \sum_{j \in A_1 \cap P_i} \omega_{ij}|\Delta Q_j| + \sum_{j \in A_1 \cap R_i} \omega_{ij}|\Delta Q_j| \tag{12}$$

$$= \sum_{j \in A_1 \cap P_i} \omega_{ij}|\Delta Q_j| + \sum_{j \in R_i} \omega_{ij}|\Delta Q_j| - \sum_{j \in R_i \cap A_0} \omega_{ij}|\Delta Q_j|, \tag{13}$$

where (13) follows from the fact that, under the assumption on A, $\Omega = A_0 \cup A_1$ is a partition of the set Ω, and hence $R_i = (R_i \cap A_0) \cup (R_i \cap A_1)$ and $R_i \cap A_1 = R_i \setminus (R_i \cap A_0)$. Then

$$\begin{aligned} h&\left(\sum_{j \in A_1} \omega_{ij}|\Delta Q_j| - \hat{V}_i^*\right) \\ &= h\left(\sum_{j \in A_1 \cap P_i} \omega_{ij}|\Delta Q_j| + \sum_{j \in A_0 \cap R_i} |\omega_{ij}||\Delta Q_j| - \left[\hat{V}_i^* - \sum_{j \in R_i} \omega_{ij}|\Delta Q_j|\right]\right) = \hat{f}_i(A), \end{aligned}$$

as desired. □

Define a collection $\mathcal{B}$ of subsets of $\overline{\Omega}$ by $A \in \mathcal{B}$ iff $|A \cap \{v_{i0}, v_{i1}\}| = 1$ for all $i \in \Omega$. By Lemma 4, the problem of selecting a set of buses to inject reactive power is equivalent to

$$\min\{\hat{f}(A) : A \in \mathcal{B}\}. \tag{14}$$

Our approach in constructing approximation algorithms is to prove that (14) consists of minimizing a supermodular objective function subject to a matroid basis constraint.

Lemma 5. *There exists a matroid $\mathcal{M}$ such that $A \in \mathcal{B}$ iff A is a basis of $\mathcal{M}$.*

Proof. Define $\Omega_i = \{v_{i0}, v_{i1}\}$. The sets $\Omega_1, \ldots, \Omega_n$ form a partition of $\overline{\Omega}$, and hence the constraint $A \in \mathcal{M}$ iff $|A \cap \Omega_i| \leq 1$ defines a partition matroid $\mathcal{M}$ by Lemma 2. Furthermore, the bases of $\mathcal{M}$ are the sets A satisfying $|A \cap \Omega_i| = 1$, which are exactly the sets in $\mathcal{B}$. □

It remains to show supermodularity of $\hat{f}(A)$, which is established by the following theorem.

Theorem 1. *The function $\hat{f}(A)$ is supermodular as a function of A.*

Proof. The approach of the proof is to show a more general result, namely, that any function $g(A)$ defined by

$$g(A) = h\left(\sum_{i \in A} \alpha_i - \beta\right),$$

where $\alpha_i \geq 0$ and $\beta \in \mathbb{R}$, is supermodular as a function of A. Define

$$\rho = \frac{\sum_{i \in A \setminus B} \alpha_i}{\sum_{i \in A \Delta B} \alpha_i},$$

where Δ is the symmetric difference operator. Since all

α_i's are nonnegative, $\rho \in [0,1]$. We have

$$\sum_{i \in A} \alpha_i = \rho \sum_{i \in A \cup B} \alpha_i + (1-\rho) \sum_{i \in A \cap B} \alpha_i$$
$$\sum_{i \in B} \alpha_i = (1-\rho) \sum_{i \in A \cup B} \alpha_i + \rho \sum_{i \in A \cap B} \alpha_i$$

Hence, by convexity,

$$\begin{aligned}
&g(A) + g(B) \\
&= h\left(\sum_{i \in A} \alpha_i - \beta\right) + h\left(\sum_{i \in B} \alpha_i\right) \\
&= h\left(\rho\left(\sum_{i \in A \cup B} \alpha_i - \beta\right) + (1-\rho)\left(\sum_{i \in A \cap B} \alpha_i - \beta\right)\right) \\
&\quad + h\left((1-\rho)\left(\sum_{i \in A \cup B} \alpha_i - \beta\right) + \rho\left(\sum_{i \in A \cap B} \alpha_i - \beta\right)\right) \\
&\leq \rho h\left(\sum_{i \in A \cup B} \alpha_i - \beta\right) + (1-\rho) h\left(\sum_{i \in A \cap B} \alpha_i - \beta\right) \\
&\quad + (1-\rho) h\left(\sum_{i \in A \cup B} \alpha_i - \beta\right) + \rho h\left(\sum_{i \in A \cap B} \alpha_i - \beta\right) \\
&= g(A \cup B) + g(A \cap B),
\end{aligned}$$

establishing supermodularity of g.

