# Continuous-time Marginal Pricing of Electricity

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***Abstract*—The current practice of discrete-time electricity pricing starts to fall short in providing an accurate economic signal reflecting the continuous-time variations of load and generation schedule in power systems. This paper introduces the fundamental mathematical theory of continuous-time marginal electricity pricing. We first formulate the continuous-time unit commitment (UC) problem as a constrained variational problem, and subsequently define the continuous-time economic dispatch (ED) problem where the binary commitment variables are fixed to their optimal values. We then prove that the continuous-time marginal electricity price equals to the Lagrange multiplier of the variational power balance constraint in the continuous-time ED problem. The proposed continuous-time marginal price is not only dependent to the incremental generation cost rate, but also to the incremental ramping cost rate of the units, thus embedding the ramping costs in calculation of the marginal electricity price. The numerical results demonstrate that the continuous-time marginal price manifests the behavior of the constantly varying load and generation schedule in power systems.**

***Index Terms*—Continuous-time marginal electricity price, generation trajectory, ramping trajectory, variational problem.**

## I. Introduction

THE fundamental operation goal of a functional power system is to balance the generation resources and load in continuous time, respecting the physical characteristics and limitations of the system. In a market-based framework to operate power systems, the balancing task is performed in multiple forward and real-time markets, where energy and various ancillary service products are traded to ensure the security of operation. Different pricing schemes are utilized to price the electricity energy and the ancillary services at different time scales [1]–[3].

Among the early works on electricity pricing, the seminal work of Schweppe and colleagues [4], which provides the fundamentals for spot pricing of electricity, has been the source of inspiration for a plethora of succeeding works. In [4], the electricity price is defined in an *hourly* basis using the Lagrange multipliers of the hourly power balance constraints. When the transmission grid is considered, the Lagrange multipliers associated with the nodal power balance constraints define the locational marginal prices for electricity [1]. In [5], in conjunction with active power pricing, the reactive power is also priced using an optimal power flow model. In [6], a market is proposed for callable forward contracts treated as derivative commodities and the associated pricing method is discussed. In addition to the electricity energy, maintaining the security of power system operation requires fair and transparent schemes for procuring and pricing the ancillary services [7]–[10]. In the early work of [7], an iterative method is used to calculate the marginal up spinning reserve cost with regard to the line flow constraints. In [8], the authors argue that a socially optimal security level is obtainable through pricing incentives and providing information on the services requirements. The importance of security pricing was soon realized by newly established markets, including New Zealand and US New England markets [11], [12]. Development of the efficient solution methods for stochastic optimization models paved the way to factor the stochastic nature of security pricing in more recent works [13], [14].

Electricity prices should ideally reflect the true marginal cost of generation, taking into account all physical system constraints, and fully compensate all resources for the costs of supplying electricity [15]. However, due to the approximate modeling of power system constraints such as ramping process, as well as the non-convex startup/shutdown costs and the minimum generation constraints in unit commitment (UC) problem, the prices may not support the equilibrium solution of the market. This may result in inability of markets to cover the operating costs of some resources, where *uplift payments* are paid by system operators to make the resources financially whole and maintain the functionality of markets [15], [16]. Several methods are proposed in technical literature that account for non-convexities in deriving marginal prices. O'Neil et al. proposed that the non-convexities can be considered as separate commodities in markets and developed a method to calculate marginal prices as well as the uplift payments to the resources [17]. A convex hull pricing model is presented in [18] in which the prices minimize the uplift payments in markets. Midcontinent ISO (MISO) has recently embedded this method in its extended locational marginal pricing process [19], [20]. Recent developments include a non-convexity pricing method that guarantees non-negative revenues for generating units [21], as well as an alternative convex hull pricing scheme for energy and reserve markets using extreme-point subdifferential [22]. A detailed analysis of properties and implementation challenges of convex hull pricing is presented in [23].

Although substantial research efforts are devoted to address pricing issues in markets and deliver a consistent and transparent price signal to the participants, less attention has been given to account for inter-temporal ramping constraints in scheduling and pricing of electricity in markets. The increased sub-hourly variations of net-load due to the large-scale renewable integration questions the adequacy of current *discrete-time* scheduling and pricing methods, which does not flexibly schedule the generation fleet to ramp in sub-hourly intervals. This may leave the system with sufficient capacity but without ramping capability to respond to fast sub-hourly variations of load that may lead to *ramping scarcity events* [24], [25], with obviously undesirable economic and security consequences

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[26]. In this regard, Federal Energy Regulatory Commission (FERC) issued the order 764 and amended the pro forma Open Access Transmission Tariff in order to require the public utilities to provide sub-hourly (15-minute) schedules for the transmission customers to reflect changes in the renewable generation output [27]. In response, the independent system operators are changing their market rules to integrate the sub-hourly scheduling in the market clearing practices. In addition, the MISO and the California ISO (CAISO) are integrating new ramping services in their markets to address the ramping challenge, and avoid the ramping scarcity events [25], [28].

In fact, the ramping scarcity events and the associated price spikes are evidence of a severe bottleneck that lies in the current ramping model as the finite difference of discrete-time power samples, which poorly models the actual continuous-time ramping process of units and hardly captures the impact of load ramping on the generation schedules and electricity prices. Although using smaller time steps may reduce the approximation error, the inherent ambiguity in ramping definition still remains an issue with the discrete-time scheduling and pricing models. To address this problem, we proposed a continuous-time UC model in [29] that schedules the *continuous-time generation and ramping trajectories* of generating units to supply the continuous-time variations of load. In [29], spline function space of Bernstein polynomials are utilized to model the continuous-time trajectories and recast the continuous-time problem into a mixed-integer linear programming problem with finite-dimensional decision space.

In this paper, we base the mathematical foundation and define the theory of continuous-time marginal pricing of electricity in day-ahead markets. In Section II, we revisit the current discrete-time UC model and present the formulation of continuous-time UC problem as a constrained variational problem, where the ramping process of generating units is modeled by *continuous-time ramping trajectory*. We define the continuous-time economic dispatch (ED) problem by fixing the binary commitment variables to their optimal values in the UC problem. We then present the necessary and sufficient optimality conditions of the continuous-time ED problem, and prove in Section III that the *continuous-time marginal electricity price* is defined as the Lagrange multiplier of the continuous-time power balance constraint of the proposed ED formulation. We define the *incremental ramping cost rate* of generating units as the cost of incremental change in their ramping, and prove that the continuous-time marginal electricity price is not only a function of the incremental generation cost rate, but also of the incremental ramping cost rate of the units. The numerical results are presented in Section IV, and conclusions are drawn in Section V.

## II. Continuous-time Day-Ahead Scheduling

The goal of day-ahead power system operation is to schedule the most economical set of generating units to supply the net-load over the day-ahead scheduling horizon $\mathcal{T}=[0,T]$. The traditional scheduling practice subdivides $\mathcal{T}$ to $N$ intervals $\mathcal{T}_n=[t_n,t_{n+1}), \mathcal{T}=\cup_{n=0}^{N-1}\mathcal{T}_n$ of the same length $\Delta t=t_{n+1}-t_n$, e.g., hourly, where $t_0=0$, $t_N=T$. The resulting discrete-time generation schedules $\mathbf{G}(t_n)=(G_1(t_n),\ldots,G_K(t_n))^T$ and commitment statuses $\mathbf{I}(t_n)=(I_1(t_n),\ldots,I_K(t_n))^T$ are optimized to supply the discrete-time load samples $D(t_n)$ at minimum cost, forming the discrete-time UC problem below:

$$\min_{\mathbf{G}(t_n),\ \mathbf{I}(t_n)} \quad \sum_{n=0}^{N-1} \big(C(\mathbf{G}(t_n)) + C^I(\mathbf{I}(t_n))\big)\,\Delta t, \tag{1}$$

$$\text{s.t.} \quad f\big(\mathbf{G}(t_n),\mathbf{I}(t_n)\big)=0, \qquad (\lambda(t_n)), \quad \forall n, \tag{2}$$

$$\mathbf{h}\big(\mathbf{G}(t_n),\mathbf{I}(t_n),\Delta t\big)\le 0, \quad (\boldsymbol{\gamma}(t_n)), \quad \forall n, \tag{3}$$

where $C(\mathbf{G}(t_n))=\sum_K C_k(G_k(t_n))$ represents sum of the generation costs of the units in each interval $n$; $C^I(\mathbf{I}(t_n))=\sum_K C_k^I(I_k(t_n))$ represents sum of the startup, shutdown, and fixed costs of the units; $f(\cdot)$ represents the discrete-time power balance constraints; $\mathbf{h}(\cdot)$ represents the set of prevailing inequality constraints, including the generating units' capacity limits, ramping limits, startup and shutdown costs, and minimum on/off time constraints; $\lambda(t_n)$ and $\boldsymbol{\gamma}(t_n)$ are respectively the Lagrange multipliers associated with the equality and inequality constraints.

