Generalized Gauss maps and integrals for three-component links: toward higher helicities for magnetic fields and fluid flows

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To each three-component link in the 3-sphere we associate a ***generalized Gauss map*** from the 3-torus to the 2-sphere, and show that the pairwise linking numbers and Milnor triple linking number that classify the link up to link homotopy correspond to the Pontryagin invariants that classify its generalized Gauss map up to homotopy. We view this as a natural extension of the familiar situation for two-component links in 3-space, where the linking number is the degree of the classical Gauss map from the 2-torus to the 2-sphere. The generalized Gauss map, like its prototype, is geometrically natural in the sense that it is equivariant with respect to orientation-preserving isometries of the ambient space, thus positioning it for application to physical situations.

When the pairwise linking numbers of a three-component link are all zero, we give an integral formula for the triple linking number analogous to the Gauss integral for the pairwise linking numbers. This new integral is also geometrically natural, like its prototype, in the sense that the integrand is invariant under orientation-preserving isometries of the ambient space. Versions of this integral have been applied by Komendarczyk in special cases to problems of higher order helicity and derivation of lower bounds for the energy of magnetic fields.

We have set this entire paper in the 3-sphere because our generalized Gauss map is easiest to present here, but in a subsequent paper we will give the corresponding maps and integral formulas in Euclidean 3-space.

I. INTRODUCTION

Borromean Rings

In his senior thesis, published in 1954, John Milnor[1] classified three-component links in the 3-sphere S^3 up to ***link homotopy***, a deformation during which each component may cross itself but distinct components must remain disjoint. A complete set of invariants is given by the pairwise linking numbers p, q and r of the components, and by the residue class μ of one further integer modulo the greatest common divisor of p, q and r, Milnor's ***triple linking number***. For example, the Borromean rings shown here have $p = q = r = 0$ and $\mu = \pm 1$, where the sign depends on the ordering and orientation of the components.

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To each three-component link L in S^3 we will associate a ***generalized Gauss map*** g_L from the 3-torus T^3 to the 2-sphere S^2 in such a way that link homotopies of L become homotopies of g_L. The assignment $L \mapsto g_L$ then defines a function

$$g : \mathcal{L}_3 \longrightarrow [T^3, S^2]$$

from the set $\mathcal{L}_3$ of link homotopy classes of three-component links in S^3 to the set $[T^3, S^2]$ of homotopy classes of maps $T^3 \to S^2$, and it will be seen below that g is injective.

Maps from T^3 to S^2 were classified up to homotopy by Lev Pontryagin[2] in 1941. A complete set of invariants is given by the degrees p, q and r of the restrictions to the 2-dimensional coordinate subtori, and by the residue class $\boldsymbol{\nu}$ of one further integer modulo *twice* the greatest common divisor of p, q and r, the ***Pontryagin invariant*** of the map. This invariant is an analogue of the Hopf invariant for maps from S^3 to S^2, and is an absolute version of the relative invariant originally defined by Pontryagin for *pairs* of maps from a 3-complex to S^2 that agree on the 2-skeleton of the domain.

Our first main result, Theorem A below, equates Milnor's and Pontryagin's invariants p, q and r for L and g_L, and asserts that $2\mu(L) = \nu(g_L)$. As a consequence, the function $g : \mathcal{L}_3 \longrightarrow [T^3, S^2]$ above is one-to-one, with image the set of maps of even ν-invariant.

In the special case when $p = q = r = 0$, we derive an explicit and geometrically natural integral formula for the triple linking number, reminiscent of Gauss' classical integral formula for the pairwise linking number. This formula and variations of it are presented in Theorem B below.

In the rest of this introduction, we provide the definition of the generalized Gauss map and give careful statements of Theorems A and B. Some motivation for our work from the mathematical perspective of configuration spaces, and from the physical perspective of fluid mechanics and plasma physics, may be found in Appendix A.

The generalized Gauss map of a three-component link in the 3-sphere

Let x, y and z be three distinct points on the unit 3-sphere S^3 in $\mathbb{R}^4$. They cannot lie on a straight line in $\mathbb{R}^4$, so must span a 2-plane there. Translate this plane to pass through the origin, and then orient it so that the vectors $x - z$ and $y - z$ form a positive basis. The result is an element $G(x, y, z)$ of the Grassmann manifold $\mathrm{G}_2\mathbb{R}^4$ of all oriented 2-planes through the origin in 4-space. This procedure defines the ***Grassmann map***

$$G : \mathrm{Conf}_3 S^3 \longrightarrow \mathrm{G}_2\mathbb{R}^4$$

pictured in Figure 1, where $\mathrm{Conf}_3 S^3 \subset S^3 \times S^3 \times S^3$ is the configuration space of ordered triples of distinct points in S^3. The map G is equivariant with respect to the diagonal SO(4) action on $S^3 \times S^3 \times S^3$ and the usual SO(4) action on $\mathrm{G}_2\mathbb{R}^4$.

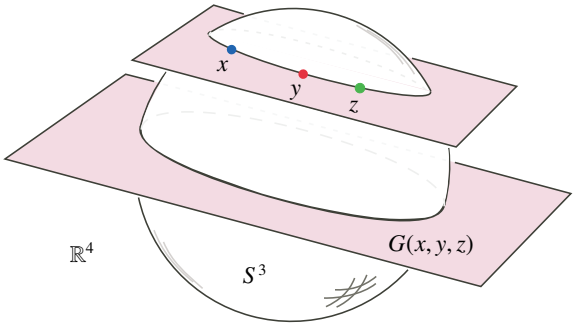


FIG. 1. The Grassmann map $(x, y, z) \mapsto G(x, y, z)$

The Grassmann manifold $\mathrm{G}_2\mathbb{R}^4$ is diffeomorphic to the product $S^2 \times S^2$ of two 2-spheres, as explained in Section IV below. Let π_+ and $\pi_- : \mathrm{G}_2\mathbb{R}^4 \to S^2$ denote the projections to the two factors. One of these will be used in the definition of the generalized Gauss map, but the choice of which one will be seen to be immaterial.

Now let L be a link in S^3 with three parametrized components

$$X = \{x(s) \mid s \in S^1\}\,,\ Y = \{y(t) \mid t \in S^1\} \text{ and } Z = \{z(u) \mid u \in S^1\}$$

as indicated schematically in Figure 2. Here and throughout, we view the parametrizing circle S^1 as the quotient $\mathbb{R}/2\pi\mathbb{Z}$, and assume implicitly that the parametrizing functions $x = x(s)$, $y = y(t)$ and $z = z(u)$ are smooth with nowhere vanishing derivatives.

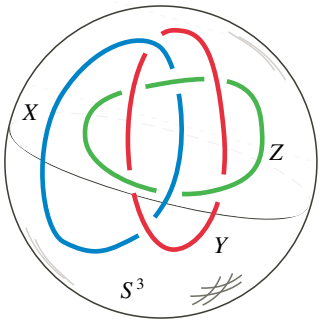


FIG. 2. The link L in S^3

We define the ***generalized Gauss map*** $g_L : T^3 \to S^2$ by the formula

$$g_L(s,t,u) \;=\; \pi_+(G(x(s),y(t),z(u)).$$

where $G : \mathrm{Conf}_3 S^3 \to \mathrm{G}_2\mathbb{R}^4$ is the Grassman map and $\pi_+ : \mathrm{G}_2\mathbb{R}^4 = S^2 \times S^2 \to S^2$ is the projection onto the first factor. Thus g_L is the composition $\pi_+ G e_L$ where $e_L : T^3 \to \mathrm{Conf}_3 S^3$, $(s,t,u) \mapsto (x(s), y(t), z(u))$ is the embedding that parametrizes the link. We regard g_L as a natural generalization of the classical Gauss map $T^2 \to S^2$ associated with a two-component link in $\mathbb{R}^3$.

It is evident that the map g_L is geometrically natural in the sense that it is equivariant with respect to the action of SO(4) on the link L and on the sphere S^2. That is, if $h \in \mathrm{SO}(4)$, then $g_{h(L)} = h \circ g_L$, where h acts on $\mathrm{G}_2\mathbb{R}^4 \cong S^2 \times S^2$ via the double covering $\mathrm{SO}(4) \to \mathrm{SO}(3) \times \mathrm{SO}(3)$, and then accordingly on each S^2 factor.

It is also clear that the homotopy class of g_L is unchanged under reparametrization of L, or more generally under any link homotopy of L. Furthermore, g_L is "symmetric" in that it transforms under any permutation of the components of L by precomposing with the corresponding permutation automorphism of T^3 multiplied by the sign of the permutation.

Statement of results

The first of our two main results gives an explicit correspondence between the Milnor link homotopy invariants of a three-component ordered, oriented link L in the 3-sphere and the Pontryagin homotopy invariants of its generalized Gauss map.

Theorem A. ***Let L be a 3-component link in S^3. Then the pairwise linking numbers p, q and r of L are equal to the degrees of its generalized Gauss map $g_L : T^3 \to S^2$ on the two-dimensional coordinate subtori of T^3, while twice Milnor's μ-invariant for L is equal to Pontryagin's ν-invariant for g_L modulo $2\gcd(p,q,r)$.***

Conventions. (1) In this paper, links in S^3 are always *ordered* (the components are taken in a specific order) and *oriented* (each component is oriented).
(2) The two-dimensional coordinate subtori of T^3 are oriented to have positive intersection with the remaining circle factors.

Our second main result is an explicit and geometrically natural formula for Milnor's triple linking number $\mu(L)$, in the case that all the pairwise linking numbers of L vanish, that is reminiscent of Gauss' classical integral formula for the linking number of a two-component link. This formula will be presented in three versions: first as an integral involving differential forms on the 3-torus, second as the same integral expressed in terms of vector fields, and finally as an infinite sum involving Fourier coefficients.

To state these formulas, we need some definitions. Let ω denote the Euclidean area 2-form on S^2, normalized to have total area 1. Then ω pulls back under the generalized Gauss map g_L to a closed

2-form ω_L on T^3, which can be converted to a divergence-free vector field $\vec{\mathbf{v}}_L$ on T^3 via the usual formula

$$\omega_L(\vec{\mathbf{a}}\,,\vec{\mathbf{b}}) \;=\; (\vec{\mathbf{a}}\times\vec{\mathbf{b}})\bullet\vec{\mathbf{v}}_L.$$

We call ω_L the ***characteristic 2-form*** of L, and $\vec{\mathbf{v}}_L$ its ***characteristic vector field***. When p, q and r are all zero, ω_L is exact and $\vec{\mathbf{v}}_L$ is in the image of curl. In Section VII we give explicit formulas for ω_L and $\vec{\mathbf{v}}_L$, and in Section VIII for the fundamental solution φ of the scalar Laplacian on T^3, which appear in the integral formulas for $\mu(L)$.

For the third version of our formula for the Milnor invariant, we need to express the characteristic 2-form and vector field in terms of Fourier series on the 3-torus. To that end, view T^3 as the quotient $(\mathbb{R}/2\pi\mathbb{Z})^3$ and write $\mathbf{x} = (s,t,u) \in \mathbb{R}^3$ for a general point there. Using the complex form of Fourier series, express

$$\omega_L \;=\; \sum_{\mathbf{n}\in\mathbb{Z}^3}\Big(c^s_{\mathbf{n}}\,dt\wedge du + c^t_{\mathbf{n}}\,du\wedge ds + c^u_{\mathbf{n}}\,ds\wedge dt\Big)\,e^{i\mathbf{n}\bullet\mathbf{x}} \;=\; \sum_{\mathbf{n}\,\in\,\mathbb{Z}^3}\mathbf{c}_{\mathbf{n}}\,e^{i\mathbf{n}\bullet\mathbf{x}}\bullet\;\star d\mathbf{x}$$

where $\mathbf{c}_{\mathbf{n}} = (c^s_{\mathbf{n}}, c^t_{\mathbf{n}}, c^u_{\mathbf{n}})$, $d\mathbf{x} = (ds,dt,du)$ and $\star d\mathbf{x} = (dt\wedge du, du\wedge ds, ds\wedge dt)$ in the final expression. It follows that $\vec{\mathbf{v}}_L = \sum_{\mathbf{n}\,\in\,\mathbb{Z}^3}\mathbf{c}_{\mathbf{n}}\,e^{i\mathbf{n}\bullet\mathbf{x}}\bullet\partial_{\mathbf{x}}$, where $\partial_{\mathbf{x}} = (\partial/\partial s, \partial/\partial t, \partial/\partial u)$.

According to Theorem A, the pairwise linking numbers p, q and r of L are equal to the degrees of its Gauss map g_L on the two-dimensional coordinate subtori of T^3, and these degrees in turn are equal to the integrals there of the characteristic 2-form ω_L. In particular, ω_L is exact if and only if these degrees are all zero. In the language of vector fields, the degree of g_L on a coordinate subtorus is the flux of the characteristic vector field $\vec{\mathbf{v}}_L$ through this subtorus, and hence $\vec{\mathbf{v}}_L$ is in the image of curl if and only if all of these fluxes are zero.

When we expand ω_L above as a Fourier series, it is only the term

$$\mathbf{c}_0\bullet\star d\mathbf{x} \;=\; c^s_0\,dt\wedge du + c^t_0\,du\wedge ds + c^u_0\,ds\wedge dt$$

which has the potential for a non-zero integral over the coordinate subtori, and it vanishes precisely when p, q and r are all zero.

Theorem B. ***If the pairwise linking numbers of a three-component link L in S^3 are all zero, then Milnor's μ-invariant of L is given by each of the following equivalent formulas***

$$\begin{aligned}\boldsymbol{\mu(L)} \;&=\; \mathbf{1/2}\int_{T^3} d^{-1}(\omega_L)\wedge\omega_L \;=\; \mathbf{1/2}\int_{T^3}\delta(\varphi * \omega_L)\wedge\omega_L && \textbf{(1)}\\ &=\; \mathbf{1/2}\int_{T^3\times T^3}\vec{\mathbf{v}}_L(\mathbf{x})\times\vec{\mathbf{v}}_L(\mathbf{y})\bullet\nabla_{\mathbf{y}}\,\varphi(\mathbf{x}-\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} && \textbf{(2)}\\ &=\; \mathbf{8\pi^3}\sum_{\mathbf{n}\neq\mathbf{0}}\mathbf{a_n}\times\mathbf{b_n}\bullet\mathbf{n}/|\mathbf{n}|^2\,. && \textbf{(3)}\end{aligned}$$

where φ is the fundamental solution of the scalar Laplacian on the 3-torus, ω_L and $\vec{\mathbf{v}}_L$ are the characteristic form and vector field of L, the sum in (3) is over all nonzero lattice points in $\mathbb{Z}^3$, and $\mathbf{a_n}$ and $\mathbf{b_n}$ are the real and imaginary parts of the Fourier coefficients $\mathbf{c_n}$ of ω_L and $\vec{\mathbf{v}}_L$.

Explanation of the notation. In (1), $d^{-1}(\omega_L)$ denotes any 1-form on T^3 whose exterior derivative is ω_L. Among such 1-forms, the specific choice $\delta(\varphi * \omega_L)$ has the smallest L^2 norm, where δ is the exterior co-derivative (L^2 adjoint of d), and where $\varphi * \omega_L$ is the convolution of φ with ω_L, discussed in detail in Sections VII and VIII.

In (2), the difference $\mathbf{x}-\mathbf{y}$ is taken in the abelian group structure on T^3, the expression $\nabla_{\mathbf{y}}\,\varphi(\mathbf{x}-\mathbf{y})$ indicates the gradient with respect to $\mathbf{y}$ while $\mathbf{x}$ is held fixed, and $d\mathbf{x}$ and $d\mathbf{y}$ are volume elements on T^3. This formula is just the vector field version of (1) in which the integral hidden in the convolution is expressed openly; we will see in Appendix A that it represents the "helicity" of the vector field $\vec{\mathbf{v}}_L$ on T^3.

Observe that the integrands in (1) and (2) are invariant under the group SO(4) of orientation-preserving rigid motions of S^3, attesting to the naturality of the formulas.

Organization of the paper

Our main narrative thread appears in the body of the paper, and may be read straight through. The background and motivation for our work from the topological perspective of configuration spaces, and from the physical perspective of fluid mechanics and plasma physics, may be found in Appendix A.

The key idea of the proof of Theorem A centers on the "delta move", a higher order variant of a crossing change. Applied to a three-component link L in the 3-sphere, we show easily that this move increases the Milnor μ-invariant by 1, and then the bulk of the proof is devoted to showing that it increases Pontryagin's ν-invariant by 2.

The foundational material for this proof is contained in Sections II and III. In particular, in Section II we discuss how to compute Milnor's μ-invariant, describe the delta move, and prove that it increases the μ-invariant by 1. In Section III, we discuss Pontryagin's homotopy classification of maps from a 3-manifold to the 2-sphere in terms of framed bordism of framed links, define the "Pontryagin link" of such a map to be the inverse image of any regular value, and then show how to convert the relative ν-invariant to an absolute one when the manifold is the 3-torus. This section concludes with a simple procedure for computing the absolute ν-invariant of any map of the 3-torus to the 2-sphere from a diagram of its Pontryagin link, whose proof appears in Appendix B.

The proof of Theorem A will occupy us in Sections IV–VI, and is organized as follows.

In Section IV, we derive an explicit formula for the generalized Gauss map g_L needed for the proof of Theorem B, but not for that of Theorem A. We then describe an alternative asymmetric form h_L of the generalized Gauss map that is more convenient for the proof of Theorem A, and that, just as for g_L, depends *a priori* on the choice of a projection to the 2-sphere. We show that the two generalized Gauss maps are homotopic to one another, and that, up to homotopy, neither in fact depends on the choice of projection. We note a close relation between h_L and the classical Gauss maps of two-component links in 3-space, and use this, together with the symmetry of g_L, to prove the first statement in Theorem A.

In Section V, we set up for the proof of the rest of Theorem A by describing the standard open-book structure on S^3 with disk pages, use a link homotopy to move a given link L into "generic position" with respect to it, and then show how to explicitly visualize the Pontryagin link of the corresponding asymmetric generalized Gauss map h_L. Finally, we use the results of Section III to develop a method for computing $\nu(h_L) = \nu(g_L)$. The key technical results in this section are the Bicycle Theorem and its corollary, the Double Crossing Formula, whose proofs appear in Appendices C and D.

In Section VI, we complete the proof of Theorem A by induction, using the methods developed in Sections II and III to first confirm the "base case", and then using the methods of Section V to carry out the inductive step, showing that the delta move increases Pontryagin's ν-invariant by 2. An entirely different, algebraic proof of Theorem A, using string links and maps of the 2-torus to the 2-sphere, may be found in our paper[3].

In Sections VII and VIII, we investigate the special case when the pairwise linking numbers p, q and r of L are all zero, and so the μ and ν-invariants are ordinary integers. In this case we will use J. H. C. Whitehead's integral formula for the Hopf invariant, adapted to maps of the 3-torus to the 2-sphere, together with a formula for the fundamental solution of the scalar Laplacian on the 3-torus as a Fourier series in three variables, to provide an explicit integral formula for $\nu(g_L)$ (Theorem B) and hence for $\mu(L)$ in light of Theorem A.

II. THE MILNOR μ-INVARIANT

Let L be a three-component link in either $\mathbb{R}^3$ or S^3, with oriented components X, Y and Z, and pairwise linking numbers p, q and r. Milnor's original definition of the triple linking number $\mu(L)$, typically denoted $\bar{\mu}_{123}(L)$, was algebraic. The formulation in his PhD thesis[4] is expressed in terms of the lower central series of the link group G, the fundamental group of the complement of L, as follows.

Choose based meridians for the link components X, Y and Z, and let x, y and z denote the corresponding elements of G. In general, these three elements do not generate G, but they do generate the quotient of G by the third term $[[G,G],G]$ in its lower central series. In this quotient group, the longitude of Z can be written as a word w in x, y and z and their inverses. Assign an integer $m_{xy}(w)$ to this word that counts with signs the number of times that x appears before y in w, allowing intervening letters. More precisely, each appearance of $x \cdots y$ or $x^{-1} \cdots y^{-1}$ contributes $+1$ to $m_{xy}(w)$, while $x \cdots y^{-1}$ or $x^{-1} \cdots y$

contributes -1. Then $\mu(L)$ is the element of $\mathbb{Z}_{\gcd(p,q,r)}$ defined by

$$\mu(L) \;=\; m_{xy}(w) \mod \gcd(p,q,r).$$

There is a geometric reformulation of this definition, found by Mellor and Melvin[5], that is more convenient for our purposes: choose Seifert surfaces F_X, F_Y and F_Z for the components of L, and move these into general position. Starting at any point on X, record its intersection with the Seifert surfaces for Y and Z by a word w_X in y and z. For example a y or y^{-1} in w_X indicates a positive or negative intersection point of X with F_Y. Set $m_X = m_{yz}(w_X)$, where the right hand side is computed as in the last paragraph, and similarly set $m_Y = m_{zx}(w_Y)$ and $m_Z = m_{xy}(w_Z)$. Finally, let t be the signed count of the number of triple points of intersection of the three Seifert surfaces. Then

$$\mu(L) \;=\; m_X + m_Y + m_Z - t \mod \gcd(p,q,r).$$

It follows from this formula that $\mu(L)$ is invariant under even permutations of the components of L, but changes sign under odd permutations. This is a well known property of Milnor's triple linking number.

We give two sample calculations of the triple linking number, using this geometric formulation, that will feature in our inductive proof of Theorem A in Section VI.

Example II.1. Let L_{pqr} be the link shown in Figure 3, with components X, Y and Z and with pairwise linking numbers p, q and r. Choose the Seifert surfaces F_X, F_Y and F_Z to be the obvious disks, essentially lying in the page, bounded by X, Y and Z.

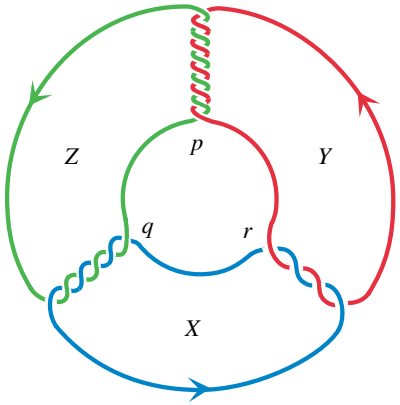


FIG. 3. The base link L_{pqr} for $p = 5$, $q = 3$, $r = -2$

Starting at appropriate points on the link components, we can read off the words

$$w_X = y^r z^q\,, \quad w_Y = z^p x^r \quad \text{and} \quad w_Z = x^q y^p.$$

Thus $m_X = qr$, $m_Y = rp$ and $m_Z = pq$. Furthermore, it is clear that there are no triple points of intersection of the three disks. Therefore

$$\mu(L_{pqr}) \;=\; qr + rp + pq - 0 \;=\; 0 \;\in\; \mathbb{Z}_{\gcd(p,q,r)}.$$

The links L_{pqr} will serve as the ***base links*** for our proof of Theorem A.

Example II.2. Consider the ***delta move*** shown in Figure 4, transforming the link L on the left to the link $\widehat{L}$ on the right. This move takes place within a 3-ball, outside of which the link is left fixed. It does not alter the pairwise linking numbers of L, and may be thought of as a higher order variant of a crossing change.