Since the function $\hat{f}_i(A)$ can be obtained from $g(A)$ by setting $\alpha_j = |\omega_{ij}|(\Delta Q_j)$ and $\beta = \hat{V}_i$, each $\hat{f}_i(A)$ is supermodular. The terms

$$\sum_{i \in A_1 \setminus O} c_i + \sum_{i \in A_0 \cap O} b_i$$

can also be shown to be supermodular. The function $\hat{f}(A)$ is therefore a sum of supermodular functions, and hence is supermodular. □

Combining Theorem 1 with the above discussion of $\mathcal{B}$, we have that Eq. (14) is a supermodular minimization problem, which can be transformed to an equivalent submodular maximization problem, subject to a matroid basis constraint.

4.3 Voltage Control Algorithm

The supermodular structure of Eq. (14) implies that there exists a polynomial-time algorithm to approximately solve (14) up to a worst-case optimality bound of 1/6 [10]. In what follows, we present a simplified algorithm that, under a mild assumption motivated by the physical properties of the power system, achieves an improved optimality bound of 1/3.

The algorithm proceeds as follows. Let $\epsilon > 0$ be a constant parameter. The set A is initialized to be equal to the current configuration of capacitor/reactor banks, so that $A = \{v_{i0} : i \in F\} \cup \{v_{i1} : i \in O\}$. At each iteration, the algorithm selects a bus $i \in \Omega$ such that

$$\hat{f}(A \setminus \{v_{i0}\} \cup \{v_{i1}\}) < (1-\epsilon)\hat{f}(A),$$

if $v_{i0} \in A$ or

$$\hat{f}(A \setminus \{v_{i1}\} \cup \{v_{i0}\}) < (1-\epsilon)\hat{f}(A)$$

if $v_{i1} \in A$. The algorithm terminates when no such bus i can be found. A pseudocode description is given as Algorithm 1.

Intuitively, at each iteration, the algorithm identifies a bus $i \in \Omega$ such that toggling its capacitor/reactor bank from off to on (or from on to off) will reduce the cost function $\hat{f}(A)$.

Algorithm 1 Algorithm for selecting a set of buses to inject reactive power.

procedure SUBMODULAR_VC($\boldsymbol{\omega}$, $\mathbf{q}$, $\mathbf{b}$, $\mathbf{c}$, O, F)
    Input: Weights ω_{ij} from inverse Jacobian J^{-1}, possible reactive power injections at each bus $\mathbf{q}$
    Switching costs $\mathbf{b}$ and $\mathbf{c}$
    Set of buses O initially injecting reactive power, $F = \Omega \setminus O$.
    Output: Set of buses S to inject reactive power.
    Initialization: $A \leftarrow \{v_{i0} : i \in F\} \cup \{v_{i1} : i \in O\}$, $A_0 \leftarrow F$, $A_1 \leftarrow O$
    while 1 **do**
        if there exists $i \in A_0$ with $\hat{f}(A \setminus \{v_{i0}\} \cup \{v_{i1}\}) < (1-\epsilon)\hat{f}(A)$ **then**
            $A \leftarrow A \setminus \{v_{i0}\} \cup \{v_{i1}\}$
            $A_1 \leftarrow A_1 \cup \{i\}$, $A_0 \leftarrow A_0 \setminus \{i\}$
        else if there exists $i \in A_1$ with $\hat{f}(A \setminus \{v_{i1}\} \cup \{v_{i0}\}) < (1-\epsilon)\hat{f}(A)$ **then**
            $A \leftarrow A \setminus \{v_{i1} \cup \{v_{i0}\}$
            $A_0 \leftarrow A_0 \cup \{i\}$, $A_1 \leftarrow A_1 \setminus \{i\}$
        else
            break
        end if
    end while
    $S \leftarrow A_1$, **return** S
end procedure

We now analyze the optimality bounds guaranteed by Algorithm 1. By construction, the algorithm converges to a ϵ-local minimum, defined as a set A satisfying $(1-\epsilon)\hat{f}(A) < \hat{f}(A \cup \{v\} \setminus \{w\})$ for any v and w such that $(A \cup \{v\} \setminus \{w\}) \in \mathcal{B}$. The goal of the analysis is to show that the cost at the local minimum is within a provable bound of the globally minimal cost. As a first step, we state the following assumption.

