FINITE RANDOM COVERINGS OF ONE–COMPLEXES AND THE EULER CHARACTERISTIC

R. KOMENDARCZYK AND J. PULLEN

Abstract. This article presents an algebraic topology perspective on the problem of finding a complete coverage probability of a one dimensional domain X by a random covering, and develops techniques applicable to the problem beyond the one dimensional case. In particular we obtain a general formula for the chance that a collection of finitely many compact connected random sets placed on X has a union equal to X. The result is derived under certain topological assumptions on the shape of the covering sets (the covering ought to be *good*, which holds if the diameter of the covering elements does not exceed a certain size), but no a priori requirements on their distribution. An upper bound for the coverage probability is also obtained as a consequence of the concentration inequality. The techniques rely on a formulation of the coverage criteria in terms of the Euler characteristic of the nerve complex associated to the random covering.

Dedicated to Professor Yuli Rudyak, on the occasion of his 65th birthday.

1. Introduction

We consider *finite random coverings* of a metric space X, i.e. finite collections of *compact random sets*: $\mathsf{A} = \{\mathsf{A}_{\{i\}}\}$, $i = 1, \ldots, n$, understood as measurable maps [17, p. 121]

$$\mathsf{A}_{\{i\}} : (\Omega, \sigma, \mathbb{P}) \longrightarrow (\mathcal{C}(X) \sqcup \{\emptyset\}, \sigma_{\text{Borel}}),$$

where $(\Omega, \sigma, \mathbb{P})$ is an underlying probability space, and $\mathcal{C}(X)$ is the set of nonempty compact subsets of X, topologized by the Hausdorff distance and given the associated Borel algebra σ_{Borel}. The set $\mathcal{C}(X) \sqcup \{\emptyset\}$ is a disjoint union with the point $\{\emptyset\}$, which plays a role of the empty set. The term *covering* may be misleading in this context, as it sometimes assumes that the union of its elements contains the domain X. In this work a covering is simply a collection of subsets of X, as it has been previously used e.g. in [14, 15].

A typical example of an infinite random covering is a *coverage process* on the Euclidean space, [17] i.e. a sequence of random sets: $\mathsf{A} = \{\xi_1 + G_1, \xi_2 + G_2, \cdots, \xi_k + G_k, \ldots\}$, where $\{G_k\}$ is a fixed family of subsets of $\mathbb{R}^n$ called *grains* of the process, and $\boldsymbol{\xi} = \{\xi_i\}$ a sequence of random vectors in $\mathbb{R}^n$. In applications G_i's are often round balls of a fixed radius, and $\boldsymbol{\xi}$ defines a Poisson process (in which case A is refereed to as a *Boolean model*, [17]). In the current paper, we make no a priori assumptions on the distribution of A except a topological requirement on the covering, namely it almost surely must be *good*, which means that each

Date: July 22, 2014.

Key words and phrases. complete coverage probability, random complexes, nerves, Vietoris-Rips complex.

intersection $\bigcap_{i\in I} \mathsf{A}_{\{i\}}$, $I = \{i_1, \dots, i_k\}$ is almost surely contractible (in general some form of convexity of $\mathsf{A}_{\{i\}}$ validates this assumption).

Problem 1. *Given a random covering* $\{\mathsf{A}_{\{i\}}\}$, $i = 1, \dots, n$ *of a metric space* X*, find a complete coverage probability:* $\mathbb{P}\big(X \subseteq |\mathsf{A}|\big)$*, where* $|\mathsf{A}| = \bigcup_i \mathsf{A}_{\{i\}}$.

Reviewing the history of Problem 1: it was first considered by Whitworth, [35] in the basic case of a finite collection of independent identically distributed fixed α–length arcs on a unit circumference circle. Much later, Stevens [32] provided a complete answer to the question of Whitworth in the form

$$\mathbb{P}(S^1 \subseteq |\mathsf{A}|) = \sum_{j=1}^{\lfloor \frac{1}{\alpha} \rfloor} (-1)^{j+1} \binom{n}{j} \Big(1 - j\alpha\Big)^{n-1} . \tag{1.1}$$

The Stevens' result was further improved by Siegel and Holst [31] where they allowed varying lengths for the arcs. In [14], Flatto obtained an asymptotic expression for coverage as $\alpha \to 0$. The extension of the circle problem to the 2-sphere S^2 was considered by Moran and Groth [28], who derived an approximation for the probability $\mathbb{P}(S^2 \subset |\mathsf{A}|)$, and later Gilbert [16] showed the bounds

$$(1-\lambda)^n \leq P(S^2 \subseteq |\mathsf{A}|) \leq \frac{4}{3} n(n-1)\lambda(1-\lambda)^{n-1},$$

where $\lambda = (\sin \frac{\alpha}{2})^2$ is the fraction of the surface of S^2 covered by spherical α–caps; i.e. caps of radius α. For $\alpha \in [\frac{\pi}{2}, \pi]$, the explicit expression for $P(S^2 \subseteq |\mathsf{A}|)$ has been found by Miles [26]. Work in [7] provides explicit formulas for the complete coverage probability for α–caps on the m-dimensional unit sphere S^m when $\alpha \in [\frac{\pi}{2}, \pi]$ and upper bounds for $\alpha \in [0, \pi)$. The literature concerning the coverage probability in the asymptotic regimes (where the diameter of grains tends to zero) is vast and we only list a small fraction here [15, 30, 2, 27, 5]. Further, the reader may consult the recent work in [7] for a more accurate account of the history of Problem 1.

In this work we focus solely on the case of finite coverings of 1-dimensional domains X which are homeomorphic to finite multigraphs, equipped with an intrinsic distance

$$d_X(x,y) = \min_{\substack{\gamma:[0,1]\mapsto X,\\ \gamma(0)=x,\ \gamma(1)=y.}} \text{length}(\gamma). \tag{1.2}$$

I.e. $d_X(x, y)$ is the length of the shortest path between x an y, which in practice is just a smallest sum of edge-lengths (and their pieces) connecting x and y (the lengths come from some choice of geometric realization of X in $\mathbb{R}^3$). Let ∂X denote the set of *leaf vertices* of X, (this notation is justified by the case when X is an interval in $\mathbb{R}$) and $\text{diam}(Y)$ the intrinsic diameter of a subset $Y \subseteq X$. Our random covering $\mathsf{A} = \{\mathsf{A}_{\{i\}}\}$ on X, will always be finite ($i = 1, \dots, n$). As already mentioned before, the basic example of a random covering is ϵ-balls: $\{B(\boldsymbol{\xi}_i, \epsilon)\}$, $i = 1, \dots, n$ in the intrinsic metric d_X, with centers $\boldsymbol{\xi}_i$ distributed in an arbitrary fashion. We approach the coverage problem by considering a random complex

$\mathcal{N}(\mathsf{A})$ directly obtained from the usual topological nerve (c.f. [33]) of realizations of A and its Euler characteristic $\boldsymbol{\chi}(\mathsf{A}) = \boldsymbol{\chi}(\mathcal{N}(\mathsf{A}))$.

Let $\mathfrak{C}_n$ be the set of labeled abstract subcomplexes on n vertices (i.e. subcomplexes of the full $(n-1)$–simplex $\boldsymbol{\Delta}_n$). By the *labeling* we understand that every subcomplex $\mathtt{s} \in \mathfrak{C}_n$ comes with an indexing of its vertices by numbers from $1, \ldots, n$. Elements $\mathtt{s}, \mathtt{r}, \mathtt{k} \in \mathfrak{C}_n$ can be identified, in a non-unique way, with subsets of the power set $2^{[n]}$, $[n] = \{1, \ldots, n\}$ (see Section 2.1.) For instance, a singleton $\mathtt{r} = \{I\}$ (where $I \subseteq \{1, \ldots, n\}$) labels a face of $\boldsymbol{\Delta}_n$. By a *finite random complex* on n vertices we understand an arbitrary discrete probability space $\mathsf{K} = \big(\mathfrak{C}_n, \mathbb{P}_{\mathsf{K}}\big)$. In order to define the random nerve $\mathcal{N}(\mathsf{A})$, one builds a distribution on $\mathfrak{C}_n$ in a way dictated by the usual nerve construction. For instance, the probability of a k–face $I = \{i_1, \ldots, i_{k+1}\}$ in $\mathfrak{C}_n$, we denote by p_I equals

$$p_I = \mathbb{P}(\{\mathtt{s} \in \mathfrak{C}_n \mid I \in \mathtt{s}\}) = \mathbb{P}\big(\mathsf{A}_{\{i_1\}} \cap \mathsf{A}_{\{i_2\}} \cap \ldots \cap \mathsf{A}_{\{i_{k+1}\}} \neq \emptyset\big). \tag{1.3}$$

Identity (1.3) can then be extended from faces to subcomplexes. I.e. given $\mathtt{s} \in \mathfrak{C}_n$

$$\begin{aligned} p_{\mathtt{s}} &= \mathbb{P}(\{\mathtt{r} \in \mathfrak{C}_n \mid \mathtt{s} \subseteq \mathtt{r}\}) = \mathbb{P}\Big(\forall_{I\in\mathtt{s}}\Big\{\bigcap_{i\in I} \mathsf{A}_{\{i\}} \neq \emptyset\Big\}\Big), \\ P_{\mathtt{s}} &= \mathbb{P}(\mathtt{s}) = \mathbb{P}\Big(\forall_{I\in\mathtt{s}}\Big\{\bigcap_{i\in I} \mathsf{A}_{\{i\}} \neq \emptyset\Big\}, \forall_{\{J\}\notin\mathtt{s}}\Big\{\bigcap_{j\in J} \mathsf{A}_{\{j\}} = \emptyset\Big\}\Big). \end{aligned} \tag{1.4}$$

Generally, we make an underlying assumption that the covering A is *good* i.e. to satisfy (almost surely) the hypotheses of the Nerve Lemma (c.f. Section 4). In Proposition 4.7 of Section 4.2, we show that a covering A of a 1-complex X is always good, if its elements $\mathsf{A}_{\{i\}}$ are connected and sufficiently small in diameter. The first version of our main theorem is stated below.

Theorem 1.1 (Coverage probability for compact connected 1–complexes X, with $\partial X = \emptyset$)**.** *Let $\mathsf{A} = \{\mathsf{A}_{\{i\}}\}$, $i = 1, \ldots, n$ be a random good covering of X (with $\partial X = \emptyset$). Then, the range of $\boldsymbol{\chi} = \boldsymbol{\chi}(\mathsf{A})$ can be restricted to*

$$\underline{m} = \chi(X) \leq \boldsymbol{\chi}(\mathsf{A}) \leq n = \overline{m}, \tag{1.5}$$

and the complete coverage probability equals

$$\begin{aligned} \mathbb{P}(X \subseteq |\mathsf{A}|) &= \mathbb{P}\big(\boldsymbol{\chi}(\mathsf{A}) = \chi(X)\big) \\ &= \sum_{\mathtt{s}\in\mathfrak{C}_n;\chi(\mathtt{s})=\chi(X)} P_{\mathtt{s}} = \sum_{\mathtt{s}\in\mathfrak{C}_n} a_{\mathtt{s}}(\boldsymbol{\chi})\, p_{\mathtt{s}}, \end{aligned} \tag{1.6}$$

where

$$a_{\mathtt{s}}(\boldsymbol{\chi}) = \sum_{k=0}^{N} v_k(\boldsymbol{\chi})\, c_{\mathtt{s},k}(\boldsymbol{\chi}), \quad N = \overline{m} - \underline{m},$$

with $p_{\mathtt{s}}$ *given in* (1.4), *and*

$$v_k(\boldsymbol{\chi}) = \frac{(-1)^k}{N!} \sum_{\underline{m}<j_1<j_2<\ldots<j_{N-k}\leq\overline{m}} j_1 j_2 \ldots j_{N-k}, \qquad v_N(\boldsymbol{\chi}) = \frac{(-1)^N}{N!}, \tag{1.7}$$

$$c_{\mathtt{s},k}(\boldsymbol{\chi}) = \begin{cases} \displaystyle\sum_{i=0}^{r^+_{top}(\mathtt{s})} \sum_{j=0}^{r^-_{top}(\mathtt{s})} (-1)^{r_{top}(\mathtt{s})-i-j} \binom{r^+_{top}(\mathtt{s})}{i} \binom{r^-_{top}(\mathtt{s})}{j} \Big(i - j + r^+_{low}(\mathtt{s}) - r^-_{low}(\mathtt{s})\Big)^k, \\ \qquad \textit{if } k \geq r(\mathtt{s}), \\ 0, \\ \qquad \textit{if } k < r(\mathtt{s}), \end{cases}$$

where $r^{\pm} = r^{\pm}(\mathtt{s})$, $r^{\pm}_{top} = r^{\pm}_{top}(\mathtt{s})$, $r^{\pm}_{low}(\mathtt{s}) = r^{\pm} - r^{\pm}_{top}(\mathtt{s})$ *stand for a number of respectively total, top and lower: even(odd) dimensional faces of* $\mathtt{s} \in \mathfrak{C}_n$, *and* $r(\mathtt{s})$ *denotes a number of all faces.*