The delta move was introduced by Matveev[6]. It was shown by Murakami and Nakanishi[7] that a suitable sequence of such moves can transform any link into any other link with the same number of components and the same pairwise linking numbers. In particular, the base link L_{pqr} above can be transformed into any other three-component link with pairwise linking numbers p, q and r by a sequence

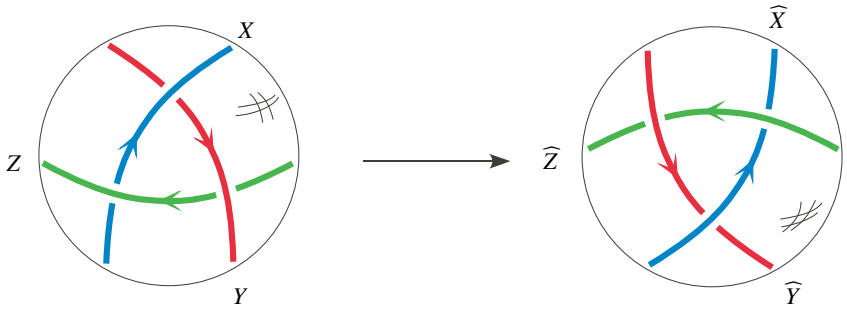


FIG. 4. The delta move $L \to \widehat{L}$

of delta moves. The inductive step of our proof of Theorem A will be based on this observation, and so we determine right now the effect of the delta move on the μ-invariant.

If the three arcs involved in the delta move *do not* come from three distinct components of the link L, then the change can be achieved by a link homotopy, and hence neither Milnor's μ-invariant nor Pontryagin's ν-invariant for g_L will change.

But if the arcs *do* come from three distinct components of L as shown in the figure, then μ increases by 1, that is $\mu(\widehat{L}) = \mu(L)+1$. This can be seen as follows, using the geometric formula for the μ-invariant.

In Figure 5, we display on the left fragments of the Seifert surfaces F_X, F_Y and F_Z before the delta move, while on the right, after the delta move, each old surface is enlarged a bit to provide new surfaces $F_{\widehat{X}}$, $F_{\widehat{Y}}$ and $F_{\widehat{Z}}$.

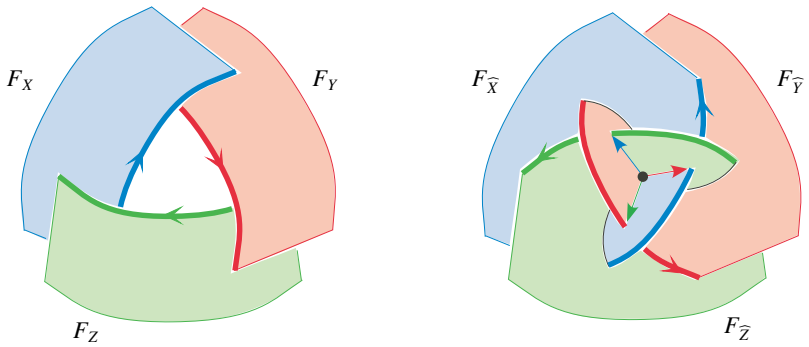


FIG. 5. A negative triple point appears

On the left, the three surface fragments are disjoint, while on the right, after their enlargement, they are not. Where these surfaces now come together, we have an additional isolated triple point, and since the normals to the surfaces at this point form a left-handed frame, this triple point gets a minus sign. Thus t drops by 1. We also see that after the delta move, the curve $\widehat{Z}$ has two extra intersections with $F_{\widehat{Y}}$, the first positive and the second negative, and no extra intersections with $F_{\widehat{X}}$. Since these new intersection points are adjacent along the curve, and of opposite signs, it follows that the new count $m_{\widehat{Z}} = m_{xy}(w_{\widehat{Z}})$ is equal to the old one m_Z. Similarly $m_{\widehat{X}} = m_X$ and $m_{\widehat{Y}} = m_Y$. Hence $\mu = m_X + m_Y + m_Z - t$ increases by 1, as claimed.

The heart of the proof of Theorem A will be to show that application of the delta move increases Pontryagin's ν-invariant by 2.

III. THE PONTRYAGIN ν-INVARIANT

Heinz Hopf[8] proved in 1931 that homotopy classes of maps from the 3-sphere to the 2-sphere are in one-to-one correspondence with the integers via his now famous Hopf invariant. Pontryagin[2] generalized this to give the homotopy classification for maps from an arbitrary finite 3-complex to the 2-sphere. It is convenient for our purposes to restrict to smooth maps from 3-manifolds to the 2-sphere,

and to use Poincaré duality to reformulate Pontryagin's result, originally presented via cohomology, in homological terms.

Definition and properties

Fix a closed, oriented, smooth 3-manifold M. The homotopy classification of maps $f : M \to S^2$ can be expressed using two differential topological invariants.

The primary invariant

$$\lambda(f) \in H_1(M) \qquad \text{(with integral coefficients understood)}$$

records the homology class of the preimage link $\mathcal{L} = f^{-1}(*)$ of any regular value $*$ of f (oriented as explained below), or equivalently the Poincaré dual of the pull-back $f^*(\omega)$ of the orientation class $\omega \in H^2(S^2)$. It is easily shown that two maps have the same primary invariant if and only if they induce the same map on homology.

The secondary invariant

$$\nu(f_0, f_1) \in \mathbb{Z}_{2d(\lambda)}$$

compares two maps f_0 and f_1 with the same primary invariant λ. Here $d(\lambda)$ is the divisibility of λ as an element of the free abelian group $H_1(M)$/torsion. Thus $d(\lambda) = 0$ if λ is of finite order, and otherwise $d(\lambda)$ is the largest positive integer d for which $\lambda = d\kappa$ for some $\kappa \in H_1(M)$.

For example, if M is the 3-torus T^3, then $\lambda(f) = (p, q, r) \in H_1(T^3) \cong \mathbb{Z}^3$ where p, q and r are the degrees of f restricted to the coordinate 2-tori, and the divisibility $d(p, q, r)$ is the greatest common divisor of p, q and r.

In the next few paragraphs, we discuss these invariants λ and ν in more detail, and provide a natural way, in the special case that M is the 3-torus, to transform ν from a relative invariant into an absolute one, meaning a function of a single map.

First recall that a *framing* of a smooth link in M is a homotopy class of trivializations of the normal bundle of the link. It can be represented by an orthonormal triple $(\mathbf{u}, \mathbf{v}, \mathbf{t})$ of vector fields along the link with respect to some Riemannian metric on M, where $\mathbf{t}$ is tangent to the link. We can use this to orient the link by $\mathbf{t}$ by insisting that the triple give the orientation on M. Conversely, if the link is already oriented by $\mathbf{t}$, then the framing can be specified by a single unit normal vector field $\mathbf{u}$, as $\mathbf{v}$ is then determined by the condition that $(\mathbf{u}, \mathbf{v}, \mathbf{t})$ be an orthonormal frame giving the orientation on M. In pictures, therefore, we often indicate a framing on an oriented link by simply drawing a thin parallel push-off of the link, recording the tips of the vectors in $\mathbf{u}$.

Now given a map $f : M \to S^2$ with regular value $*$, the link $\mathcal{L} = f^{-1}(*) \subset M$ inherits a framing by pulling back an oriented basis for the tangent space to S^2 at $*$, and acquires an orientation from this framing, as above. Equipped with this framing and orientation, $\mathcal{L}$ will be called the ***Pontryagin link*** of f at $*$.

Note that any framed oriented link $\mathcal{L}$ in M arises as the Pontryagin link of some map from M to S^2. In particular, the ***Pontryagin–Thom construction*** produces such a map, given by wrapping each normal disk fiber of a tubular neighborhood of $\mathcal{L}$ around the 2-sphere by the exponential map, using the framing to identify the fiber with the disk of radius π in the tangent space to S^2 at $*$, and sending everything outside the neighborhood to the antipode of $*$. This construction provides a one-to-one correspondence between the set $[M, S^2]$ of homotopy classes of maps $M \to S^2$, and the set $\Omega_1^{\text{fr}}(M)$ of framed bordism classes of framed oriented links in M (see e.g. Chapter 7 in Milnor[9]).

Now we return to our discussion of the invariants associated with maps $f : M \to S^2$.

An easy argument shows that the primary invariant $\lambda(f)$, the homology class of the oriented link $f^{-1}(*)$, is independent of the choice of regular value $*$, and that $\lambda(f)$ is invariant under homotopies of f. Indeed, the preimages $\mathcal{L}_0$ and $\mathcal{L}_1$ of any regular values for any pair of maps homotopic to f are bordant in $M \times [0, 1]$ (see Milnor[9] for a proof). It follows that $\mathcal{L}_0$ and $\mathcal{L}_1$ are homologous in M. Conversely, homologous links in M are bordant in $M \times [0, 1]$ by a standard argument going back to Thom[10]. This shows that the partition of $[M, S^2]$ into subsets $[M, S^2]_\lambda$ according to their primary invariants $\lambda \in H_1(M)$ corresponds to the partition of $\Omega_1^{\text{fr}} M$ into *unframed* bordism classes.

The secondary invariant associated with a pair of bordant framed links measures the obstruction to extending the framings on the links across any bordism between them. More precisely, given f_0 and f_1 in $[M, S^2]_\lambda$ with Pontryagin links $\mathcal{L}_0$ and $\mathcal{L}_1$, and an oriented surface $\mathcal{F} \subset M \times [0, 1]$ with

$\partial\mathcal{F} = \mathcal{L}_1 \times 1 - \mathcal{L}_0 \times 0$, the framings on $\mathcal{L}_0$ and $\mathcal{L}_1$ combine to give a normal framing of $\mathcal{F}$ along its boundary. The obstruction to extending this framing across $\mathcal{F}$ is measured by its relative Euler class $e(\mathcal{F})$ in $H^2(\mathcal{F}, \partial\mathcal{F}; \pi_1 \mathrm{SO}(2)) = \mathbb{Z}$, which in homological terms is the intersection number of $\mathcal{F}$ with a generic perturbation of itself that is directed by the given framings along $\partial\mathcal{F}$. This class depends on the choice of the regular values of f_0 and f_1 used to define the Pontryagin links, and on the choice of the bordism $\mathcal{F}$ between the links. But the residue class of $e(\mathcal{F})$ mod $2d(\lambda)$ does not depend on these choices; see Gompf[11] and Cencelj, Repovš and Skopenkov[12] for details, and also see Auckly and Kapitanksi[13]. This residue class

$$\nu(f_0, f_1) \;=\; e(\mathcal{F}) \bmod 2d(\lambda) \;\in\; \mathbb{Z}_{2d(\lambda)}$$

will be referred to as the ***relative Pontryagin ν-invariant*** of f_0 and f_1.

Converting the relative Pontryagin *ν*-invariant into an absolute invariant

The task of converting ν into an *absolute* invariant, that is, changing it from a function of two variables to a function of one variable, requires the choice of a ***base map*** f_λ in each subset $[M, S^2]_\lambda$ of homotopy classes of maps with primary invariant λ. One can then define the ***absolute Pontryagin ν-invariant*** by

$$\nu(f) \;=\; \nu(f, f_\lambda)$$

for any $f \in [M, S^2]_\lambda$. Whether such choices can be made in a topologically meaningful way depends on the manifold M.

For example, Pontryagin[2] (page 356) explicitly cautioned against trying to make this choice when $M = S^1 \times S^2$. In this case there are, up to homotopy, exactly two maps to S^2 with primary invariant $1 \in H_1(S^1 \times S^2) = \mathbb{Z}$, meaning degree 1 on the cross-sectional 2-spheres in $S^1 \times S^2$. One of these is the projection $f_0(\theta, x) = x$ to the S^2 factor, and the other is the twist map $f_1(\theta, x) = \mathrm{rot}_\theta(x)$ that rotates the S^2 factor once while traversing the S^1 factor; here rot_θ indicates rotation of S^2 through an angle θ about its polar axis. Note that the double twist $f_2(\theta, x) = \mathrm{rot}_{2\theta}(x)$ is homotopic to f_0.

What Pontryagin observed is that f_0 and f_1 differ by an automorphism of $S^1 \times S^2$, and so there is no natural way to choose which one should serve as the base map for the homology class $[S^1 \times S^2, S^2]_1$. More precisely, the automorphism h given by $h(\theta, x) = (\theta, \mathrm{rot}_\theta(x))$ satisfies $f_0 h = f_1$, while $f_1 h = f_2$, which is homotopic to f_0. Thus the maps f_0 and f_1 have equal topological status, and so neither is more basic than the other.

A key feature of this example is the existence of a homotopically nontrivial automorphism of $S^1 \times S^2$ that induces the identity on homology. In general, if a 3-manifold M supports such an automorphism h, then $\lambda(fh) = \lambda(f)$ for any map $f : M \to S^2$. Hence the existence of h provides a potential obstruction to the natural choice of base maps in $[M, S^2]_\lambda$ for all λ. The 3-torus has no such automorphisms, due to the fact that its higher homotopy groups vanish.

For each triple of integers p, q and r, we now show how to pick out a specific map $f_{pqr} : T^3 \to S^2$ having these preassigned cross-sectional degrees, which can serve in a topologically meaningful way as the base map for the set $[T^3, S^2]_{(p,q,r)}$ of all such maps.

We will describe f_{pqr} by specifying its Pontryagin link $\mathcal{L}_{pqr}$, as follows. Choose three pairwise disjoint circles C_s, C_t and C_u that are cosets of the coordinate circle subgroups

$$S^1_s = S^1 \times 0 \times 0\,,\; S^1_t = 0 \times S^1 \times 0 \;\text{ and }\; S^1_u = 0 \times 0 \times S^1$$

of the 3-torus (where as usual $S^1 = \mathbb{R}/2\pi\mathbb{Z}$) with disjoint tubular neighborhoods $\mathcal{N}_s$, $\mathcal{N}_t$ and $\mathcal{N}_u$. Equip these circles with their ***coordinate framings*** induced from the Lie framing $(\partial_s, \partial_t, \partial_u)$ of the tangent bundle of T^3, that is, (∂_t, ∂_u) for C_s, (∂_u, ∂_s) for C_t and (∂_s, ∂_t) for C_u. Then construct $\mathcal{L}_{pqr}$ from p parallel copies of C_s in $\mathcal{N}_s$ (meaning p distinct cosets of S^1_s lying in $\mathcal{N}_s$), q copies of C_t in $\mathcal{N}_t$ and r copies of C_u in $\mathcal{N}_u$, all with their coordinate framings, as indicated in Figure 6. Thus f_{pqr} wraps the disk fibers of the tubes $\mathcal{N}_s$, $\mathcal{N}_t$ and $\mathcal{N}_u$ around S^2 by maps of degree p, q and r, and is constant elsewhere.

Here and below, the 3-torus is pictured as the cube $[0, 2\pi]^3$ in *stu*-space with opposite faces identified. The axes correspond to the coordinate circles S^1_s, S^1_t and S^1_u, and the coordinate framings are the "blackboard" framings, i.e. those given by parallel push-offs in the projection shown. The link $\mathcal{L}_{53-2}$ is shown in the figure. Note that by construction $\mathcal{L}_{000}$ is empty, so f_{000} is constant.

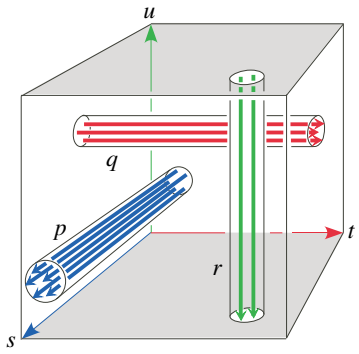


FIG. 6. Pontryagin link $\mathcal{L}_{pqr}$ for the base map f_{pqr}

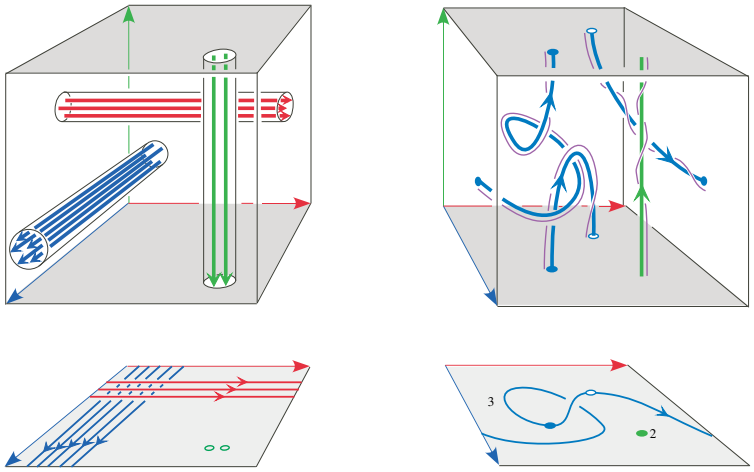
Computing the absolute Pontryagin ν-invariant

Our goal is to give a simple procedure for computing the Pontryagin ν-invariant of a map $f : T^3 \to S^2$ from a "toral diagram" of its Pontryagin link $\mathcal{L}$.

By definition, a ***toral diagram*** of $\mathcal{L}$ consists of

1. a classical oriented link diagram $\mathcal{D}$ in the 2-torus T^2 with ***crossings*** $\mathcal{C}$,
2. a finite set of signed points in T^2, the ***marked points***, partitioned into the ***isolated*** ones $\mathcal{M} \subset T^2 - \mathcal{D}$, and the ***internal*** ones $\mathcal{N} \subset \mathcal{D} - \mathcal{C}$, and
3. integer ***framings*** for each component of $\mathcal{D}$ and each point in $\mathcal{M}$.

For example, the link $\mathcal{L}_{pqr}$ above is duplicated in Figure 7(a) with its toral diagram beneath it, and another example is given in Figure 7(b).



(a) The link $\mathcal{L}_{pqr}$ and its diagram (b) Another example

FIG. 7. Toral diagrams of framed links in T^3

It is understood that $\mathcal{L}$ should project to $\mathcal{D} \cup \mathcal{M}$ under the projection $T^3 \to T^2$ sending (s, t, u) to (s, t), and so points in $\mathcal{M}$ correspond to vertical components $* \times * \times S^1$ in $\mathcal{L}$, oriented up or down according to the signs. The points in $\mathcal{N}$ correspond to transverse intersections of $\mathcal{L}$ with the ***horizontal 2-torus*** $T^2 \times 0$ – shaded in the figures – where the sign $+1$ or -1 indicates whether the curve points up or down near the intersection.

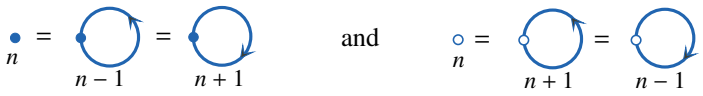
To explain the crossings $\mathcal{C}$, view T^3 as the cube $[0, 2\pi]^3$ in stu-space with opposite faces identified, as before, and T^2 as the square $[0, 2\pi]^2$ in the st-plane with opposite sides identified. Above $\mathcal{D} - \mathcal{N}$, the

link $\mathcal{L}$ resides in $T^2 \times (0, 2\pi)$. At a crossing, the over-crossing strand is the one with the larger vertical u-coordinate in the cube.

Finally, the framings specify a push-off of $\mathcal{L}$ by comparison with the blackboard framing of $\mathcal{D} \cup \mathcal{M}$ in T^2. These "blackboard" framings are obtained by pushing $\mathcal{D} \cup \mathcal{M}$ off itself in the direction of a normal vector field in T^2, and then lifting these push-offs to a collection of framing curves for the components of $\mathcal{L}$ that we call their 0-*framings*. For vertical circles, these are just the coordinate framings. Now the n-framing on any given component of $\mathcal{L}$ is the one obtained from the 0-framing by adding n full twists, right or left handed according as n is positive or negative.

In these figures we typically indicate the framing on the link in T^3 by a thin push-off, and use the convention that the unlabeled components of the diagram have framing zero, that is, the blackboard framing. We also denote the positive marked points in the diagram with solid dots, and the negative ones with hollow dots.

We have allowed isolated marked points in the definition above so that projections of links such as $\mathcal{L}_{pqr}$ will qualify as toral diagrams, and to facilitate our later work. Note, however, that a generic isotopy of $\mathcal{L}$ will eliminate the set $\mathcal{M}$ of isolated marked points, converting an n-framed isolated marked point – corresponding to a vertical circle in $\mathcal{L}$ – into a small $(n \pm 1)$-framed circle with one internal marked point – corresponding to a spiral perturbation of the vertical circle. More precisely, using the notation above of solid dots for positive marked points and hollow dots for negative ones, we have



as is readily verified by a suitable picture in the 3-torus.

Now suppose we are given a toral diagram of a Pontryagin link $\mathcal{L}$ for a map $f : T^3 \to S^2$. We say that the diagram ***represents*** f, and seek to compute $\nu(f)$ from it.

First observe that the primary invariant

$$\lambda(f) \;=\; (p, q, r)$$

is easily read from the diagram. Indeed p and q (which are the degrees of the projections of $\mathcal{L}$ to the horizontal circle factors S^1_s and S^1_t) are just the intersection numbers of $\mathcal{D}$ with S^1_t and $-S^1_s$, and r (the degree of the projection of $\mathcal{L}$ to the vertical circle S^1_u, or equivalently the intersection number of $\mathcal{L}$ with the horizontal 2-torus) is the sum of the signs of all the marked points. We call p and q the ***horizontal winding numbers*** of the diagram, and r its ***vertical winding number***. For what follows, we will also need to consider the vertical winding of the individual components of $\mathcal{L}$. Each such component $\mathcal{L}_i$ projects either to some subset $\mathcal{D}_i$ of $\mathcal{D}$, or to a point $\mathcal{M}_i$ in $\mathcal{M}$ (if $\mathcal{L}_i$ is vertical). In either case, we call this projected image the "i^{th} component" of the diagram, and write r_i for the sum of all the marked points that lie on it. Clearly r_i is just the vertical winding number of $\mathcal{L}_i$, and $\sum r_i = r$, the total vertical winding number of the diagram.

To compute $\nu(f)$, we will transform $\mathcal{L}$ by a sequence of framed bordisms into $\mathcal{L}_{pqr}$ with some extra twists in the framing. The number of twists is by definition $\nu(f)$; counting these twists ultimately yields the simple formula for $\nu(f)$ that will be given in Proposition III.1.

To cleanly state this formula, we assume from the outset that our diagram has no crossings. There is no loss of generality in doing so since the crossings can be eliminated by ***saddle bordisms*** of $\mathcal{L}$, as illustrated in Figure 8 (viewing $\mathcal{L}$ from the top, looking down on a crossing in the horizontal torus) which of course do not change $\nu(f)$. Furthermore, we will see that the Pontryagin link of a "generic link" (to be defined in Section V), has a toral diagram without crossings.

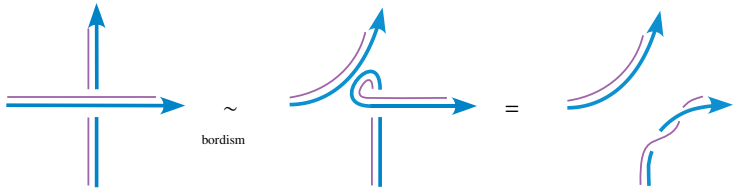


FIG. 8. Using saddles to eliminate crossings

Each such saddle bordism is achieved by adding a 1-handle, as indicated on the left side of Figure 9, and drawn in full, suppressing one dimension in T^3, on the right side.