ASSUMPTION 1. *If A is a ϵ-local optimum of the function $\hat{f}(A)$, then $\hat{f}(A) < \hat{f}(\overline{\Omega} \setminus A)$.*

Assumption 1 states that, at any local optimum of $\hat{f}(A)$, the total cost is less than the cost that would be achieved by switching all capacitors that are off in configuration A to be on, and vice versa. We justify this assumption by noting that making this switch would likely involve reducing reactive power near buses that are currently below their desired voltages, while injecting reactive power at buses that are already above their desired voltages. Furthermore, the configuration represented by $\overline{\Omega} \setminus A$ would likely incur a significant switching cost.

Under this assumption, we have the following optimality result.

THEOREM 2. *Let M satisfy $\hat{f}(A) \leq M$ for all $A \subseteq \overline{\Omega}$. Define S to be the set chosen by Algorithm 1, and let S^* be the optimal solution to* $\min\{f(S) : S \subseteq \Omega\}$. *Then*

$$M - f(S) \geq \left(\frac{1}{3+\epsilon}\right)(M - f(S^*)). \tag{15}$$

PROOF. Let $A^* = \{v_{i0} : i \notin S^*\} \cup \{v_{i1} : i \in S^*\}$ and $A = \{v_{i0} : i \notin S\} \cup \{v_{i1} : i \in S\}$. Define a function $\hat{g}(A) = M - \hat{f}(A)$, so that $\hat{g}$ is a nonnegative submodular function. By Lemma 1,

$$(2+\epsilon)\hat{g}(A) \geq \hat{g}(A \cup A^*) + \hat{g}(A \cap A^*).$$

Applying Assumption 1 yields

$$(3+\epsilon)\hat{g}(A) \geq \hat{g}(A \cup A^*) + \hat{g}(\overline{\Omega} \setminus A) + \hat{g}(A \cap A^*). \tag{16}$$

By submodularity and nonnegativity of $\hat{g}$, we have

$$\begin{aligned} &\hat{g}(A \cup A^*) + \hat{g}(\overline{\Omega} \setminus A) \\ &\geq \hat{g}(A^c \cap (A \cup A^*)) + \hat{g}(A^c \cup (A \cup A^*)) \\ &= \hat{g}(\overline{\Omega}) + \hat{g}(A^* \setminus A) \geq \hat{g}(A^* \setminus A). \end{aligned}$$

Applying this inequality to (16) yields

$$\begin{aligned} (3+\epsilon)\hat{g}(A) &\geq \hat{g}(A^* \setminus A) + \hat{g}(A \cap A^*) \\ &\geq \hat{g}(A^*) + \hat{g}(\emptyset) \geq \hat{g}(A^*) \end{aligned}$$

by submodularity of $\hat{g}$. Substitution of the definition of $\hat{g}$ then gives (15). □

4.4 Voltage Control Under Heavy Loading

Under certain operating conditions, the function $\overline{f}(O')$ defined in Section 4.1 possesses additional structure that can be exploited to improve the optimality bounds and remove the necessity of Assumption 1. In this subsection, we consider the case where the inequality

$$2V_i \geq (V_i + V_j)\cos\theta_{ij} \tag{17}$$

holds at all neighboring buses i and j. This condition holds under heavy loading conditions, when V_i and V_j are within their normal range (between 0.95 and 1.05 pu) and θ_{ij} is greater than 13 degrees due to real power flows between buses. The properties of the inverse Jacobian matrix under this heavy loading condition are described in the following lemma.

LEMMA 6. *If Eq. (17) holds and $|\theta_i - \theta_j| < \frac{\pi}{2}$ at all neighboring buses i and j, then all entries of J^{-1} are nonnegative.*

PROOF. Recall that an M-matrix is a matrix with non-positive off diagonal entries and whose eigenvalues have positive real parts. Since the inverse of an M-matrix is nonnegative, it suffices to show that the Jacobian is an M-matrix when (17) holds. First, note that the off-diagonal entries are given by

$$J_{ij} = -V_i B_{ij}\cos\theta_{ij} < 0$$

for each pair of neighboring buses.