The most common day-ahead pricing practice includes solving the mixed-integer linear programming (MILP) UC problem (1)-(3), fixing the binary commitment variables to their optimal values and sequentially solving the ensuing $N$ single-period linear programming (LP) economic dispatch (ED) problems [17]. The optimal Lagrange multipliers $\lambda(t_n)$ of the solution of the ED problems present the day-ahead marginal prices at discrete times $t_n$. In addition, non-convexity prices are determined using the Lagrange multipliers associated with the equality constraints fixing the commitment variables [17]. Alternative methods include the convex hull pricing in which the prices minimize the uplift payments in markets [20], [23]

### A. Continuous-time UC model

The discrete-time UC problem (1)-(3) schedules for the discrete-time samples of units' generation, implying that units shall follow piecewise constant generation trajectories from one schedule to the next [30]. This follows that the units' ramping is modeled as the finite difference between the consecutive generation samples. Clearly, the discrete-time generation schedules and the resulting rampings does not appropriately utilize the flexibility of generating units to compensate the faster variations of net-load that may lead to the ramping scarcity events. In addition, we argue that calculating marginal prices using the sequential solution of single-period ED problems may not appropriately factor the impacts of ramping constraints in day-ahead prices, and may result in prices that does not reflect the true marginal generation cost.

As an alternative to the discrete-time modeling approach, let us assume that the generating units are modeled by continuous-time generation trajectories $\mathbf{G}(t)=(G_1(t),\ldots,G_K(t))^T$ and continuous-time binary commitment variables $\mathbf{I}(t)=(I_1(t),\ldots,I_K(t))^T$, which are scheduled to balance the continuous-time net-load trajectory $D(t)$ at minimum cost [29]. In the continuous-time modeling approach, the finite difference ramping model tends to derivative as the length of time intervals $\Delta t$ approaches to zero, and allows us to define the *continuous-time ramping trajectory* of unit $k$, $\dot{G}_k(t)$, as the time derivative of its generation trajectory:

$$\dot{G}_k(t) \triangleq \lim_{\Delta t \to 0} \frac{G_k(t_{n+1}) - G_k(t_n)}{\Delta t} = \frac{dG_k(t)}{dt}. \quad (4)$$

Defining the explicit continuous-time ramping trajectories $\dot{\mathbf{G}}(t) = \big(\dot{G}_1(t), \ldots, \dot{G}_K(t)\big)^T$, we can assume that in contrast to the cost function $C_k(G_k(t))$ that is only a function of generation trajectory, the units are allowed to submit a joint generation and ramping cost function $C_k(G_k(t), \dot{G}_k(t))$ in dollar per unit of time, which is a function of both generation and ramping trajectories [31]. Integration of an explicit ramping cost would allow the units to compensate the additional wear and tear cost that they may incur due to more frequent ramping [32]–[34]. In addition, continuous-time modeling of $\mathbf{G}(t)$, $\dot{\mathbf{G}}(t)$ and $\mathbf{I}(t)$ allows us to formulate the *continuous-time UC* problem as follows:

$$\min_{\mathbf{G}(t),\mathbf{I}(t)} \int_{\mathcal{T}} \Big( C\big(\mathbf{G}(t), \dot{\mathbf{G}}(t)\big) + C^I\big(\mathbf{I}(t)\big) \Big)\, dt, \quad (5)$$

$$\text{s.t.} \quad f\big(\mathbf{G}(t), \mathbf{I}(t)\big) = 0, \qquad (\lambda(t)), \quad t \in \mathcal{T}, \quad (6)$$

$$\mathbf{h}\big(\mathbf{G}(t), \dot{\mathbf{G}}(t), \mathbf{I}(t)\big) \le 0, \quad (\boldsymbol{\gamma}(t)), \quad t \in \mathcal{T}, \quad (7)$$

where $C(\mathbf{G}(t), \dot{\mathbf{G}}(t)) = \sum_K C_k(G_k(t), \dot{G}_k(t))$, and $\lambda(t)$ and $\boldsymbol{\gamma}(t)$ are respectively the continuous-time Lagrange multiplier trajectories associated with continuous-time equality and inequality constraints (6), (7). The ability to capture the ideal flexibility of generating units through continuous-time UC model (5)-(7) would allow us to flexibly schedule the units to balance the continuous-time shape of net-load over $\mathcal{T}$. However, the continuous-time UC model (5)-(7) is an infinite-dimensional computationally-intractable variational problem. In our previous work [29], we proposed a solution method for the continuous-time UC problem (5)-(7) where spline function space of Bernstein polynomials are utilized to model the continuous-time trajectories and recast the variational problem into a MILP problem with finite-dimensional decision space. Coefficients of projecting the continuous-time trajectories in the function space of Bernstein polynomials represent the decision variables of the resulting MILP problem [29].

In this paper, we aim to develop the fundamental mathematical theory to define the continuous-time marginal price associated with the continuous-time UC model. Assume that we obtain the optimal solutions $\mathbf{G}^*(t)$, $\dot{\mathbf{G}}^*(t)$ and $\mathbf{I}^*(t)$ of the continuous-time UC using the function space method proposed in [29]. We adapt the approach used in [17] and fix binary variables in the continuous-time UC (5)-(7) to their optimal values $\mathbf{I}^*(t)$, and define the *continuous-time ED* problem:

$$\min_{\mathbf{G}(t)} \quad J(\mathbf{G}(t)) = \int_{\mathcal{T}} C\big(\mathbf{G}(t), \dot{\mathbf{G}}(t)\big) dt, \quad (8)$$

$$\text{s.t.} \quad \mathbf{1}^T \mathbf{G}(t) = D(t), \qquad (\lambda(t)), \quad t \in \mathcal{T}, \quad (9)$$

$$\underline{\mathbf{G}}(t) \le \mathbf{G}(t) \le \overline{\mathbf{G}}(t), \quad (\underline{\boldsymbol{\nu}}(t), \overline{\boldsymbol{\nu}}(t)), \quad t \in \mathcal{T}, \quad (10)$$

$$\underline{\dot{\mathbf{G}}}(t) \le \dot{\mathbf{G}}(t) \le \overline{\dot{\mathbf{G}}}(t), \quad (\underline{\boldsymbol{\mu}}(t), \overline{\boldsymbol{\mu}}(t)), \quad t \in \mathcal{T}, \quad (11)$$

$$\mathbf{G}(0) = \mathbf{G}^0, \quad (12)$$

where $\underline{\mathbf{G}}(t) = \big(\underline{G}_1 I_1^*(t), \ldots, \underline{G}_K I_K^*(t)\big)^T$ and $\overline{\mathbf{G}}(t) = \big(\overline{G}_1 I_1^*(t), \ldots, \overline{G}_K I_K^*(t)\big)^T$ are respectively the constant continuous-time lower and upper capacity bounds, with $\underline{G}_k$ and $\overline{G}_k$ representing the minimum and maximum capacities of unit $k$; $\underline{\dot{\mathbf{G}}}(t) = \big(\underline{\dot{G}}_1 I_1^*(t), \ldots, \underline{\dot{G}}_K I_K^*(t)\big)^T$ and $\overline{\dot{\mathbf{G}}}(t) = \big(\overline{\dot{G}}_1 I_1^*(t), \ldots, \overline{\dot{G}}_K I_K^*(t)\big)^T$ are respectively the constant continuous-time lower and upper ramping bounds, where $\underline{\dot{G}}_k$ and $\overline{\dot{G}}_k$ represent the minimum and maximum ramping limits of unit $k$. Note that the cost term $C^I(\mathbf{I}^*(t))$ in the UC objective functional (5) becomes constant as we fix the integer variables, and thus does not appear in the objective funcional (8). In addition, the minimum on/off time constraints that are purely dependent on the integer variables would become redundant as we fix the integer variables, and thus are not included in the continuous-time ED problem.