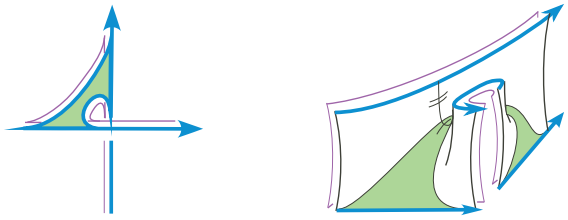


FIG. 9. The saddle bordism

The effect of the saddle bordism on the diagram is easy to describe. If it is used to eliminate a self-crossing of an n_i-framed component $\mathcal{D}_i$ (meaning $\mathcal{D}_i$ is the projection of a component $\mathcal{L}_i$ of $\mathcal{L}$) then $\mathcal{D}_i$ splits into two components whose framings must add up to $n_i \pm 1$, depending on the sign of the crossing. Beyond this condition, the framings on the new components can be chosen arbitrarily since twists in the original framing can be shifted along $\mathcal{L}_i$ at will. If the saddle bordism is used to eliminate a crossing between distinct components $\mathcal{D}_i$ and $\mathcal{D}_j$ of $\mathcal{D}$ with framings n_i and n_j, then the result is a single component with framing $n_i + n_j \pm 1$.

Once we have a toral diagram *without crossings* representing f, the extra data needed to compute $\nu(f)$ is the list of ***vertical winding numbers*** r_i of its components together with one additional integer $n = \sum n_i$, the sum of all the component framings n_i, which we call the ***total framing*** and place as a label next to the diagram.

To state the formula efficiently, we will use one more list of numbers that is easily read from the diagram. First pick a basepoint $*$ in T^2 away from $\mathcal{D} \cup \mathcal{M}$, and then for each i, choose an arc γ_i that runs from the i^{th} component of $\mathcal{D} \cup \mathcal{M}$ to $*$. Now define the ***depth*** of the i^{th} component to be

$$d_i \;=\; 2\,\gamma_i \bullet \mathcal{D} \mod 2\gcd(p,q,r)\,,$$

that is, twice the intersection number in T^2 of the arc γ_i with the union $\mathcal{D}$ of the closed curves in the diagram. This intersection number must be properly interpreted for components of $\mathcal{D}$. In this case the initial point of γ_i lies on $\mathcal{D}$, contributing $\pm\frac{1}{2}$ to the intersection number, and thus ± 1 to d_i. It follows that d_i is always an odd integer for components of $\mathcal{D}$, and an even integer for components of $\mathcal{M}$, meaning isolated marked points. An example is shown in Figure 10.

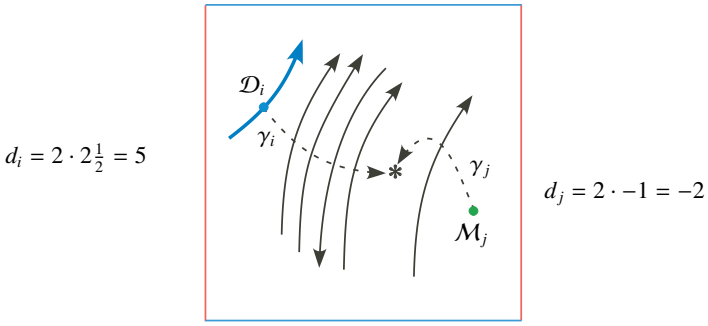


FIG. 10. Depths of components in the diagram

Of course the depth d_i depends on the choice of arc γ_i, but only modulo $2\gcd(p,q)$, and so it is certainly well defined modulo $2\gcd(p,q,r)$. Furthermore, in the formula for $\nu(f)$ below, the depths appear only as coefficients in the sum $\sum r_i d_i$. It is readily seen that this sum changes by a multiple of $\sum 2r_i = 2r$ when moving the basepoint $*$, and so it is well defined modulo $2\gcd(p,q,r)$, independent of the choices of $*$ and of the arcs γ_i.

We can now state the main result of this section.

Proposition III.1. ***Let $f : T^3 \to S^2$ be a smooth map whose Pontryagin link $\mathcal{L}$ is represented by a toral diagram in T^2 without crossings, with horizontal winding numbers p and q, vertical winding number r, and total framing n. Then the primary and secondary Pontryagin invariants of f are given by***

$$\begin{aligned} \lambda(f) &= (p, q, r) \quad \textit{and} \\ \nu(f) &= n + pq + \textstyle\sum r_i d_i \mod 2\gcd(p, q, r) \end{aligned}$$

where r_i and d_i are the vertical winding numbers and depths (with respect to any chosen basepoint) of the components of the diagram.

The proof is given in Appendix B. The following application will arise in the proof of Theorem A in Section VI.

Example III.2. If f is represented by the $\pm(pq + r)$-framed (p, q) torus link in T^2, with one component K of vertical winding number r, and the rest of vertical winding number zero, then $\lambda(f) = (p, q, r)$, and choosing a base point adjacent to K,

$$\nu(f) \;=\; (pq + r) + pq + (1 \cdot r + 0 + 0) \;=\; 2(pq + r) \;=\; 0 \;\in \mathbb{Z}_{2\gcd(p,q,r)}.$$

IV. EXPLICIT FORMULAS FOR THE GENERALIZED GAUSS MAP

In this section we give an explicit diffeomorphism from the Grassmann manifold $\mathrm{G}_2\mathbb{R}^4$ of oriented 2-planes through the origin in 4-space to a product of two 2-spheres, and then use it to give a formula for the Gauss map $g_L : T^3 \to S^2$ of a three-component link L in the 3-sphere. For simplicity, we have dropped the adjective "generalized" and henceforth simply refer to $g_L : T^3 \to S^2$ as the "Gauss map", while continuing to call its prototype $T^2 \to S^2$ the "classical Gauss map".

We also describe an alternative form

$$h_L : T^3 \to S^2$$

of the Gauss map, homotopic to g_L but more convenient for the proof of Theorem A that we will give in Section VI. The formula for h_L will reveal a close connection between the Gauss map and the classical Gauss map $T^2 \to S^2$ for two-component links in 3-space.

Throughout we regard $\mathbb{R}^4$ as the algebra of ***quaternions***, with orthonormal basis $1, i, j, k$ and unit sphere S^3 (oriented so that i, j, k is a positive frame for the tangent space to S^3 at the point 1), and $\mathbb{R}^3$ as the subspace of ***pure imaginary quaternions*** spanned by i, j and k, with unit sphere S^2. Thus S^3 is viewed as the multiplicative group of unit quaternions, and S^2 as the subset of pure imaginary unit quaternions.

For any quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$, let $\bar{q}$ denote its conjugate $q_0 - q_1 i - q_2 j - q_3 k$, which coincides with q^{-1} when $q \in S^3$, and let $\mathrm{Re}\,(q) = q_0$ and $\mathrm{Im}\,(q) = q_1 i + q_2 j + q_3 k$ denote its real and imaginary parts.

The Grassmann manifold $\mathrm{G}_2\mathbb{R}^4$

It is well known that $\mathrm{G}_2\mathbb{R}^4$ can be identified with a product of two 2-spheres. In particular, we will use the diffeomorphism

$$\pi : \mathrm{G}_2\mathbb{R}^4 \longrightarrow S^2 \times S^2$$

that maps the oriented plane $\langle a, b\rangle$, spanned by an orthonormal 2-frame (a, b) in $\mathbb{R}^4$, to the point $(b\bar{a}, \bar{a}b)$ in $S^2 \times S^2$. Note that both coordinates

$$\pi_+\langle a, b\rangle \;=\; b\bar{a} \qquad \text{and} \qquad \pi_-\langle a, b\rangle \;=\; \bar{a}b$$

do in fact lie in S^2 since right and left multiplication by $\bar{a}$ are orthogonal transformations of $\mathbb{R}^4$, carrying the orthonormal frame (a, b) to the orthonormal frames $(1, b\bar{a})$ and $(1, \bar{a}b)$.

To see that π is well-defined, consider any other orthonormal basis for the plane $\langle a, b\rangle$. It must be of the form (ac, bc) for some c in the circle subgroup $C_{\bar{a}b}$ through $\bar{a}b$, since this group acts on the plane by

rotations. Thus it suffices to check that $bc\,\overline{ac} = b\bar{a}$, which is immediate, and that $\overline{ac}\,bc = \bar{a}b$, which is true since c commutes with $\bar{a}b$.

To see that π is in fact a diffeomorphism, we can simply write down the inverse. It maps a pair (x, y) in $S^2 \times S^2$ to the plane $\langle c, cy\rangle$ where c is the midpoint of any geodesic arc from x to y on S^2. This can be verified by a straightforward calculation using the fact that conjugation by a pure imaginary quaternion rotates the 2-sphere about that quaternion by π radians, so $cy\bar{c} = x$.

The Gauss map g_L

Recall from the introduction that the Grassmann map $G : \mathrm{Conf}_3 S^3 \to \mathrm{G}_2\mathbb{R}^4$ sends a triple (x, y, z) of distinct points in S^3 to the plane they span in $\mathbb{R}^4$, translated to pass through the origin, and oriented so that

$$G(x, y, z) \;=\; \langle x - z, y - z\rangle.$$

We have extended notation so that for *any* two linearly independent vectors a and b in $\mathbb{R}^4$, the symbol $\langle a, b\rangle$ denotes the oriented plane they span. Then, given a three-component link L in S^3, its Gauss map $g_L : T^3 \to S^2$ is defined using the Grassmann map G and the projection $\pi_+ : \mathrm{G}_2\mathbb{R}^4 \to S^2$ by the formula

$$g_L(s, t, u) \;=\; \pi_+ G(x, y, z) \;=\; \pi_+\langle x - z, y - z\rangle$$

where $x = x(s)$, $y = y(t)$ and $z = z(u)$ parametrize the components of L.

To make this explicit, and to show that using π_- in place of π_+ in the definition would not change the homotopy class of g_L, we need expressions for $\pi_\pm\langle a, b\rangle$ when a and b are arbitrary linearly independent vectors in $\mathbb{R}^4$, but not necessarily orthonormal.

For example, if a and b are orthogonal, then $b\bar{a}$ and $\bar{a}b$ are still pure imaginary, and so need only be normalized to give $\pi_+\langle a, b\rangle$ and $\pi_-\langle a, b\rangle$.

For a general pair of linearly independent vectors a and b, the vector $c = b - (b\bullet a/a\bullet a)a$, where $\bullet$ is the dot product in $\mathbb{R}^4$, is orthogonal to a and satisfies $\langle a, c\rangle = \langle a, b\rangle$. Therefore $\pi_+\langle a, b\rangle = \pi_+\langle a, c\rangle$, which equals the unit normalization of the vector $c\bar{a} = b\bar{a} - b\bullet a$. But $b\bullet a = \mathrm{Re}\,(b\bar{a})$, and so $\pi_+\langle a, b\rangle$ is the unit normalization of $\mathrm{Im}\,(b\bar{a})$. Similarly $\pi_-\langle a, b\rangle$ is the unit normalization of $\mathrm{Im}\,(\bar{a}b)$. Therefore, for *any* two linearly independent vectors a and b, we have $\pi_\pm\langle a, b\rangle = (a, b)_\pm/|(a, b)_\pm|$ where $(\ ,\ )_\pm$ are the skew symmetric bilinear forms on $\mathbb{R}^4$ defined by

$$(a, b)_+ \;=\; \mathrm{Im}\; b\bar{a} \qquad \text{and} \qquad (a, b)_- \;=\; \mathrm{Im}\; \bar{a}b\,.$$

It follows that $g_L(s, t, u)$ is the unit normalization of the vector

$$\begin{aligned} F(x, y, z) \;&=\; (x - z, y - z)_+ \;=\; (x, y)_+ + (y, z)_+ + (z, x)_+ \\ &=\; (ix\bullet y + iy\bullet z + iz\bullet x\,,\ jx\bullet y + jy\bullet z + jz\bullet x\,,\ kx\bullet y + ky\bullet z + kz\bullet x) \end{aligned}$$

where the last expression follows from the formula $(a, b)_+ = (ia\bullet b, ja\bullet b, ka\bullet b)$. This formula is seen as follows. By definition, $(a, b)_+ = \mathrm{Im}\,(b\bar{a}) = (i\bullet b\bar{a})i + (j\bullet b\bar{a})j + (k\bullet b\bar{a})k$. But $q\bullet b\bar{a} = qa\bullet b$ for any q (in particular i, j or k) since right multiplication by the unit quaternion $a/|a|$ is an isometry, and so $(a, b)_+ = (ia\bullet b)i + (ja\bullet b)j + (ka\bullet b)k = (ia\bullet b, ja\bullet b, ka\bullet b)$. Summarizing, we have shown:

Proposition IV.1. ***The Gauss map $g_L : T^3 \to S^2$ of a three-component link L in S^3 is given by the formula***

$$g_L(s, t, u) \;=\; F(x, y, z)/|F(x, y, z)|$$

where $x = x(s)$, $y = y(t)$ and $z = z(u)$ parametrize the components of L, and $F : \mathrm{Conf}_3 S^3 \to \mathbb{R}^3$ is the function defined above.

This formula for g_L will be used in our proof of Theorem B in Sections VII and VIII. For Theorem A it will be more convenient, for the most part, to use an alternative form of the Gauss map that we introduce next.

The asymmetric Gauss map h_L

Given a three-component link L in the 3-sphere, we define below an asymmetric version

$$h_L : T^3 \longrightarrow S^2$$

of the Gauss map in which the last component of L plays a distinguished role, and show that it is homotopic to the earlier defined symmetric Gauss map g_L. As will be seen, the map h_L can be viewed as a parametrized family of classical Gauss maps for twisted versions of the first two components of L, parametrized by the third component.

For notational economy, we use $[q]$ to denote the unit normalization $q/|q|$ of a nonzero quaternion q.

The key motivation for the definition of h_L is that the Grassmann map $G : \mathrm{Conf}_3 S^3 \to \mathrm{G}_2\mathbb{R}^4$ factors up to homotopy through the ***Stiefel manifold*** $\mathrm{V}_2\mathbb{R}^4$ of orthonormal 2-frames in 4-space. More precisely, let pr_z denote ***stereographic projection*** of $S^3 - \{z\}$ onto $z^\perp$, the 3-plane through the origin in $\mathbb{R}^4$ orthogonal to z. Then we define the ***Stiefel map***

$$H : \mathrm{Conf}_3 S^3 \longrightarrow \mathrm{V}_2\mathbb{R}^4 \quad , \quad (x, y, z) \longmapsto (z, [\mathrm{pr}_z x - \mathrm{pr}_z y])$$

(recall that the square brackets signify unit normalization), and will show that it is a homotopy equivalence whose composition with the canonical projection

$$P : \mathrm{V}_2\mathbb{R}^4 \longrightarrow \mathrm{G}_2\mathbb{R}^4 \quad , \quad (z, v) \longmapsto \langle z, v \rangle$$

is homotopic to the Grassmann map G.

To see this, first observe that there is a deformation retraction of $\mathrm{Conf}_3 S^3$ to its subspace

$$V \;=\; \{(v, -z, z) \mid v \perp z\}$$

defined as follows. Start with $(x, y, z) \in \mathrm{Conf}_3 S^3$, and consider the points $\mathrm{pr}_z x$ and $\mathrm{pr}_z y$ in $z^\perp$. Translation in $z^\perp$ moves $\mathrm{pr}_z y$ to the origin, and then dilation in $z^\perp$ makes the translated $\mathrm{pr}_z x$ into a unit vector. Conjugating this motion by pr_z moves x to $[\mathrm{pr}_z x - \mathrm{pr}_z y]$, moves y to $-z$, and leaves z fixed, as pictured in Figure 11, thus defining a deformation retraction of $\mathrm{Conf}_3 S^3$ to its subspace V, sending (x, y, z) to $([\mathrm{pr}_z x - \mathrm{pr}_z y], -z, z)$.

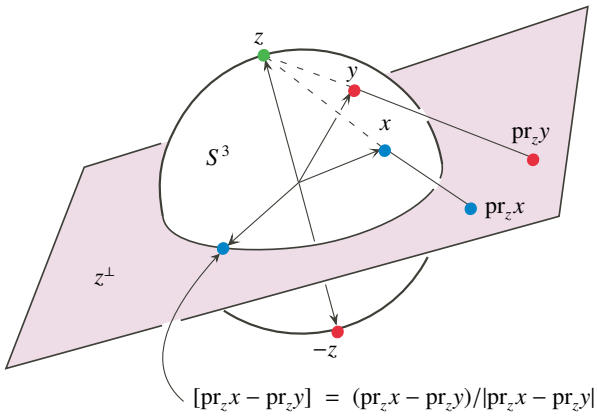


FIG. 11. The deformation retraction $\mathrm{Conf}_3 S^3 \to V$

Identifying V with $\mathrm{V}_2\mathbb{R}^4$ via $(v, -z, z) \leftrightarrow (z, v)$, this shows that H is a homotopy equivalence with homotopy inverse I given by $I(z, v) = (v, -z, z)$. The calculation

$$GI(z, v) \;=\; \langle v - z, -2z \rangle \;=\; \langle z, v \rangle \;=\; P(z, v)\,,$$

shows that $GI = P$, and so $G \simeq PH$ as claimed.

Now observe that there are two natural homeomorphisms from the Stiefel manifold $\mathrm{V}_2\mathbb{R}^4$ to $S^3 \times S^2$ that arise from viewing S^3 as the unit quaternions and S^2 as the pure imaginary unit quaternions, namely $(z, v) \mapsto (z, v\bar{z})$ and $(z, v) \mapsto (z, \bar{z}v)$. These yield projections $\pi_\pm : \mathrm{V}_2\mathbb{R}^4 \to S^2$ given by

$$\pi_+(z, v) \;=\; v\bar{z} \qquad \text{and} \qquad \pi_-(z, v) = \bar{z}v.$$

These are just the lifts to $\mathrm{V}_2\mathbb{R}^4$ of the previously defined projections $\pi_\pm : \mathrm{G}_2\mathbb{R}^4 \to S^2$ with the same names.

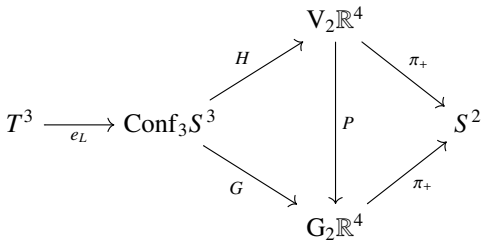
Define the ***asymmetric Gauss map*** $h_L : T^3 \to S^2$ to be the composition $\pi_+ He_L$:

$$h_L(s,t,u) \;=\; [\mathrm{pr}_z x - \mathrm{pr}_z y]\,\bar{z}$$

where as usual $x = x(s)$, $y = y(t)$ and $z = z(u)$ parametrize the components of L, and the square brackets indicate unit normalization.

Proposition IV.2. ***The two versions g_L and h_L of the Gauss map of a three-component link L in S^3 are homotopic. Furthermore, these maps are independent, up to homotopy, of the choice of projections π_+ or π_- used in their definitions.***

Proof. By definition $g_L = \pi_+ G e_L$ and $h_L = \pi_+ H e_L$, shown in the diagram below as the maps from left to right across the bottom and top, respectively.



Here $e_L(s,t,u) = (x(s), y(t), z(u))$ records the parametrization of the link, and G and H are the Grassmann and Stiefel maps with their associated projections π_+. It was shown above that the left triangle in the diagram commutes up to homotopy, and the right triangle commutes on the nose. Therefore h_L and g_L are homotopic.

Now the same argument shows that the maps $h'_L = \pi_- He_L$ and $g'_L = \pi_- Ge_L$ are homotopic. Hence to complete the proof, it suffices to show that h_L and h'_L are homotopic. To do so, view $\mathrm{V}_2\mathbb{R}^4$ as the unit tangent bundle of S^3, with projection p to the base S^3 given by $p(z,v) = z$. Then the composition pHe_L is null-homotopic, since it maps onto the third component of L, and so the map He_L is homotopic to a map into any S^2-fiber of the bundle $\mathrm{V}_2\mathbb{R}^4 \to S^3$. We choose the fiber over $z = 1$, where the two projections $\pi_+(z,v) = v\bar{z}$ and $\pi_-(z,v) = \bar{z}v$ coincide. It follows that h_L and h'_L are homotopic. □

A Gaussian view of the asymmetric Gauss map

The formula above for h_L involves first normalizing a vector in $z^\perp$, and then multiplying by $\bar{z}$ to move it to the unit sphere S^2 in the pure imaginary quaternion 3-space $\mathbb{R}^3$. We would like to express this directly as the normalization of a vector in $\mathbb{R}^3$.

A geometric argument shows that stereographic projection pr_a is given by

$$\mathrm{pr}_a b \;=\; (\mathrm{Im}\; b\bar{a})a/(1 - \mathrm{Re}\; b\bar{a})$$

and it follows that $(\mathrm{pr}_a b)c = \mathrm{pr}_{ac}\, bc$ for any three unit quaternions a, b and c with $a \neq b$. Hence the formula $h_L(s,t,u) = [\mathrm{pr}_z x - \mathrm{pr}_z y]\,\bar{z}$ can be rewritten as

$$h_L(s,t,u) \;=\; [\mathrm{pr}_1\, x\bar{z} - \mathrm{pr}_1\, y\bar{z}] \;=\; [\mathrm{pr}_{-1}(-y\bar{z}) - \mathrm{pr}_{-1}(-x\bar{z})]$$

where, as usual, $x = x(s)$, $y = y(t)$ and $z = z(u)$ parametrize L, and the square brackets indicate unit normalization in $\mathbb{R}^3$. We favor the last expression because stereographic projection from -1 is orientation-preserving, while from 1 it is orientation-reversing.

Now for any two distinct unit quaternions a and z, introduce the abbreviation

$$a_z \;=\; \mathrm{pr}_{-1}(-a\bar{z}) \;\in\; \mathbb{R}^3\,,$$

and so in particular $a_{-1} = \mathrm{pr}_{-1}\, a$. Then we can write

$$h_L(s,t,u) \;=\; [y_z - x_z]\,.$$

For fixed u, this is just the classical Gauss map for the two-component link $X_z \cup Y_z \subset \mathbb{R}^3$ that is the image of the first two components $X \cup Y$ of L under the map $a \mapsto a_z$. Thus the asymmetric Gauss map h_L can be viewed as a one-parameter family of classical Gauss maps for images in 3-space of $X \cup Y$. The third component Z provides the parameter and determines the axes about which these images are gradually twisted.

In the next section, we will explain this perspective more carefully. But we can see right now that it yields an easy proof of the first part of Theorem A, equating the pairwise linking numbers of the link L with the degrees of the restriction of its Gauss map g_L to the coordinate 2-tori.

Proof of the first statement in Theorem A

It can be arranged by an isotopy of L that $z(0) = -1$. Then the restriction of h_L to the coordinate 2-torus $S^1 \times S^1 \times 0$ is precisely the classical Gauss map of the stereographic image $X_{-1} \cup Y_{-1}$ of $X \cup Y$, whose degree is equal to the linking number $\mathrm{Lk}(X, Y)$ since pr_{-1} is orientation-preserving. Since g_L is homotopic to h_L, the same is true for g_L. But then it follows from the symmetry of g_L that $\mathrm{Lk}(X, Z)$ and $\mathrm{Lk}(Y, Z)$ are given by the degrees of g_L on $S^1 \times 0 \times S^1$ and on $0 \times S^1 \times S^1$, respectively.

The proof of the second statement in Theorem A – which relates the triple linking number of L to the Pontryagin ν-invariant of its Gauss map – is more delicate. It will occupy us for the next two sections.