If the diameter of $\mathsf{A}_{\{i\}}$ *is smaller than* $\frac{1}{6}\mathcal{C}$ *almost surely (where* $\mathcal{C}$ *is a length of the shortest cycle in* X, *known as* girth*), then* $p_{\mathtt{s}}$ *further simplifies as*

$$p_{\mathtt{s}} = \mathbb{P}\big(\forall_{(i,j)\in E(\mathtt{s}),i<j}\big\{\mathsf{A}_{\{i\}} \cap \mathsf{A}_{\{j\}} \neq \emptyset\big\}\,\big), \tag{1.8}$$

where $E(\mathtt{s})$ *is the edge set of* $\mathtt{s}$*.*

An extension of the above result to the case of a 1–complex X with no assumptions on ∂X is provided in Theorem 5.1 of Section 5. One obvious corollary of the above result is the fact that the complete coverage probability of any good random covering $\{\mathsf{A}_{\{i\}}\}$ is determined by finitely many numbers, which is not obvious when considering e.g. *vacancy*, i.e. the volume of $X - |\mathsf{A}|$ c.f. [17]. The complexity of computing $\mathbb{P}(X \subseteq |\mathsf{A}|)$ via the formula of Theorem 1.1 is not addressed here. However, one may expect that, due to the size of the set $\mathfrak{C}_n$, computation of coefficients $a_{\mathtt{s}}(\boldsymbol{\chi})$ or the set $\{\mathtt{s} \in \mathfrak{C}_n \mid \boldsymbol{\chi}(\mathtt{s}) = \chi(X)\}$ is double exponentially hard in n. On a positive note, coefficients $a_{\mathtt{s}}(\boldsymbol{\chi})$ are independent of the underlying distribution vector $(p_{\mathtt{s}})$, therefore once computed for a certain size problem can be reapplied as $(p_{\mathtt{s}})$ changes. The vector $(p_{\mathtt{s}})$ can be conveniently estimated numerically (e.g. via the standard maximum likelihood estimation, c.f. [23]) but again in the simplest case of Equation (1.8) it is of exponential size: $2^{\binom{n}{2}}$. Therefore, in practical situations the formula derived in Theorem 1.1 can apply to the covering problems with small n.

In a longer perspective, one may be interested asymptotic distributions of $\boldsymbol{\chi}(\mathsf{A})$ (as $n \to \infty$) which would lead to parametric estimators or useful bounds for $\mathbb{P}(X \subseteq |\mathsf{A}|)$. Currently available results (e.g. in [21, 20]) concern sparse regimes and they are not applicable, unless we allow the diameter of random sets in $\mathsf{A} = \{\mathsf{A}_{\{i\}}\}$ to tend to 0 sufficiently fast as n tends to infinity (see e.g. [15]). Concerning the question of useful bounds for $\mathbb{P}(X \subseteq |\mathsf{A}|)$, as a first step we derive an upper bound for the coverage probability, via the concentration inequality [3] in the following

Theorem 1.2. *Let* $\mathsf{A} = \{\mathsf{A}_{\{i\}}\}$, $i = 1, \ldots, n$ *be a random good covering of* X, *then*

$$\mathbb{P}(X \subseteq |\mathsf{A}|) \leq \exp\Big(\frac{-\mu_0^2}{2n(|\chi_{rel}(X, \partial X)| + 2)^2}\Big), \tag{1.9}$$

where μ_0 *denotes the expected value of the relative Euler characteristic* $\boldsymbol{\chi}_{rel}(\mathsf{A}, \mathsf{A}_{\partial X})$ *of the random pair* $(\mathcal{N}(\mathsf{A}), \mathcal{N}(\mathsf{A}_{\partial X}))$.

Although, Theorem 1.1 is restricted to the case of 1–complexes, the question of complete coverage probability for such spaces is not without a practical meaning. One may consider the 1–complex to be e.g. a system of streets in the city or underground channels. In such cases random coverings can be associated with sensing regions of e.g. vehicles equipped with sensors (c.f. [8]). Beyond 1–complexes, techniques of algebraic topology provide coverage criteria for higher dimensional objects. For instance, if an underlying space X is an m-dimensional manifold a necessary and sufficient condition for coverage is nonvanishing of the m-th(top) Betti number of the nerve $\mathcal{N}(\mathsf{A})$. We aim to develop these ideas in subsequent papers.

The article is organized as follows: In Section 2 we further discuss the general setup of random complexes and their associated invariants – mainly $\boldsymbol{\chi}(\mathsf{A})$. In Section 3, we derive relevant formulas for distributions of the random relative Euler characteristic, see Corollary 3.5. Further, in Section 4, we prove several basic topological results showing that the Euler characteristic of the nerve of a covering determines complete coverage of a 1–complex. In Proposition 4.7, we also provide a sufficient condition for a covering to be good (in terms of the girth of a 1–complex). We collect relevant facts and prove Theorem 1.1 in Section 5. The upper bound for $\mathbb{P}(X \subseteq |\mathsf{A}|)$ of Theorem 1.2 is shown in Section 6.

Acknowledgments. We wish to thank Gustavo Didier for discussions and Tewodros Amdeberhan for many useful suggestions and help with combinatorial aspects of Section 3.3. We are also grateful to Robert Ghrist for introducing us to the subject both via his work [8, 10, 9] with vin de Silva on coverage problems in sensor networks, and inspiring conversations during the *First National Forum of Young Topologists* at Tulane University in 2009. The first author would like to thank the organizers of *Applied Topology Conference* in Bedlewo, 2013. Both authors wish to thank the anonymous referee for corrections to the manuscript.

Theorem 1.1 is a part of the doctoral thesis of [29] of the second author. Both authors acknowledge the support of NSF DMS #1043009. The first author was also partially supported by DARPA YFA N66001-11-1-4132.

2. Random complexes and their topological invariants.

2.1. **Random complexes.** We refer the reader to [18] for background on algebraic topology. Consider $\boldsymbol{\Delta}_n$ to be a full simplex on n–vertices indexed by $1, \ldots, n$ (geometrically $\boldsymbol{\Delta}_n$ is the convex hull of n points given by the standard basis vectors in $\mathbb{R}^{n+1}$). Recall that a d-dimensional face in $\boldsymbol{\Delta}_n$ can indexed by the collection of its $d+1$ vertices: $I = \{i_1, i_2, \ldots, i_{d+1}\}$, where $1 \leq i_1 < i_2 < \ldots < i_{d+1} \leq n$. Denote the set of all faces of $\mathbf{\Delta}_n$ by $f(n)$, and particularly, d–dimensional faces by $f_d(n)$. I.e.

$$f(n) = \{I \mid I \subseteq 2^{[n]}\}, \qquad f_d(n) = \{I \mid I \subseteq 2^{[n]}, |I| = d+1\}. \tag{2.1}$$

Consider the set of all *labeled sub-complexes* $\mathfrak{C}_n$ of $\mathbf{\Delta}_n$ union a special point $\{\emptyset\}$ playing a role of the empty set. By a *labeled sub-complex* we understand a subcomplex of $\mathbf{\Delta}_n$, determined by all its faces with labeling given by vertices of $\mathbf{\Delta}_n$. A natural set to consider for enumerating labeled subcomplexes is the power set $2^{f(n)}$ of $f(n)$, which is further denoted by $\mathfrak{P}_n$ (we assume $\mathfrak{P}_n$ contains the empty set). Here and thereafter, we use notation $\mathtt{s}$, $\mathtt{r}$, $\mathtt{k}$ for elements of both $\mathfrak{C}_n$ and $\mathfrak{P}_n$.

Clearly, there is a surjective correspondence $\Pi : \mathfrak{P}_n \mapsto \mathfrak{C}_n$ which to a given subset $\mathtt{s} \in \mathfrak{P}_n$ assigns a subcomplex $\Pi(\mathtt{s})$ in $\mathfrak{C}_n$ given by the union of elements I of $\mathtt{s}$ and their subsets (i.e. the lattice of subsets associated to the faces of subcomplex $\mathtt{s}$). Π is clearly not bijective, however, with certain choices we may easily build right inverses. In particular, we will be interested in two cases, which we refer to as the *antichain* and *chain* representations: $\widehat{\cdot} : \mathfrak{C}_n \mapsto \mathfrak{P}_n$, $\widetilde{\cdot} : \mathfrak{C}_n \mapsto \mathfrak{P}_n$. The antichain representative $\widehat{\mathtt{s}} \in \mathfrak{P}_n$ of $\mathtt{s} \in \mathfrak{C}_n$, contains only its top dimensional faces, also known as *facets*, i.e.

$$\widehat{\mathtt{s}} = \{I \in \mathtt{s} \mid \text{such that for any } J \in \mathtt{s}, J \neq I \text{ either } J \subset I \text{ or } (J \not\subseteq I \text{ and } I \not\subseteq J)\}.$$

The chain representative $\widetilde{\mathtt{s}}$ of $\mathtt{s} \in \mathfrak{C}_n$ is obtained from the antichain representative by adding all remaining subfaces of $\mathtt{s}$. Clearly, $\widetilde{\mathtt{s}} = \mathtt{s}$ if $\mathtt{s} \in \mathfrak{C}_n$ thus $\mathfrak{C}_n = \widetilde{\mathfrak{C}}_n$, we also have projections $\widehat{\Pi} : \mathfrak{P}_n \mapsto \widehat{\mathfrak{C}}_n$, $\widetilde{\Pi} : \mathfrak{P}_n \mapsto \widetilde{\mathfrak{C}}_n = \mathfrak{C}_n$, where $\widetilde{\Pi} = \Pi$. Note that the cardinality of $\widehat{\mathfrak{C}}_n$, and therefore $\mathfrak{C}_n$ and $\widetilde{\mathfrak{C}}_n$, is given by the Dedekind number $M(n)$, c.f. [22]. For any $\mathtt{s} \in \mathfrak{C}_n$, we call the elements of $\widehat{\mathtt{s}}$, *top faces* or *facets* of $\mathtt{s}$.

Recall from Section 1 that by *finite random complex* on n vertices we understand a discrete probability space $\mathsf{K} = \big(\mathfrak{C}_n, \mathbb{P}_{\mathsf{K}}\big)$. It is easy to see that $\mathbb{P}_{\mathsf{K}}$ satisfies the following equivalent conditions[1] (for $I' \subseteq I$)

(A) $\mathbb{P}_{\mathsf{K}}\big(I \mid (I')^c\big) = \mathbb{P}_{\mathsf{K}}\big(\{\mathtt{s} \in \mathfrak{C}_n \,|\, I \in \mathtt{s}\} \mid \{\mathtt{r} \in \mathfrak{C}_n \,|\, \{I'\} \notin \mathtt{r}\}\big) = 0,$
(B) $\mathbb{P}_{\mathsf{K}}\big(I' \mid I\big) = \mathbb{P}\big(\{\mathtt{s} \in \mathfrak{C}_n \,|\, \{I'\} \in \mathtt{s}\} \mid \{\mathtt{r} \in \mathfrak{C}_n \,|\, I \in \mathtt{r}\}\big) = 1.$

In short **(A)** says that if a subface I' of I has not occurred then I cannot occur either; equivalently, **(B)** says that I' occurs whenever I has occurred. We say that K is supported on a subcomplex, $\mathtt{k} \in \mathfrak{C}_n$ if and only if for any $I \notin \mathtt{k}$ we have $\mathbb{P}_{\mathsf{K}}(\{I\}) = 0$. Given random complexes K and L on n-vertices, the joint probability space $(\mathsf{K}, \mathsf{L}) := (\mathfrak{C}_n \times \mathfrak{C}_n, \mathbb{P}_{\mathsf{K},\mathsf{L}})$ is a *random pair* if and only if L is *almost surely a subcomplex* of K, i.e. the following condition holds

(C) for every $(\mathtt{s}, \mathtt{r}) \in \mathfrak{C}_n \times \mathfrak{C}_n$ such that $\mathtt{r} \not\subseteq \mathtt{s}$ we have $\mathbb{P}_{\mathsf{K},\mathsf{L}}(\,\mathtt{s}, \mathtt{r}\,) = 0$.