By the Gershgorin Circle Theorem, a sufficient condition for positivity of the eigenvalues is that the matrix columns are diagonally dominant, or equivalently,

$$\sum_{j \in N(i)} 2V_i B_{ij} - B_{ij}V_j\cos\theta_{ij} - \sum_{j \in N(i)} B_{ij}V_i\cos\theta_{ij} > 0.$$

If Eq. (17) holds, then each term of the summation is positive, completing the proof. □

Lemma 6 provides needed additional structure to show that the metric $\overline{f}(O')$ is supermodular as a function of O'.

PROPOSITION 1. *If (17) holds and $\cos\theta_{ij} < \frac{\pi}{2}$ at all neighboring buses i and j, then the function $\overline{f}(O')$ is supermodular as a function of O'.*

PROOF. From the proof of Theorem 1, we have that the function

$$g(A) = h\left(\sum_{j \in A}\alpha_j - \beta\right)$$

is supermodular when $\alpha_i \geq 0$ for all i. Letting $\alpha_j = \omega_{ij}|\Delta Q_j|$, we have that $\overline{f}$ can be written as a sum of functions of this form, where nonnegativity is guaranteed by Lemma 6. Hence $\overline{f}(O')$ is supermodular. □

Supermodularity of O' eliminates the need for the matroid basis constraint in (14), and implies that

$$\min\{\overline{f}(O') : O' \subseteq \Omega\}$$

consists of unconstrained minimization of a supermodular objective function. Removing the matroid constraint enables an optimality bound of 1/2, without the need for Assumption 1, using a randomized greedy algorithm. The randomized greedy algorithm is described as Algorithm 2 [1].

Algorithm 2 Selecting a set of buses for voltage control under heavy loading.

procedure VC_HEAVY($\boldsymbol{\omega}$, $\mathbf{q}$, $\mathbf{b}$, $\mathbf{c}$, O, F)
  Input: Weights ω_{ij} from inverse Jacobian J^{-1}, possible reactive power injections at each bus $\mathbf{q}$
  Switching costs $\mathbf{b}$ and $\mathbf{c}$
  Set of buses O initially injecting reactive power, $F = \Omega \setminus O$.
  Output: Set of buses S to inject reactive power.
  Initialization: $U_0 \leftarrow \emptyset$, $V_0 \leftarrow \Omega$
  for $i = 1, \ldots, n$ **do**
    $\mu_i \leftarrow f(U_{i-1} \cup \{i\}) - f(U_{i-1})$
    $\nu_i \leftarrow f(V_{i-1} \setminus \{i\}) - f(V_{i-1})$
    $\mu_i' \leftarrow \max\{\mu_i, 0\}$, $\nu_i' \leftarrow \max\{\nu_i, 0\}$
    $r \leftarrow 0$ with probability $\frac{\mu_i'}{\mu_i' + \nu_i'}$, $r \leftarrow 1$ else
    if $r == 0$ **then**
      $U_i \leftarrow U_{i-1} \cup \{i\}$, $V_i \leftarrow V_{i-1}$
    else
      $U_i \leftarrow U_{i-1}$, $V_i \leftarrow V_{i-1} \setminus \{i\}$
    end if
  end for
  $S \leftarrow U_n$, **return** S
end procedure

LEMMA 7. *Let $\hat{O}$ denote the set chosen by Algorithm 2, and let $O^* = \arg\min\{\overline{f}(O') : O' \subseteq \Omega\}$. Let M be an upper bound on $\overline{f}(O')$ for all $O' \subseteq \Omega$. Then*

$$M - \overline{f}(\hat{O}) \geq \frac{1}{2}(M - \overline{f}(O^*).$$

The lemma is a special case of [1, Lemma 3.1].

4.5 Complexity Analysis and Comparison

We now discuss the complexity of our approach and compare with the current state of the art [14]. The complexity of Algorithm 1 is described by the following proposition.