V. THE PONTRYAGIN ν-INVARIANT OF THE GENERALIZED GAUSS MAP

Fix a link L in S^3 with three components X, Y and Z parametrized by $x = x(s)$, $y = y(t)$ and $z = z(u)$. Recalling that S^3 is regarded as the group of unit quaternions, the asymmetric Gauss map $h_L : T^3 \to S^2$ is given by

$$h_L(s, t, u) \;=\; [y_z - x_z]$$

where a_z is an abbreviation for the vector $\mathrm{pr}_{-1}(-a\bar{z})$ in pure imaginary quaternion 3-space $\mathbb{R}^3$, and the square brackets indicate unit normalization. To carry out the proof of Theorem A, we need a way to compute the absolute Pontryagin ν-invariant of h_L. A procedure for doing so is described here.

Throughout this section and the next, $\mathbb{R}^3$ is pictured in the usual way, with the ij-plane horizontal and the k-axis pointing straight up as in Figure 12. In particular, we view k as the north pole of the unit sphere S^2.

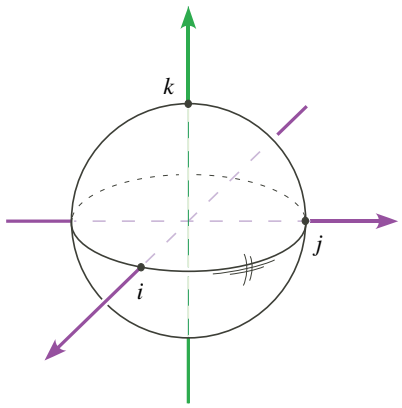


FIG. 12. Pure imaginary quaternion 3-space

Outline of the procedure for computing $\nu(h_L)$

First, we will move L by a link homotopy into a favorable position – referred to as ***generic*** below – so that, in particular, the north pole k of S^2 is a regular value of h_L.

Then we will construct a toral diagram (in the sense of Section III) for the associated ***Pontryagin link***

$$\mathcal{L} \;=\; h_L^{-1}(k) \;\subset\; T^3\,.$$

Roughly speaking, our approach is as follows.

By definition of h_L, the Pontryagin link $\mathcal{L}$ consists of all $(s,t,u) \in T^3$ for which the vector in $\mathbb{R}^3$ from x_z to y_z points straight up, where $x = x(s)$, $y = y(t)$ and $z = z(u)$. The genericity of L will imply that for some points (s,t) in the 2-torus T^2 there is a *unique* $u = u(s,t) \in S^1$ that will make this happen, while for all other points (s,t), *no* u will work. Furthermore, the set $\mathcal{D}$ of all points of the first kind, called ***isogonal points*** for reasons explained below, is a smooth 1-dimensional submanifold of T^2 whose components we call ***icycles***.

Thus $\mathcal{L}$ is the graph of the function $u(s,t)$ over the collection $\mathcal{D}$ of icycles in T^2:

$$\mathcal{L} \;=\; \{(s,t,u) \in T^3 \mid (s,t) \in \mathcal{D} \text{ and } u = u(s,t)\}\,.$$

When suitably oriented and decorated with framing and vertical winding numbers, $\mathcal{D}$ will be the desired toral diagram of $\mathcal{L}$.

In particular, the icycles in $\mathcal{D}$ correspond to certain oriented cycles of vectors directed from X to Y, which we call ***bicycles***, that are easily spotted from a picture of L. Each bicycle has a ***longitudinal*** and ***meridional degree***, recording how much it turns and spins relative to the standard open book structure on S^3. The framing and vertical winding number of the corresponding icycle are determined by these degrees.

Therefore a diagram for $\mathcal{L}$ can be constructed once we identify the bicycles in L. With this diagram in hand, the methods of Section III can then be used to compute $\nu(h_L)$.

We now give the details of this procedure. There are three geometries involved: spherical geometry in S^3, where the link L lives, Euclidean geometry in $\mathbb{R}^3$, the setting for the asymmetric Gauss map, and hyperbolic geometry in the complex upper half plane $\mathbb{H}$, which turns out to be for us the natural geometry on the pages of the "standard" open book decomposition of $\mathbb{R}^3$. We begin with an explicit construction of this open book, which provides a framework for the discussion that follows.

The standard open books in $\mathbb{R}^3$ and S^3

Consider the great circle K in S^3 through 1 and k, and the orthogonal great circle C through i and j. Orient both circles by left complex multiplication by K (i.e. from 1 toward k on K, and i toward j on C) so that their linking number is $+1$.

Stereographic projection from -1 carries K onto the k-axis in $\mathbb{R}^3$, and fixes C, which now appears as the unit circle in the ij-plane. The complement $\mathbb{V}$ of the k-axis in $\mathbb{R}^3$ is naturally identified with the product of a circle S^1 (viewed as the quotient $\mathbb{R}/2\pi\mathbb{Z}$) with the complex upper half plane $\mathbb{H}$ (which for later purposes will be viewed as the hyperbolic plane) via the diffeomorphism

$$\mathbb{V} \;\longrightarrow\; S^1 \times \mathbb{H}$$

with coordinates $\ell : \mathbb{V} \longrightarrow S^1$ and $m : \mathbb{V} \longrightarrow \mathbb{H}$ given by

$$\ell(q) \;=\; \arg(q_1 + q_2 i) \qquad \text{and} \qquad m(q) \;=\; q_3 + |q_1 + q_2 i|\, i\,.$$

for $q = q_1 i + q_2 j + q_3 k \in \mathbb{V} = \mathbb{R}^3 - k$-axis. In other words, if $q = (r,\theta,z)$ in cylindrical coordinates, then $\ell(q) = \theta$ and $m(q) = z + ri$. We call ℓ and m the ***longitudinal*** and ***meridional projections*** in $\mathbb{R}^3$, and refer to $\ell(q)$ as the ***polar angle*** of q.

The longitudinal projection defines the ***standard open book*** in $\mathbb{R}^3$, with binding the k-axis, and with pages $P_\theta = \ell^{-1}(\theta)$ for $\theta \in S^1$. The pages are just the oriented vertical half-planes bounded by the k-axis, each indexed by its polar angle as shown in Figure 13. The meridional projection serves to identify each page P_θ with the hyperbolic plane $\mathbb{H}$, with "center" $i_\theta = i\cos\theta + j\sin\theta$ corresponding to i. The union of all the page centers is the unit circle C.

The longitudinal and meridional projections in $\mathbb{R}^3$ lift to S^3 (in the complement of K) by composing with stereographic projection from -1, given by $\mathrm{pr}_{-1} q = (\mathrm{Im}\, q)/(1 + \mathrm{Re}\, q) = (q_1 i + q_2 j + q_3 k)/(1 + q_0)$. Relying on the context, we continue to denote them by ℓ and m, and to refer to $\ell(q)$ as the polar angle of q. Explicitly, for $q = q_0 + q_1 i + q_2 j + q_3 k \in S^3 - K$,

$$\ell(q) \;=\; \arg(q_1 + q_2 i) \qquad \text{and} \qquad m(q) \;=\; \frac{q_3 + |q_1 + q_2 i|\, i}{1 + q_0}\,.$$

Just as in $\mathbb{R}^3$, the longitudinal projection in S^3 defines the ***standard open book*** in S^3, with binding K, and with pages $H_\theta = \ell^{-1}(\theta)$. The pages are now open great hemispheres in S^3 with centers (i.e. poles) i_θ along the great circle C, and with K as equator. By design, pr_{-1} maps each hemispherical page H_θ in S^3 onto the corresponding half-planar page P_θ in $\mathbb{R}^3$.

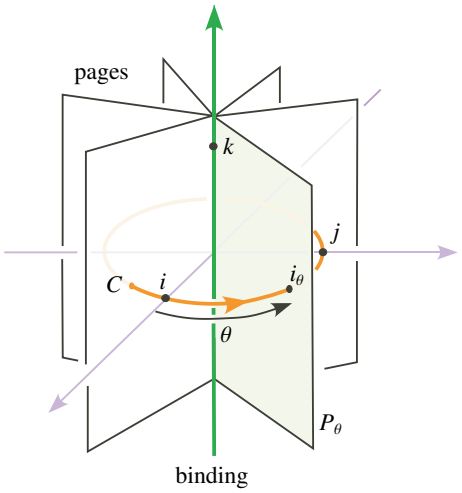


FIG. 13. The standard open book in $\mathbb{R}^3$

Generic links

A three-component link $L = X \cup Y \cup Z$ in S^3 is ***generic*** if

1. Z coincides with the oriented binding K of the standard open book, and
2. X and Y wind "generically" around Z.

More precisely, the second condition requires the restriction to $X \cup Y$ of the longitudinal projection $\ell : S^3 - K \longrightarrow S^1$, which sends each point to its polar angle, to be a Morse function with just one critical point per critical value. Geometrically, this means X and Y are transverse to the pages of the standard open book in S^3, except for finitely many pages where exactly one of them turns around at a single point. These will be called the ***critical points*** of L, while all other points on $X \cup Y$ will be called ***regular points***. Each regular point w has a sign, denoted **sign(*w*)**, when viewed as an intersection point of L with the page containing w. Thus $\text{sign}(w) = +1$ or -1 according to whether L is oriented in the direction of increasing or decreasing polar angle near w.

An example of a generic link is shown in Figure 14. It has four critical points, two on X and two on Y, indicated by dots in the picture.

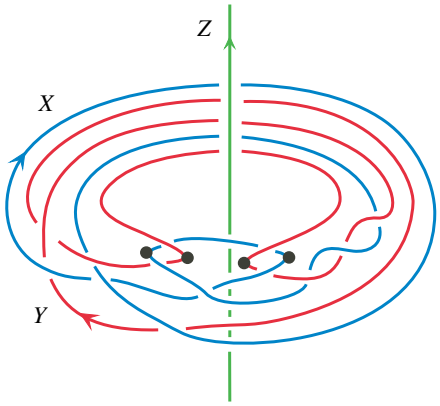


FIG. 14. A generic link

Any three-component link in S^3 is evidently link homotopic to a generic one: first unknot the last component by a link homotopy and move it to coincide with K, and then adjust the first two components by a small isotopy to satisfy the genericity condition (2). *From this point on we assume that L is generic without further mention.* As a consequence, we will show the following:

(a) The north pole $k \in S^2$ is a regular value of the Gauss map h_L.

(b) There is a simple method for constructing a toral diagram for the associated Pontryagin link $\mathcal{L} = h_L^{-1}(k)$ from a picture of L.

This is the content of the "bicycle theorem" below. To state it precisely, we need to introduce the key notion of a *bicycle* in L, and its associated *icycle* in T^2.

Bicycles and icycles

Assume, as always, that the components X, Y and Z of L are parametrized by smooth functions $x = x(s)$, $y = y(t)$ and $z = z(u)$ with nowhere vanishing derivatives. In particular, points (s,t) in the 2-torus parametrize pairs of points $x \in X$ and $y \in Y$. Suppose that x and y have the same polar angle θ, or equivalently that they lie in a common page H_θ of the standard open book in S^3. Then we call (s,t) an ***isogonal point*** in T^2, and call (x,y) a ***page vector*** in L with polar angle θ, reflecting the fact that the vector in $\mathbb{R}^3$ from $\mathrm{pr}_{-1}(x)$ to $\mathrm{pr}_{-1}(y)$ lies entirely on the half-planar page P_θ.

A page vector (x,y) will be called ***critical*** if x or y is a critical point of L, and ***regular*** if both x and y are regular. A regular page vector is ***positive*** if the oriented strands of L through x and y point in the same direction, meaning $\mathrm{sign}(x) = \mathrm{sign}(y)$, and is ***negative*** if they point in opposite directions. These notions are illustrated in Figure 15, in which the vectors labeled 1 and 2 are positive regular page vectors, 3 is negative regular, and 4 is critical.

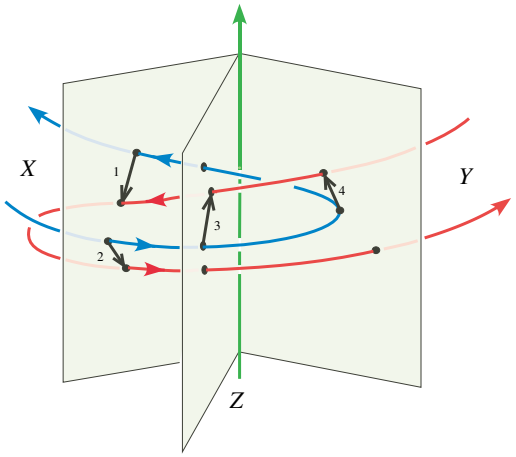


FIG. 15. Page vectors

Now consider the spaces

$$\mathcal{D} = \{\text{isogonal points in } T^2\} \qquad \text{and} \qquad \mathcal{P} = \{\text{page vectors in } L\}.$$

By definition $\mathcal{D}$ parametrizes $\mathcal{P}$. The genericity of L implies that $\mathcal{P}$ consists of a finite number of disjoint cycles of page vectors, and that $\mathcal{D}$ consists of a finite collection of smooth simple closed curves in T^2. Orient $\mathcal{P}$ so that it points to the right (meaning in the direction of increasing polar angle) at each positive regular page vector in it, and to the left at each negative regular page vector. This gives a well-defined orientation on $\mathcal{P}$, inducing one on $\mathcal{D}$ as well. We call these the ***preferred orientations*** on $\mathcal{P}$ and $\mathcal{D}$.

Definition. A ***bicycle*** (or "bi-cycle") in L is a component $\mathcal{P}_i$ of $\mathcal{P}$, that is, an oriented cycle of page vectors. Each bicycle is parametrized by a component $\mathcal{D}_i$ of $\mathcal{D}$, which we call its associated ***icycle*** (or "i-cycle").

Some examples of bicycles and their associated icycles

We first draw in Figure 16 four local pictures of a bicycle near a regular page vector, and below them, their parametrizing icycles. It is understood that these pictures take place somewhere in front of the upward pointing Z axis. The four cases represent the possible directions of X and Y relative to the page containing the vector. In each case, the orientation of the bicycle is indicated by a squiggly arrow.

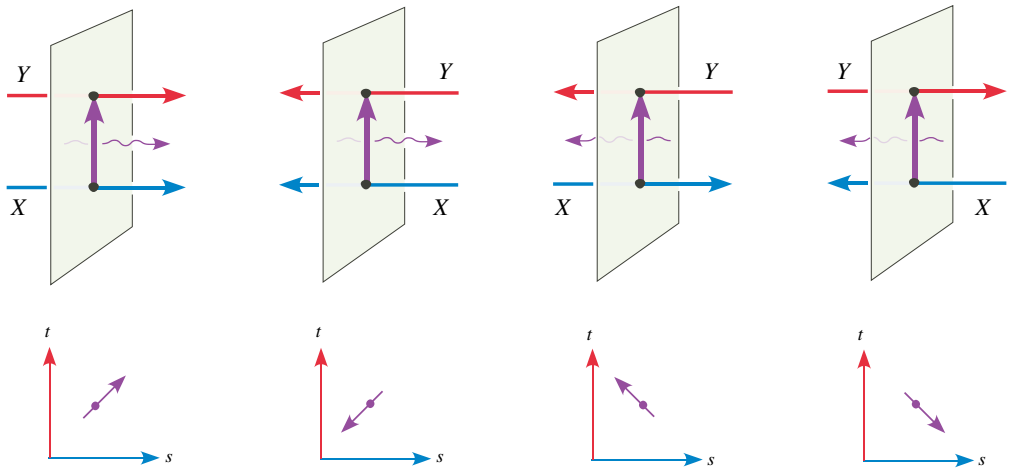
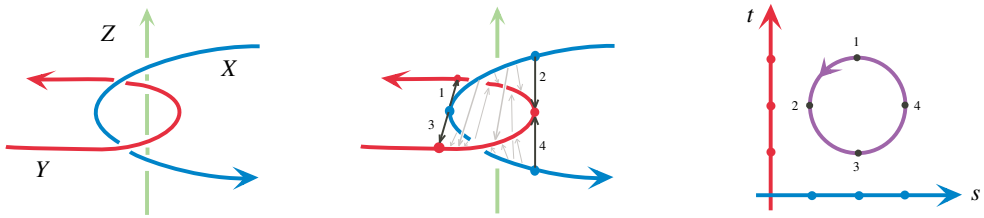


FIG. 16. Local pictures of a bicycle and its corresponding icycle

For an example of a full bicycle, consider the "clasp" between X and Y pictured in Figure 17(a). This gives rise to the bicycle in Figure 17(b), passing successively through the vectors labeled $1, 2, 3, 4$ and then back to 1. The associated icycle is a counterclockwise circle in T^2, shown in Figure 17(c). The route taken by this bicycle is "short" in the sense that it does not wind around the binding, although it does spin within the pages.



(a) the clasp (b) the bicycle (c) the icycle

FIG. 17. Bicycle arising from a clasp

As another example, the link shown in Figure 14 and reproduced in Figure 18(a) below has three bicycles. Two of them are short, arising from the clasps as in the previous example, while the remaining long one oscillates back and forth in the longitudinal direction, eventually making one full revolution around the binding. It is an instructive exercise left to the reader to recover the plot of the associated icycles in Figure 18(b), in which the trivial circles labeled A and B correspond to the clasps in L with the same labels, C labels the icycle that parametrizes the long bicycle, and the corners of the square parametrize the pair (x, y) indicated by the dots in Figure 18(a).

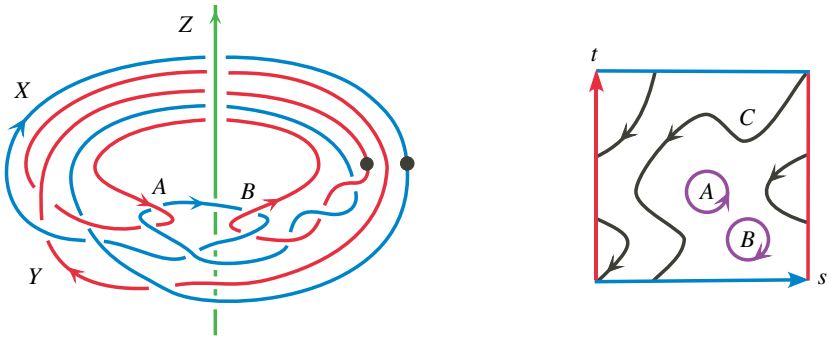
Before proceeding, we remind the reader that our interest in icycles associated to L stems from the fact that – when suitably decorated – they give a toral diagram for a Pontryagin link of the Gauss map h_L. This is the content of the bicycle theorem.

The Bicycle Theorem

The longitudinal and meridional projections on $S^3 - K$, defined earlier, induce projections by the same name on the space $\mathcal{P}$ of page vectors,

$$S^1 \xleftarrow{\ell} \mathcal{P} \xrightarrow{m} \mathbb{H},$$

given by $\ell(x, y) = \ell(x) = \ell(y)$ and $m(x, y) = \arg(m(y) - m(x))$. In other words $\ell(x, y)$ is the polar angle that parametrizes the common hemispherical page in S^3 containing x and y, and $m(x, y)$ is the argument of the vector from $m(x)$ to $m(y)$ in $\mathbb{H}$.



(a) the generic link L (b) the icycles of L

FIG. 18. A generic link and its icycles

Using these projections, we define the ***longitudinal*** and ***meridional degrees*** of a bicycle $\mathcal{P}_i$ in L by

$$\ell_i \;=\; \deg(\ell\,|\,\mathcal{P}_i) \qquad \text{and} \qquad m_i \;=\; \deg(m\,|\,\mathcal{P}_i).$$

These integer invariants record, respectively, the number of times $\mathcal{P}_i$ travels around the binding Z, and the number of times its vectors spin around in the pages as it goes.

For example, any bicycle arising from a clasp between X and Y has zero longitudinal degree, while its meridional degree can be ± 1. In particular, the one shown in Figure 17 has meridional degree -1, while the ones labeled A and B in Figure 18(a) have degrees 1 and -1, respectively. The long bicycle in Figure 18(a), labeled C in Figure 18(b), has longitudinal degree 2 and meridional degree 1.

For any icycle $\mathcal{D}_i$ in T^2, parametrizing a bicycle $\mathcal{P}_i$ in L, define the ***framing*** n_i and ***vertical winding number*** r_i of $\mathcal{D}_i$ by

$$n_i \;=\; -\ell_i - m_i \qquad \text{and} \qquad r_i \;=\; m_i$$

where ℓ_i and m_i are the longitudinal and meridional degrees of $\mathcal{P}_i$.

We can now state the main result of this section.

Bicycle Theorem V.1. ***Let L be a generic link in S^3. Then***

(a) The north pole $k \in S^2$ is a regular value of $h_L : T^3 \to S^2$.

(b) The collection $\mathcal{D}$ of icycles of L, together with their framings and vertical winding numbers as defined above, forms a toral diagram for the associated Pontryagin link $\mathcal{L} = h_L^{-1}(k)$.

The proof will be given in Appendix C. We now illustrate how the bicycle theorem can be used to compute the Pontryagin invariant of the Gauss map of a generic link.

Computing $\nu(h_L)$ for a generic link L using the bicycle theorem

As a first example, again consider the link L pictured in Figure 18(a). As noted above, it has three bicyles A, B, C, with longitudinal degrees $0, 0, 2$, meridional degrees $1, -1, 1$, and so by definition, framings $-1, 1, -3$ and vertical winding numbers $1, -1, 1$.

By the bicycle theorem, the Pontryagin link for the Gauss map h_L has toral diagram as shown in Figure 18(b) with vertical winding numbers 1, -1 and 1 on the icycles A, B and C, and with global framing $n = -1 + 1 - 3 = -3$. Thus the total vertical winding number is $r = 1 - 1 + 1 = 1$ and from the diagram we compute the horizontal winding numbers to be $p = -1$ and $q = -2$. (These values for the winding numbers of the diagram are confirmed by the calculations $p = \mathrm{Lk}(Y, Z) = -1$, $q = \mathrm{Lk}(X, Z) = -2$ and $r = \mathrm{Lk}(X, Y) = 1$.) Thus the invariant $\nu(h_L)$ is well defined modulo $2 = 2\gcd(-1, -2, 1)$.

Using a base point in the lower right corner of the diagram, and straight line paths from the icycles to the base point, the depths of the icycles A, B and C are 1, -1 and -1. Thus by Proposition III.1 we conclude that

$$\nu(h_L) \;=\; n + pq + \textstyle\sum r_i d_i \;=\; -3 + 2 + 1 \;=\; 0 \;\in\; \mathbb{Z}_2.$$

Although the purpose of this example is to illustrate how the bicycle theorem is used for computations, we note that Theorem A (yet to be proved) yields the same result here effortlessly, since it implies that the Pontryagin invariant of the Gauss map of any three-component link in S^3 is even. Therefore, when the pairwise linking numbers p, q and r are relatively prime, as they are in this case, the computation is modulo $2\,\gcd(p,q,r) = 2$, and so the Pontryagin invariant is zero.

Double crossing changes

For our next example we analyze the effect on $\nu(h_L)$ of changing two crossings of opposite signs between the first two components of a generic link $L = X \cup Y \cup Z$. This will be a key step in our inductive proof of Theorem A.

To this end, choose a positive and a negative crossing between X and Y in a suitable projection of L, and let (x_+, y_+) and (x_-, y_-) be the corresponding page vectors. Changing both of the crossings yields a new link $\widehat{L}$, with the same pairwise linking numbers as L. This is illustrated in Figure 19 where L is the Borromean rings, shown on the left, and $\widehat{L}$ is the unlink, shown on the right.