[1] because events $\{\mathtt{s} \in \mathfrak{C}_n \,|\, I \in \mathtt{s}\}$ and $\{\mathtt{r} \in \mathfrak{C}_n \,|\, \{I'\} \notin \mathtt{r}\}$ are disjoint

For a given K (or (K,L)) it will be convenient to consider Bernoulli random variables which are *indicator functions of faces* in K, i.e. for $I\in f(n)$ we define

$$e_I:\mathfrak{C}_n\longrightarrow\{0,1\},\qquad e_I(\mathtt{s})=\begin{cases}1, & I\in\mathtt{s},\\ 0, & \text{otherwise},\end{cases}\tag{2.2}$$

For any $\mathtt{s}\in\mathfrak{C}_n$, we set $e_{\mathtt{s}}=\prod_{I\in\mathtt{s}}e_I$ an *indicator function of the subcomplex* $\mathtt{s}$. Clearly $e_{\mathtt{s}}$ takes value 1 on $\mathtt{r}$ if and only if $\mathtt{s}\subseteq\mathtt{r}$. Let us define vectors $(p_{\mathtt{s}})$ and $(P_{\mathtt{s}})$ (directly related to vectors in (1.4), when the underlying random complex is $\mathcal{N}(\mathsf{A})$):

$$\begin{aligned}p_{\mathtt{s}}&=\mathbb{P}\Big(e_{\mathtt{s}}=\prod_{I\in\mathtt{s}}e_I=1\Big),\\ P_{\mathtt{s}}&=\mathbb{P}\Big(\big(\prod_{I\in f(n),I\in\mathtt{s}}e_I\prod_{J\in f(n),J\notin\mathtt{s}}(1-e_J)\big)=1\Big).\end{aligned}\tag{2.3}$$

Clearly, a random complex K is fully determined by indicator functions $\{e_I\}$ of faces and their joint distribution. In the next section e_Is will serve as formal indeterminates for functions defining topological random variables on K, such as the Euler characteristic. The main use of conditions **(A)**(**(B)**) and **(C)** is to define (in Section 2.3) a natural polynomial ring for random topological invariants such as $\boldsymbol{\chi}(\mathsf{K})$.

Remark 2.1. In general, one could consider a more flexible model of a finite random complex with $(\mathfrak{P}_n,\bar{\mathbb{P}})$ as the underlying probability space. It can be thought of as a distribution on open faces of $\boldsymbol{\Delta}_n$ (i.e. interiors of faces) with an exception of the zero dimension (the vertices.) In this model it is possible, for instance, for an edge to occur without its vertices (i.e. **(A)** can be violated).

Remark 2.2. We may easily generalize the definition of the random complex K to the case $n=\infty$, and thus removing dependence on n in the definition. This is done by considering all labeled subcomplexes $\mathfrak{C}_\infty$ of the infinite simplex $\boldsymbol{\Delta}_\infty=\bigcup_n\boldsymbol{\Delta}_n$, and regarding a *random complex* K as a probability space $(\mathfrak{C}_\infty,\mathbb{P}_{\mathsf{K}})$. Such random complex is finite provided the support of K is contained in $\boldsymbol{\Delta}_n$ for sufficiently big n.

2.2. **Topological invariants in the random setting.** Recall that, thanks to the Poincare-Euler formula [18], the Euler characteristic of a general n-complex K is given by

$$\chi(K)=\sum_{j=0}^{n}(-1)^j\dim C_j(K;\mathbb{R}),\tag{2.4}$$

where $\dim C_j(K;\mathbb{R})$ denotes the dimension, as a vector space, of the real coefficient jth chain group $C_j(K;\mathbb{R})$, and equals (in the absolute case) to the number of j-dimensional faces $\boldsymbol{f}_j(K)$ of K. We will also need a relative version of χ. Given a pair (K,L) where L is a subcomplex of K we have

$$\chi_{rel}(K,L) = \sum_{j=0}^{n} (-1)^j \dim C_j(K,L;\mathbb{R}), \tag{2.5}$$

where $\dim C_j(K,L;\mathbb{R})$ denotes the dimension of the jth relative chain group $C_j(K,L;\mathbb{R}) = C_j(K;\mathbb{R})/C_j(L;\mathbb{R})$, as a real vector space (c.f. [18]). Note that

$$\dim C_j(K,L;\mathbb{R}) = \boldsymbol{f}_j(K) - \boldsymbol{f}_j(L). \tag{2.6}$$

Invariants $\chi = \chi(K)$ and $\chi_{rel} = \chi_{rel}(K,L)$ can be expressed in terms of Betti numbers $\{\beta_k(K)\}$, $\{\beta_k(K,L)\}$ of the chain complexes $C_*(K)$ and $C_*(K,L)$, (c.f. [18]). Specifically,

$$\chi = \sum_{j=0}^{n} (-1)^j \beta_j(K), \qquad \chi_{rel} = \sum_{j=0}^{n} (-1)^j \beta_j(K,L). \tag{2.7}$$

2.3. **Random polynomials.** Given a random complex K let us treat the indicator functions of faces $\{e_I\}$ (or in a case of a random pair $\{e_I, w_J\}$) as formal indeterminates and consider a polynomial ring in e_I (without loss of generality we work over $\mathbb{R}$):

$$\mathbb{R}[e_I] := \mathbb{R}[e_{\{1\}}, \ldots, e_{\{n\}}, e_{\{1,2\}}, \ldots, e_{\{i_1,\ldots,i_k\}}, \ldots, e_{\{1,\ldots,n\}}],$$

or $\mathbb{R}[e_I, w_J]$ in the case of random pairs. Observe that any random variable $\mathtt{X}$ on K is given as such polynomial, i.e.

$$\mathtt{X} = \sum_{\mathtt{s}\in\mathfrak{C}_n} \mathtt{X}_{\mathtt{s}} \Big(\prod_{I\in f(n), I\in\mathtt{s}} e_I \prod_{J\in f(n), J\notin\mathtt{s}} (1 - e_J) \Big), \tag{2.8}$$

where $\mathtt{X}_{\mathtt{s}}$ is a value of $\mathtt{X}$ at $\mathtt{s} \in \mathfrak{C}_n$. Based on (2.4) we may express the random Euler characteristic $\boldsymbol{\chi} = \boldsymbol{\chi}(\mathsf{K})$

$$\begin{aligned} &\boldsymbol{\chi} : (\mathfrak{C}_n, \mathbb{P}_{\mathsf{K}}) \longrightarrow \mathbb{Z}, \\ &\boldsymbol{\chi}(\mathtt{s}) = \chi(\mathtt{s}), \end{aligned} \tag{2.9}$$

as the following polynomial in $\mathbb{R}[e_I]$:

$$\boldsymbol{\chi} = \sum_{I\in f(n)} (-1)^{|I|-1} e_I. \tag{2.10}$$

Lemma 2.3. *Given a random complex $\mathsf{K} = (\mathfrak{C}_n, \mathbb{P}_{\mathsf{K}})$ and its collection of the indicator functions $\{e_I\}$, consider $\mathtt{Q}, \mathtt{Q}' \in \mathbb{R}[e_I]$ as two representatives of the same coset in $\mathbb{R}[e_I]/\mathcal{I}$ where $\mathcal{I}$ is an ideal generated by the following relations*

$$\{e_J\, e_I = e_J \mid \textit{for all } I \subseteq J\}, \tag{2.11}$$

(in particular: $e_I^2 = e_I$). Then $\mathtt{Q} = \mathtt{Q}'$ almost surely.

Proof. It suffices to show that $\mathbb{P}(e_J\, e_I = e_J) = 1$ for any I, J where $I \subseteq J$. We have

$$\mathbb{P}(e_J\, e_I = 0) = \mathbb{P}(e_J = 0, e_I = 1) + \mathbb{P}(e_J = 1, e_I = 0) + \mathbb{P}(e_J = 0, e_I = 0).$$

Thanks to **(A)** : $\mathbb{P}(e_J = 1, e_I = 0) = 0$, thus

$$\mathbb{P}(e_J\, e_I = 0) = \mathbb{P}(e_J = 0, e_I = 1) + \mathbb{P}(e_J = 0, e_I = 0) = \mathbb{P}(e_J = 0),$$

and $\mathbb{P}(e_J\, e_I = 1) = 1 - \mathbb{P}(e_J\, e_I = 0) = 1 - \mathbb{P}(e_J = 0) = \mathbb{P}(e_J = 1)$. □

We will further denote the quotient ring $\mathbb{R}[e_I]/\mathcal{I}$ by $\mathbb{R}_{\mathcal{I}}[e_I]$. Clearly, $\mathbb{R}_{\mathcal{I}}[e_I]$ has an additive basis of monomials indexed by the chain representatives: $\mathsf{s} \in \mathfrak{C}_n$:

$$e_{\mathsf{s}} = \prod_{I \in \mathsf{s}} e_I. \tag{2.12}$$

In the case of pairs (K, L) we have a pair of sets of face indicator functions $\{e_I, w_J\}$ corresponding to K and L respectively. Then, it is relevant to consider a polynomial ring $\mathbb{R}[e_I, w_J]$ modulo relations in (2.11) and additionally (thanks to property **(C)**):

$$\begin{aligned} &\{w_J\, w_I = w_J \mid \text{for all } I \subseteq J\},\\ &\{w_I = w_I e_J, \mid \text{for all } J \subseteq I\}. \end{aligned} \tag{2.13}$$

The resulting quotient ring will be denoted by $\mathbb{R}_{\mathcal{I}}[e_I, w_J]$, and the analogous statement as Lemma 2.3 is true for random variables expressed as representatives in $\mathbb{R}_{\mathcal{I}}[e_I, w_J]$. An important for us example of a polynomial in $\mathbb{R}_{\mathcal{I}}[e_I, w_J]$ is the relative Euler characteristic

$$\begin{aligned} &\boldsymbol{\chi}_{rel}(\mathsf{K}, \mathsf{L}) : (\mathfrak{C}_n \times \mathfrak{C}_n, \mathbb{P}_{\mathsf{K}}) \longrightarrow \mathbb{Z},\\ &\boldsymbol{\chi}_{rel}(\mathsf{s}, \mathsf{s}') = \chi_{rel}(\mathsf{s}, \mathsf{s}'),\ \text{if } \mathsf{s}' \subseteq \mathsf{s}, \quad \text{i.e. the relative Euler characteristic of } (\mathsf{s}, \mathsf{s}')\\ &\qquad\qquad = 0,\ \text{if } \mathsf{s}' \not\subseteq \mathsf{s}. \end{aligned} \tag{2.14}$$

Note, that thanks to **(C)**, the set of pairs $(\mathsf{s}, \mathsf{s}')$ such that $\mathsf{s}' \not\subseteq \mathsf{s}$ is of measure zero in (K, L) and thus the value of $\boldsymbol{\chi}_{rel} = \boldsymbol{\chi}_{rel}(\mathsf{K}, \mathsf{L})$ on such pairs is irrelevant. Thanks to (2.6), the polynomial expression for $\boldsymbol{\chi}_{rel}$ is given as follows

$$\boldsymbol{\chi}_{rel} = \sum_{I \in f(n)} (-1)^{|I|-1}(e_I - w_I). \tag{2.15}$$

3. Moments and distributions of the random Euler characteristic.

We begin with basic review of the *method of moments* for the finite range discrete random variable $\mathtt{X}$, and provide a specific formulation based on the recent work in [12]. Alternatively, one could use factorial moments (see e.g. [4, p. 17]), however they do not offer any advantage in the setting of the random Euler characteristic.