The optimization problem (8)-(12) is a constrained variational problem, where (8) represents the objective functional to be minimized over $\mathcal{T}$. The continuous-time power balance constraint is formulated in (9), and (10)-(11) confine the generation and ramping trajectories between their minimum and maximum limits over the scheduling horizon. The vector of the generation trajectories at time zero, $\mathbf{G}(0)$, is set to the vector of initial values, $\mathbf{G}^0$, in (12). The distinct feature of the constraints (9)-(12) is that they are enforced in every instant of time over the scheduling horizon $\mathcal{T}$ and are called variational constraints. As a result, the associated Lagrange multipliers $\lambda(t)$, $\underline{\boldsymbol{\nu}}(t)$, $\overline{\boldsymbol{\nu}}(t)$, $\underline{\boldsymbol{\mu}}(t)$ and $\overline{\boldsymbol{\mu}}(t)$ are also continuous-time trajectories defined over $\mathcal{T}$. Using the optimality conditions of the continuous-time ED problem developed next, we will prove in Section III that the Lagrange multiplier trajectory $\lambda(t)$ defines the continuous-time marginal electricity price.

### B. Derivation of Optimality Conditions

Here we intend to derive the necessary and sufficient optimality conditions for the continuous-time ED problem (8)-(12). The underlying assumptions for the derivations are: 1) generation trajectories $\mathbf{G}(t)$ are assumed to be $C^1$ (continuously differentiable) functions of $t$. The physical implication is that the inertia of the rotating parts of generating units avert the abrupt changes in generation. The proposed solution method in [29] ensures $C^1$ continuity of the continuous-time capacity bounds $\underline{\mathbf{G}}(t)$ and $\overline{\mathbf{G}}(t)$ in (10), thus ensuring $C^1$ continuity of $\mathbf{G}(t)$ over $\mathcal{T}$ including startup and shutdown intervals; 2) cost functions of generation units are independent of each other; 3) the cost functions are $C^1$ and monotonically increasing convex functions of their arguments; 4) the cost functions are not explicit functions of $t$. We have derived the optimality conditions of a generic constrained variational problem in Appendix A. In the following, we present the optimality conditions for the problem (8)-(12), and where required, we refer to the corresponding derivation in the Appendix A.

Let us first form the Lagrangian associated with the variational problem in (8)-(12) as:

$$\begin{aligned}
&\mathcal{L}\big(\mathbf{G}(t), \dot{\mathbf{G}}(t), \lambda(t), \underline{\boldsymbol{\nu}}(t), \overline{\boldsymbol{\nu}}(t), \underline{\boldsymbol{\mu}}(t), \overline{\boldsymbol{\mu}}(t)\big) = \\
&\quad C\big(\mathbf{G}(t), \dot{\mathbf{G}}(t)\big) + \lambda(t)\big(D(t) - \mathbf{1}^T \mathbf{G}(t)\big) \\
&\quad + \underline{\boldsymbol{\nu}}^T(t)\big(\underline{\mathbf{G}}(t) - \mathbf{G}(t)\big) + \overline{\boldsymbol{\nu}}^T(t)\big(\mathbf{G}(t) - \overline{\mathbf{G}}(t)\big) \\
&\quad + \underline{\boldsymbol{\mu}}^T(t)\big(\underline{\dot{\mathbf{G}}}(t) - \dot{\mathbf{G}}(t)\big) + \overline{\boldsymbol{\mu}}^T(t)\big(\dot{\mathbf{G}}(t) - \overline{\dot{\mathbf{G}}}(t)\big). \quad (13)
\end{aligned}$$

The necessary optimality conditions of the continuous-time ED problem (8)-(12) are derived below.

*1) Euler-Lagrange Equations:* As proved in Theorem A.1, the optimal trajectories $\mathbf{G}^*(t)$ of the continuous-time ED should solve the Euler-Lagrange equations (44). Let us calculate two terms $\frac{\partial \mathcal{L}}{\partial \mathbf{G}(t)}$ and $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{G}}(t)}$ for the Lagrangian (13):

$$\frac{\partial \mathcal{L}}{\partial \mathbf{G}(t)} = \frac{\partial C(\mathbf{G}(t),\dot{\mathbf{G}}(t))}{\partial \mathbf{G}(t)} - \lambda(t)\mathbf{1} - \underline{\boldsymbol{\nu}}(t) + \overline{\boldsymbol{\nu}}(t), \quad (14)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{G}}(t)} = \frac{d}{dt}\left(\frac{\partial C(\mathbf{G}(t),\dot{\mathbf{G}}(t))}{\partial \dot{\mathbf{G}}(t)}\right) - \dot{\underline{\boldsymbol{\mu}}}(t) + \dot{\overline{\boldsymbol{\mu}}}(t). \quad (15)$$

Using (14), (15), the Euler-Lagrange equations are derived as:

$$\frac{\partial C(\mathbf{G}(t),\dot{\mathbf{G}}(t))}{\partial \mathbf{G}(t)} - \frac{d}{dt}\left(\frac{\partial C(\mathbf{G}(t),\dot{\mathbf{G}}(t))}{\partial \dot{\mathbf{G}}(t)}\right)$$
$$- \lambda(t)\mathbf{1} - \underline{\boldsymbol{\nu}}(t) + \overline{\boldsymbol{\nu}}(t) + \dot{\underline{\boldsymbol{\mu}}}(t) - \dot{\overline{\boldsymbol{\mu}}}(t) = 0. \quad (16)$$

The Euler-Lagrange equations represent the first-order necessary condition for local optimum of the variational problems, analogous to the condition that the partial derivatives are zero at a local extreme point in static optimization. The Euler-Lagrange equations (16) represent a set of $K$ differential equations that is to be solved over the entire scheduling horizon $\mathcal{T}$ in order to calculate the optimal generation trajectories and the Lagrange multiplier trajectories. Solution of the $K$ Euler-Lagrange equations (16) would require $2K$ boundary values. The first $K$ boundary values are provided by the initial values of generation trajectories in (12), and the second $K$ boundary values are set by the *transversality conditions* (17) below.

*2) Transversality Conditions:* In the power system operation problems, the generation trajectories are usually free-ended; this means there is not any specific boundary value condition for single units that needs to be met at the end of the scheduling horizon. In this case, as mentioned in the Remark in Appendix A.1), the optimal generation trajectories should also satisfy the *transversality conditions* expressed as follows:

$$\left.\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{G}}(t)}\right|_{t=T} = 0. \quad (17)$$

*3) Complimentarity Slackness Conditions:* As discussed in Appendix A.2), the inequality constraints (10) and (11) and the associated Lagrange multipliers should satisfy the complimentarity slackness conditions as follows:

$$\underline{\nu}_k(t)(\underline{G}_k(t) - G_k(t)) = 0, \;\; \underline{\nu}_k(t) \geq 0, \;\; \forall k, \forall t \in \mathcal{T}, \quad (18)$$
$$\overline{\nu}_k(t)(G_k(t) - \overline{G}_k(t)) = 0, \;\; \overline{\nu}_k(t) \geq 0, \;\; \forall k, \forall t \in \mathcal{T}, \quad (19)$$
$$\underline{\mu}_k(t)(\underline{\dot{G}}_k(t) - \dot{G}_k(t)) = 0, \;\; \underline{\mu}_k(t) \geq 0, \;\; \forall k, \forall t \in \mathcal{T}, \quad (20)$$
$$\overline{\mu}_k(t)(\dot{G}_k(t) - \overline{\dot{G}}_k(t)) = 0, \;\; \overline{\mu}_k(t) \geq 0, \;\; \forall k, \forall t \in \mathcal{T}. \quad (21)$$

The complimentarity slackness conditions ensure that the Lagrange multiplier associated with an inequality constraint is either zero when the constraint is not binding, or is a non-negative number when the constraint is binding.