PROPOSITION 2. *Let $f_{min} = \min\{f(S) : S \subseteq \Omega\}$ and $f_{max} = \max\{f(S) : S \subseteq \Omega\}$. The runtime of the algorithm is bounded above by $O(nT_0)$, where*

$$T_0 = \lceil \frac{\log\left[\frac{f_{min}}{f_{max}}\right]}{\log(1-\epsilon)} \rceil.$$

PROOF. Let S_i denote the set S after i iterations of the algorithm. By construction, $f(S_i) < (1-\epsilon)f(S_{i-1})$, and hence $f(S_i) < (1-\epsilon)^i f(S_0)$. Let T denote the number of iterations before the algorithm terminates. By definition,

$$f_{min} \leq f(S_T) < (1-\epsilon)^T f(S_0) \leq (1-\epsilon)^T f_{max}.$$

Rearranging terms yields $T \leq T_0$. Each iteration requires at most n evaluations of the objective function $f(S)$ in order to identify a bus to activate or deactivate its capacitor/reactor bank, implying that the computation is $O(nT_0)$ in the worst case. $\square$

Our Algorithm 2 provides lower linear computational complexity in the number of buses ($O(n)$) by inspection. For comparison, the approach of [14] is based on enumerating all possible control actions until an action is found that resolves any voltage instability. The number of such actions, denoted $m = 2^k$, will satisfy $m \ll 2^n$. If actions are sampled uniformly at random, the expected complexity is $O(2^n/2^k) = O(2^{n-k})$, which is significantly larger than the linear complexity provided by Algorithms 1 and 2.

5. NUMERICAL STUDY

This section presents simulation results for our proposed submodular approach to voltage control. The system topology, bus angles, and line reactances are from the IEEE 30-bus test data. Initial voltages are varied around the reference value $\mathbf{v_{ref}} = \mathbf{1}$ p.u. The acceptable voltage range is defined as (0.95,1.05) p.u., while the desired voltage range is set to be (0.98, 1.02) p.u. The parameter $\epsilon = 0$ from Algorithm 1.

In our simulation, we assume each bus is a PQ bus with a capacitor bank ready to switch on/off. We set the possible reactive power injection ΔQ at each bus to be 0.005 p.u. In most cases, we choose the trade-off factor $\lambda \geq 5$ to emphasize the penalty cost of voltage violation. For switching costs at bus i, we set $b_i = 1$ and $c_i = 1$ for switching capacitor/reactor banks off and on respectively.

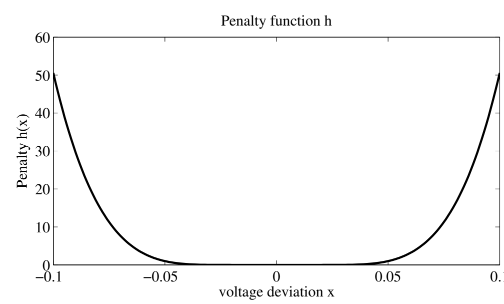

Figure 1: Convex penalty function $h(x)$ for voltage deviation.

The convex penalty function is defined as

$$h(x) = \begin{cases} \alpha(x - V_{max})^4, & x > V_{max} \\ 0, & V_{min} < x \leq V_{max} \\ \beta(x - V_{min})^4, & x \leq V_{min} \end{cases}$$

where V_{min} and V_{max} are the lower and upper bounds on the desired voltage region (Figure 1).

To evaluate our approach, we considered two test cases, namely, voltage control under normal operating condi-