3.1. **Method of moments.** First, we need basic information on the *Vandermonde matrix* $\mathcal{V}$ (c.f. [25]). Given a fixed sequence of real numbers $\mathbf{x} = \{x_0, x_1, \ldots, x_N\}$, $\mathcal{V}$ is an $(N+1) \times (N+1)$ matrix explicitly given as follows

$$\mathcal{V} = \mathcal{V}(\mathbf{x}) = \begin{pmatrix} 1 & x_0 & \cdots & x_0^N \\ 1 & x_1 & \cdots & x_1^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & \cdots & x_N^N \end{pmatrix}.$$

Note that $\mathcal{V}$ is invertible provided the x_i's are distinct (c.f. [25]). A closed form of $\mathcal{V}^{-1}$ has been derived in [12] in terms of the elementary symmetric polynomials. Denote by $\mathfrak{e}_i(j)(\mathbf{x})$ the ith–elementary symmetric polynomial in variables: $x_0, \cdots, \widehat{x}_j, \cdots, x_N$ for $j = 0, \cdots, N$, where $\widehat{x}_j$ means that x_j is omitted. Specifically

$$\mathfrak{e}_i(j)(\mathbf{x}) = \begin{cases} 1 & \text{if } i = 0 \\ \displaystyle\sum_{1 \le l_1 < l_2 < \ldots < l_i \le N; l_k \neq j} x_{l_1} x_{l_2} \ldots x_{l_i} & \text{if } i > 0 \end{cases}. \tag{3.1}$$

By [12, p. 647], we have

$$\mathcal{V}(\mathbf{x})^{-1} = (v_{ki}(\mathbf{x})), \qquad \text{where} \quad v_{ki}(\mathbf{x}) = (-1)^{N+k} \frac{\mathfrak{e}_{N-k}(i)(\mathbf{x})}{\prod_{j=0, j\neq i}^{N} (x_i - x_j)}, \tag{3.2}$$

for $i = 0, \ldots, N$, $k = 0, \ldots, N$. In the case $\mathbf{x}$ is an integer interval $[\underline{m}, \ldots, \overline{m}]$, $\underline{m}, \overline{m} \in \mathbb{Z}$, $\underline{m} \le \overline{m}$ of size $N = \overline{m} - \underline{m}$ we obtain

$$v_{ki}(\mathbf{x}) = v_{ki}(\underline{m}, \overline{m}) = \frac{(-1)^{i+k}}{N!} \binom{N}{i} \mathfrak{e}_{N-k}(i)(\underline{m}, \ldots, \overline{m}). \tag{3.3}$$

Lemma 3.1. *Let* $\mathtt{X}$ *be a discrete random variable of a finite range* $\mathbf{x} = \{x_0, x_1, \ldots, x_N\}$, *and let* $\mu_k = \mathbb{E}(\mathtt{X}^k)$ *denote the k-th moment of* $\mathtt{X}$. *Given the vector* $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_N)$ *we can recover the distribution of* $\mathtt{X}$ *explicitly as follows*

$$p_i = \mathbb{P}(\mathtt{X} = x_i) = \sum_{k=0}^{N} v_{ki}\, \mu_k, \qquad i = 0, \ldots, N, \tag{3.4}$$

where $v_{ki} = v_{ki}(\mathbf{x})$ *are the Vandermonde coefficients.*

Proof. By definition we have a linear system of N equations

$$\mu_k = \sum_{i=0}^{N} x_i^k p_i, \qquad \text{for} \quad k = 0, 1, \ldots, N.$$

In matrix form this system reads: $\mathbf{p}\mathcal{V} = \boldsymbol{\mu}$ where $\mathbf{p} = (p_0, \ldots, p_N)$, and $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_N)$. Since all x_i's are distinct $\det(\mathcal{V}) = \prod_{i \neq j} (x_i - x_j) \neq 0$. Thus $\mathcal{V}$ is invertible and we have the unique solution $\mathbf{p} = \boldsymbol{\mu}\mathcal{V}^{-1}$. Identity (3.4) is now a direct consequence of (3.2). □

Our goal for the next subsection is to provide expressions for distributions of polynomial random variables in $\mathbb{R}_{\mathcal{I}}[e_I]$.

3.2. **Distributions of random polynomials.** Since the differences between $\mathbb{R}_{\mathcal{I}}[e_I]$ and $\mathbb{R}_{\mathcal{I}}[e_I, w_J]$ are mostly notational, we choose to work with the former. Recall from Section 2.3 that any representative polynomial in $\mathbb{R}[e_I]$ is a linear combination of monomials $e_{\mathtt{k}}$ from (2.12)

$$\mathtt{Q} = \sum_{\mathtt{k}\in\mathfrak{P}_n} c_{\mathtt{k}}\, e_{\mathtt{k}}, \qquad c_{\mathtt{k}} \in \mathbb{R}, \tag{3.5}$$

where the constant coefficient $c_0 = c_\emptyset$ is indexed by the empty set. Note that if $\mathtt{Q} \in \mathbb{R}_{\mathcal{I}}[e_I]$ then, thanks to the relations in $\mathbb{R}_{\mathcal{I}}[e_I]$, we may always pick expansions of $\mathtt{Q}$ in terms of the antichain or chain representatives i.e.

$$\mathtt{Q} = \sum_{\widehat{\mathtt{s}}\in\widehat{\mathfrak{C}}_n} c_{\widehat{\mathtt{s}}}\, e_{\widehat{\mathtt{s}}}, \qquad \text{or} \qquad \mathtt{Q} = \sum_{\widetilde{\mathtt{s}}\in\widetilde{\mathfrak{C}}_n} c_{\widetilde{\mathtt{s}}}\, e_{\widetilde{\mathtt{s}}} = \sum_{\mathtt{s}\in\mathfrak{C}_n} c_{\mathtt{s}}\, e_{\mathtt{s}}, \tag{3.6}$$

where in the second expansion we just applied our convention from Section 2.1 to identify elements of $\mathfrak{C}_n$ with their chain representatives. We refer to 3.6(left) as the *antichain representative* and 3.6(right) as the *chain representative* of $\mathtt{Q}$ in $\mathbb{R}_{\mathcal{I}}[e_I]$. Note that from Lemma 2.3 it is irrelevant which expansion of $\mathtt{Q}$ we choose. Below, we outline a strategy to determine coefficients $c_{\mathtt{k}}$ of (3.5) via the inclusion–exclusion principle.

Recall, the general form of the *inclusion–exclusion principle*, [24]: Given a finite set F and functions $f, g : 2^F \longrightarrow \mathbb{R}$,

$$g(S') = \sum_{S: S\subseteq S'} f(S), \qquad S' \subseteq F, \tag{3.7}$$

we have

$$f(S') = \sum_{S: S\subseteq S'} (-1)^{|S'|-|S|} g(S), \qquad S' \subseteq F. \tag{3.8}$$

Recall the following notation: given $\mathtt{Q} \in \mathbb{R}[e_I]$ and $\mathtt{s} \in \mathfrak{P}_n$ define

$$\mathtt{Q}(\mathtt{s}) := \mathtt{Q}(\{e_I = 1 \mid I \in \mathtt{s}\}). \tag{3.9}$$

I.e. $\mathtt{Q}(\mathtt{s})$ is a polynomial obtained from $\mathtt{Q}$ by substituting $e_I = 1$ for all $I \in \mathtt{s}$, and $\mathtt{Q}(\mathtt{s})(0)$ its constant coefficient.

Lemma 3.2. *Consider any representative* $\mathtt{Q} \in \mathbb{R}_{\mathcal{I}}[e_I]$ *in a general form* (3.5). *For any* $\mathtt{k} \in \mathfrak{P}_n$ *the coefficient* $c_{\mathtt{k}}$ *of* $\mathtt{Q}$ *in the expansion* (3.5) *is given as follows*

$$c_{\mathtt{k}}(\mathtt{Q}) = \sum_{\mathtt{r}\in\mathfrak{P}_n, \mathtt{r}\subseteq\mathtt{k}} (-1)^{|\mathtt{k}|-|\mathtt{r}|}\, \mathtt{Q}(\mathtt{r})(0). \tag{3.10}$$

In the case $\mathtt{Q}$ *is represented by the chain expansion (right)*(3.6), *for any* $\mathtt{s} \in \mathfrak{C}_n$, $\mathtt{s} \neq \{\emptyset\}$ *we have*

$$c_{\mathtt{s}}(\mathtt{Q}) = \sum_{\mathtt{r}\in\mathfrak{C}_n, \mathtt{r}\subseteq\mathtt{s}} (-1)^{|\mathtt{s}|-|\mathtt{r}|}\, (\mathtt{Q}(\mathtt{r})(0) - c_0), \tag{3.11}$$

where $c_0 = c_\emptyset = \mathtt{Q}(0)$ *is the constant term of* $\mathtt{Q}$.

Proof. In the inclusion–exclusion principle set $F = \mathtt{k}$. Then any subset $S \subseteq F$ is just a subset of faces $\mathtt{r}$ of $\mathtt{k}$, i.e. $\mathtt{r} \in \mathfrak{P}_n$ and $\mathtt{r} \subseteq \mathtt{k}$. Directly from (3.5) and (3.9) for any $\mathtt{r} \subseteq \mathtt{k}$, we have

$$\mathtt{Q}(\mathtt{r})(0) = \sum_{\mathtt{r}' \subseteq \mathtt{r}} c_{\mathtt{r}'}$$

thus setting $g(\mathtt{r}) = \mathtt{Q}(\mathtt{r})(0)$ and $f(\mathtt{r}) = c_{\mathtt{r}}$, Equation (3.10) follows from (3.8). To obtain (3.11) consider the polynomial $\bar{\mathtt{Q}} = \mathtt{Q} - c_0$. If $\mathtt{r} \subseteq \mathtt{k}$ and $\mathtt{r} \neq \widehat{\mathtt{r}}$, then $\bar{\mathtt{Q}}(\mathtt{r})(0) = 0$. Therefore, for $\mathtt{s} \in \mathfrak{C}_n$, Equation (3.10) yields

$$c_{\mathtt{s}}(\bar{\mathtt{Q}}) = \sum_{\mathtt{r} \in \mathfrak{P}_n, \mathtt{r} \subseteq \mathtt{s}} (-1)^{|\mathtt{s}|-|\mathtt{r}|} \bar{\mathtt{Q}}(\mathtt{r})(0) = \sum_{\mathtt{r} \in \mathfrak{C}_n, \mathtt{r} \subseteq \mathtt{s}} (-1)^{|\mathtt{s}|-|\mathtt{r}|} \bar{\mathtt{Q}}(\mathtt{r})(0).$$

Because $c_{\mathtt{s}}(\mathtt{Q}) = c_{\mathtt{s}}(\bar{\mathtt{Q}})$ for $\mathtt{s} \neq \emptyset$, the identity in (3.11) follows. □

For a polynomial random variable $\mathtt{Q} \in \mathbb{R}[e_I]$ in a general form (3.5), define constants

$$\underline{m}(\mathtt{Q}) = \sum_{\mathtt{s} \in \mathfrak{P}_n} c^-_{\mathtt{s}}, \quad c^-_{\mathtt{s}} = \min\{c_{\mathtt{s}}, 0\}, \qquad \overline{m}(\mathtt{Q}) = \sum_{\mathtt{s} \in \mathfrak{P}_n} c^+_{\mathtt{s}}, \quad c^+_{\mathtt{s}} = \max\{c_{\mathtt{s}}, 0\}. \tag{3.12}$$

Denote the coefficients of the general expansion (3.5) of the chain representative of the k-th power $(\mathtt{Q})^k$ by $c_{\mathtt{s},k}(\mathtt{Q})$, i.e.

$$\mathtt{Q}^k = \sum_{\mathtt{s} \in \mathfrak{C}_n} c_{\mathtt{s},k}(\mathtt{Q})\, e_{\mathtt{s}} \ . \tag{3.13}$$

We summarize efforts of this section by stating the following result which is a direct consequence of Lemma 3.1 and Lemma 3.2.