*4) Original Problem Constraints:* The optimal trajectories, of course, should satisfy all the problem constraints (9)-(12).

Suppose that the optimal trajectories $\mathbf{G}^*(t)$, $\dot{\mathbf{G}}^*(t)$ satisfy the Euler-Lagrange equations (16), the transversality condition (17), and the complimentarity slackness conditions (18)-(21). The convexity assumption of cost functions $C_k(G_k(t),\dot{G}_k(t))$ provides the *sufficient condition* that the trajectories are globally optimal solution of the problem (8)-(12).

## III. Continuous-time Marginal Electricity Price

We define the continuous-time marginal electricity price in the following theorem.

**Theorem III.1** (Continuous-time Marginal Electricity Price)**.** *Let $\mathbf{G}^*(t)$ and $J(\mathbf{G}^*(t))$ be the optimal generation trajectories and the optimal objective functional value of the problem* (8)-(12)*. The optimal Lagrange multiplier trajectory $\lambda(t)$ associated with the variational power balance constraint* (9) *is the rate at which the objective functional is changed due to an incremental change in load at time $t$, and is defined as the continuous-time marginal electricity price.*

*Proof.* Let $D(t)$ be incremented by an infinitesimally small and localized $C^1$ trajectory, $\delta D(t)$, which takes positive values in $(\tau, \tau{+}\delta t)$ and vanishes to zero at $t{=}\tau$ and $t{=}\tau{+}\delta t$, where $\tau \in \mathcal{T}$. This incremental variation is sufficiently small that an optimal solution still exists and involves the same binding inequality constraints, i.e., the incremental load variation $\delta D(t)$ results in an incremental change to the optimal trajectories $\mathbf{G}^*(t)$ and $\dot{\mathbf{G}}^*(t)$, the operation costs $C(\mathbf{G}(t),\dot{\mathbf{G}}(t))$, and the total objective functional $J$. Thus, we express the optimal value of the objective functional as a continuously differentiable function of load trajectory, i.e., $J^*(D(t))$. The task here is to calculate the rate of change of $J^* \equiv J^*(D(t))$ due to the load variation $\delta D(t)$. We first calculate the incremental change in $J^*$ due to incremental load variation $\delta D(t)$:

$$\begin{aligned}\triangle J^* &= J^*\left(D(t)+\delta D(t)\right) - J^*(D(t)) \\ &= \frac{\partial J^*(D(t))}{\partial D(t)}\delta D(t) + \mathbf{O}(\|\delta D(t)\|),\end{aligned} \quad (22)$$

where $\|\delta D(t)\|$ is the $L_\infty$ norm of $\delta D(t)$, and $\mathbf{O}(\|\delta D(t)\|)$ denotes its higher order functions that tend to zero faster than $\|\delta D(t)\|$. We neglect this term in the right hand side of (22) and substitute $J^*(D(t)) = \int_\mathcal{T} \mathcal{L}^* dt$, where the Lagrangian is defined in (13):

$$\triangle J^* = \int_\mathcal{T}\Bigg(\frac{\partial C(\mathbf{G}(t),\dot{\mathbf{G}}(t))}{\partial D(t)} + \lambda(t) - \lambda(t)\mathbf{1}^T\frac{\partial \mathbf{G}(t)}{\partial D(t)} +$$
$$(\overline{\boldsymbol{\nu}}(t) - \underline{\boldsymbol{\nu}}(t))^T\frac{\partial \mathbf{G}(t)}{\partial D(t)} + (\overline{\boldsymbol{\mu}}(t) - \underline{\boldsymbol{\mu}}(t))^T\frac{\partial \dot{\mathbf{G}}(t)}{\partial D(t)}\Bigg)\delta D(t)dt. \quad (23)$$

Applying the total derivative to the first term of (23) and rearranging the terms, we have:

$$\triangle J^* = \int_\mathcal{T}\Bigg(\left[\frac{\partial C(\mathbf{G}(t),\dot{\mathbf{G}}(t))}{\partial \mathbf{G}(t)} - \lambda(t)\mathbf{1} + \overline{\boldsymbol{\nu}}(t) - \underline{\boldsymbol{\nu}}(t)\right]^T\frac{\partial \mathbf{G}(t)}{\partial D(t)}$$
$$+ \left[\frac{\partial C(\mathbf{G}(t),\dot{\mathbf{G}}(t))}{\partial \dot{\mathbf{G}}(t)} + \overline{\boldsymbol{\mu}}(t) - \underline{\boldsymbol{\mu}}(t)\right]^T\frac{\partial \dot{\mathbf{G}}(t)}{\partial D(t)} + \lambda(t)\Bigg)\delta D(t)dt. \quad (24)$$

The incremental load trajectory $\delta D(t)$ takes positive values in $(\tau, \tau+\delta t)$ and equals zero in $[0,\tau]\cup[\tau+\delta t, T]$. Thus, $\delta D(t)$ uniformly tends to $\|\delta D(t)\|$ in $(\tau, \tau+\delta t)$ when $\|\delta D(t)\|$ is sufficiently small. Using the integration by parts, changing the limits of the integral, and taking $\|\delta D(t)\|$ out of the integral, (24) is rewritten as:

$$\begin{aligned}\triangle J^* = \|\delta D(t)\| \int_{\tau}^{\tau+\delta t} \Bigg( \lambda(t) + \Bigg[ \frac{\partial C(\mathbf{G}(t),\dot{\mathbf{G}}(t))}{\partial \mathbf{G}(t)} - \lambda(t)\mathbf{1} + \overline{\boldsymbol{\nu}}(t) \\ - \underline{\boldsymbol{\nu}}(t) - \frac{d}{dt}\Bigg( \frac{\partial C(\mathbf{G}(t),\dot{\mathbf{G}}(t))}{\partial \dot{\mathbf{G}}(t)} + \overline{\boldsymbol{\mu}}(t) - \underline{\boldsymbol{\mu}}(t) \Bigg) \Bigg]^T \frac{\partial \mathbf{G}(t)}{\partial D(t)} \Bigg) dt \\ + \Bigg( \frac{\partial C(\mathbf{G}(t),\dot{\mathbf{G}}(t))}{\partial \dot{\mathbf{G}}(t)} + \overline{\boldsymbol{\mu}}(t) - \underline{\boldsymbol{\mu}}(t) \Bigg)^T \frac{\partial \mathbf{G}(t)}{\partial D(t)} \Bigg|_{t=\tau}^{t=\tau+\delta t} \|\delta D(t)\|. \end{aligned} \tag{25}$$

The second term in the right hand side integral of (25) repeats the Euler-Lagrange equation (16) and thus is zero. The last term also becomes zero when $\|\delta D(t)\|$ tends to zero. Thus, (25) becomes:

$$\triangle J^* = \|\delta D(t)\| \int_{\tau}^{\tau+\delta t} \lambda(t) dt = \lambda(\tau)\|\delta D(t)\|\delta t. \tag{26}$$

Dividing (26) by the product of $\|\delta D(t)\|$ and $\delta t$ and taking the limits we reach the theorem result and conclude our proof:

$$\lim_{\substack{\delta t\to 0 \\ \|\delta D(t)\|\to 0}} \frac{\triangle J^*}{\|\delta D(t)\|\delta t} = \lim_{\delta t \to 0} \frac{\delta J^*}{\delta t} = \lambda(\tau), \tag{27}$$

where $\delta J^*$ is the first variation of the optimal objective functional with respect to the incremental variation in load:

$$\delta J^* = \lim_{\|\delta D(t)\|\to 0} \frac{\triangle J^*}{\|\delta D(t)\|}. \tag{28}$$

□

The continuous-time marginal electricity price $\lambda(t)$ defined in Theorem III.1, in dollar per MW in unit of time, represents the cost of supplying the incremental load variation at time $t$.