(a) the Borromean rings L (b) the unlink $\widehat{L}$

FIG. 19. A double crossing change

We then say that $\widehat{L}$ is obtained from L by a ***double crossing change***, and propose to use the bicycle theorem to compute the resulting change $\Delta\nu = \nu(h_{\widehat{L}}) - \nu(h_L)$ in Pontryagin invariants.

As it turns out, there is a simple formula for $\Delta\nu$ involving the link L_0 obtained from L by "smoothing" both crossings in the usual way:

The smoothed link L_0 has three components, P and Q (replacing X and Y) and Z. The component P goes from + to − along Y, and then back from − to + along X, where we retain the + and − labels after smoothing, while Q goes from + to − along X, and then back from − to + along Y. This is illustrated in Figure 20 for the double crossing change shown in Figure 19, where we use X_{+-} to denote the arc on X from + to −, and so forth. Now, as a consequence of the Bicycle Theorem V.1, we have the following formula for $\Delta\nu$, whose proof appears in Appendix D:

Corollary V.2. (Double Crossing Formula) ***If $L = X \cup Y \cup Z$ is transformed into $\widehat{L}$ by a double crossing change, and P and Q are the components of the associated smoothing of $X \cup Y$, as explained above, then the corresponding Pontryagin invariants change by***

$$\Delta\nu \;=\; 2\,\mathrm{Lk}(P,Z) \;=\; -2\,\mathrm{Lk}(Q,Z) \;\in\; \mathbb{Z}_{2\,\gcd(p,q,r)}$$

where p, q and r are the pairwise linking numbers of the components of L.

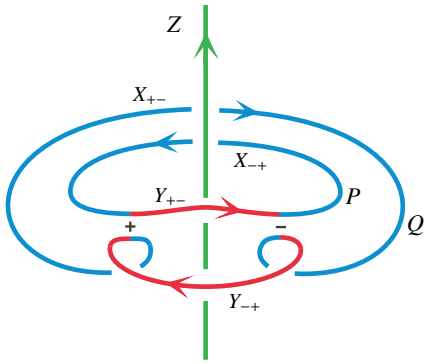
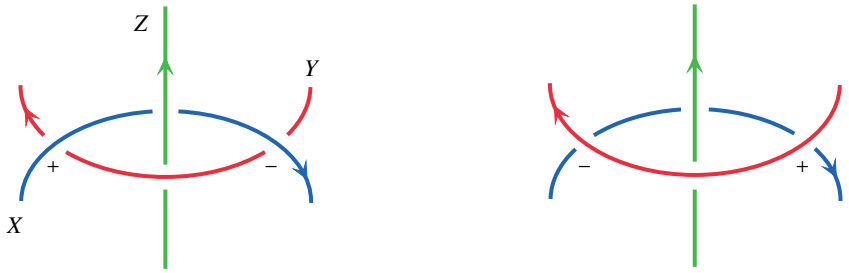


FIG. 20. The smoothed link $L_0 = P \cup Q \cup Z$

For the inductive step of the proof of Theorem A in the next section, we will need to apply this formula for $\Delta\nu$ under the double crossing change $L \to \widehat{L}$ shown in Figure 21 (which will be seen to be equivalent to a delta move). It is understood that L and $\widehat{L}$ should coincide outside the picture, where in fact they can be arbitrary. Indeed, if not generic outside the ball, they can be adjusted by a link homotopy to become so, and then the methods described above apply. Since the component Q of the smoothed link L_0 is just a meridian of Z with $\mathrm{Lk}(Q, Z) = -1$, it follows that $\Delta\nu = 2$.

Thus we have proved the following:



(a) the link L (b) the link $\widehat{L}$

FIG. 21. A simple double crossing change

Corollary V.3. ***If two links L and $\widehat{L}$ coincide outside a 3-ball, and appear in the ball as shown in Figure 21, then $\nu(h_{\widehat{L}}) = \nu(h_L) + 2$.***

VI. PROOF OF THEOREM A

Let L be a three-component link in the 3-sphere with Gauss map $g_L : T^3 \to S^2$. Theorem A asserts that the degrees of g_L on the coordinate 2-tori of T^3 are equal to the pairwise linking numbers p, q and r of the components of L, and that Pontryagin's absolute ν-invariant of g_L is equal to twice Milnor's triple linking number $\mu(L)$ mod $2\gcd(p, q, r)$.

The first statement was proved easily at the end of Section IV. We now prove the second statement by an inductive argument, relying heavily on the techniques and results of the last section. Throughout, we will use the asymmetric Gauss map $h_L : T^3 \to S^2$ in place of g_L, since these two maps are homotopic.

Proof of the base case of Theorem A

In Example II.1 we introduced the "base links" L_{pqr} with pairwise linking numbers p, q and r and $\mu(L_{pqr}) = 0$. See Figure 3. We will show that Theorem A holds for these links.

To accomplish this, we must show that for each p, q and r, the associated Gauss map $h_{L_{pqr}}$ is homotopic to the base map $f_{pqr} : T^3 \to S^2$ used in Section III to convert the relative Pontryagin ν-invariant to an absolute ν-invariant. In other words, we must show that $\nu(h_{L_{pqr}}) = 0$.

First move L_{pqr} by a link homotopy into generic position, and let $L = X \cup Y \cup Z$ be the resulting link. Then Z coincides with the binding K of the standard open book, while X and Y wind around Z in a generic fashion, q and p times respectively, linking each other r times along the way. In fact this winding can be made *monotonic*, to appear as shown in Figure 22 for the case $(p, q, r) = (5, 3, -2)$.

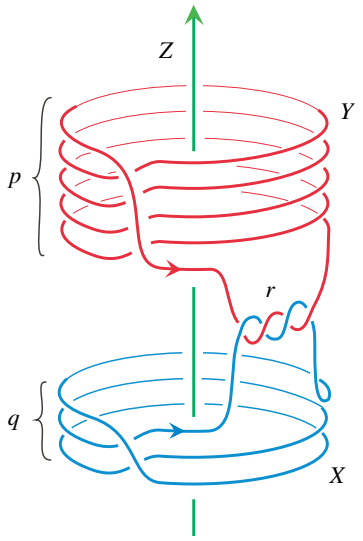


FIG. 22. A generic link L representing L_{pqr}

Observe that one need not actually construct a link homotopy carrying L_{pqr} to L, but need only check that the pairwise and triple linking numbers of L_{pqr} and L coincide, and then appeal to Milnor's classification. The verification that $\mu(L) = 0$ is completely analogous to the calculation for $\mu(L_{pqr})$, using the obvious stacked disks, joined by half twisted bands, as Seifert surfaces for X and Y, and a hemispherical page in the open book on S^3 as a Seifert surface for Z.

Now the bicycles in L are easily identified. There are $d = \gcd(p, q)$ of them, all of longitudinal degree $\ell = \operatorname{lcm}(p, q)$, and all but one of meridional degree 0, the remaining one being of degree r. The parametrizing icycles are all parallel to a $(p/d, q/d)$ torus knot in the 2-torus, with vertical winding numbers all equal to zero, except one equal to r. By the bicycle theorem, these icyles form a diagram for the Pontryagin link of h_L, with global framing $-(pq + r)$. This is exactly the situation described in Example III.2, and so

$$\nu(h_{L_{pqr}}) = \nu(h_L) = 0 \in \mathbb{Z}_{2\gcd(p,q,r)}$$

as asserted.

Proof of the inductive step of Theorem A

First recall from Example II.2 that any three-component link L with pairwise linking numbers p, q and r is link homotopic to a link obtained from L_{pqr} by a sequence of delta moves of the type shown in Figure 4 (or its inverse), by a result of Murakami and Nakanishi[7]. It was shown in that example that each such move increases $\mu(L)$ by 1.

Now observe that such a delta move $\Delta : L \to \widehat{L}$ can be viewed as a double crossing change $\mathbb{D}$, of the kind shown in Figure 21, composed with an isotopy, as indicated in Figure 23. By Corollary V.3, this double crossing change increases $\nu(L)$ by 2, and so Δ does the same.

Since this argument applies to an arbitrary delta move, it follows that $\nu(L) = 2\mu(L)$ for *all* three-component links L. This completes the proof of Theorem A.

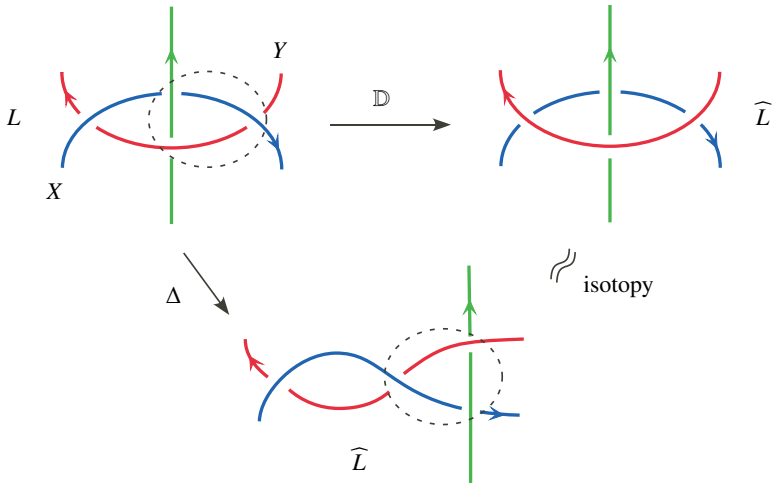


FIG. 23. Another view of the delta move

VII. PROOF OF THEOREM B, FORMULAS (1) AND (2)

Theorem B provides three formulas for Milnor's μ-invariant of a three-component link L in S^3 whose pairwise linking numbers vanish:

(1) (differential forms) $\mu(L) \;=\; 1/2 \int_{T^3} d^{-1}(\omega_L) \wedge \omega_L \;=\; 1/2 \int_{T^3} \delta(\varphi * \omega_L) \wedge \omega_L$

(2) (vector fields) $\mu(L) \;=\; 1/2 \int_{T^3 \times T^3} \vec{\mathbf{v}}_L(\mathbf{x}) \times \vec{\mathbf{v}}_L(\mathbf{y}) \bullet \nabla_{\mathbf{y}} \varphi(\mathbf{x}-\mathbf{y})\; d\mathbf{x}\, d\mathbf{y}$

(3) (Fourier series) $\mu(L) \;=\; 8\pi^3 \sum_{\mathbf{n}\neq\mathbf{0}} \mathbf{a_n} \times \mathbf{b_n} \bullet \mathbf{n}/|\mathbf{n}|^2$

The notation is explained in the introduction.

In the subsections that follow, we obtain explicit expressions for the characteristic 2-form ω_L and vector field $\vec{\mathbf{v}}_L$, and for the fundamental solution φ to the scalar Laplacian on T^3, and then establish formula (1) by making use of J. H. C. Whitehead's integral formula for the Hopf invariant. Then, we show how to obtain formula (2) from formula (1). Finally, in Section VIII we review the calculus of differential forms and Fourier analysis on T^3, in order to make explicit the role and form of φ and obtain formula (3).

Explicit formula for the 2-form ω_L and the vector field $\vec{\mathbf{v}}_L$ on T^3

Recall the formula for the Gauss map $g_L\colon T^3 \to S^2$ of the link L given in Proposition IV.1,

$$g_L(s,t,u) \;=\; \frac{F(x,y,z)}{|F(x,y,z)|},$$

where $x = x(s)$, $y = y(t)$ and $z = z(u)$ parametrize the components of L, and where F was given in Section IV as

$$F(x,y,z) \;=\; (ix\bullet y + iy\bullet z + iz\bullet x\,,\; jx\bullet y + jy\bullet z + jz\bullet x\,,\; kx\bullet y + ky\bullet z + kz\bullet x).$$

Here we view x, y and z as quaternions for the multiplication and as vectors in $\mathbb{R}^4$ when performing the dot product. For simplicity, we write $g_L = F/|F|$, suppressing the appearance of the evaluation map $e_L(s,t,u) = (x(s), y(t), z(u))$.

Again let ω be the Euclidean area 2-form on the unit 2-sphere $S^2 \subset \mathbb{R}^3$, normalized so that the total area is 1 instead of 4π. If $\mathbf{p}$ is a point of S^2, and $\mathbf{a}$ and $\mathbf{b}$ are tangent vectors to S^2 at $\mathbf{p}$, then

$$\omega_{\mathbf{p}}(\mathbf{a},\mathbf{b}) \;=\; \frac{1}{4\pi}(\mathbf{a}\times\mathbf{b})\bullet\mathbf{p}.$$

This 2-form ω on S^2 extends to a closed 2-form $\overline{\omega}$ in $\mathbb{R}^3 - \mathbf{0}$ given by

$$\overline{\omega}_{\mathbf{p}}(\mathbf{a}, \mathbf{b}) \;=\; \frac{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{p}}{4\pi\,|\mathbf{p}|^3},$$

which is the pullback of ω from S^2 to $\mathbb{R}^3 - \mathbf{0}$ via the map $\mathbf{p} \mapsto \mathbf{p}/|\mathbf{p}|$. Hence the pullback $g_L^*\omega$ of ω from S^2 to T^3 via $g_L = F/|F|$ is the same as the pullback $F^*\overline{\omega}$ of $\overline{\omega}$ from $\mathbb{R}^3 - \mathbf{0}$ to T^3 via F.

Write

$$g_L^*\omega \;=\; F^*\overline{\omega} \;=\; a(s,t,u)\; dt \wedge du + b(s,t,u)\; du \wedge ds + c(s,t,u)\; ds \wedge dt.$$

Then we have

$$\begin{aligned} a(s,t,u) \;&=\; F^*\overline{\omega}(\partial_t, \partial_u) \;=\; \overline{\omega}(F_*\partial_t, F_*\partial_u) \\ &=\; \overline{\omega}(F_t, F_u) \;=\; \frac{(F_t \times F_u) \cdot F}{4\pi\,|F|^3}, \end{aligned}$$

and likewise for $b(s,t,u)$ and $c(s,t,u)$, where the subscripts on F denote partial derivatives.

Therefore, the characteristic 2-form of the link L is

$$\omega_L \;=\; g_L^*\omega \;=\; \frac{F_t \times F_u \cdot F}{4\pi\,|F|^3}\; dt \wedge du + \frac{F_u \times F_s \cdot F}{4\pi\,|F|^3}\; du \wedge ds + \frac{F_s \times F_t \cdot F}{4\pi\,|F|^3}\; ds \wedge dt,$$

and its corresponding characteristic vector field is

$$\vec{\mathbf{v}}_L \;=\; \frac{F_t \times F_u \cdot F}{4\pi\,|F|^3}\; \partial_s + \frac{F_u \times F_s \cdot F}{4\pi\,|F|^3}\; \partial_t + \frac{F_s \times F_t \cdot F}{4\pi\,|F|^3}\; \partial_u.$$

Proof of Theorem B, formula (1)

Let L be a three-component link in S^3 with pairwise linking numbers p, q and r all zero. By the first part of Theorem A these numbers are the degrees of the Gauss map $g_L\colon T^3 \to S^2$ on the 2-dimensional coordinate subtori. Since these degrees are all zero, g_L is homotopic to a map $g\colon T^3 \to S^2$ which collapses the 2-skeleton of T^3 to a point:

$$g_L \;\simeq\; g\colon T^3 \xrightarrow{\;\sigma\;} S^3 \xrightarrow{\;f\;} S^2,$$

where σ is the collapsing map. By the second part of Theorem A, Milnor's μ-invariant of L is equal to half of Pontryagin's ν-invariant $\nu(g_L)$ (comparing g_L to the constant map f_{000} as explained in Section III), which in turn is just the Hopf invariant of $f\colon S^3 \to S^2$,

$$\mu(L) \;=\; \tfrac{1}{2}\nu(g_L) \;=\; \tfrac{1}{2}\,\mathrm{Hopf}(f).$$

We can thus use J. H. C. Whitehead's integral formula for the Hopf invariant as the first ingredient in our formula for the μ invariant. Starting from Hopf's definition of his invariant of a map $f\colon S^3 \to S^2$ as the linking number between the inverse images of two regular values, Whitehead[14] found an integral formula for $\mathrm{Hopf}(f)$ as follows.

Let ω be the area 2-form on S^2, normalized so that $\int_{S^2} \omega = 1$. Its pullback $f^*\omega$ is a closed 2-form on S^3, which is exact because $H^2(S^3;\mathbb{R}) = 0$. Hence $f^*\omega = d\alpha$ for some 1-form α on S^3, and Whitehead showed that the Hopf invariant of f is given by the formula

$$\mathrm{Hopf}(f) \;=\; \int_{S^3} \alpha \wedge f^*\omega \;=\; \int_{S^3} d^{-1}(f^*\omega) \wedge f^*\omega\,,$$

the value of the integral being independent of the choice of α.

One way to make Whitehead's formula explicit, and to make the integrand geometrically natural, is to choose $\alpha = \alpha_0$, the 1-form of *least* L^2 *norm* for which $d\alpha_0 = f^*\omega$. Although we do not follow this approach here, we show explicitly how to do so in Appendix F since this may be of independent interest.

Instead, we are going to pull the whole situation back to T^3 and perform our calculations there. In particular, the formula for Hopf(f) above pulls back to the formula

$$\nu(g_L) \;=\; \int_{T^3} \overline{\alpha} \wedge g^*\omega,$$

where $\overline{\alpha}$ is any 1-form on T^3 such that $d\overline{\alpha} = g^*\omega$. One such choice would be $\overline{\alpha} = \sigma^*\alpha_0$. However, since we do not have an explicit analytic formula for σ, and since the formula for α_0 is complicated, we will pursue a slightly different approach.

Taking advantage of the fact that g_L is homotopic to g, we also have

$$\nu(g_L) \;=\; \int_{T^3} \alpha \wedge g_L^*\omega \;=\; \int_{T^3} \alpha \wedge \omega_L$$

for any 1-form α on T^3 such that $d\alpha = \omega_L$. Therefore, what we really need is a canonical way to produce such a 1-form α on T^3. We will prove in Proposition VIII.2 that

$$\alpha_L \;=\; \delta(\varphi * \omega_L)$$

has the desired property that $d\alpha_L = \omega_L$. Moreover, if $\widetilde{\alpha}$ is any other 1-form such that $d\widetilde{\alpha} = \omega_L$, then $|\,\alpha_L\,| \le |\,\widetilde{\alpha}\,|$ in $L^2(T^3)$, with equality if and only if $\widetilde{\alpha} = \alpha_L$.

We thus obtain the explicit integral formula for Milnor's μ-invariant of the three component link L:

$$\mu(L) \;=\; \tfrac{1}{2}\nu(g_L) \;=\; \frac{1}{2}\int_{T^3} \delta(\varphi * \omega_L) \wedge \omega_L,$$

which is formula (1) of Theorem B.

Proof of Theorem B, formula (2)

The 1-form $\delta(\varphi * \omega_L)$ on T^3 which appears in formula (1) of Theorem B converts to the vector field $\nabla\times(\varphi * \vec{\mathbf{v}}_L)$, which can be regarded as the magnetic field on T^3 due to the current flow $\vec{\mathbf{v}}_L$. The customary minus sign here is now hidden in the definitions of the Laplacian and its inverse (Green's operator).

The integral formula for Milnor's μ-invariant given in formula (1),

$$\mu(L) \;=\; \frac{1}{2}\int_{T^3} \delta(\varphi * \omega_L) \wedge \omega_L\,,$$

then converts to the formula

$$\mu(L) \;=\; \frac{1}{2}\int_{T^3} \Big(\nabla \times (\varphi * \vec{\mathbf{v}}_L)\Big) \bullet \vec{\mathbf{v}}_L \, d\,\mathrm{vol}$$

in the language of vector fields. To obtain formula (2) of Theorem B, we first expand out the convolution integral

$$(\varphi * \vec{\mathbf{v}}_L)(\mathbf{x}) \;=\; \int_{T^3} \vec{\mathbf{v}}_L(\mathbf{y})\, \varphi(\mathbf{x}-\mathbf{y})\, d\mathbf{y}.$$

Then we compute its curl:

$$\Big(\nabla_{\mathbf{x}} \times (\varphi * \vec{\mathbf{v}}_L)\Big)(\mathbf{x}) \;=\; \int_{T^3} \nabla_{\mathbf{x}} \times \Big(\vec{\mathbf{v}}_L(\mathbf{y})\, \varphi(\mathbf{x}-\mathbf{y})\Big)\, d\mathbf{y} \;=\; -\int_{T^3} \vec{\mathbf{v}}_L(\mathbf{y}) \times \nabla_{\mathbf{x}}\, \varphi(\mathbf{x}-\mathbf{y})\, d\mathbf{y}\,,$$

using the product formula $\nabla \times (f\vec{\mathbf{a}}) = f(\nabla \times \vec{\mathbf{a}}) - \vec{\mathbf{a}} \times \nabla f$. Note that $\nabla_{\mathbf{x}} \times \vec{\mathbf{v}}_L(\mathbf{y}) = \mathbf{0}$ since, from the point of view of the variable $\mathbf{x}$, the vector field $\vec{\mathbf{v}}_L(\mathbf{y})$ is constant.

Inserting the expression for the curl of $\varphi * \vec{\mathbf{v}}_L$ into the formula for $\mu(L)$, we get

$$\begin{aligned}
\mu(L) \;&=\; -\frac{1}{2}\int_{T^3\times T^3} \Big(\vec{\mathbf{v}}_L(\mathbf{y}) \times \nabla_{\mathbf{x}}\, \varphi(\mathbf{x}-\mathbf{y})\Big) \,\bullet\, \vec{\mathbf{v}}_L(\mathbf{x})\, d\mathbf{x}\, d\mathbf{y} \\
&=\; -\frac{1}{2}\int_{T^3\times T^3} \vec{\mathbf{v}}_L(\mathbf{x}) \times \vec{\mathbf{v}}_L(\mathbf{y}) \,\bullet\, \nabla_{\mathbf{x}}\, \varphi(\mathbf{x}-\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} \\
&=\; \frac{1}{2}\int_{T^3\times T^3} \vec{\mathbf{v}}_L(\mathbf{x}) \times \vec{\mathbf{v}}_L(\mathbf{y}) \,\bullet\, \nabla_{\mathbf{y}}\, \varphi(\mathbf{x}-\mathbf{y})\, d\mathbf{x}\, d\mathbf{y}\,,
\end{aligned}$$

where at the last step we hid the minus sign by taking the gradient of φ with respect to $\mathbf{y}$ instead of $\mathbf{x}$, and obtained formula (2) of Theorem B.

VIII. FOURIER SERIES AND THE PROOF OF THEOREM B, FORMULA (3)

Fourier series and the fundamental solution of the Laplacian

In the proofs of formulas (1) and (2) of Theorem B, we needed to find a 1-form α on T^3 whose exterior derivative is the exact 2-form ω_L associated with the three-component link L in S^3. We asserted that we can choose $\alpha = \delta(\varphi * \omega_L)$, where φ is the fundamental solution of the scalar Laplacian on T^3. Furthermore, we asserted that this choice of α is canonical in the sense that it has the smallest L^2 norm among all possible choices. We justify these assertions in this section and lay the groundwork for the proof of formula (3) of Theorem B by studying the calculus of differential forms on $T^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ in terms of their Fourier series.