Theorem 3.3. *Given* $\mathtt{Q}$ *as a chain representative in* $\mathbb{R}_{\mathcal{I}}[e_I]$, *suppose that the set of realizations of* $\mathtt{Q}$ *is in the integer interval* $[\underline{m}, \overline{m}]$. *Then the distribution of* $\mathtt{Q}$ *and its moments are given as follows*

$$\begin{aligned} \mu_k = \mathbb{E}(\mathtt{Q}^k) &= \sum_{\mathtt{s} \in \mathfrak{C}_n} (\mathtt{Q}(\mathtt{s})(0))^k P_{\mathtt{s}} = \sum_{\mathtt{s} \in \mathfrak{C}_n} c_{\mathtt{s},k}(\mathtt{Q}) p_{\mathtt{s}}, \\ \mathbb{P}(\mathtt{Q} = \underline{m} + j) &= \sum_{\mathtt{s} \in \mathfrak{C}_n; \mathtt{Q}(s)(0) = \underline{m}+j} P_{\mathtt{s}} = \sum_{\mathtt{s} \in \mathfrak{C}_n} a_{\mathtt{s},j}(\mathtt{Q})\, p_{\mathtt{s}}, \qquad j \in [0, N], \quad N = \overline{m} - \underline{m} \\ \textit{for} \qquad a_{\mathtt{s},j}(\mathtt{Q}) &= \sum_{k=0}^{N} v_{kj}(\mathtt{Q})\, c_{\mathtt{s},k}(\mathtt{Q}), \end{aligned} \tag{3.14}$$

where $v_{kj}(\mathtt{Q})$ *were defined in* (3.2). *Further,* $c_0 = \mathtt{Q}(0)$ *and* $c_{0,k} = c_0^k$, *and for* $\mathtt{s} \neq \emptyset$:

$$c_{\mathtt{s},k}(\mathtt{Q}) = \sum_{\mathtt{r} \in \mathfrak{C}_n; \mathtt{r} \subseteq \mathtt{s}} (-1)^{|\mathtt{s}|-|\mathtt{r}|} (\mathtt{Q}(\mathtt{r})(0) - c_0)^k. \tag{3.15}$$

Proof. Since e_{s} are Bernoulli random variables

$$\mu_k = \mathbb{E}(\mathsf{Q}^k) = \sum_{\mathsf{s}\in\mathfrak{C}_n} c_{\mathsf{s},k}(\mathsf{Q})\,\mathbb{E}(e_{\mathsf{s}}) = \sum_{\mathsf{s}\in\mathfrak{P}_n} c_{\mathsf{s},k}(\mathsf{Q})\,p_{\mathsf{s}},$$

thus (3.14) is an immediate consequence of (3.4). Formula (3.15) follows from (3.11) applied to Q^k. □

3.3. **Formulas for $\boldsymbol{\chi}(\mathsf{K})$, $\boldsymbol{f}_d(\mathsf{K})$ and $\boldsymbol{\chi}_{rel}(\mathsf{K},\mathsf{L})$.** In this section we aim to provide slightly more tractable formulas for the coefficients $c_{\mathsf{s},k}(\,\cdot\,)$ and the integer ranges $[\underline{m}(\,\cdot\,), \overline{m}(\,\cdot\,)]$ for the polynomials $\boldsymbol{\chi} = \boldsymbol{\chi}(\mathsf{K})$, $\boldsymbol{f}_d = \boldsymbol{f}_d(\mathsf{K})$ and $\boldsymbol{\chi}_{rel} = \boldsymbol{\chi}_{rel}(\mathsf{K},\mathsf{L})$, where K is a given random complex on n vertices. Thanks to Theorem 3.3, it will provide us with a more precise characterization of distributions for these polynomials.

We begin with the case of $\boldsymbol{f}_d(\mathsf{K})$. Clearly, the range of $\boldsymbol{f}_d$ is contained in between

$$\underline{m}(\boldsymbol{f}_d) = 0, \qquad \text{and} \qquad \overline{m}(\boldsymbol{f}_d) = \binom{n}{d+1}. \tag{3.16}$$

For a subcomplex $\mathsf{s} \in \mathfrak{C}_n$ and its corresponding antichain $\widehat{\mathsf{s}}$, recall the following notation

$$\begin{aligned}
r^+_{top} &= r^+_{top}(\mathsf{s}) = \{\text{numer of even dimensional faces in } \widehat{\mathsf{s}}\},\\
r^-_{top} &= r^-_{top}(\mathsf{s}) = \{\text{numer of odd dimensional faces in } \widehat{\mathsf{s}}\},\\
r^+_{low} &= r^+_{low}(\mathsf{s}) = \{\text{numer of even dimensional faces in } \mathsf{s} - \widehat{\mathsf{s}}\},\\
r^-_{low} &= r^-_{low}(\mathsf{s}) = \{\text{numer of odd dimensional faces in } \mathsf{s} - \widehat{\mathsf{s}}\},\\
r_{top} &= r_{top}(\mathsf{s}) = r^+_{top} + r^-_{top} = |\widehat{\mathsf{s}}|,\\
r_{low} &= r_{low}(\mathsf{s}) = |\mathsf{s}| - |\widehat{\mathsf{s}}|, \qquad r = r(\mathsf{s}) = r_{top} + r_{low} = |\mathsf{s}|.
\end{aligned} \tag{3.17}$$

Given a random complex K, a basic example of interest is the number of its d–dimensional faces

$$\boldsymbol{f}_d = \sum_{\{I\}\in\mathfrak{C}_n;\,|I|=d+1} e_I, \tag{3.18}$$

and the *Euler characteristic of* K. By the Euler–Poincare formula (see Equation (2.4), c.f. [18]) we have the following relation between (3.18) and (2.9)

$$\boldsymbol{\chi} = \sum_{d=0}^{n-1} (-1)^d \boldsymbol{f}_d. \tag{3.19}$$

Moreover,

$$\boldsymbol{\chi}(\mathsf{s})(0) = \chi(\mathsf{s}) = r^+(\mathsf{s}) - r^-(\mathsf{s}).$$

Proposition 3.4. *We have the following formulas for the coefficients of $\boldsymbol{f}_d$ and $\boldsymbol{\chi}$:*

$$c_{\mathsf{s},k}(\boldsymbol{f}_d) = \sum_{i=1}^{r_{top}(\mathsf{s})} (-1)^{r_{top}(\mathsf{s})-i} \binom{r_{top}(\mathsf{s})}{i} i^k, \tag{3.20}$$

$$
\begin{aligned}
c_{\mathtt{s},k}(\boldsymbol{\chi}) &= \sum_{\mathtt{l}\in\mathfrak{C}_n;\mathtt{l}\subseteq\mathtt{s}} (-1)^{|\mathtt{s}|-|\mathtt{l}|}(\boldsymbol{\chi}(\mathtt{l})(0))^k = \sum_{\mathtt{l}\in\mathfrak{C}_n;\mathtt{l}\subseteq\mathtt{s}} (-1)^{|\mathtt{s}|-|\mathtt{l}|}\big(r^+(\mathtt{l})-r^-(\mathtt{l})\big)^k \\
&= \sum_{i=0}^{r^+_{top}(\mathtt{s})}\sum_{j=0}^{r^-_{top}(\mathtt{s})} (-1)^{r_{top}(\mathtt{s})-i-j}\binom{r^+_{top}(\mathtt{s})}{i}\binom{r^-_{top}(\mathtt{s})}{j}\big(i-j+r^+_{low}(\mathtt{s})-r^-_{low}(\mathtt{s})\big)^k
\end{aligned}
\tag{3.21}
$$

Proof of Formula (3.20). Applying (3.15) directly to $\boldsymbol{f}_d$ we obtain the first identity in (3.20). For the second equation in (3.20), let $\mathtt{l}\in\mathfrak{P}_n$ be the set of all d–faces. Since $\boldsymbol{f}_d=\sum_{I\in\mathtt{l}} e_I$, for any $\mathtt{k}\subseteq\mathtt{l}$, Equation (3.10) implies

$$
c_{\mathtt{k}}((\boldsymbol{f}_d)^k) = \sum_{\mathtt{r}\in\mathfrak{P}_n;\mathtt{r}\subseteq\mathtt{k}} (-1)^{|\mathtt{k}|-|\mathtt{r}|}\,(\boldsymbol{f}_d(\mathtt{r})(0))^k = \sum_{i=1}^{|\mathtt{k}|}(-1)^{|\mathtt{k}|-i}\binom{|\mathtt{k}|}{i} i^k. \tag{3.22}
$$

Considering $\boldsymbol{f}_d$ as an element of $\mathbb{R}_{\mathcal{I}}[e_I]$ and choosing a chain representative for $\boldsymbol{f}^k_d$, we conclude that its coefficients $c_{\mathtt{s},k}(\boldsymbol{f}^k_d)$ vanish unless the corresponding antichain $\widehat{\mathtt{s}}$ consists of purely d–faces. In the latter case we obtain from (3.22)

$$
c_{\mathtt{s},k}((\boldsymbol{f}_d)^k) = c_{\mathtt{k}}((\boldsymbol{f}_d)^k), \qquad \text{for}\quad \mathtt{k}=\widehat{\mathtt{s}},
$$

which implies the identity in (3.20) via the notation of (3.17). □

Next, we turn to the random polynomial $\boldsymbol{\chi}=\boldsymbol{\chi}(\mathsf{K})$. The range of $\boldsymbol{\chi}(\mathsf{K})$ is contained in $[\underline{m}(\boldsymbol{\chi}),\overline{m}(\boldsymbol{\chi})]$ where

$$
\underline{m}(\boldsymbol{\chi}) = -\sum_{r;0<2r+1\leq n}\binom{n}{2r+1}, \qquad \text{and} \qquad \overline{m}(\boldsymbol{\chi}) = \sum_{r;0<2r\leq n}\binom{n}{2r}. \tag{3.23}
$$

If K is supported on some subcomplex $\mathtt{k}\in\mathfrak{C}_n$, smaller than the full n–simplex, the above range can be narrowed to

$$
\underline{m}(\boldsymbol{\chi}(\mathsf{K})) = -\sum_{0\leq 2r+1\leq \dim(\mathtt{k})}\boldsymbol{f}_{2r+1}(\mathtt{k}), \qquad \overline{m}(\boldsymbol{\chi}(\mathsf{K})) = \sum_{0\leq 2r\leq \dim(\mathtt{k})}\boldsymbol{f}_{2r}(\mathtt{k}).
$$

Proof of Formula (3.21). Applying (3.15) to $\mathtt{Q}=\boldsymbol{\chi}$ directly, one obtains the first part of (3.21). To obtain the second part we choose to present a different argument for the purpose of cross verification. Recall that given indeterminates $x_1,\ldots,x_m$, we have the following multinomial formula (c.f. [13])

$$
(x_1+x_2+\ldots+x_m)^k = \sum_{\substack{\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_m),\\ |\boldsymbol{\alpha}|=k}} \binom{k}{\boldsymbol{\alpha}} x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_m^{\alpha_m}, \tag{3.24}
$$

where $\binom{k}{\boldsymbol{\alpha}}=\frac{k!}{\alpha_1!\alpha_2!\ldots\alpha_m!}$, $\alpha_i\geq 0$, $|\boldsymbol{\alpha}|=\sum_i\alpha_i$ and $\boldsymbol{\alpha}$ form all possible partitions of k. Let $\boldsymbol{\alpha}$ have coordinates indexed by $f(n)$ (i.e. faces of $\boldsymbol{\Delta}_n$). A direct application of (3.24) to (2.10) yields

$$(\boldsymbol{\chi})^k = \sum_{\substack{\boldsymbol{\alpha}=(\alpha_I),\\ |\boldsymbol{\alpha}|=k}} \binom{k}{\boldsymbol{\alpha}} \prod_{I\in f(n)} \Big((-1)^{|I|-1} e_I\Big)^{\alpha_I} = \sum_{\substack{\boldsymbol{\alpha}=(\alpha_I),\\ |\boldsymbol{\alpha}|=k}} \Big((-1)^{\sum_{I\in \mathsf{s}(\boldsymbol{\alpha})}(|I|-1)\alpha_I}\Big)\binom{k}{\boldsymbol{\alpha}} e_{\mathsf{s}(\boldsymbol{\alpha})}, \tag{3.25}$$

where we denoted

$$\mathsf{s}(\boldsymbol{\alpha}) = \{I \in f(n) \mid \alpha_I > 0\}. \tag{3.26}$$

Observe that for any $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$,

$$e_{\mathsf{s}(\boldsymbol{\alpha})} = e_{\mathsf{s}(\boldsymbol{\alpha}')}, \qquad \text{in} \quad \mathbb{R}_{\mathcal{I}}[e_I], \tag{3.27}$$

if and only if the corresponding antichains are the same i.e. $\widehat{\mathsf{s}(\boldsymbol{\alpha})} = \widehat{\mathsf{s}(\boldsymbol{\alpha}')}$. Fix a chain representative of some complex $\mathsf{s} \in \mathfrak{C}_n$ and let $\widehat{\mathsf{s}}$ be the corresponding antichain. Clearly, $\widehat{\mathsf{s}} \subseteq \mathsf{s}$, consider partitions $\boldsymbol{\alpha}$ of k which are in the form $\boldsymbol{\alpha} = \boldsymbol{\beta} + \boldsymbol{\gamma}$ where $\boldsymbol{\beta} = (\beta_I)$, satisfies: $\beta_I > 0$ for $I \in \widehat{\mathsf{s}}$ and $\beta_I = 0$ for $I \in \mathsf{s} - \widehat{\mathsf{s}}$, and $\boldsymbol{\gamma} = (\gamma_I)$ satisfies: $\gamma_I \geq 0$ for $I \in \mathsf{s} - \widehat{\mathsf{s}}$ and $\gamma_I = 0$ for $I \in \widehat{\mathsf{s}}$. The following claim immediately follows

Claim: Given $\mathsf{s} \in \mathfrak{C}_n$ and any partition $\boldsymbol{\alpha}$ of k indexed by $f(n)$, we have $\widetilde{\Pi}(\mathsf{s}(\boldsymbol{\alpha})) = \mathsf{s}$ if and only if $\boldsymbol{\alpha}$ has the above decomposition: $\boldsymbol{\beta} + \boldsymbol{\gamma}$.