### *A. Calculation of Continuous-time Marginal Electricity Price*

As shown for the hypothetical generation and ramping trajectories in Fig. 1, at every time instant $t \in \mathcal{T}$, units supplying the load may belong to one of the groups below:

– *Unconstrained Units*: the units that their capacity and ramping constraints are not binding and thus can flexibly change their generation and ramping (e.g., time periods 1, 3, 5, 7 in Fig. 1). According to the complimentary slackness conditions (18)-(21), the multipliers $\underline{\nu}_k(t)$, $\overline{\nu}_k(t)$, $\underline{\mu}_k(t)$ and $\overline{\mu}_k(t)$ are zero for the unconstrained units. We show these units by the time-varying set $K_t^u \equiv K^u(t)$.
– *Ramp-constrained Units*: the units with binding down/up ramping constraints (11) that can change their generation with the constant limited down/up ramp rate (e.g., time periods 4, 8 in Fig. 1). According to the complimentary slackness conditions (20)-(21), the multipliers $\underline{\mu}_k(t)$ or $\overline{\mu}_k(t)$ are non-negative numbers for these units. We show these units by the time-varying set $K_t^r \equiv K^r(t)$.
– *Capacity-constrained Units*: the units with binding maximum/minimum capacity constraints (10) that cannot increase/decrease their generation (e.g., time periods 2, 6 in Fig. 1). According to the complimentary slackness conditions (18)-(19), the multipliers $\overline{\nu}_k(t)$ or $\underline{\nu}_k(t)$ are non-negative numbers for these units. We show these units by time-varying set $K_t^c \equiv K^c(t)$.

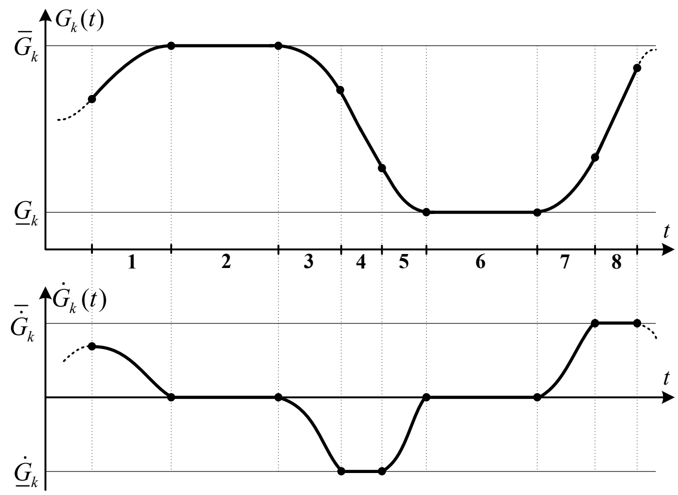


Fig. 1. Operating states of generating units

We aim to derive the value of $\lambda(t)$ from (23). The incremental load variation $\delta D(t)$ at time $t \in \mathcal{T}$ is compensated flexibly by the unconstrained units, and by the ramp-constrained units with a constant rate. Capacity-constrained generating units, however, cannot contribute to compensate the incremental load variation, meaning that $\frac{\partial \mathbf{G}(t)}{\partial D(t)}$ is zero for these units. Further, the Lagrange multipliers $\underline{\nu}_k(t)$ and $\overline{\nu}_k(t)$ are equal to zero for the unconstrained and ramp-constrained units. Thus, the term $(\overline{\boldsymbol{\nu}}(t) - \underline{\boldsymbol{\nu}}(t))^T \frac{\partial \mathbf{G}(t)}{\partial D(t)}$ in (23) is uniformly equal to zero over $\mathcal{T}$. With similar reasoning, the term $\big(\overline{\boldsymbol{\mu}}(t) - \underline{\boldsymbol{\mu}}(t)\big)^T \frac{\partial \dot{\mathbf{G}}(t)}{\partial D(t)}$ in (23) would be equal to zero over $\mathcal{T}$. Besides, power balance constraint (9) requires that $\mathbf{1}^T \frac{\partial \mathbf{G}(t)}{\partial D(t)}$ be equal to 1 in (23). Exerting these substitutions in (23), we follow similar technique used in the proof of Theorem III.1 (i.e., changing the integral limits, replacing $\delta D(t)$ with $\|\delta D(t)\|$, and taking the limits where $\|\delta D(t)\|$ and $\delta t$ tend to zero), and derive the closed-form value of $\lambda(t)$ as:

$$\begin{aligned}\lambda(t) = \left( \frac{\partial C(\mathbf{G}(t),\dot{\mathbf{G}}(t))}{\partial \mathbf{G}(t)} \right)^T \frac{\partial \mathbf{G}(t)}{\partial D(t)} \\ + \left( \frac{\partial C(\mathbf{G}(t),\dot{\mathbf{G}}(t))}{\partial \dot{\mathbf{G}}(t)} \right)^T \frac{\partial \dot{\mathbf{G}}(t)}{\partial D(t)}. \end{aligned} \tag{29}$$

Defining $IC_k^G(t)$ as the *incremental generation cost rate* and $IC_k^{\dot{G}}(t)$ as the *incremental ramping cost rate* of unit $k$:

$$IC_k^G(t) \triangleq \frac{\partial C_k(G_k(t),\dot{G}_k(t))}{\partial G_k(t)}, \tag{30}$$

$$IC_k^{\dot{G}}(t) \triangleq \frac{\partial C_k(G_k(t),\dot{G}_k(t))}{\partial \dot{G}_k(t)}, \tag{31}$$

we can further expand (29), eliminate the zero terms, and rearrange the remaining terms in summations as follows:

$$\lambda(t) = \sum_{k \in (K_t^u \cup K_t^r)} IC_k^G(t) \frac{\partial G_k(t)}{\partial D(t)} + \sum_{k \in K_t^u} IC_k^{\dot{G}}(t) \frac{\partial \dot{G}_k(t)}{\partial D(t)}, \quad t \in \mathcal{T}, \tag{32}$$

where $\frac{\partial G_k(t)}{\partial D(t)}$ and $\frac{\partial \dot{G}_k(t)}{\partial D(t)}$ are respectively the generation and ramping variations of unit $k$ contributing towards balancing the incremental load variation at time $t$. Similar to the current definition of incremental generation cost rate, the incremental ramping cost rate $IC_k^{\dot{G}}(t)$ represents the cost of incremental change in ramping of unit $k$ at time $t$. Accordingly, the continuous-time marginal electricity price $\lambda(t)$ in (32) equals to the weighted average of the incremental generation cost rates of the unconstrained and the ramp-constrained units, plus the weighted average of the incremental ramping cost rates of the unconstrained units.

**Corollary 1.** *In the presence of explicit terms in cost function for valuating the ramping of the generating units, the continuous-time marginal electricity price is not only a function of the generation and the incremental generation cost rate of the units, but also a function of the ramping and the incremental ramping cost rate of the units. This result emphasizes that if the continuous-time generation and ramping trajectories of the units are explicitly modeled and valuated in the economic operation planning of power systems, the resulting Lagrange multiplier trajectory $\lambda(t)$ of the variational power balance constraint embeds the impact of ramping costs in the continuous-time marginal price of electricity.*

The corollary 1 of (32) presents a formal mathematical approach to factor the ramping costs and capability of generating units in power systems operation. In addition, (32) presents a mathematically proven approach to merit units based on their ramping capability, and award those that are better able to assist power systems in compensating the net-load ramping.