We will prove the following two results:

Proposition VIII.1. ***The fundamental solution of the scalar Laplacian on the 3-torus $T^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ is given by the formula***

$$\varphi(\mathbf{x}) \;=\; \frac{1}{8\pi^3} \sum_{\mathbf{n}\neq\mathbf{0}} e^{i\mathbf{n}\bullet\mathbf{x}}/|\mathbf{n}|^2.$$

The function φ is C^∞ at all points $\mathbf{x} \in T^3$ except $\mathbf{0}$, where it becomes infinite.

Proposition VIII.2. ***If ω is any exact differential form on T^3 with C^∞ coefficients, then***

$$\alpha \;=\; \delta(\varphi * \omega)$$

is a C^∞ differential form satisfying $d\alpha = \omega$. Furthermore, if $d\widetilde{\alpha} = \omega$ as well, then $|\alpha|_{L^2} \le |\widetilde{\alpha}|_{L^2}$, with equality if and only if $\widetilde{\alpha} = \alpha$.

Before diving into calculations, we pause for some words of explanation: We write $\Omega^k(T^3)$ for the space of C^∞ k-forms on T^3. With d as the exterior differentiation operator taking $\Omega^k(T^3)$ to $\Omega^{k+1}(T^3)$, and δ the co-differentiation map adjoint to d in the L^2 sense, the Laplacian of a k-form α is

$$\Delta\alpha \;=\; (d\delta + \delta d)\alpha.$$

This definition gives us the "geometer's sign convention" for the Laplacian on functions (0-forms):

$$\Delta f \;=\; -\left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}\right).$$

Proof of Proposition VIII.1

The *fundamental solution* of the scalar Laplacian is a function φ, convolution with which "inverts" the Laplacian to the extent that this is possible. On T^3, only functions that integrate to zero are in the range of the Laplacian, and so "the" fundamental solution of Δ on T^3 is the function φ which satisfies

$$\int_{T^3} \varphi \, d\mathrm{vol} \;=\; 0 \qquad \text{and} \qquad \Delta(\varphi * f) \;=\; f$$

for all $f \in C^\infty(T^3)$ such that $\displaystyle\int_{T^3} f \, d\mathrm{vol} = 0$.

Even though we have expressed φ in terms of complex exponentials, the value of φ is real for real values of $\mathbf{x}$ because of the symmetry of the coefficients.

Figure 24 shows the graph of the corresponding fundamental solution

$$\varphi(\mathbf{x}) \;=\; \frac{1}{4\pi^2} \sum_{\mathbf{n}\neq\mathbf{0}} e^{i\mathbf{n}\cdot\mathbf{x}}/|\mathbf{n}|^2$$

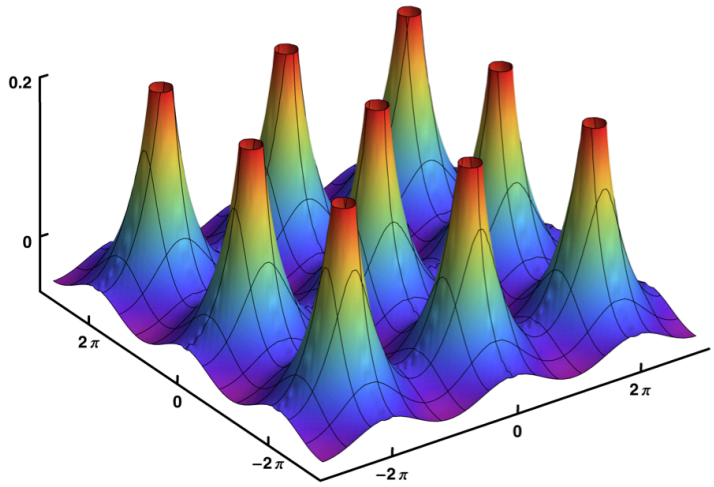


FIG. 24. Fundamental solution of the scalar Laplacian on $S^1 \times S^1$

of the scalar Laplacian on the 2-torus $S^1 \times S^1$, summed for $|\mathbf{n}| \leq 10$, and displayed over the range $[-3\pi, 3\pi] \times [-3\pi, 3\pi]$.

With respect to the L^2 inner product

$$\langle f, g \rangle \;=\; \int_{T^3} f(\mathbf{x})\, \overline{g(\mathbf{x})}\, d\mathbf{x}\,,$$

the set of complex exponentials $\{e^{i\mathbf{n}\cdot\mathbf{x}} \mid \mathbf{n} \in \mathbb{Z}^3\}$ is an orthogonal set, and $\langle e^{i\mathbf{n}\cdot\mathbf{x}}, e^{i\mathbf{n}\cdot\mathbf{x}} \rangle = 8\pi^3$ for all $\mathbf{n}$.

Any function f in $L^2(T^3)$ can be expanded into a Fourier series

$$f \;\sim\; \sum_{\mathbf{n}\in\mathbb{Z}^3} c_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{x}}$$

where

$$c_{\mathbf{n}} \;=\; \frac{1}{8\pi^3}\langle f, e^{i\mathbf{n}\cdot\mathbf{x}} \rangle \;=\; \frac{1}{8\pi^3} \int_{T^3} f(\mathbf{x})\, e^{-i\mathbf{n}\cdot\mathbf{x}}\, d\mathbf{x}.$$

Moreover, by Plancharel's theorem, the $L^2(T^3)$-norm of f is equal to the ℓ^2-norm of the sequence of coefficients:

$$|f|_{L^2} \;=\; \int_{T^3} |f(x)|^2\, d\mathbf{x} \;=\; \sum_{\mathbf{n}\in\mathbb{Z}^3} 8\pi^3 |c_{\mathbf{n}}|^2.$$

This shows that the function φ of Proposition VIII.1 is in $L^2(T^3)$, since $|c_{\mathbf{n}}|^2$ is of order $1/|\mathbf{n}|^4$, while the number of lattice points at distance $|\mathbf{n}|$ from $\mathbf{0}$ is of order $|\mathbf{n}|^2$.

Because the range of the scalar Laplacian is the (closed) subspace of functions f for which

$$\int_{T^3} f(\mathbf{x})\, d\mathbf{x} \;=\; 0\,,$$

we will denote this subspace of $L^2(T^3)$ by $L^2_0(T^3)$, and similarly for other function spaces, where a zero subscript indicates that all functions in the space have average value zero. In terms of Fourier coefficients, $f \in L^2_0(T^3)$ if and only if the Fourier coefficient $c_{\mathbf{0}} = c_{(0,0,0)} = 0$.

For functions f and g in $C^\infty(T^3)$, their ***convolution*** $f * g$ is defined by

$$\big(f * g\big)(\mathbf{x}) \;=\; \int_{T^3} f(\mathbf{y})\, g(\mathbf{x}-\mathbf{y})\, d\mathbf{y}\,,$$

and also lies in $C^\infty(T^3)$. One checks easily that $e^{i\mathbf{m}\cdot\mathbf{x}} * e^{i\mathbf{n}\cdot\mathbf{x}} = 0$ if $\mathbf{m} \neq \mathbf{n}$, and $e^{i\mathbf{n}\cdot\mathbf{x}} * e^{i\mathbf{n}\cdot\mathbf{x}} = 8\pi^3 e^{i\mathbf{n}\cdot\mathbf{x}}$. Therefore, if f and g have Fourier series $\sum c_\mathbf{n} e^{i\mathbf{n}\cdot\mathbf{x}}$ and $\sum d_\mathbf{n} e^{i\mathbf{n}\cdot\mathbf{x}}$, then the Fourier series of $f * g$ is

$$8\pi^3 \sum_{\mathbf{n}\in\mathbb{Z}^3} c_\mathbf{n} d_\mathbf{n} e^{i\mathbf{n}\cdot\mathbf{x}}.$$

From this it follows that the operation of convolution satisfies $f * g = g * f$ and $(f * g) * h = f * (g * h)$ whenever all the convolutions are defined. Furthermore,

$$\frac{\partial}{\partial x_k}(f * g) \;=\; \frac{\partial f}{\partial x_k} * g \;=\; f * \frac{\partial g}{\partial x_k}.$$

For $u \in C^\infty(T^3)$, with

$$u(\mathbf{x}) \;=\; \sum_{\mathbf{n}\in\mathbb{Z}^3} b_\mathbf{n} e^{i\mathbf{n}\cdot\mathbf{x}}\,,$$

we have

$$\Delta u(\mathbf{x}) \;=\; \sum_{\mathbf{n}\neq\mathbf{0}} |\mathbf{n}|^2 b_\mathbf{n} e^{i\mathbf{n}\cdot\mathbf{x}}.$$

Because the $(0,0,0)$-coefficient of Δu is zero, this shows why in order to solve $\Delta u = f$ we must restrict f to be in $C_0^\infty(T^3)$. And for $f \in C_0^\infty(T^3)$, with

$$f(\mathbf{x}) \;=\; \sum_{\mathbf{n}\neq\mathbf{0}} c_\mathbf{n} e^{i\mathbf{n}\cdot\mathbf{x}},$$

the unique solution $u \in C_0^\infty(T^3)$ of $\Delta u = f$ has $b_\mathbf{n} = c_\mathbf{n}/|\mathbf{n}|^2$, and so, since

$$\varphi(\mathbf{x}) \;=\; \frac{1}{8\pi^3} \sum_{\mathbf{n}\neq\mathbf{0}} e^{i\mathbf{n}\cdot\mathbf{x}}/|\mathbf{n}|^2\,,$$

we conclude that $u = \varphi * f$. This completes the proof of Proposition VIII.1 except for the assertion about the smoothness of φ away from its singularity, which follows from well-known local regularity theorems for elliptic partial differential equations (see, for example, Folland[15], Theorem 6.33).

Fourier series and the calculus of differential forms on the 3-torus

To prove Proposition VIII.2, we express the calculus of differential forms on T^3 in terms of Fourier series. Except for the fundamental solution φ of the (scalar) Laplacian on T^3, we will assume all functions and forms are C^∞.

We continue to express functions (0-forms) as Fourier series: for $\mathbf{x} = (s,t,u)$ and $\mathbf{n} = (n_1,n_2,n_3)$, we write

$$f(\mathbf{x}) \;=\; \sum_{\mathbf{n}\in\mathbb{Z}^3} c_\mathbf{n} e^{i\mathbf{n}\cdot\mathbf{x}} \;\in\; \Omega^0(T^3).$$

Likewise, we employ the notation described in the introduction and express 1-forms using $\mathbf{c}_\mathbf{n} = (c^s_\mathbf{n}, c^t_\mathbf{n}, c^u_\mathbf{n})$ and $d\mathbf{x} = (ds, dt, du)$. We write

$$\alpha(\mathbf{x}) \;=\; \sum_{\mathbf{n}\in\mathbb{Z}^3} \mathbf{c}_\mathbf{n} e^{i\mathbf{n}\cdot\mathbf{x}} \bullet d\mathbf{x} \;\in\; \Omega^1(T^3).$$

With $\star d\mathbf{x} = (dt\wedge du, du\wedge ds, ds\wedge dt)$ as before, we can express a 2-form as

$$\beta(\mathbf{x}) \;=\; \sum_{\mathbf{n}\in\mathbb{Z}^3} \mathbf{c}_\mathbf{n} e^{i\mathbf{n}\cdot\mathbf{x}} \bullet \star\, d\mathbf{x} \;\in\; \Omega^2(T^3).$$

Finally, with $dV = ds\wedge dt\wedge du$, a 3-form can be written as

$$\gamma(\mathbf{x}) \;=\; \sum_{n\in\mathbb{Z}^3} c_\mathbf{n} e^{i\mathbf{n}\cdot\mathbf{x}} dV \;\in\; \Omega^3(T^3).$$

It is straightforward to express both the exterior derivative d and the codifferential δ in terms of Fourier coefficients. With f, α, β and γ as above, we have

$$\begin{aligned}
df &= \sum ic_{\mathbf{n}}\mathbf{n}e^{i\mathbf{n}\cdot\mathbf{x}}\bullet d\mathbf{x} && \text{and} \quad \delta f = 0,\\
d\alpha &= \sum i\mathbf{n}\times\mathbf{c}_{\mathbf{n}}e^{i\mathbf{n}\cdot\mathbf{x}}\bullet \star\, d\mathbf{x} && \text{and} \quad \delta\alpha = -\sum i\mathbf{n}\bullet\mathbf{c}_{\mathbf{n}}e^{i\mathbf{n}\cdot\mathbf{x}}\\
d\beta &= \sum i\mathbf{n}\bullet\mathbf{c}_{\mathbf{n}}e^{i\mathbf{n}\cdot\mathbf{x}}dV && \text{and} \quad \delta\beta = \sum i\mathbf{n}\times\mathbf{c}_{\mathbf{n}}e^{i\mathbf{n}\cdot\mathbf{x}}\bullet d\mathbf{x}\\
d\gamma &= 0 && \text{and} \quad \delta\gamma = -\sum ic_{\mathbf{n}}\mathbf{n}e^{i\mathbf{n}\cdot\mathbf{x}}\bullet \star\, d\mathbf{x}
\end{aligned}$$

Thus, exterior differentiation and co-differentiation are expressed in terms of vector algebraic operations on the Fourier coefficients. From these expressions, we conclude the following about the kernel and image of d and δ:

- For 0-forms,
$$\begin{aligned}
\ker d &= \{f \mid c_{\mathbf{n}} = 0 \text{ for } \mathbf{n}\neq\mathbf{0},\ \text{while}\ \ c_{\mathbf{0}} \text{ is arbitrary}\}\\
\operatorname{im} d &= \{0\}\\
\ker\delta &= \{\text{all } f\}\\
\operatorname{im}\delta &= \{f \mid c_{\mathbf{0}} = 0\}
\end{aligned}$$

- For 1-forms,
$$\begin{aligned}
\ker d &= \{\alpha \mid \mathbf{c}_{\mathbf{n}} = \lambda_{\mathbf{n}}\mathbf{n} \text{ for } \mathbf{n}\neq\mathbf{0},\ \text{while } \mathbf{c}_{\mathbf{0}} \text{ is arbitrary}\}\\
\operatorname{im} d &= \{\alpha \mid \mathbf{c}_{\mathbf{n}} = \lambda_{\mathbf{n}}\mathbf{n} \text{ for } \mathbf{n}\neq\mathbf{0},\ \mathbf{c}_{\mathbf{0}} = \mathbf{0}\}\\
\ker\delta &= \{\alpha \mid \mathbf{c}_{\mathbf{n}}\bullet\mathbf{n} = 0 \text{ for } \mathbf{n}\neq\mathbf{0},\ \text{while } \mathbf{c}_{\mathbf{0}} \text{ is arbitrary}\}\\
\operatorname{im}\delta &= \{\alpha \mid \mathbf{c}_{\mathbf{n}}\bullet\mathbf{n} = 0 \text{ for } \mathbf{n}\neq\mathbf{0},\ \mathbf{c}_{\mathbf{0}} = \mathbf{0}\}
\end{aligned}$$

- For 2-forms,
$$\begin{aligned}
\ker d &= \{\beta \mid \mathbf{c}_{\mathbf{n}}\bullet\mathbf{n} = 0 \text{ for } \mathbf{n}\neq\mathbf{0},\ \text{while } \mathbf{c}_{\mathbf{0}} \text{ is arbitrary}\}\\
\operatorname{im} d &= \{\beta \mid \mathbf{c}_{\mathbf{n}}\bullet\mathbf{n} = 0 \text{ for } \mathbf{n}\neq\mathbf{0},\ \mathbf{c}_{\mathbf{0}} = \mathbf{0}\}\\
\ker\delta &= \{\beta \mid \mathbf{c}_{\mathbf{n}} = \lambda_{\mathbf{n}}\mathbf{n} \text{ for } \mathbf{n}\neq\mathbf{0},\ \text{while } \mathbf{c}_{\mathbf{0}} \text{ is arbitrary}\}\\
\operatorname{im}\delta &= \{\beta \mid \mathbf{c}_{\mathbf{n}} = \lambda_{\mathbf{n}}\mathbf{n} \text{ for } \mathbf{n}\neq\mathbf{0},\ \mathbf{c}_{\mathbf{0}} = \mathbf{0}\}
\end{aligned}$$

- For 3-forms,
$$\begin{aligned}
\ker d &= \{\text{all } \gamma\}\\
\operatorname{im} d &= \{\gamma \mid \mathbf{c}_{\mathbf{0}} = 0\}\\
\ker\delta &= \{\gamma \mid \mathbf{c}_{\mathbf{n}} = 0 \text{ for } \mathbf{n}\neq\mathbf{0},\ \text{while}\ \ \mathbf{c}_{\mathbf{0}} \text{ is arbitrary}\}\\
\operatorname{im}\delta &= \{0\}
\end{aligned}$$

Therefore $\operatorname{im} d \subset \ker d$ and $\operatorname{im}\delta \subset \ker\delta$, and the following orthogonal (with respect to the L^2 inner product) decompositions hold:

$$\Omega^k(T^3) = \ker\delta \oplus \operatorname{im} d = \operatorname{im}\delta \oplus \ker d$$

for $k = 0, \ldots, 3$. The k-forms in $\ker d \cap \ker\delta$ are the forms whose Fourier series contain only constant terms. These are called *harmonic* k-forms because they are in the kernel of the Laplacian $\Delta = d\delta + \delta d$, so we write $\mathrm{Har}^k(T^3) = \ker d \cap \ker\delta$. We then have the Hodge decomposition

$$\Omega^k(T^3) = \operatorname{im}\delta \oplus \mathrm{Har}^k(T^3) \oplus \operatorname{im} d.$$

The Laplacian $\Delta\colon \Omega^k(T^3) \to \Omega^k(T^3)$ preserves this Hodge decomposition, taking $\operatorname{im}\delta$ bijectively to itself, killing $\mathrm{Har}^k(T^3)$, and taking $\operatorname{im} d$ bijectively to itself.

Proof of Proposition VIII.2

We must show that if ω is any C^∞ exact differential k-form on T^3, then the $(k-1)$-form $\alpha = \delta(\varphi * \omega)$ is C^∞ and satisfies $d\alpha = \omega$, and that if $d\widetilde{\alpha} = \omega$ as well, then $|\alpha|_{L^2} \le |\widetilde{\alpha}|_{L^2}$, with equality if and only if $\widetilde{\alpha} = \alpha$. We will carry this out specifically for 2-forms, because it is the case we use in Theorem B. The proofs for 1-forms and 3-forms are essentially the same.

For a 2-form

$$\beta \;=\; \sum_{\mathbf{n}\in\mathbb{Z}^3} \mathbf{c_n} e^{i\mathbf{n}\cdot\mathbf{x}} \bullet \;\star\, d\mathbf{x}\,,$$

its Laplacian is given by

$$\Delta\beta \;=\; \sum_{\mathbf{n}\neq\mathbf{0}} |\mathbf{n}|^2 \mathbf{c_n} e^{i\mathbf{n}\cdot\mathbf{x}} \bullet \;\star\, d\mathbf{x}.$$

We can use convolution with the fundamental solution φ of the scalar Laplacian to express the Green's operator,

$$\mathrm{Gr}(\beta) \;=\; \varphi * \beta \;=\; \sum_{\mathbf{n}\neq\mathbf{0}} \frac{\mathbf{c_n}}{|\mathbf{n}|^2} e^{i\mathbf{n}\cdot\mathbf{x}} \bullet \;\star\, d\mathbf{x}.$$

Clearly the Laplacian Δ and the Green's operator are inverses of one another when applied to 2-forms β with $\mathbf{c_0} = \mathbf{0}$, equivalently, to 2-forms β orthogonal to $\mathrm{Har}^2(T^3)$.

To prove Proposition VIII.2, assume that β is exact, that is, $\beta \in \mathrm{im}\, d$. Then $\mathrm{Gr}(\beta)$ is also exact, hence certainly closed, and therefore

$$\beta \;=\; \Delta\,\mathrm{Gr}(\beta) \;=\; (d\delta + \delta d)\,\mathrm{Gr}(\beta) \;=\; d\delta\,\mathrm{Gr}(\beta) \;=\; d\delta(\varphi * \beta).$$

Thus $\alpha = \delta(\varphi * \beta)$ satisfies $d\alpha = \beta$.

If $\widetilde{\alpha}$ is any other 2-form satisfying $d\widetilde{\alpha} = \beta$, then $\widetilde{\alpha}$ differs from α by some closed 2-form, which must be L^2-orthogonal to α since $\alpha \in \mathrm{im}\,\delta$. Thus, $|\alpha|_{L^2} \le |\widetilde{\alpha}|_{L^2}$ by the Pythagorean theorem.

Since the Fourier coefficients of a C^∞ 2-form decrease faster than any negative power of $|\mathbf{n}|$, we have that $\mathrm{Gr}(\beta)$ and $\delta\,\mathrm{Gr}(\beta)$ will be C^∞ if β is. This completes the proof of Proposition VIII.2.

Proof of Theorem B, formula (3)

This formula expresses Milnor's invariant $\mu(L)$ in terms of the Fourier coefficients of ω_L:

$$\mu(L) \;=\; 8\pi^3 \sum_{\mathbf{n}\neq\mathbf{0}} \frac{\mathbf{a_n}\times\mathbf{b_n}\cdot\mathbf{n}}{|\mathbf{n}|^2}$$

where

$$\omega_L \;=\; \sum_{\mathbf{n}\neq\mathbf{0}} \mathbf{c_n} e^{i\mathbf{n}\cdot\mathbf{x}} \cdot \;\star\, d\mathbf{x}$$

with $\mathbf{c_n} = \mathbf{a_n} + i\mathbf{b_n}$ and with $\mathbf{a_n}$ and $\mathbf{b_n}$ real.

The formula for ω_L is summed over $\mathbf{n} \neq \mathbf{0}$ because, as we saw in the Introduction just before the statement of Theorem B, the hypothesis of pairwise linking numbers zero is equivalent to the vanishing of the coefficient $\mathbf{c_0}$ in the Fourier expansion of ω_L.

We begin with formula (1) of Theorem B for $\mu(L)$:

$$\mu(L) \;=\; \frac{1}{2}\int_{T^3} \delta(\varphi * \omega_L) \wedge \omega_L.$$

We have

$$\varphi * \omega_L \;=\; \sum_{\mathbf{n}\neq\mathbf{0}} \frac{\mathbf{c_n} e^{i\mathbf{n}\cdot\mathbf{x}}}{|\mathbf{n}|^2} \cdot \;\star\, d\mathbf{x}\,,$$

and hence, from our table of derivatives and co-derivatives in terms of Fourier series,

$$\delta(\varphi * \omega_L) \;=\; \sum_{\mathbf{n}\neq\mathbf{0}} \frac{(\mathbf{n}\times\mathbf{c_n})\, i\, e^{i\mathbf{n}\cdot\mathbf{x}}}{|\mathbf{n}|^2}\cdot d\mathbf{x}.$$

To compute $\delta(\varphi * \omega_L)\wedge\omega_L$, we use the fact that if $\mathbf{v}\cdot d\mathbf{x}$ is a 1-form and $\mathbf{w}\cdot\star d\mathbf{x}$ is a 2-form, then

$$(\mathbf{v}\cdot d\mathbf{x})\wedge(\mathbf{w}\cdot\star d\mathbf{x}) \;=\; (\mathbf{v}\cdot\mathbf{w})\, dV.$$

Thus

$$\delta(\varphi * \omega_L)\wedge\omega_L \;=\; \sum_{\mathbf{m}\neq\mathbf{0}}\sum_{\mathbf{n}\neq\mathbf{0}} \frac{(\mathbf{n}\times\mathbf{c_n})\cdot\mathbf{c_m}\, i e^{i(\mathbf{m}+\mathbf{n})\cdot\mathbf{x}}}{|\mathbf{n}|^2}\, dV.$$

When we insert this double sum into the integral $\mu(L) = \frac{1}{2}\int_{T^3}\delta(\varphi * \omega_L)\wedge\omega_L$, most of the terms in the summation will integrate to zero, leaving only the terms where $\mathbf{m}+\mathbf{n} = \mathbf{0}$. In those cases, $e^{i(\mathbf{m}+\mathbf{n})\cdot\mathbf{x}} = e^0 = 1$ integrates to $8\pi^3$, the volume of T^3.