Therefore, the $c_{\mathsf{s},k}(\boldsymbol{\chi})$ coefficient of the chain representative of $(\boldsymbol{\chi})^k$ is a sum of coefficients of $e_{\mathsf{s}(\boldsymbol{\alpha})}$ for all $\boldsymbol{\alpha}$ in the form $\boldsymbol{\beta} + \boldsymbol{\gamma}$. Applying notation (3.17) we may express it as

$$(\boldsymbol{\chi})^k = \sum_{\mathsf{s}\in\mathfrak{C}_n} c_{\mathsf{s},k}(\boldsymbol{\chi})\, e_{\mathsf{s}}, \qquad \text{where} \tag{3.28}$$

$$c_{\mathsf{s},k}(\boldsymbol{\chi}) = \begin{cases} \displaystyle\sum_{\substack{(\boldsymbol{\beta},\boldsymbol{\gamma})=(\beta_1,\ldots,\beta_{r_{top}},\gamma_1,\ldots,\gamma_{r_{low}}),\\ |\boldsymbol{\beta}|+|\boldsymbol{\gamma}|=k,\, \beta_i>0,\gamma_j\geq 0}} (-1)^{\sum_{i=1}^{r_{top}}(|I_i|-1)\beta_i+\sum_{j=1}^{r_{low}}(|J_j|-1)\gamma_j}\binom{k}{\boldsymbol{\beta},\boldsymbol{\gamma}}, & \text{if } k\geq r,\\ 0, & \text{otherwise,}\end{cases}$$

where we indexed the faces of $\widehat{\mathsf{s}}$ in s by $\{I_i\}$, $i = 1, \ldots, r_{top}$ and faces of $\mathsf{s} - \widehat{\mathsf{s}}$ in s by $\{J_j\}$, $j = 1, \ldots, r_{low}$. To set up the inclusion–exclusion principle, note that the sum for $c_{\mathsf{s},k}(\boldsymbol{\chi})$ is a part of the larger sum (where we allow $\beta_i \geq 0$, and $(\boldsymbol{\beta},\boldsymbol{\gamma}) = (\beta_1, \ldots, \beta_{r_{top}}, \gamma_1, \ldots, \gamma_{r_{low}})$):

$$\sum_{\substack{(\boldsymbol{\beta},\boldsymbol{\gamma})\\ |\boldsymbol{\beta}|+|\boldsymbol{\gamma}|=k,\, \beta_i\geq 0,\gamma_j\geq 0}} (-1)^{\sum_{i=1}^{r_{top}}(|I_i|-1)\beta_i+\sum_{j=1}^{r_{low}}(|J_j|-1)\gamma_j}\binom{k}{\boldsymbol{\beta},\boldsymbol{\gamma}} = \Big(\sum_{i=1}^{r_{low}}(-1)^{(|I_i|-1)} + \sum_{j=1}^{r_{top}}(-1)^{(|J_j|-1)}\Big)^k.$$

We stratify the above sum with respect to number of β_i's strictly greater than zero, and set up the inclusion–exclusion as follows. Let $F = \{1, \ldots, r_{top}\}$ and define for any $S \subseteq F$, functions f, g (in (3.7), (3.8)) as

$$f(S)=\sum_{\substack{(\boldsymbol{\beta},\boldsymbol{\gamma})=(\{\beta_i\},\{\gamma_j\}),|\boldsymbol{\beta}|+|\boldsymbol{\gamma}|=k,\gamma_j\geq 0,\\ \beta_i>0,\text{ if } i\in S,\ \beta_i=0\text{ if } i\notin S.}}(-1)^{\sum_{i=1}^{r_{top}}(|I_i|-1)\beta_i+\sum_{j=1}^{r_{low}}(|J_j|-1)\gamma_j}\binom{k}{\boldsymbol{\beta},\boldsymbol{\gamma}},$$

$$g(S)=\Big(\sum_{i\in S}(-1)^{(|I_i|-1)}+\sum_{j=1}^{r_{low}}(-1)^{(|J_j|-1)}\Big)^k.$$

Observe that $\sum_{j=1}^{r_{low}}(-1)^{(|J_j|-1)}=r^+_{low}-r^-_{low}$, which yields

$$g(S)=\Big(|S^+|-|S^-|+r^+_{low}-r^-_{low}\Big)^k.$$

where $|S^+|(|S^-|)$ denotes number of even(odd) dimensional faces of $\widehat{\mathtt{s}}$ indexed by S. By (3.8) we obtain

$$f(F)=\sum_{S:S\subseteq F}(-1)^{r_{top}-|S|}\Big(|S^+|-|S^-|+r^+_{low}-r^-_{low}\Big)^k.$$

Since there are r^+_{top} even dimensional faces and r^-_{top} odd dimensional faces in $\widehat{\mathtt{s}}$, for a fixed $i\in[0,r^+_{top}]$ and $j\in[0,r^-_{top}]$ there are exactly $\binom{r^+_{top}}{i}\binom{r^-_{top}}{j}$ subsets $S\subseteq F$ satisfying $i=|S^+|$, $j=|S^-|$. Thus the second part of (3.21) now follows from $f(F)=c_{\mathtt{s},k}(\boldsymbol{\chi})$. □

As the last case of interest, we consider is the relative Euler characteristic $\boldsymbol{\chi}_{rel}=\boldsymbol{\chi}_{rel}(\mathsf{K},\mathsf{L})$ of a random pair (K,L). Denoting the characteristic functions of K by $\{e_I\}$ and of L by $\{w_J\}$, (2.5) and (2.6) imply the following polynomial expression

$$\boldsymbol{\chi}_{rel}=\sum_{d=0}^{n-1}(-1)^k\Big(\sum_{I\in f_d(n)}(e_I-w_I)\Big). \tag{3.29}$$

Analogously, as in the absolute case, the distribution of (K,L) is determined by

$$p_{\mathtt{s},\mathtt{r}}=\mathbb{P}(e_{\mathtt{s}}=1,w_{\mathtt{r}}=1)=\mathbb{P}(e_{\mathtt{s}}w_{\mathtt{r}}=1). \tag{3.30}$$

The maximal constants for the range of $\boldsymbol{\chi}_{rel}(\mathsf{K},\mathsf{L})$ are

$$\underline{m}(\boldsymbol{\chi}_{rel})=\underline{m}(\boldsymbol{\chi})-\overline{m}(\boldsymbol{\chi}),\quad\text{and}\quad\overline{m}(\boldsymbol{\chi}_{rel})=\overline{m}(\boldsymbol{\chi})-\underline{m}(\boldsymbol{\chi}). \tag{3.31}$$

For convenience we state the following corollary of Theorem 3.3:

Corollary 3.5 (Distribution of $\boldsymbol{\chi}_{rel}(\mathsf{K},\mathsf{L})$)**.** *Given a random pair* (K,L)*, the distribution of* $\boldsymbol{\chi}_{rel}$ *on* $[\underline{m}(\boldsymbol{\chi}_{rel}),\overline{m}(\boldsymbol{\chi}_{rel})]$ *is given as follows, for* $j\in[0,N]$, $N=\overline{m}(\boldsymbol{\chi}_{rel})-\underline{m}(\boldsymbol{\chi}_{rel})$

$$\mathbb{P}\big(\boldsymbol{\chi}_{rel}=\underline{m}(\boldsymbol{\chi}_{rel})+j\big)=\sum_{(\mathtt{s},\mathtt{r})\in\mathfrak{C}_n\times\mathfrak{C}_n}a_{\mathtt{s},\mathtt{r},j}(\boldsymbol{\chi}_{rel})\,p_{\mathtt{s},\mathtt{r}}, \tag{3.32}$$

$$a_{\mathtt{s},\mathtt{r},j}(\boldsymbol{\chi}_{rel})=\sum_{k=0}^{N}\big(v_{kj}(\boldsymbol{\chi}_{rel})\,c_{\mathtt{s},\mathtt{r},k}(\boldsymbol{\chi}_{rel})\big),$$

where (using the notation of (3.17)*)*

$$\mathbb{E}((\boldsymbol{\chi}_{rel}(\mathsf{K},\mathsf{L}))^k) = \sum_{(\mathtt{s},\mathtt{r})\in\mathfrak{C}_n\times\mathfrak{C}_n} c_{\mathtt{s},\mathtt{r},k}\, p_{\mathtt{s},\mathtt{r}}, \qquad c_{\mathtt{s},\mathtt{r},k} = c_{\mathtt{s},\mathtt{r},k}(\boldsymbol{\chi}_{rel}) \tag{3.33}$$

$$c_{\mathtt{s},\mathtt{r},k} = \begin{cases} \displaystyle\sum_{\substack{i\in[0,r^+_{top}(\mathtt{s})],j\in[0,r^-_{top}(\mathtt{s})],\\ i'\in[0,r^+_{top}(\mathtt{r})],j'\in[0,r^-_{top}(\mathtt{r})]}} (-1)^{r_{top}(\mathtt{s})+r_{top}(\mathtt{r})-i-j-i'-j'} \binom{r^+_{top}(\mathtt{s})}{i}\binom{r^-_{top}(\mathtt{s})}{j}\binom{r^+_{top}(\mathtt{r})}{i'}\binom{r^-_{top}(\mathtt{r})}{j'}\cdot \\ \qquad\cdot\Big((i-j)+(i'-j')+(r^+_{low}(\mathtt{s})-r^-_{low}(\mathtt{s}))+(r^+_{low}(\mathtt{r})-r^-_{low}(\mathtt{r}))\Big)^k, \ \textit{for} \quad k\geq r, \\ 0, \qquad \textit{for} \quad k<r. \end{cases}$$

The proof is as fully analogous the previous arguments and is omitted. Note that the expression for $c_{\mathtt{s},\mathtt{r},k}(\boldsymbol{\chi}_{rel})$ in (3.33) simplifies to (3.21) whenever $\mathsf{L}=\emptyset$.

4. Coverings of one–complexes and the Euler characteristic.

Given a deterministic *covering* of a finite simplicial complex X, i.e. a collection of compact connected subsets $A=\{A_{\{i\}}\}$, we can define its *nerve*, $\mathcal{N}(A)$ as a finite complex where vertices $\{i\}$ are just elements $A_{\{i\}}$ of the covering and a k-face $I=\{i_1,\ldots,i_{k+1}\}$ belongs to $\mathcal{N}(A)$, if and only if $A_{\{i_1\}}\cap A_{\{i_2\}}\cap\ldots\cap A_{\{i_{k+1}\}}\neq\emptyset$ (c.f. [33]).