**Corollary 2.** *In the presence of explicit ramping costs,* (32) *defines a new criterion for the* marginal *generating unit in power system operation, where units merit the others and become marginal not only for their less incremental generation cost rate, but also for their less incremental ramping cost rate. Suppose there are two units with the same incremental generation cost rate. The unit with less incremental ramping cost rate would offer an overall less incremental cost, and thus becomes marginal and sets a lower price for electricity in* (32). *In addition,* (32) *mathematically explains the situation when the unavailability of ramping capacity from the cheaper units would increase the marginal electricity price.*

Assuming that the cost function of units is merely a function of their generation trajectory, the second term in (32) is zero and the continuous-time marginal electricity price becomes:

$$\lambda(t) = \sum_{k \in (K_t^u \cup K_t^r)} IC_k^G(t) \frac{\partial G_k(t)}{\partial D(t)}, \quad \forall t \in \mathcal{T}, \tag{33}$$

which states that the marginal electricity price at the continuous-time $t$ equals to the weighted average of the incremental generation cost rates of the unconstrained and ramp-constrained generating units. While the current discrete-time calculation of marginal prices using sequential solution of single-period ED problems may fall short to discern the price implications of inter-temporal ramping constraints as well as the fast sub-hourly rampings of the net-load, the continuous-time marginal electricity price $\lambda(t)$ in (32) (and in (33)) would accurately reflect the underlying ramping constraints that govern the operation of generating units. Thus, the continuous-time marginal electricity price would provide a more accurate price signal based on the actual continuous-time loading and ramping condition of power systems.

Calculation of the continuous-time marginal electricity price in (32) is an ex-post analysis to the continuous-time UC problem (5)-(7). We first solve the continuous-time UC using the MILP formulation developed in [29]. We then fix the binary variables to the optimal values $\mathbf{I}^*(t)$, and solve the ensuing LP formulation of the continuous-time ED (8)-(12) in the function space of Bernstein polynomials. The optimal solution of Bernstein coefficients would be utilized to reconstruct the optimal generation and ramping trajectories $\mathbf{G}^*(t)$ and $\dot{\mathbf{G}}^*(t)$ of the units, which then are plugged in to (32) in order to calculate the continuous-time marginal electricity price.

### B. Calculation of the Other Lagrange Multipliers

In addition to the Lagrange multiplier $\lambda(t)$, we can also calculate the Lagrange multipliers associated with the binding ramping or capacity constraints of the units.

*1) Lagrange Multipliers of the Capacity Constraints:* Suppose that, at time $t$, generating unit $k$ has reached one of its minimum or maximum generation capacity limits. The corresponding non-negative Lagrange multipliers $\underline{\nu}_k(t)$ or $\overline{\nu}_k(t)$ of the binding capacity constraints can be calculated using the Euler-Lagrange equation (16) as:

$$\underline{\nu}_k(t) = IC_k^G(t) - \frac{d}{dt}\left(IC_k^{\dot{G}}(t)\right) - \lambda(t), \tag{34}$$

$$\overline{\nu}_k(t) = \lambda(t) - IC_k^G(t) + \frac{d}{dt}\left(IC_k^{\dot{G}}(t)\right), \tag{35}$$

where $\lambda(t)$ is calculated in (32). The multipliers $\underline{\nu}_k(t)$ and $\overline{\nu}_k(t)$ are respectively the sensitivity of the optimal cost functional $J^*$ to the incremental changes in the value of minimum and maximum capacities of unit $k$ at time $t$. In (34), the positivity of $\underline{\nu}_k(t)$ implies that when the unit is scheduled at minimum capacity at time $t$, its incremental generation cost rate minus the time derivative of its incremental ramping cost rate is more than the marginal price at that time. In addition, the positivity of $\overline{\nu}_k(t)$ in (34) implies that the incremental generation cost rate of the unit generating at maximum capacity minus the time derivative of its incremental ramping cost rate is less than the marginal price at that time.

*2) Lagrange Multipliers of the Ramping Constraints:* Suppose that generation unit $k$ reaches one of its down or up ramping limits at time $t_r$, but not any of the capacity constraints. This case represents the situation that leads to higher electricity prices, when a cheaper unit still has enough generation capacity, but not enough ramping capability to provide the ramping requirement of net-load. In this case, the continuous-time ED problem would schedule additional unit(s) to cater for the ramping requirement of the system, and thus the marginal price would be set by the more expensive unit. From the generating units point of view, the implication is that owning cheaper generation capacity does not ensure to stay competitive in the electricity market, and the units may lose the opportunity to generate because they do not offer competitive ramping capability in the market. This would provide a natural competency for the generating units (and possibly storage devices) with higher ramping capability.

The corresponding non-negative Lagrange multipliers $\underline{\mu}_k(t)$ or $\overline{\mu}_k(t)$ of the binding ramping constraints can be calculated using the Euler-Lagrange equation (16) for $t \geq t_r$:

$$\underline{\mu}_k(t) = IC_k^{\dot{G}}(t) - \int_{t_r}^{t} \left(IC_k^G(t) - \lambda(t)\right) dt, \tag{36}$$

$$\overline{\mu}_k(t) = \int_{t_r}^{t} \left(IC_k^G(t) - \lambda(t)\right) dt - IC_k^{\dot{G}}(t), \tag{37}$$

where $\lambda(t)$ is calculated in (32). The multipliers $\underline{\mu}_k(t)$ and $\overline{\mu}_k(t)$ are respectively the sensitivity of the optimal cost functional $J^*$ to the incremental changes in the value of the down/up ramping limits of unit $k$ at time $t$.

Note that the case of both generation capacity and ramping constraints being binding only happens in single instants of time, i.e., when a generation trajectory ramps up from its minimum generation with its maximum ramping up capability, or when a generation unit ramps down from its maximum generation with its maximum ramping down capability.

## IV. Numerical Results

The generating units of the IEEE Reliability Test System (RTS) [35] are used to implement the proposed continuous-time pricing model and compare it to the discrete-time approach. The continuous-time load is constructed using the five-minute net-load forecast data of CAISO for Jan. 4, 2016 [36], which is scaled down for the IEEE-RTS peak load of 2850 MW. The value of continuous-time load at mid-point of each hour is used as the hourly load in the traditional hourly UC models. Likewise, the value of continuous-time load at mid-point of each half-hourly interval is considered as the half-hourly load. As seen in Fig. 2, the hourly and half-hourly approximations of the load miss the sub-hourly load variations and rampings. Three cases are studied below.

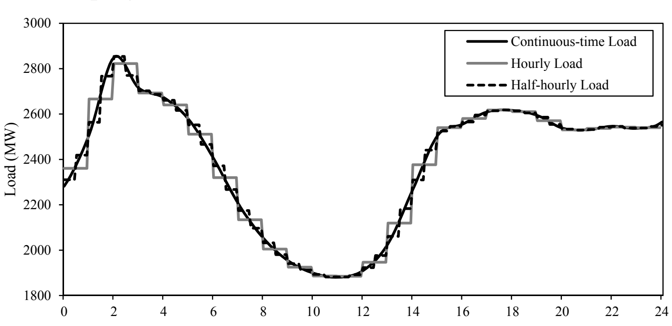


Fig. 2. Continuous-time, hourly, and half-hourly load curves

*Case 1*: In Case 1, we compare the results of continuous-time scheduling and pricing of electricity, with those of the traditional hourly and half-hourly approaches. The ramping costs of the cost functions are assumed to be zero in this case. We first run the continuous-time, hourly and half-hourly UC models to determine the optimal schedule and commitment status of the units. The total operation costs for the three models are respectively $\$478,071$, $\$476,902$ and $\$476,895$. The continuous-time UC dispatches more energy than the other models conveying the highest operation cost. Even though the half-hourly UC dispatches more energy compared to the hourly counterpart, additional commitment variables of the half-hourly model provide higher flexibility to reduce the fixed operation costs through half-hourly startup and shutdowns. Therefore the operation costs are almost the same for the hourly and half-hourly models.

In order to calculate the continuous-time marginal price, we fix the binary variables in the continuous-time UC model to their optimum values, solve the ensuing continuous-time ED problem, and calculate the continuous-time price using (33). In addition, we fix the binary variables to their optimum values in the hourly and half-hourly UC models, and sequentially solve 24 and 48 single-period ED problems, where the Lagrange multipliers of the power balance constraints form the discrete-time marginal prices. The continuous-time and discrete-time (hourly and half-hourly) marginal prices are compared in Fig. 3. As expected, the price curves in Fig. 3 follow the shapes of load curves in Fig. 2. While the hourly and half-hourly prices provide a single price for each time interval, the continuous-time price changes constantly over time reflecting the time-varying load and generation schedule. The continuous-time UC model commits an additional 12 MW unit at hour 3, which is used to supply the sub-hourly load not captured by the hourly and half-hourly models. This results in a minor peak in the continuous-time price at the time which is not captured by the other two price curves in Fig. 3.