Thus

$$\mu(L) \;=\; 4\pi^3\sum_{\mathbf{n}\neq\mathbf{0}}\frac{i(\mathbf{n}\times\mathbf{c_n})\cdot\mathbf{c_{-n}}}{|\mathbf{n}|^2} \;=\; 4\pi^3\sum_{\mathbf{n}\neq\mathbf{0}}\frac{i(\mathbf{c_n}\times\mathbf{c_{-n}})\cdot\mathbf{n}}{|\mathbf{n}|^2}$$

and it remains to simplify this last series.

Since $\mathbf{c_n} = \mathbf{a_n} + i\mathbf{b_n}$ and ω_L is real-valued, we have $\mathbf{c_{-n}} = \bar{\mathbf{c}}_\mathbf{n} = \mathbf{a_n} - i\mathbf{b_n}$. Hence $\mathbf{c_n}\times\mathbf{c_{-n}} = -2i\,\mathbf{a_n}\times\mathbf{b_n}$, and therefore

$$\mu(L) \;=\; 4\pi^3\sum_{\mathbf{n}\neq\mathbf{0}}\frac{i(\mathbf{c_n}\times\mathbf{c_{-n}})\cdot\mathbf{n}}{|\mathbf{n}|^2} \;=\; 8\pi^3\sum_{\mathbf{n}\neq\mathbf{0}}\frac{(\mathbf{a_n}\times\mathbf{b_n})\cdot\mathbf{n}}{|\mathbf{n}|^2},$$

completing the proof of formula (3) and, with it, that of Theorem B.

Appendix A: Background and Motivation

Configuration spaces.

To study the linking of simple closed curves in a 3-manifold M from the perspective of homotopy theory, it is convenient to ignore the knotting of individual components and focus on the relation of ***link homotopy***. For some background on this notion, see Milnor[1,4] and, for example, Massey[16], Casson[17], Turaev[18], Porter[19], Fenn[20], Orr[21], Cochran[22], and Habegger and Lin[23].

Configuration spaces come into the picture as follows. Let L be an ordered, oriented link in M with n components $X, Y, \dots$ parametrized by $x = x(s), y = y(t), \dots$ for $s, t, \dots$ in S^1. Then consider the ***evaluation map***

$$e_L : T^n \;\longrightarrow\; \mathrm{Conf}_n M\,, \qquad (s,t,\dots)\longmapsto(x,y,\dots)$$

from the n-torus $T^n = S^1\times\cdots\times S^1$ to the configuration space $\mathrm{Conf}_n M$ of ordered n-tuples of distinct points in M. Since link homotopies of L become homotopies of e_L, the assignment $L\mapsto e_L$ induces a map from the set $\mathcal{L}_n(M)$ of link homotopy classes of n-component links in M to the set $[T^n, \mathrm{Conf}_n M]$ of homotopy classes of maps from T^n to $\mathrm{Conf}_n M$,

$$e : \mathcal{L}_n(M) \;\longrightarrow\; [T^n, \mathrm{Conf}_n M].$$

We can think of the map e as defining a representation from the world of link homotopy to the world of homotopy, and the basic question is whether or not this representation is faithful, that is, one-to-one.

For two-component links in $\mathbb{R}^3$, the representation e is faithful. The set $\mathcal{L}_2(\mathbb{R}^3)$ is in one-to-one correspondence with the integers via the classical linking number, while the set $[T^2, \mathrm{Conf}_2\mathbb{R}^3]$ is also in one-to-one correspondence with the integers via the Brouwer degree, since $\mathrm{Conf}_2\mathbb{R}^3$ deformation

retracts to S^2. The correspondence e is bijective since the linking number of a two-component link equals the degree of its Gauss map

$$T^2 \longrightarrow S^2 \quad , \quad (s,t) \longmapsto \frac{y-x}{|y-x|}.$$

The profit is the famous integral formula of Gauss[24],

$$\mathrm{Lk}(X,Y) \;=\; \frac{1}{4\pi}\int_{T^2} \frac{dx}{ds}\times\frac{dy}{dt}\bullet\frac{x-y}{|x-y|^3}\;ds\,dt \;=\; \int_{T^2}\frac{dx}{ds}\times\frac{dy}{dt}\bullet\nabla_y\,\varphi(y-x)\,ds\,dt,$$

where $\varphi(x) = 1/4\pi|x|$ is the fundamental solution of the scalar Laplacian in $\mathbb{R}^3$. The integrand is natural, in the sense that it is invariant under the group of orientation-preserving rigid motions of $\mathbb{R}^3$, acting on the link.

By contrast, for two-component links in S^3 the representation e is not faithful. The configuration space Conf_2S^3 has the homotopy type of S^3, and hence all maps of T^2 to Conf_2S^3 are homotopically trivial. In particular, the analogue of Gauss's linking integral in S^3, with an integrand which is geometrically natural in the above sense, cannot be obtained by this route. Nevertheless, such an integral formula was found by DeTurck and Gluck[25] using an alternative route via electrodynamics on the 3-sphere, and independently by Kuperberg[26] via the calculus of double forms.

For higher-dimensional two-component links, the same dichotomy holds. In $\mathbb{R}^n$, Scott[27] and Massey and Rolfsen[28] showed that link homotopy classes of links whose components are copies of S^k and S^{n-k-1} are in bijective correspondence with homotopy classes of maps from $S^k\times S^{n-k-1}$ to the configuration space $\mathrm{Conf}_2\mathbb{R}^n \simeq S^{n-1}$, and consequently with $\pi_{n-1}(S^{n-1}) \cong \mathbb{Z}$. Finding a geometric linking integral is straightforward, as the Gauss linking integral and its proof easily generalize to this setting. Shifting the scene to S^n does not change the link homotopy story, but all maps from $S^k\times S^{n-k-1}$ to the configuration space $\mathrm{Conf}_2S^n \simeq S^n$ are homotopically trivial. Geometrically natural linking integrals still exist in this situation, as demonstrated by DeTurck and Gluck[29] and Shonkwiler and Vela-Vick[30], but finding them requires new techniques.

For n-component ***homotopy Brunnian*** links in $\mathbb{R}^3$ – meaning links that become trivial up to link homotopy when any single component is removed – and analogous links in higher dimensions, Koschorke[31] showed that the representation e is again faithful. This provided the first proof (up to sign) of our Theorem A for the case when the pairwise linking numbers are zero.

The content of the present paper is that, for arbitrary three-component links L in S^3, the representation e is faithful, and that we are led thereby to a natural integral for Milnor's triple linking number when the pairwise linking numbers vanish. The relevant configuration space Conf_3S^3 is easily seen to deformation retract to $S^3\times S^2$, where the S^3 coordinate records one of the three points in each triple (see Section IV). It follows that the evaluation map $e_L : T^3 \to \mathrm{Conf}_3S^3$ is homotopic to a map of T^3 into an S^2 fiber, and this turns out to be, up to homotopy, our generalized Gauss map g_L.

Theorem A asserts, among other things, that L and L' are link homotopic if and only if g_L and $g_{L'}$ are homotopic, and hence that

$$e : \mathcal{L}_3(S^3) \longrightarrow [T^3, \mathrm{Conf}_3S^3]$$

is faithful. Furthermore, it was observed above that if $h \in \mathrm{SO}(4)$ is an orientation preserving isometry of S^3, then $e_{h(L)} = h\cdot e_L$ where $\cdot$ is the diagonal action, and so the integrands in the formulas for Milnor's triple linking number $\mu(L)$ in Theorem B are invariant under the action of SO(4).

In this paper, our constructions and theorems are set specifically in S^3; the corresponding results in $\mathbb{R}^3$ will be treated in a sequel.

Fluid mechanics and plasma physics.

The ***helicity*** of a vector field V defined on a bounded domain Ω in $\mathbb{R}^3$ is given by the formula

$$\mathrm{Hel}(V) \;=\; \frac{1}{4\pi}\int_{\Omega\times\Omega} V(x)\times V(y)\bullet\frac{x-y}{|x-y|^3}\;dx\,dy \;=\; \int_{\Omega\times\Omega} V(x)\times V(y)\bullet\nabla_y\,\varphi(x-y)\,dx\,dy.$$

where, as above, φ is the fundamental solution of the scalar Laplacian in $\mathbb{R}^3$, and dx and dy are volume elements.

Woltjer[32] introduced this notion during his study of the magnetic field in the Crab Nebula, and showed that the helicity of a magnetic field remains constant as the field evolves according to the equations of ideal magnetohydrodynamics, and that it provides a lower bound for the field energy during such evolution. The term "helicity" was coined by Moffatt[33], who also derived the above formula from Woltjer's original expression.

There is no mistaking the analogy with Gauss's linking integral, and no surprise that helicity is a measure of the extent to which the orbits of V wrap and coil around one another. Since its introduction, helicity has played an important role in astrophysics and solar physics, and in plasma physics here on earth.

Looking back at Theorem B, we see that the integral in our formula for Milnor's μ-invariant of a three-component link L in the 3-sphere expresses the helicity of the associated vector field $\bar{\mathbf{v}}_L$ on the 3-torus.

Our study was motivated by a problem proposed by Arnold and Khesin[34] regarding the search for "higher helicities" for divergence-free vector fields. In their own words:

> ***The dream is to define such a hierarchy of invariants for generic vector fields such that, whereas all the invariants of order $\leq k$ have zero value for a given field and there exists a nonzero invariant of order $k+1$, this nonzero invariant provides a lower bound for the field energy.***

Since the helicity integral above is analogous to the Gauss linking integral, the general hope is that higher helicities will be analogous to higher order linking invariants. An alternative approach to helicity may be found in the work of Cantarella and Parsley[35].

The formulation in Theorems A and B has led to partial results that address the case of vector fields on invariant domains such as flux tubes modeled on the Borromean rings; see Komendarczyk[36,37].

Other integral formulas. Previous integral formulas for Milnor's triple linking number and attempts to define a higher order helicity can be found in the work of Massey[16,38], Monastyrsky and Retakh[39], Berger[40,41], Guadagnini, Martellini and Mintchev[42], Evans and Berger[43], Akhmetiev and Ruzmaiken[44,45], Arnold and Khesin[34], Laurence and Stredulinsky[46], Leal[47], Hornig and Mayer[48], Rivière[49], Khesin[50], Bodecker and Hornig[51], Auckly and Kapitanski[13], Akhmetiev[52], and Leal and Pineda[53].

The principal sources for these formulas are Massey triple products in cohomology, quantum field theory in general, and Chern–Simons theory in particular. A common feature of these integral formulas is that choices must be made to fix the domain of integration and the value of the integrand.

Appendix B: Proof of Proposition III.1

This proposition gives a formula for the primary and secondary invariants $\lambda(f)$ and $\nu(f)$, of a smooth map $f : T^3 \to S^2$ whose Pontryagin link $\mathcal{L}$ is represented by a toral diagram in T^2 *without crossings*, with horizontal winding numbers p and q, vertical winding number r, and total framing n:

$$\lambda(f) \;=\; (p,q,r) \qquad \text{and} \qquad \nu(f) \;=\; n + pq + \sum r_i d_i \mod 2\gcd(p,q,r)$$

where r_i and d_i are the vertical winding numbers and depths (with respect to any chosen basepoint) of the components of the diagram.

The formula for $\lambda(f)$ was derived in Section III and, as explained there, is at least well defined modulo $2\gcd(p,q,r)$, independent of the choice of basepoint. So it remains to verify that this is the correct formula for $\nu(f)$.

First observe that, in the absence of crossings, the integers r_i are all that are needed to recover the link $\mathcal{L}$ up to *isotopy*. In particular, each nonvertical component of $\mathcal{L}$ can be taken to wind monotonically around the last circle factor of T^3. Furthermore, equipped with the total framing n, we can recover the framed link $\mathcal{L}$ up to *framed bordism*. To see this, note that a pair of saddle bordisms can be used as shown in Figure 25 to transfer a twist in the framing from any component of $\mathcal{L}$ to any other, and so we can distribute the framings in any desired way among the components.

We now propose to transform $\mathcal{L}$ by a sequence of framed bordisms into $\mathcal{L}_{pqr}$ with some extra twists in the framing. These correspond to homotopies of f and so do not change the Pontryagin invariant $\nu(f)$. We will carry this out on a diagrammatic level, transforming our given toral diagram, with total framing n, to the diagram for $\mathcal{L}_{pqr}$ shown in Figure 7(a) (reproduced below for the reader's convenience) with

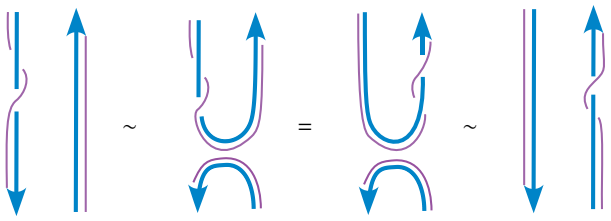


FIG. 25. Using saddles to transfer twists

some total framing, which by definition will equal $\nu(f)$. So we must simply keep track of the change in the total framing as we proceed.

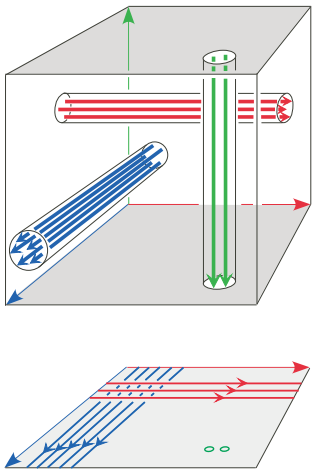


FIG. 7(a). The link $\mathcal{L}_{pqr}$ and its diagram

We first use saddle bordisms to replace the vertical winding of each non-vertical component $\mathcal{L}_i$ by $|r_i|$ zero-framed vertical components, at the cost of adding r_i to the total framing. The new vertical circles are "adjacent" to $\mathcal{D}_i$ – meaning displaced slightly to the *right* of it in the projection – and oriented up or down according to the sign of r_i. This is illustrated in Figure 26. The net effect on the diagram is to reduce all the vertical winding numbers to zero on the components of $\mathcal{D}$ (i.e. to eliminate the internal marked points $\mathcal{N}$), to add $\sum |r_j|$ isolated marked points, and to add $\sum r_j$ to the total framing of the diagram, where the sums are only over the non-vertical components of $\mathcal{L}$.

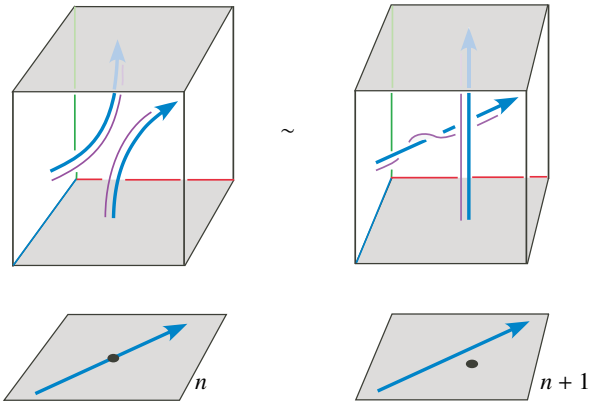


FIG. 26. Using saddles to replace vertical winding by vertical circles

Since our target is the base link $\mathcal{L}_{pqr}$ (with extra twists), the vertical circles must be gathered together. This is the purpose of our base point $*$, which will serve as a gathering spot, and the arcs γ_i, which will

serve as the paths in T^2 along which to move the vertical circles which at the moment are adjacent to the $\mathcal{L}_i$. At each intersection of γ_i with a component of $\mathcal{D}$, these vertical circles must cross the corresponding strand of $\mathcal{L}$. This crossing can be accomplished by a pair of saddle bordisms as indicated in Figure 27, adding ± 2 to the diagram framing. On the diagrammatic level, this can viewed as a two step process, running the bordism shown in Figure 26 backwards and then forwards, in order to move an isolated marked point across a component of $\mathcal{D}$, as shown at the bottom of the figure.

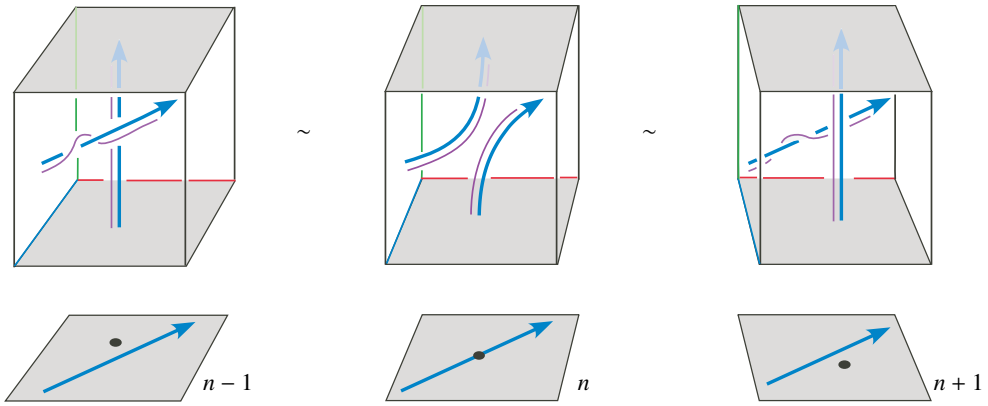


FIG. 27. Using saddles to gather the vertical circles

Now it is easy to check that the act of removing the vertical winding of the nonvertical components of $\mathcal{L}$, and then gathering all the resulting vertical circles (including the original ones corresponding to $\mathcal{M}$) will add $\sum r_i d_i$ to the total framing of the diagram. Visually, this process can be thought of as a "migration" to the base point of all the marked points in the diagram.

Next use disk bordisms to remove the null homotopic components of $\mathcal{D}$ (arranging for them to be 0-framed by storing their twists elsewhere, as explained above) and similarly use annular bordisms to remove any pair of curves that are parallel but oppositely oriented. The diagram now consists of a (p, q) torus link in T^2 (meaning c parallel copies of the (p', q') torus knot, where $p = cp'$ and $q = cq'$ with p' and q' relatively prime) together with r isolated marked points near $*$. The total framing, which was originally equal to n, has changed to $n + \sum r_i d_i$.

Finally we transform the (p, q) torus link by using saddle bordisms (reversing the process described above for removing crossings) into p copies of S^1_s and q copies of S^1_t, grouped as in $\mathcal{L}_{pqr}$, at the cost of adding pq to the total framing. Thus we have arrived at $\mathcal{L}_{pqr}$, having changed the total framing from n to $n + pq + \sum r_i d_i$. This verifies the stated formula for $\nu(f)$, and completes the proof of Proposition III.1. □

Appendix C: Proof of the Bicycle Theorem V.1

The Bicycle Theorem asserts that for a generic link L in S^3 with asymmetric characteristic map $h_L : T^3 \to S^2$ (defined in Section IV), the north pole $k \in S^2$ is a regular value of h_L, and the collection $\mathcal{D}$ of icycles of L together with their framings and vertical winding numbers (as defined in Section V) form a toral diagram for the associated Pontryagin link $\mathcal{L} = h_L^{-1}(k)$.

To prove this, choose a parametrization $x = x(s)$, $y = y(t)$ and $z = z(u)$ for the three components X, Y and Z of L. Then Z coincides with the binding K of the standard open book, which is the circle subgroup of S^3 containing the quaternion k. Looking back at the formula for the Gauss map

$$h_L(s, t, u) \;=\; (y_z - x_z)/|y_z - x_z| \quad \text{where} \quad a_z = \mathrm{pr}_{-1}(-a\bar{z})\,,$$

we see that we must understand the K-action sending $z \in K$ to the automorphism $a \mapsto a\bar{z}$ of S^3, and the induced K-action on $\mathbb{R}^3$ via stereographic projection. Visualizing how these actions transform the pages of the standard open books will aid us in our subsequent arguments, and we do this next.

The geometry of the *K*-action

For any $z = \cos\alpha + k\sin\alpha$ in the binding K, right multiplication by $\bar{z} = \cos\alpha - k\sin\alpha$ is an isometry of S^3 that rotates K by $-\alpha$ radians, likewise rotates the orthogonal great circle C of page centers by α radians, and so advances each hemispherical page H_θ to the page $H_{\theta+\alpha}$ while simultaneously rotating the page by $-\alpha$ radians about its center i_θ. Therefore, as z traverses K, any given page turns once around K, successively occupying the positions vacated by the other pages. During this time, the page spins once negatively about its center, so that in total it is following a left-handed screw motion along C.

This turning of the hemispherical pages about K in S^3 is transferred by stereographic projection to a turning of the half-planar pages about the k-axis in $\mathbb{R}^3$, while the *spherical* rotations of the pages in S^3 become *hyperbolic* rotations of the pages in $\mathbb{R}^3$ about their centers. This is a consequence of the conformality of stereographic projection, which implies that the page identification $H_\theta \leftrightarrow P_\theta$ is conformal. Once again, the net effect is a left-handed screw motion along C.

This description of the K-action has the following technical consequence that is critical for our study of the Gauss map of a generic link.

Lemma C.1. (Twist Lemma) ***Let x and y be distinct points in S^3 lying in the complement of the binding K of the standard open book.***

(a) If x and y lie on different pages, then for $z \in K$, the vector $y_z - x_z$ never lies on a page of the corresponding open book in $\mathbb{R}^3$, and in particular never points straight up.

(b) If x and y lie on the same page, then as z traverses K, the vectors $y_z - x_z$ lie on successive pages in $\mathbb{R}^3$, turning once positively around the binding, and spinning once counterclockwise without backtracking within the pages as they go. In particular, $y_z - x_z$ points straight up for a unique $z = \tau(x, y) \in K$.

Proof. Since the K-action carries pages to pages, x and y will lie on the same page in S^3 if and only if x_z and y_z lie on the same page in $\mathbb{R}^3$ for all $z \in K$. Part (a) of the lemma is now obvious. Using the meridional projection, and the description of the K-action above, part (b) of the lemma translates into the following statement. For any pair of distinct points x and y in the upper half-plane model of the hyperbolic plane $\mathbb{H}$, the function

$$d(\alpha) \;=\; \arg(\mathrm{rot}_\alpha y - \mathrm{rot}_\alpha x)$$

is strictly increasing, where $\mathrm{rot}_\alpha : \mathbb{H} \to \mathbb{H}$ is hyperbolic rotation about i by α radians. Since we wish to prove this for all x and y, it suffices to show that $d'(0) > 0$ (because $d'(\alpha)$ for one choice of x and y is equal to $d'(0)$ for some other choice of x and y).