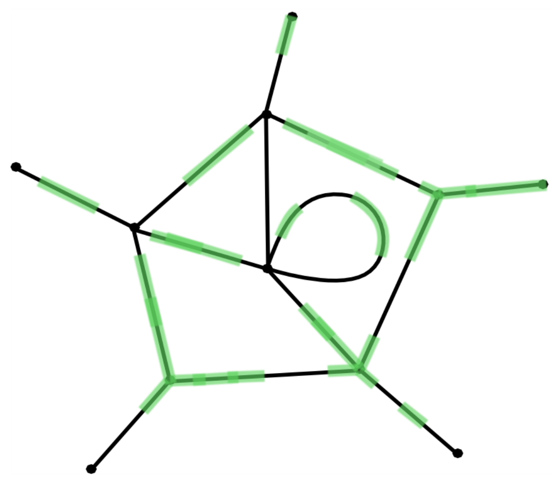

Figure 4: An example of a 1-complex with marked realization of a good cover.

The following result, due to Borsuk [6], is of fundamental importance in algebraic topology

Lemma 4.1 (The Nerve Lemma [6])**.** *Let* $A=\{A_{\{i\}}\}$ *be a covering of* X *and* $\mathcal{N}(A)$ *the associated nerve. If all intersections* $A_{\{i_1\}}\cap A_{\{i_2\}}\cap\ldots\cap A_{\{i_{k+1}\}}$*, for* $k>0$ *are contractible, then* $\mathcal{N}(A)$ *has a homotopy type of the subspace* $|A|=\bigcup_i A_{\{i\}}$ *of* X*.*

Recall that a subset of X is contractible if it can be deformed continuously to a point [18]. If $A=\{A_{\{i\}}\}$ satisfies the assumption of this lemma then we call it a *good* covering (of X).

In the remainder of this section we collect elementary facts from algebraic topology and show how the Euler characteristic of $\mathcal{N}(A)$ provides a criteria for a good deterministic covering $A = \{A_{\{i\}}\}$, to completely cover a connected 1–complex X, the proofs are basic and are either omitted or deferred to Appendix A.

4.1. **Coverage and the nerve complex.** We assume throughout that X is a connected 1–complex (c.f. [18, p. 103]) homeomorphic to a multi-graph, and denote ∂X the set of leaf vertices of X.

Proposition 4.2. *Let $\{A_{\{i\}}\}$ be a good covering of X, $|A| = \bigcup_i A_{\{i\}}$, denote $U = |A|$ and $V = \overline{|A|^c}$. Then,*

$$\beta_1(X) \geq \beta_1(U), \tag{4.1}$$

and

$$\chi(X) \leq \chi(U). \tag{4.2}$$

Moreover, if the inequality in (4.1) *is strict then* (4.2) *is also strict.*

By the Nerve Lemma, an obvious necessary condition for $X \subseteq |A|$ is

$$\chi(X) = \chi(|A|) = \chi(\mathcal{N}(A)). \tag{4.3}$$

If $\partial X = \emptyset$, we have the following

Corollary 4.3. *Suppose X satisfies $\partial X = \emptyset$, then* (4.3) *implies $X \subseteq |A|$.*

When $\partial X \neq \emptyset$, the condition (4.3) is insufficient; however we may adjust it by using the relative version $\chi_{rel}(X, \partial X)$ of the Euler characteristic (2.5). Note that for the pair $(X, \partial X)$, $\chi_{rel}(X, \partial X)$ reduces to

$$\chi_{rel}(X, \partial X) = \chi(X) - \#\{\partial X\},$$

where $\#\{\partial X\}$ is a number of points in ∂X. By [18, p. 102] we may consider the quotient complex $X' = X/\partial X$ which is a 1–complex ([18, p. 103]) with $\partial X' = \emptyset$, and

$$\chi_{rel}(X, \partial X) = \chi(X/\partial X).$$

Let $q : X \mapsto X'$ be the quotient projection, then the covering A of X projects to the covering A' of X'. It is not true that A' is automatically a good covering of X', one may easily find examples where this is the case. However, the following fact is available (proof left to the reader)

Lemma 4.4. *Given $A = \{A_{\{i\}}\}$ is a good covering of X, let for every i the intersection $A_{\{i\}} \cap \partial X$ be either empty or a point (in other words $A_{\partial X} = \{A_{\{i\}} \cap \partial X\}$ is a good covering of ∂X). Then the quotient covering A' of X' is also good.*

Consequently, we say that A is a *good covering of the pair* $(X, \partial X)$ provided A is good for X and $A_{\partial X}$ is good for ∂X. Then by the above lemma A' is good for X' and Corollary 4.3 says that A' covers X', if and only if $\chi(|A'|) = \chi(X')$. It leads us to the following generalization of Corollary 4.3.

Lemma 4.5. *Given a good covering* $A = \{A_{\{i\}}\}$ *of* $(X, \partial X)$ *let* $|A| = \bigcup_i A_{\{i\}}$. *Then* $X \subseteq |A|$, *if and only if*

$$\chi_{rel}(\mathcal{N}(A), \mathcal{N}(A_{\partial X})) = \chi_{rel}(X, \partial X) \tag{4.4}$$

or equivalently

$$\chi(|A|) = \chi(X) - \#\{\partial X\} + \#\{|A| \cap \partial X\}. \tag{4.5}$$

Remark 4.6. Equivalently, the coverage condition for $(X, \partial X)$ can be obtained by looking at the covering $\widehat{A}$, equal to a union of A and the boundary vertices: $\partial X = \{x_1, \ldots, x_{\#\{\partial X\}}\}$. Then $\widehat{A}$ is good if satisfies the conditions of Lemma 4.4

$$\begin{aligned}\chi(|\widehat{A}|) &= \chi(|A| \cup \partial X) = \chi(|A|) + \chi(\partial X) - \chi(|A| \cap \partial X)\\ &= \chi(|A|) + \#\{\partial X\} - \#\{|A| \cap \partial X\},\end{aligned}$$

which together with (4.3) leads us to (4.4).

4.2. **Coverage of X by ε-balls. Vietoris–Rips complex.** A special case of interest (see e.g. [34, 10]) is when a connected 1–complex X ought to be covered by ε-size neighborhoods, and ε can be sufficiently small. In such cases the topology of $\mathcal{N}(A)$ simplifies and one may work with Vietoris–Rips complex [19], as we show in the following paragraphs.
Recall that given a simplicial complex K its *Vietoris–Rips complex* $\mathcal{R}(K)$, [19] is defined to be a maximal simplicial complex (with respect to inclusion) which has the same 1-skeleton as K. In practice, this means that $\mathcal{R}(K)$ is obtained by filling every k-clique in the graph $K^{(1)}$ with a $(k-1)$-dimensional face, e.g. 3-cycles are filled with 2-simplices in $\mathcal{R}(K)$, etc.

We will consider a finite covering $A = \{A_{\{1\}}, \ldots, A_{\{n\}}\}$ of (X, d_X) by closed ε-balls. Possible shapes of such balls for ε sufficiently small are depicted on Figure 4.2.

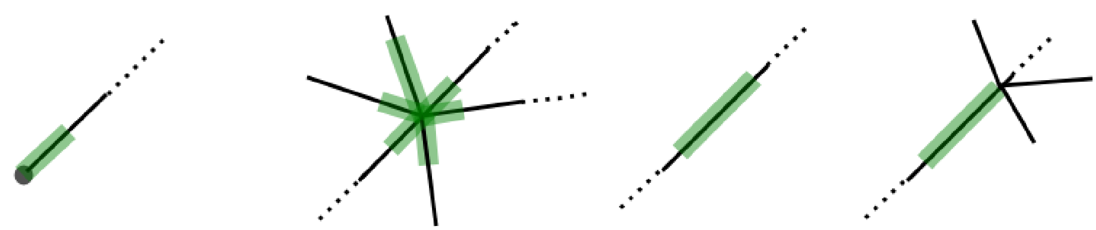

Figure 4.2: Possible shapes of closed ε-balls in X with the intrinsic distance d_X.

Let us denote by $\mathcal{R}(A)$ the Vietoris–Rips complex of the nerve of the cover, and record the following

Proposition 4.7. *Suppose* $\mathcal{C}$ *is the girth of* X', *i.e. the length of the shortest cycle in the quotient complex* $X' = X/\partial X$. *Then,*

(*i*) *if* $\varepsilon < \frac{1}{4}\mathcal{C}$, *the covering* A *by* ε*-balls in* (X, d_X) *is a good cover.*
(*ii*) *if* $\varepsilon < \frac{1}{6}\mathcal{C}$, *the nerve* $\mathcal{N}(A)$ *of* A *equals* $\mathcal{R}(A)$.

Proof. For (i) we must show that every k-fold intersection $A_{\{i_1\}} \cap A_{\{i_2\}} \cap \ldots \cap A_{\{i_k\}}$ has a homotopy type of a point. Because $\mathrm{diam}(A_{\{i\}}) < \mathcal{C}$, $A_{\{i\}}$ is a connected tree and therefore contractible, which shows the claim for $k = 1$. For $k = 2$, first suppose that a nonempty intersection $A_{\{i\}} \cap A_{\{j\}}$ is disconnected i.e. $\dim(\widetilde{H}_0(A_{\{i\}} \cap A_{\{j\}})) \geq 1$ (where $\widetilde{H}_*(\,\cdot\,)$ denotes the reduced homology groups c.f. [18]). Since $A_{\{i\}}$ and $A_{\{j\}}$ are connected, the reduced Mayer-Vietoris sequence for $A_{\{i\}} \cap A_{\{j\}}$ then simplifies to

$$0 \longrightarrow \widetilde{H}_1(A_{\{i\}} \cup A_{\{j\}}) \longrightarrow \widetilde{H}_0(A_{\{i\}} \cap A_{\{j\}}) \longrightarrow \widetilde{H}_0(A_{\{i\}}) \oplus \widetilde{H}_0(A_{\{j\}}) \cong \{0\},$$

We obtain $\widetilde{H}_1(A_{\{i\}} \cup A_{\{j\}}) \cong \widetilde{H}_0(A_{\{i\}} \cap A_{\{j\}}) \cong \mathbb{R}^k$ for some $k \geq 1$, which implies that $A_{\{i\}} \cup A_{\{j\}}$ contains a nontrivial cycle. This however contradicts the fact that $\mathrm{diam}(A_{\{i\}} \cup A_{\{j\}}) \leq 4\,\varepsilon < \mathcal{C}$. Thus k has to vanish and $A_{\{i\}} \cap A_{\{j\}}$ must be connected, contain no cycle, and is therefore contractible. Now, for an induction step with respect to k, it suffices to apply the previous step to $A' = A_{\{i_1\}} \cap A_{\{i_2\}} \cap \ldots \cap A_{\{i_k\}}$ and $A'' = A_{\{i_{k+1}\}}$.

Before proving (ii), recall the 1-dimensional version of Helly's Theorem (c.f. [11]) implies that given a finite collection of intervals $\{C_1, C_2, \ldots, C_n\}$ on $\mathbb{R}$, if the intersection of each pair is nonempty, i.e. $C_i \cap C_j \neq \emptyset$, for every $1 \leq i, j \leq n$, then $\bigcap_{i=1}^{n} C_i \neq \emptyset$.

First consider the case of 3-fold intersections, i.e. supposing that $A_{\{j\}} \cap A_{\{k\}} \neq \emptyset$, $1 \leq k \neq j \leq 3$. We aim to show that $A_{\{1\}} \cap A_{\{2\}} \cap A_{\{3\}} \neq \emptyset$. Observe that $V = A_{\{1\}} \cup A_{\{2\}} \cup A_{\{3\}}$ is connected and by the argument of (i) it must be a connected tree, i.e. contains no cycles. Let $p_{1,2}$, $p_{2,3}$, $p_{1,3}$ be distinct points in V such that $p_{i,j} \in A_{\{i\}} \cap A_{\{j\}}$. Note that for each pair: $p_{i,j}$, $p_{s,t}$ there exists a path in V connecting these points. We now consider two cases: (1) one of these paths, we denote by l, contains all three points $p_{i,j}$, then the collection $\{C_i\}$, $C_i = l \cap A_{\{i\}}$, $i = 1, 2, 3$ satisfies the assumptions of Helly's Theorem which implies the claim. (2) none of the paths between paris of $p_{i,j}$'s contain the third point. Consider two shortest paths: l_1 between $p_{1,2}$ and $p_{2,3}$, and l_2 between $p_{1,2}$ and $p_{2,3}$ then $l_{1,2} = l_1 \cap l_2$ is a segment between $p_{1,2}$ and some vertex of $v \in V$. The vertex v has to be in one of $A_{\{j\}}$'s, w.l.o.g. suppose $v \in A_{\{2\}}$ (as other cases are analogous.) Then if v is also in $A_{\{1\}}$ or $A_{\{3\}}$ we can take $p_{1,2}$ or $p_{2,3}$ equal to v and use (1). If $v \notin A_{\{1\}}$ and $v \notin A_{\{3\}}$ then we observe that either $A_{\{1\}}$ or $A_{\{3\}}$ is disconnected which is not the case. This concludes the proof of (ii) for the 3-fold case, the general case can be obtained by induction. □

5. Complete coverage probability.

In this section we interpret results of Sections 4.1–4.2 in the random setting.