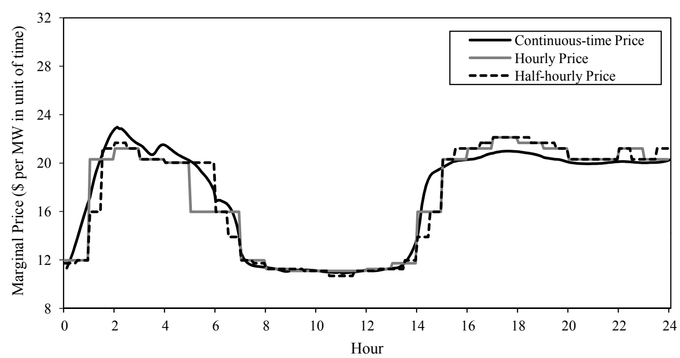


Fig. 3. Marginal electricity prices: Case 1

*Case 2*: In Case 2, we study the impacts of ramping scarcity on the marginal prices of electricity. For this reason, we reduce the ramping limits of the units by a factor of 2.5, rerun the models and recalculate the continuous-time and discrete-time prices. The operation costs are increased in this case by $\$1,084$, $\$420$, and $\$473$ respectively for the continuous-time, hourly and half-hourly UC models, as compared to Case 1. In this case, the continuous-time UC commits additional fast-ramping yet more expensive 20MW units to supply the ramping requirement of the load at hour 3 that stems from sensitivity of the model to the units' tight ramping constraints. Thus, Fig. 4 shows an spike in the continuous-time marginal price trajectory, which is due to the utilization of more expensive units. Finite difference ramping model in the hourly and half-hourly UC, however, is unable to fully recognize the unavailability of adequate ramping resources, and schedules the same generation fleet as in Case 1. While the half-hourly model is more discerning to ramping limitation and schedules the fast-ramping 100MW units half hour more than Case 1 to supply the load ramping when it sharply descends at hour 6, the hourly model merely modifies generation of the units without altering their commitment. Despite the minimal changes in the generation schedules, the hourly and half-hourly marginal prices in Fig. 4 are the same as the prices in Case 1, turning a blind eye to the ramping limitations.

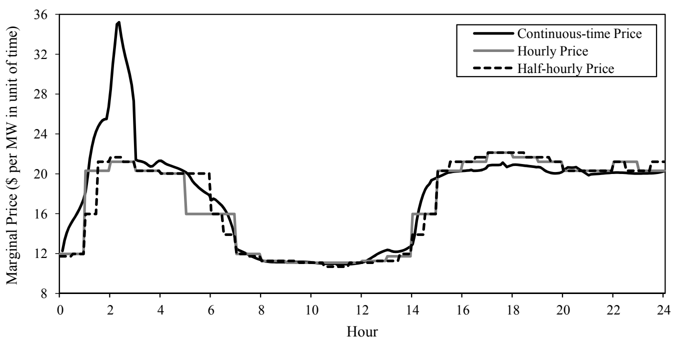


Fig. 4. Marginal electricity prices: Case 2

*Case 3*: In Case 3, we examine the behavior of the continuous-time marginal price in the presence of explicit ramping costs in the objective function, while the ramping limitations are the same as Case 1. The 197 MW, 100 MW, and 20 MW units offer ramping costs in their cost functions and the remaining units provide ramping with zero cost. The ramping cost coefficients are assumed to be five percent of the corresponding generation cost coefficients. The total operation cost in this case is higher than Case 1 by $\$2,689$, which is due to the additional cost of ramping procurement from the units. The continuous-time marginal price trajectory of Case 3 is decomposed in Fig. 5 to its components as in (32). The second component of the marginal price, which reflects the cost of ramping, is non-zero during hours 1-7 and 15-24 due to the deployment of 197 MW and 100 MW units.

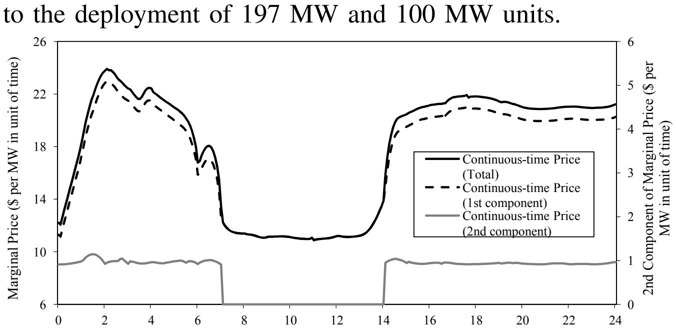


Fig. 5. Continuous-time marginal electricity price: Case 3

## V. Conclusions

This paper presents the mathematical foundation and defines the theory of continuous-time marginal pricing of electricity in day-ahead markets. Using the necessary optimality conditions of the underlying variational problem, we prove that the Lagrange multiplier associated with the variational power balance constraint, in the solution of the continuous-time UC problem where the commitment variables are fixed to their optimal values, presents the continuous-time marginal electricity price. We proved that the continuous-time marginal price is a function of both the incremental generation cost rate and the incremental ramping cost rate of the units contributing towards supplying the incremental load at every instant of time. While the current discrete-time calculation of marginal prices using sequential solution of single-period ED problems may fall short to discern the price implications of inter-temporal ramping constraints as well as the fast sub-hourly rampings of the net-load, the continuous-time marginal electricity price would accurately reflect the underlying ramping constraints that govern the operation of generating units. The numerical results demonstrate that the continuous-time marginal electricity price provides a more accurate price signal reflecting the time-varying loading condition and generation schedules. Although the half-hourly model outperforms the hourly model in the simulation results, it still falls short in reflecting the continuous-time variations of load and the corresponding generation schedules in the marginal prices.

In this paper, we put forth the theory of continuous-time marginal pricing in day-ahead markets, however, design of the market rules, bidding structure, settlement process, and estimating the incremental ramping cost rates of generating units are open research questions to be addressed in future works. In addition, the proposed continuous-time marginal price may not necessarily minimize the uplift payments associated with the non-convexities in the UC problem. More research may be required to enhance the continuous-time pricing model that not only accurately embeds the continuous-time variations of loads, but also minimizes the possible uplift payments.

## Acknowledgment

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## Appendix A
## Necessary Optimality Conditions for Constrained Variational Problems

Suppose that we are interested in finding the optimal values of $C^1$ (continuously differentiable) decision variable trajectories $\mathbf{x}(t) = (x_1(t), \ldots, x_K(t))^T$ that minimize a $C^1$ (continuously differentiable) function $F(\mathbf{x}(t), \dot{\mathbf{x}}(t))$ over a time horizon $\mathcal{T} = [0, T]$, subject to the applicable constraints. This optimization problem can be formulated as a *constrained variational problem* as follows:

$$\min_{\mathbf{x}(t)} \quad J(\mathbf{x}(t)) = \int_{\mathcal{T}} F(\mathbf{x}(t), \dot{\mathbf{x}}(t)) dt \tag{38}$$

$$\text{s.t.} \quad \mathbf{f}(\mathbf{x}(t), \dot{\mathbf{x}}(t)) = 0, \qquad \forall t \in \mathcal{T} \quad (\boldsymbol{\lambda}(t)) \tag{39}$$

$$\mathbf{h}(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \leq 0, \qquad \forall t \in \mathcal{T} \quad (\boldsymbol{\nu}(t)) \tag{40}$$

$$\mathbf{x}(0) = \mathbf{a}, \ \mathbf{x}(T) = \mathbf{b}, \tag{41}$$

where $J(\mathbf{x}(t))$ is the objective functional, and $\mathbf{f}$ and $\mathbf{h}$ are $C^1$ functions respectively representing the vectors of variational (pointwise) equality and inequality constraints; $\mathbf{a}$ and $\mathbf{b}$ are the vectors of constant boundary values of the trajectories; $\boldsymbol{\lambda}(t)$ and $\boldsymbol{\nu}(t)$ are the vectors of piecewise-continuous Lagrange multiplier trajectories associated with the equality and inequality constraints. The optimal values of the trajectories that minimize $J(\mathbf{x}(t))$ is shown by $\mathbf{x}^*(t)$, and belong to the set of admissible functions $\mathcal{X}$ that satisfy the problem constraints in (39)-(41). In the following we first derive the necessary optimality conditions for the problem with variational equality constraints. We then approach the problem with variational inequality constraints.