To prove this, we transfer the problem to the Poincaré disk $\mathbb{D}$ using the conformal map

$$f : \mathbb{D} \;\longrightarrow\; \mathbb{H} \quad , \quad f(z) \;=\; \frac{1-z}{1+z}\, i$$

that sends 0 to i. Hyperbolic rotation of $\mathbb{H}$ about i by any angle is conjugate by f to euclidean rotation of $\mathbb{D}$ by the same angle. Thus we must show that for any pair of distinct points x and y in $\mathbb{D}$, the function

$$g(\alpha) \;=\; \arg(f(e^{i\alpha}y) - f(e^{i\alpha}x))$$

has positive derivative at $\alpha = 0$. Noting that $f(a) - f(b) = 2i(b-a)/(1+a)(1+b)$ and using the fact that the argument function converts products into sums and quotients into differences, we find that $g(\alpha)$ differs by a constant from the function

$$h(\alpha) \;=\; \alpha - \arg(1 + e^{i\alpha}x) - \arg(1 + e^{i\alpha}y),$$

and so it remains to show that $h'(0) > 0$. But a simple geometric argument using the central angle theorem from elementary plane geometry shows that, for any given $z \in \mathbb{D}$, the derivative of the function $\arg(1 + e^{i\alpha}z)$ at $\alpha = 0$ is strictly less than $1/2$. Therefore $h'(0) > 1 - 1/2 - 1/2 \;=\; 0$ as desired. □

It follows from the twist lemma that, for any point (s,t) in the 2-torus that is isogonal for our generic link L (meaning that $x = x(s)$ and $y = y(t)$ have the same polar angle, and hence lie on the same page of the open book), there exists a unique $u = u(s,t) \in S^1$ for which the vector $y_z - x_z$ points straight up, namely the u for which $z = z(u) = \tau(x,y)$. It follows that the link

$$\mathcal{L} \;=\; h_L^{-1}(k) \;=\; \{(s,t,u) \in T^3 \mid (s,t) \in \mathcal{D} \text{ and } u = u(s,t)\}\,,$$

is the graph of the function $u(s,t)$ over the collection $\mathcal{D}$ of isogonal curves in T^2, as asserted in the introduction.

We now use the twist lemma to prove part (a) of the bicycle theorem, asserting that k is a regular value of h_L. This will endow the components of $\mathcal{L}$ with orientations (and thus vertical winding numbers) and framings, as defined in Section III. The proof of the bicycle theorem will then be completed by showing that these agree with the preferred orientations, vertical winding numbers and framings of the components of $\mathcal{D}$, also defined in Section III.

Why is k a regular value of h_L?

Suppose that $h_L(s,t,u) = k$. This means that the vector from x_z to y_z points straight up, and so lies in some page P of the standard open book in $\mathbb{R}^3$. We must show that at (s,t,u) the vectors $\partial_s h_L$, $\partial_t h_L$ and $\partial_u h_L$ span the tangent space to S^2 at k.

We are guided by Figure 28, depicting a neighborhood of the vertical vector $\vec{\mathbf{v}}$ from x_z to y_z in $\mathbb{R}^3$, lying on the page P. In this figure, X_z and Y_z denote the images of X and Y under the z-action, one or both of which must be transverse to P since L is generic. We arbitrarily depict X_z tangent to P and Y_z transverse to it.

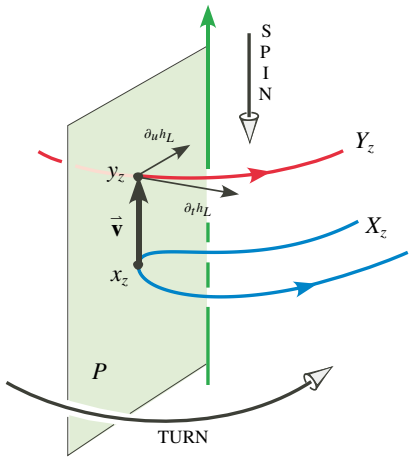


FIG. 28. k is regular

The hollow arrows in the figure indicate the left-handed screw motion of the K-action. In particular, as the u parameter increases along K, this action (the stereographic image of right multiplication by $-\bar{z}$) turns the pages in the indicated direction while spinning them in a left-handed fashion about their centers.

As the t parameter along Y increases, the tip of $\vec{\mathbf{v}}$ moves to the right along Y_z, so that $\partial_t h_L$ also points to the right, transverse to P. For clarity, we draw this partial derivative vector at the tip of $\vec{\mathbf{v}}$, even though it is really tangent to S^2 at its north pole k.

As the u parameter along Z increases, two things happen which affect $\vec{\mathbf{v}}$. The page containing it turns around the binding, but such an action keeps $\vec{\mathbf{v}}$ vertical, so has no infinitesimal effect. In addition, the page spins around its center, shifting the binding in a downward direction. Thus by the twist lemma, the vector $\partial_u h_L$ is nonzero and tangent to the page, pointing toward the binding when placed with its initial point at y_z.

It follows that $\partial_t h_L$ and $\partial_u h_L$ are linearly independent, confirming that (s,t,u) is a regular point of the map h_L. Since (s,t,u) was chosen arbitrarily in $h_L^{-1}(k)$, we see that k is a regular value of h_L, proving part (a) of the Bicycle Theorem.

Finishing the proof of the Bicycle Theorem

Starting with a three-component link L in generic position in S^3, we have shown above that the Pontryagin link $\mathcal{L}$ of its Gauss map h_L is given by

$$\mathcal{L} \;=\; h_L^{-1}(k) \;=\; \{(s,t,u) \in T^3 \mid (s,t) \in \mathcal{D} \text{ and } u = u(s,t)\}\,.$$

It remains to prove that the preferred orientation, vertical winding number r_i and framing n_i of each isogonal curve $\mathcal{D}_i$, as specified in Figure 16 and just before the statement of the bicycle theorem, agree with the orientation, vertical winding number and framing of the component $\mathcal{L}_i$ of the Pontryagin link that lies over $\mathcal{D}_i$.

To see that the orientations agree, choose a regular page vector (x, y) parametrized by an isogonal point (s, t) in $\mathcal{D}_i$, and consider the orientation of Y relative to the page containing x and y, as recorded by the sign of y.

Figure 29 shows the associated vertical vector (x_z, y_z) in 3-space for the case when $\text{sign}(y) = +1$, causing the displaced link component Y_z to point to the *right*. Arguing as above (see Figure 28) we see that $\partial_t h_L$ also points to the right, while $\partial_u h_L$ points toward the binding as always. Thus $\partial_t h_L$ and $\partial_u h_L$ form a positive basis for the tangent plane to S^2 at k. Since ∂_s, ∂_t and ∂_u form a positive basis for the tangent space to T^3 at each of its points, it follows that the link component $\mathcal{L}_i$ must be oriented in the direction of *increasing* s near the associated point (s, t, u) in the 3-torus, and this agrees with the orientation assigned to $\mathcal{D}_i$ in Figure 16.

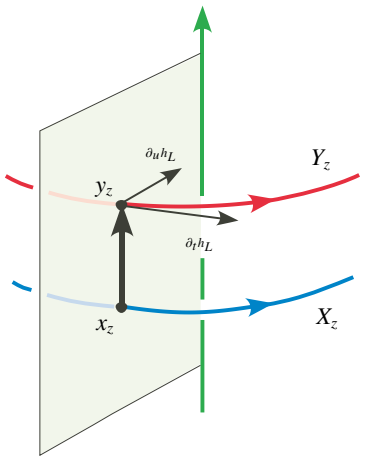


FIG. 29. Orientation of the Pontryagin link

Similarly, if $\text{sign}(y) = -1$ then $\mathcal{L}_i$ points in the direction of *decreasing* s near (s, t, u), which again is seen from Figure 16 to agree with the preferred orientation on $\mathcal{D}_i$.

Next consider the vertical winding number r_i of $\mathcal{D}_i$. By definition r_i is equal to the meridional degree m_i of the associated bicycle $\mathcal{P}_i$, that is, the number of times that the page vectors spin in the pages of the open book as the bicycle is traversed. By the twist lemma, this is equal to the number of times that z spins around K, or equivalently that $u = u(s, t)$ spins around the circle as (s, t) traverses $\mathcal{D}_i$, which is the vertical winding number of $\mathcal{L}_i$.

Finally consider the framing n_i of $\mathcal{D}_i$. By definition $n_i = -\ell_i - m_i$, where ℓ_i and m_i are the longitudinal and meridional degrees of the associated bicycle $\mathcal{P}_i$. This integer specifies a normal vector field $\vec{\mathbf{n}}$ to $\mathcal{L}_i$ by adding n_i full twists (positive or negative according to the sign of n_i) to the lift $\vec{\mathbf{z}}$ of a normal vector field to $\mathcal{D}_i$ in T^2, the "zero" or "blackboard" framing.

We must show that the vector field $\vec{\mathbf{n}}$ coincides with the Pontryagin framing of $\mathcal{L}_i$. In other words, the differential of h_L carries it onto a homotopically trivial loop of nonzero tangent vectors to S^2 at k. To see this, first note that $\vec{\mathbf{z}}$ is homotopic to the vertical vector field ∂_u, remaining nonzero and transverse to $\mathcal{L}_i$ during the homotopy. Now since $\partial_u h_L$ is always tangent to the page containing x_z and y_z, as shown above, it is clear that each longitudinal circuit of the bicycle $\mathcal{P}_i$ causes $dh_L(\vec{\mathbf{z}})$ to spin once in the same direction about the binding. By the twist lemma, each meridional spin of the bicycle also causes $dh_L(\vec{\mathbf{z}})$ to spin once about the binding. Thus $dh_L(\vec{\mathbf{z}})$ spins $\ell_i + m_i$ times as the bicycle is traversed. Adding

$n_i = -\ell_i - m_i$ full twists to the framing negates this spinning, and so $dh_L(\vec{\mathbf{n}})$ does not spin at all, up to isotopy, as asserted.

This completes the proof of the bicycle theorem.

Appendix D: Proof of the Double Crossing Formula, Corollary V.2

Recall that this is a formula for the change $\Delta\nu$ in the Pontryagin ν-invariant of a link $L = X \cup Y \cup Z$ in S^3 when a pair of crossings of opposite signs between X and Y are changed. Smoothing both crossings, the components X and Y are transformed into a pair of knots P and Q, as explained in detail above the statement of Corollary V.2 and illustrated in Figure 20. The double crossing formula then states that

$$\Delta\nu \;\equiv\; 2\,\mathrm{Lk}(P,Z) \;\equiv\; -2\,\mathrm{Lk}(Q,Z) \mod 2\,\gcd(p,q,r)$$

where p, q and r are the pairwise linking numbers of the components of L.

To prove this, first note that

$$\mathrm{Lk}(P \cup Q, Z) \;=\; \mathrm{Lk}(X \cup Y, Z) \;=\; p + q \;\equiv\; 0 \mod \gcd(p,q,r)$$

so it suffices to establish the first equality.

We may assume that the two page vectors $\vec{\mathbf{v}}_+ = (x_+, y_+)$ and $\vec{\mathbf{v}}_- = (x_-, y_-)$ associated with the chosen crossings lie on distinct hemispherical pages H_+ and H_- of the standard open book in S^3, and that these pages contain no critical points in the link. These two page vectors may lie in distinct bicycles in L, or they may lie in the same bicycle.

Suppose first that $\vec{\mathbf{v}}_+$ and $\vec{\mathbf{v}}_-$ lie in distinct bicycles $\mathcal{P}_+$ and $\mathcal{P}_-$ in L. When we change L to $\widehat{L}$, these bicycles will change, but the icycles $\mathcal{D}_+$ and $\mathcal{D}_-$ in the 2-torus that parametrize them will stay the same. However, their vertical winding numbers r_+ and r_- (which record the meridional degrees of $\mathcal{P}_+$ and $\mathcal{P}_-$) and their framings n_+ and n_- (which record the negative of the sum of the meridional and longitudinal degrees of $\mathcal{P}_+$ and $\mathcal{P}_-$) will change. In particular, we claim that

$$\widehat{r}_+ \;=\; r_+ - 1 \qquad \text{and} \qquad \widehat{r}_- \;=\; r_- + 1\,,$$

and consequently $\widehat{n}_+ = n_+ + 1$ and $\widehat{n}_- = n_- - 1$ since the longitudinal degrees of the bicycles clearly do not change. The figure below helps us to see this.

In this figure, we start with a positive crossing in L and change it to a negative crossing in $\widehat{L}$. Since the strands of X and Y are both pointing to the right, the bicycle $\mathcal{P}_+$ is moving from left to right. During this motion the page vectors in $\mathcal{P}_+$ undergo half a *counter-clockwise* rotation with respect to the preferred orientation on the pages. In the corresponding picture for $\widehat{L}$, we see half a *clockwise* rotation. Therefore the bicycle in $\widehat{L}$ has one more full clockwise rotation in the pages than in L, and so $\widehat{r}_+ = r_+ - 1$ as claimed.

If, for example, we switched the arrow on the strand of Y, we would have a negative crossing, but then the corresponding bicycle would be moving from right to left, and we would see that $\widehat{r}_- = r_- + 1$ as claimed. With this guidance, we leave the remaining cases to the reader.

Now suppose that the page vectors $\vec{\mathbf{v}}_+$ and $\vec{\mathbf{v}}_-$ lie in the same bicycle $\mathcal{P}_+ = \mathcal{P}_-$. Then the above changes in vertical winding number r for the corresponding icycle $\mathcal{D}_+ = \mathcal{D}_-$ will cancel, and so we see that neither the vertical winding number nor the framing of the icycle will change, that is $\widehat{r}_\pm = r_\pm$ and $\widehat{n}_\pm = n_\pm$.

Note that in either case, whether $\vec{\mathbf{v}}_+$ and $\vec{\mathbf{v}}_-$ lie in the same or different bicycles, the *total framing* n of the diagram (the sum of the framings of the icycles) does not change.

At this point we recall Proposition III.1, which tells us that

$$\nu(h_L) \;=\; n + pq + \textstyle\sum r_i d_i\,.$$

In passing from L to $\widehat{L}$ via the double crossing change, we have just seen that the total framing n does not change, and the pairwise linking numbers p and q certainly do not change. We have also seen that the icycles stay the same, so their depths d_i do not change. Only the winding numbers r_+ and r_- of the (possibly equal) icycles $\mathcal{D}_+$ and $\mathcal{D}_-$ may change. In particular, they also do not change when the page vectors

$$\vec{\mathbf{v}}_+ \;=\; (x(s_+), y(t_+)) \qquad \text{and} \qquad \vec{\mathbf{v}}_- \;=\; (x(s_-), y(t_-))$$

lie in the same bicycle, and so $\Delta\nu = 0$ in this case, while they change to $\widehat{r}_+ = r_+ - 1$ and $\widehat{r}_- = r_- + 1$ when $\vec{\mathbf{v}}_+$ and $\vec{\mathbf{v}}_-$ lie in distinct bicycles, in which case we have

$$\Delta\nu \;=\; \nu(h_{\widehat{L}}) - \nu(h_L) \;=\; (\widehat{r}_- - r_-)d_- + (\widehat{r}_+ - r_+)d_+ \;=\; d_- - d_+\,.$$

Thus in either case it remains to prove that $d_- - d_+ = 2\,\mathrm{Lk}(P, Z)$.

To show this, we must compute the depths d_+ and d_- of $\mathcal{D}_+$ and $\mathcal{D}_-$. This requires a choice of base point, and then a choice of paths γ_+ and γ_- from $\mathcal{D}_+$ and $\mathcal{D}_-$ to this base point. We let (s_+, t_-) be the base point, and use the vertical path γ_+ from (s_+, t_+) to (s_+, t_-) and the horizontal path γ_- from (s_-, t_-) to (s_+, t_-).

The depth d_- of the component $\mathcal{D}_-$ of $\mathcal{D}$ counts the intersections of γ_- with $\mathcal{D}$, which means that it counts the times that $(x(s), y(t_-))$ is a page vector for $s_- \le s \le s_+$, assuming parametrizations set up so that $s_- < s_+$. Since $y(t_-)$ already lies in the page H_-, this means we are counting the times that $x(s)$ also lies in H_- for $s_- \le s \le s_+$. In other words we are counting the intersection number $X_{-+} \bullet H_-$. Examining Figure 16, we find that the signs of the intersection points of γ_- with $\mathcal{D}$ *agree* with the signs of the corresponding intersection points of X_{-+} with H_-, and hence

$$d_- \;=\; 2\,X_{-+} \bullet H_- \mod 2\gcd(p, q, r).$$

In a similar fashion, intersections of γ_+ with $\mathcal{D}$ correspond to intersections of Y_{+-} with H_+, but in this case, the signs of corresponding points of intersection are *opposites*, and so the depth of $\mathcal{D}_+$ is

$$d_+ \;=\; -2\,Y_{+-} \bullet H_+ \mod 2\gcd(p, q, r).$$

Therefore the assertion that $d_- - d_+ = 2\mathrm{Lk}(P, Z)$, which will complete the proof of the corollary, reduces to the identity

$$X_{-+} \bullet H_- + Y_{+-} \bullet H_+ \;=\; \mathrm{Lk}(P, Z)\,,$$

which follows easily from the fact that $P = X_{-+} \cup Y_{+-}$. Figure 30 helps us to see this.

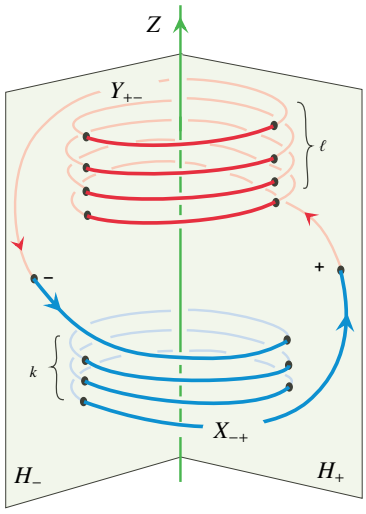


FIG. 30. The linking of $P = X_{-+} \cup Y_{+-}$ and Z

In the figure, the arc X_{-+} winds around the vertical Z axis k times and the arc Y_{+-} winds around it ℓ times, while the entire loop $P = X_{-+} \cup Y_{+-}$ winds around it $k + \ell + 1$ times. By our half-counting convention, we have

$$X_{-+} \bullet H_- \;=\; \tfrac{1}{2} + k \qquad \text{and} \qquad Y_{+-} \bullet H_+ \;=\; \tfrac{1}{2} + \ell$$

and the result follows. □

Appendix E: Numerical computation

Matlab was used to calculate an approximation to Milnor's μ-invariant as given by formula (3) of Theorem B for the three-component link L in S^3 parametrized by

$$\begin{aligned}
\mathbf{x}(s) &= ((1 - p^2 + (p^2 - q^2)\sin^2 s)^{1/2},\ p\cos s,\ q\sin s,\ 0) \\
\mathbf{y}(t) &= ((1 - p^2 + (p^2 - q^2)\sin^2 t)^{1/2},\ 0,\ p\cos t,\ q\sin t) \\
\mathbf{z}(u) &= ((1 - p^2 + (p^2 - q^2)\sin^2 u)^{1/2},\ q\sin u,\ 0,\ p\cos u)
\end{aligned}$$

for $p = .7$, $q = .4$, and $s, t, u \in [0, 2\pi]$; this is a concrete realization of the Borromean rings with $\mu = -1$, whose stereographic projection is shown in Figure 31. In particular, we used Matlab to approximate the Fourier coefficients $\mathbf{c_n}$ of its characteristic form ω_L for $\mathbf{n} \in [-64, 64]^3$, using subdivisions of the s, t and u intervals into 256 subintervals to estimate the associated integrals. The approximation of μ we obtained in this way was -0.999999999987.

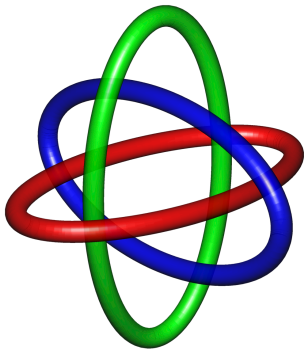


FIG. 31. Borromean rings

Appendix F: Whitehead's integral formula for the Hopf invariant

In this appendix, we make J. H. C. Whitehead's integral formula for the Hopf invariant of a map $f\colon S^3 \to S^2$ explicit. Recall from Section VII Whitehead's formula,

$$\mathrm{Hopf}(f) \;=\; \int_{S^3} \alpha \wedge \omega_f,$$

where $\omega_f = f^*\omega$ is the pullback to S^3 of the normalized area form ω on S^2, and α is any 1-form on S^3 such that $d\alpha = \omega_f$. To make this formula explicit requires a way to produce such an α.

To do this, we first write Whitehead's formula in the language of vector fields,

$$\mathrm{Hopf}(f) \;=\; \int_{S^3} \vec{\mathbf{a}} \bullet \vec{\mathbf{v}}_f \, d\,\mathrm{vol},$$

where the 2-form ω_f on S^3 has been converted to the vector field $\vec{\mathbf{v}}_f$ in the usual way, and where $\vec{\mathbf{a}}$ is any vector field on S^3 such that $\nabla \times \vec{\mathbf{a}} = \vec{\mathbf{v}}_f$.

An explicit recipe for $\vec{\mathbf{a}}$ was given by DeTurck and Gluck[25]:

$$\vec{\mathbf{a}}(\mathbf{y}) \;=\; \mathrm{BS}(\vec{\mathbf{v}}_f)(\mathbf{y}) \;=\; \int_{S^3} P_{\mathbf{yx}}\, \vec{\mathbf{v}}_f(\mathbf{x}) \times \nabla_{\mathbf{y}}\, \varphi(\mathbf{x},\mathbf{y})\, d\mathbf{x}.$$

Here BS is the Biot–Savart operator for vector fields on the 3-sphere. In the last integral, $P_{\mathbf{yx}}$ indicates parallel transport in S^3 along the geodesic segment from $\mathbf{x}$ to $\mathbf{y}$, the function φ is given by

$$\varphi(\alpha) \;=\; -\frac{1}{4\pi^2}(\pi-\alpha)\csc\alpha,$$

and $\varphi(\mathbf{x},\mathbf{y})$ is an abbreviation for $\varphi(\alpha(\mathbf{x},\mathbf{y}))$, with $\alpha(\mathbf{x},\mathbf{y})$ the geodesic distance on S^3 between $\mathbf{x}$ and $\mathbf{y}$. The significance of the function φ is that it is the fundamental solution of a shifted Laplacian on S^3:

$$-\Delta\varphi - \varphi = \delta,$$

where δ is the Dirac delta function.

The above formula for $\mathrm{BS}(\vec{\mathbf{v}}_f)$ is the analogue on S^3 of the classical Biot–Savart formula from electrodynamics in $\mathbb{R}^3$, expressing the magnetic field $\mathrm{BS}(\vec{\mathbf{v}}_f)$ in terms of the current flow $\vec{\mathbf{v}}_f$. The equation $\nabla \times \mathrm{BS}(\vec{\mathbf{v}}_f) = \vec{\mathbf{v}}_f$ is just one of Maxwell's equations, transplanted to S^3.

Inserting this formula for $\mathrm{BS}(\vec{\mathbf{v}})$ into the previous formula for the Hopf invariant and performing simple manipulations, we get

$$\mathrm{Hopf}(f) \;=\; -\int_{S^3\times S^3} P_{\mathbf{yx}}\, \vec{\mathbf{v}}_f(\mathbf{x}) \times \vec{\mathbf{v}}_f(\mathbf{y}) \bullet \nabla_{\mathbf{y}}\, \varphi(\mathbf{x},\mathbf{y})\, d\mathbf{x}\, d\mathbf{y},$$

the explicit version of Whitehead's integral formula on the 3-sphere S^3.

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The Borromean rings shown on the first page are from a panel in the carved walnut doors of the Church of San Sigismondo in Cremona, Italy. The photograph is courtesy of Peter Cromwell[55].

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