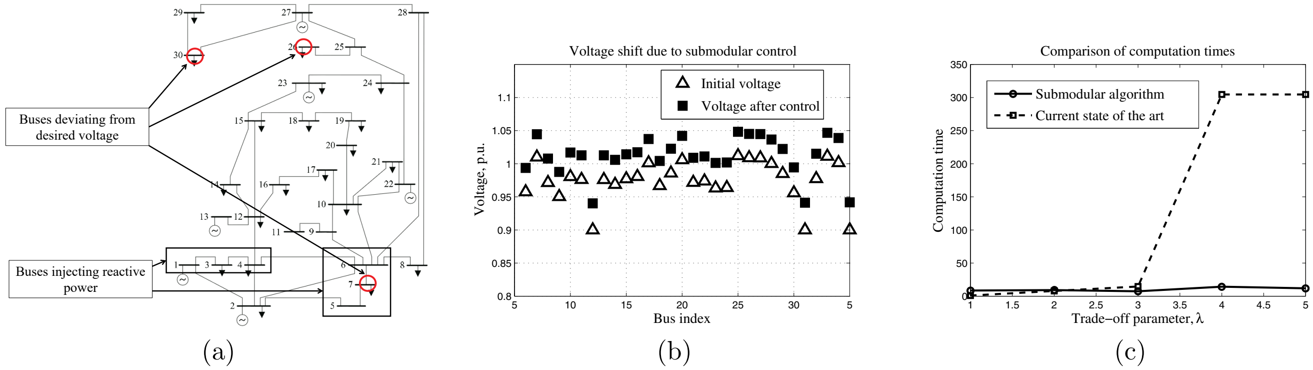

Figure 2: Simulation study of our proposed submodular optimization approach for voltage control. (a) IEEE 30 bus network topology considered in this study. Circled buses have initial voltages that are below the normal operating range. Squared buses are selected by our algorithm to inject reactive power. (b) Change in voltage due to reactive power injections. The reactive power injections are sufficient to drive deviating buses to their normal operating regions. (c) Comparison of time required to compute the optimal voltage control strategy for submodular approach and the current state of the art. The existing approach requires significantly greater computation overhead as the switching cost per bus increases.

tions and voltage control when one or more buses deviate from their desired voltages. We first considered voltage control when all buses are within the normal operating region of $(0.95, 1.05)$ p.u., but some buses deviate from the desired range of $(0.98, 1.02)$ p.u. The parameter $\lambda = 10$ for this case, and no capacitor/reactor banks were assumed to be active. The submodular optimization approach recommended switching six capacitor banks from off to on, thus reducing the total voltage deviation from 4.563 to 0.7805. The maximum deviation of any bus from the desired voltage was reduced from 0.5 to 0.4.

We next consider the case where one or more voltages are deviating from their normal operating ranges. As shown in Figure 2(a), buses 7, 26, and 30 have voltages that drop to 0.9 p.u., which must be mitigated through reactive power injection. Initially, the set $O = \emptyset$. The submodular control approach selected buses $\{1, 3, 4, 5, 6, 7\}$ to inject reactive power in order to restore the overall system voltage. We observe that, while buses 26 and 30 experienced lower voltages, they were not selected to inject reactive power. For this test case, there were sufficiently few reactive power losses between the selected buses and the low-voltage buses to enable a return to voltage stability. At the same time, the chosen set of actions ensured that the voltages of the remaining buses did not rise *above* their desired operating region.

The impact of the chosen control action is shown in Figure 2(b). The minimum voltage of any bus increased from 0.9 to 0.94, so that all voltages are close to the desired operating region, while the maximum voltage of 1.05 is within the desired limits. The total voltage deviation was reduced from 154 to 12.

A comparison between the submodular approach to voltage control and the exhaustive search approach [14] is shown in Figure 2(c). Our implementation of the exhaustive search first computes the objective function for control actions at a single bus, followed by all control actions involving two buses, and so on. The number of computations of the objective function to select a voltage control strategy that achieves a local minimum of the objective function $f(S)$ is shown. As the trade-off parameter λ increases, the number of buses that inject reactive power at the local optimum increases. This leads to a corresponding increase in the search space, and hence the complexity, of the exhaustive search algorithm. The execution time of the submodular approach, however, does not depend on the system parameters.

6. CONCLUSIONS AND FUTURE WORK

We considered the problem of voltage control in power systems. Current approaches to voltage control rely on enumerating possible control actions across all buses, and hence are computationally intensive and may lead to inefficient voltage control strategies with large switching costs. In this paper, we formulated a discrete optimization problem of selecting a subset of buses to inject reactive power (e.g., through activating capacitor/reactor banks or transformer tap changes) in order to minimize the deviation from the desired voltage and switching costs. Our main contribution was to prove that this joint cost function is supermodular as a function of the set of buses that inject reactive power.

We demonstrated that the voltage control problem is equivalent to submodular maximization with a matroid basis constraint, leading to efficient approximation algorithms with provable optimality bounds. These algorithms enable selection of control actions with less computation than current approaches that enumerate all control actions. This reduced complexity enables searching for control strategies that minimize switching costs in addition to preventing voltage instability. Evaluation of the submodular algorithms on larger power systems will be studied in our future work.

Injecting reactive power at load buses is the standard approach to ensuring voltage stability. Demand response mechanisms that are being deployed in the future smart grid provide an additional method of voltage control by reducing reactive power demand. Our future work will focus on scalable and effective voltage control through demand response, as well as joint consideration of demand response and traditional reactive power injection. We will also investigate efficient distributed algorithms that can be implemented at the individual buses.

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