*1) Variational problems with equality constraints:* Assume that the objective functional (38) is only constrained to the variational equality constraints (39). Let us augment the objective functional (38) as:

$$\min_{\mathbf{x}(t)} \; J(\mathbf{x}(t)) = \int_{\mathcal{T}} \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \boldsymbol{\lambda}(t)) dt, \tag{42}$$

where the Lagrangian function is defined as:

$$\begin{aligned} \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \boldsymbol{\lambda}(t)) = F(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \\ + \boldsymbol{\lambda}^T(t)\mathbf{f}(\mathbf{x}(t), \dot{\mathbf{x}}(t)). \end{aligned} \tag{43}$$

The necessary optimality conditions for the equality-constrained variational problem that minimizes $J(\mathbf{x}(t))$ subject to (39) would be equivalent to those for the unconstrained variational problem (42) [37]. The necessary optimality conditions for the problem (42) are provided in Theorem A.1 [37].

**Theorem A.1.** *Suppose that $F(\mathbf{x}(t), \dot{\mathbf{x}}(t))$ and $\mathbf{f}(\mathbf{x}(t), \dot{\mathbf{x}}(t))$ are $C^1$ functions. A necessary condition for $\mathbf{x}^*(t) \in \mathcal{X}$ to minimize the objective functional* (42) *is that $\mathbf{x}^*(t)$ is a solution to the differential equations:*

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}(t)} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}(t)} = 0, \tag{44}$$

*that is called Euler-Lagrange equations.*

*Proof.* The Euler-Lagrange equation is a result of the first order optimality condition of variational problems, which states that the *first variation* of the augmented objective functional, $\delta J^*(\mathbf{x}(t))$, should be zero at the optimal value of the trajectories [37]. Let $\mathbf{x}(t)$ be an admissible function, and $\delta\mathbf{x}(t)$, called *variation* of $\mathbf{x}(t)$, is an infinitesimally small deviation from $\mathbf{x}(t)$, i.e., $\| \delta\mathbf{x}(t) \| \ll \| \mathbf{x}(t) \|$. We first calculate the variation of $J^*(\mathbf{x}(t))$ with respect to $\delta\mathbf{x}(t)$ as:

$$\begin{aligned} \Delta J^*(\mathbf{x}(t)) &= J^*(\mathbf{x}(t) + \delta\mathbf{x}(t)) - J^*(\mathbf{x}(\mathbf{t})) \\ &= \int_{\mathcal{T}} \mathcal{L}\big((\mathbf{x}(t) + \delta\mathbf{x}(t)), (\dot{\mathbf{x}}(t) + \delta\dot{\mathbf{x}}(t)), \boldsymbol{\lambda}(t)\big) dt \\ &- \int_{\mathcal{T}} \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t), \boldsymbol{\lambda}(t)) dt. \end{aligned} \tag{45}$$

Let us recast (45) in the following by linearizing $\mathcal{L}\big((\mathbf{x}(t)+\delta\mathbf{x}(t)), (\dot{\mathbf{x}}(t)+\delta\dot{\mathbf{x}}(t)), \boldsymbol{\lambda}(t)\big)$ using the first-order Taylor expansion:

$$\Delta J^*(\mathbf{x}(t)) = \int_{\mathcal{T}} \left( \left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}(t)}\right)^T \delta\mathbf{x}(t) + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}(t)}\right)^T \delta\dot{\mathbf{x}}(t) \right) dt. \tag{46}$$

Using integration by parts, the second term in the right hand side integral of (46) turns into:

$$\begin{aligned} \int_{\mathcal{T}} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}(t)}\right)^T \delta\dot{\mathbf{x}}(t) dt = - \int_{\mathcal{T}} \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}(t)}\right)^T \delta\mathbf{x}(t) dt \\ + \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}(t)}\right)^T \delta\mathbf{x}(t)\Big|_{t=0}^{t=T}. \end{aligned} \tag{47}$$

Since the boundary values are constant, the second term in the right hand side of (47) equals to zero, for the variations $\delta\mathbf{x}(t)|_{t=0}$ and $\delta\mathbf{x}(t)|_{t=T}$ are zero. Substituting (47) in (46):

$$\Delta J(\mathbf{x}(t)) = \int_{\mathcal{T}} \left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}(t)} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}(t)}\right)^T \delta\mathbf{x}(t) dt. \tag{48}$$

Using (48), we force the first variation of the objective functional, $\delta J^*(\mathbf{x}(t))$, to be equal to zero at the optimal solution:

$$\begin{aligned} \delta J^*(\mathbf{x}(t)) &= \lim_{\|\delta\mathbf{x}(t)\| \to 0} \frac{\triangle J^*}{\|\delta\mathbf{x}(t)\|} \\ &= \lim_{\|\delta\mathbf{x}(t)\| \to 0} \int_{\mathcal{T}} \left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}(t)} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}(t)}\right)^T \frac{\delta\mathbf{x}(t)}{\|\delta\mathbf{x}(t)\|} dt = 0. \end{aligned} \tag{49}$$

Since $\delta\mathbf{x}(t)$ is chosen arbitrarily, the differential equation in the integral (48) should vanish to zero:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}(t)} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}(t)} = 0. \tag{50}$$

This concludes our proof. $\square$

*Remark:* Suppose the boundary values $\mathbf{x}(T)$ are not specified and the trajectories are free at the end of the horizon. In this case, the second term in the right hand side of (47) is not equal to zero anymore, and, in addition to Euler-Lagrange equation, the optimal trajectories $\mathbf{x}^*(t)$ should also satisfy the following equation that is called *transversality condition*:

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}(t)}\Big|_{t=T} = 0. \tag{51}$$

*2) Variational problems with equality and inequality constraints:* Let us assume that, in addition to the equality constraints, the objective functional (38) is also constrained to the inequality constraints (40). In this case, we define the Lagrangian function as follows:

$$\begin{aligned} \mathcal{L}\big(\mathbf{x}(t), \dot{\mathbf{x}}(t), \boldsymbol{\lambda}(t), \underline{\boldsymbol{\nu}}(t), \overline{\boldsymbol{\nu}}(t), \underline{\boldsymbol{\mu}}(t), \overline{\boldsymbol{\mu}}(t)\big) = F(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \\ + \boldsymbol{\lambda}^T(t)\mathbf{f}(\mathbf{x}(t), \dot{\mathbf{x}}(t)) + \boldsymbol{\nu}^T(t)\mathbf{h}(\mathbf{x}(t), \dot{\mathbf{x}}). \end{aligned} \tag{52}$$

In addition to the Euler-Lagrange equations (44) and the transversality conditions (51), the variational problems with inequality constraints should satisfy additional necessary conditions. The fundamental concept here is that the inequality constraints only restrict the domain of feasibility in the set of admissible functions $\mathcal{X}$ when they are binding. The binding inequality constraint $h(\mathbf{x}(t), \dot{\mathbf{x}}(t)) \leq 0$ act like the equality constraints $h(\mathbf{x}(t), \dot{\mathbf{x}}(t)) = 0$, except that their Lagrange multipliers are non-negative. This is stated in the following equations that is know as *complimentarity slackness condition*:

$$\nu(t)h(\mathbf{x}(t), \dot{\mathbf{x}}(t)) = 0, \quad \nu(t) \geq 0. \tag{53}$$

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