5.1. **Random coverings and the random nerve.** Suppose $\mathsf{A} = \{\mathsf{A}_{\{i\}}\}$ is a random covering of a metric space X. We define the *nerve* $\mathcal{N}(\mathsf{A})$ of A by defining a probability measure $\mathbb{P}_{\mathsf{A}}$ on $\mathfrak{C}_n$ via the process elucidated in Section 1 in (1.3) and (1.4). Observe that given a subspace $Y \subseteq X$ we obtain an induced random covering A_Y from A:

$$\mathsf{A}_Y = \{\mathsf{A}_{\{1\}} \cap Y, \mathsf{A}_{\{2\}} \cap Y, \ldots, \mathsf{A}_{\{n\}} \cap Y\}$$

The definition of $\mathbb{P}_{\mathsf{A}}$ extends to pairs $(\mathcal{N}(\mathsf{A}), \mathcal{N}(\mathsf{A}_Y))$ in an obvious way. In particular given $(\mathtt{s}, \mathtt{r}) \in \mathfrak{C}_n \times \mathfrak{C}_n$, we set

$$\begin{aligned} p_{\mathtt{s},\mathtt{r}} &= \mathbb{P}(\{(\mathtt{k}, \mathtt{l}) \in \mathfrak{C}_n \times \mathfrak{C}_n \mid \mathtt{s} \subseteq \mathtt{k}, \mathtt{r} \subseteq \mathtt{l}\}) \\ &= \mathbb{P}\big(\forall_{I \in \mathtt{s}}\big\{\bigcap_{i \in I} \mathsf{A}_{\{i\}} \neq \emptyset\big\}, \forall_{\{J\} \in \mathtt{r}}\big\{\bigcap_{j \in J} \mathsf{A}_{\{j\}} \cap Y \neq \emptyset\big\}\big). \end{aligned} \tag{5.1}$$

Clearly, $\mathcal{N}(\mathsf{A})$ is a random complex, and $(\mathcal{N}(\mathsf{A}), \mathcal{N}(\mathsf{A}_Y))$ is a random pair. We say a finite random covering $\{\mathsf{A}_{\{i\}}\}_{i=1,\ldots,n}$ of X is *good* if and only if it is a good covering on X almost surely. Further, we say a random covering $\mathsf{A} = \{\mathsf{A}_{\{i\}}\}$ of a pair $(X, \partial X)$ is good provided it is a good covering of X and $\mathsf{A}_{\partial X}$ is a good covering of ∂X. $|\mathsf{A}|$ will denote the random set $\bigcup_i \mathsf{A}_{\{i\}}$.

5.2. **Proof of the extended version of Theorem 1.1.** Let $\boldsymbol{\chi}_{rel}(\mathsf{A}, \mathsf{A}_{\partial X})$ be the relative Euler characteristic of the pair $(\mathcal{N}(\mathsf{A}), \mathcal{N}(\mathsf{A}_{\partial X}))$. We may now state Theorem 1.1 for a general 1–complex X.

Theorem 5.1 (Coverage probability of a 1-complex X with $\partial X \neq \emptyset$)**.** *Let $\mathsf{A} = \{\mathsf{A}_{\{i\}}\}$, $i = 1, \ldots, n$ be a random good covering of the pair $(X, \partial X)$. Then, the range of $\boldsymbol{\chi}_{rel}(\mathsf{A}, \mathsf{A}_{\partial X})$ can be restricted to*

$$\underline{m} = \chi_{rel}(X, \partial X) \leq \boldsymbol{\chi}_{rel}(\mathsf{A}, \mathsf{A}_{\partial X}) \leq n = \overline{m}, \tag{5.2}$$

and the complete coverage probability equals

$$\begin{aligned} \mathbb{P}(X \subseteq |\mathsf{A}|) &= \mathbb{P}\big(\boldsymbol{\chi}_{rel}(\mathsf{A}, \mathsf{A}_{\partial X}) = \chi_{rel}(X, \partial X)\big), \\ &= \sum_{(\mathtt{s},\mathtt{r}) \in \mathfrak{C}_n \times \mathfrak{C}_n} a_{\mathtt{s},\mathtt{r}}(\boldsymbol{\chi}_{rel})\, p_{\mathtt{s},\mathtt{r}}, \end{aligned} \tag{5.3}$$

where $a_{\mathtt{s},\mathtt{r}}(\boldsymbol{\chi}_{rel}) = a_{\mathtt{s},\mathtt{r},0}(\boldsymbol{\chi}_{rel})$ are defined in (3.32) *of Corollary 3.5, and $p_{\mathtt{s},\mathtt{r}}$ in* (5.1)*.*

Proof. Under the given assumptions, Lemma 4.5 implies

$$\mathbb{P}(X \subseteq |\mathsf{A}|) = \mathbb{P}\big(\boldsymbol{\chi}_{rel}(\mathsf{A}, \mathsf{A}_{\partial X}) = \chi_{rel}(X, \partial X)\big). \tag{5.4}$$

At this point the formula (3.32) of Corollary 3.5 can be applied to the random pair $(\mathcal{N}(\mathsf{A}), \mathcal{N}(\mathsf{A}_{\partial X}))$ to give an exact expression for $\mathbb{P}\big(\boldsymbol{\chi}_{rel}(\mathsf{A}, \mathsf{A}_{\partial X}) = \chi_{rel}(X, \partial X)\big)$. In this particular case the range of $\boldsymbol{\chi}(\mathsf{A}, \mathsf{A}_{\partial X})$ is given by (5.2), where the lower bound follows from Proposition 4.2, and the upper bound corresponds to the case when elements of the covering A are pairwise disjoint and contained in $X - \partial X$, i.e. $\mathcal{N}(\mathsf{A})$ is just n distinct points. The formula for $p_{\mathtt{s}}$ in (1.8) is a direct consequence of Proposition 4.7, (see also Remark 5.3). □

Remark 5.2. Note that $\mathcal{N}(\mathsf{A}_{\partial X})$ generally contains high dimensional faces and therefore the chain expansion of $\boldsymbol{\chi}^k_{rel}$ in $\mathbb{R}_{\mathcal{I}}[e_I, w_J]$ involves monomials in $e_{\mathtt{s}}$ and $w_{\mathtt{r}}$. To simplify this expansion one may observe that $\mathcal{N}(\mathsf{A}_{\partial X})$ has a homotopy type of finitely many points or is empty. Specifically, from (4.5) we have

$$\boldsymbol{\chi}_{rel}(\mathsf{A}, \mathsf{A}_{\partial X}) = \boldsymbol{\chi}(\mathsf{A}) - \#\{\mathsf{A} \cap \partial X\}.$$

The random variable $\#\{\mathsf{A}\cap\partial X\}$ (counting points in $\mathsf{A}_{\partial X}$) can be expressed as follows:

$$\boldsymbol{\chi}_{rel}(\mathsf{A},\mathsf{A}_{\partial X}) = \boldsymbol{\chi}(\mathsf{A}) - \sum_{i=1}^{q} w_{\{i\}}. \tag{5.5}$$

where $\{1,\ldots,q\}$ label points of ∂X and $\{w_{\{i\}}\}_{i=1,\ldots,q}$ are the indicator functions of points in $\mathsf{A}_{\partial X}$. Consequently, we may derive expressions for powers $\boldsymbol{\chi}^k_{rel}$ as polynomials in $\mathbb{R}[e_I, w_{\{i\}}]$. These expansions of $\boldsymbol{\chi}^k_{rel}$ involve products of $e_{\mathbf{s}}$ and $w_{\{i\}}$ only, which may provide a different way to express $\mathbb{P}(X\subseteq|A|)$.

Remark 5.3. In order to be more explicit about how the computation of $p_{\mathbf{s},\mathbf{r}}$ simplifies in the case the nerve $\mathcal{N}(\mathsf{A})$ equals the Vietoris–Rips complex $\mathcal{R}(\mathsf{A})$, let us suppose $\mathsf{A}_{\{i\}}$ are ε-radius closed balls in X with random centers $\xi_i\in X$. In $\mathcal{R}(\mathsf{A})$ any simplex indexed by $I=\{i_1,i_2,\ldots,i_k\}$ is determined by its edges, and an edge $\{i,j\}$ in $\mathcal{R}(\mathsf{A})$ occurs if and only if $|\xi_i-\xi_j|\leq 2\varepsilon$ (where $|\cdot-\cdot|$ is a short notation for the distance $d_X(\cdot,\cdot)$ on X). For instance, we have

$$p_I = \mathbb{P}(\mathsf{A}_{\{i_1\}}\cap\mathsf{A}_{\{i_2\}}\cap\ldots\cap\mathsf{A}_{\{i_k\}}\neq\emptyset) = \mathbb{P}\big(|\xi_{i_s}-\xi_{i_t}|\leq 2\varepsilon \mid \forall_{s,t}\ s\neq t\big).$$

Enumerate points in ∂X as follows $\{x_1,x_2,\ldots,x_M\}$, $M=\#\{\partial X\}$. Now, $p_{\mathbf{s},\mathbf{r}}$ given in (5.1) is just a volume of the set

$$A_{\mathbf{s},\mathbf{r}} = \{(\xi_1,\ldots,\xi_n)\in X^n \mid \forall_{I\in\mathbf{s}}\forall_{\substack{s,t\in I,\\ s\neq t}}\ |\xi_s-\xi_t|\leq 2\varepsilon,\ \forall_{I\in\mathbf{r}}\,\exists_{1\leq s\leq M}\,\forall_{i\in I}\ |\xi_i-x_s|\leq\varepsilon\},$$

which in the case $\mathbb{P}=d\xi_1\,d\xi_2\ldots d\xi_n$ (i.e. ξ_i's are independent) can be computed via ordinary calculus techniques or estimated numerically. These formulas further simplify, if $\partial X=\emptyset$, but we do not attempt these computations here.

6. Proof of Theorem 1.2

In this section we use the method of finite differences, c.f. [1], to give an upper bound for the complete coverage probability in terms of the expected Euler characteristic and prove Theorem 1.2. Let $\{\mathsf{A}_{\{i\}}\}$, $i=1,\ldots,n$ be a finite good covering of X, consider the following shifted version of the relative Euler characteristic $\boldsymbol{\chi}_{rel}(\mathsf{A},\mathsf{A}_{\partial X})$ of $(\mathcal{N}(\mathsf{A}),\mathcal{N}(\mathsf{A}_{\partial X}))$:

$$\boldsymbol{\chi}_0 = \boldsymbol{\chi}_{rel}(\mathsf{A},\mathsf{A}_{\partial X}) - \underline{m},$$

where $\underline{m}=\chi_{rel}(X,\partial X)$. From (2.7) we obtain

$$\boldsymbol{\chi}_{rel}(\mathsf{A},\mathsf{A}_{\partial X}) = \boldsymbol{\beta}_0 - \boldsymbol{\beta}_1, \tag{6.1}$$

where $\boldsymbol{\beta}_*=\boldsymbol{\beta}_*(\mathsf{A},\mathsf{A}_{\partial X})$ stand for the random relative Betti numbers. Recall that $\{e_I, w_J\}$, $I,J\in f(n)$ stand for the indicator functions of faces in $(\mathcal{N}(\mathsf{A}),\mathcal{N}(\mathsf{A}_{\partial X}))$.

We will consider a filtration by random vectors V_i denoting $(e_{I(i)}, f_{J(i)})$ where $I(i),J(i)\in f(n)$ are subsets of $\{1,\ldots,i\}$. Note that V_i reveals subcomplexes in $\mathfrak{C}_n$ spanned by vertices

1 through i. By analogy to the setting of Erdős–Rényi model [1], we set up a *vertex exposure martingale*, associated with $\boldsymbol{\chi}_0$ and $\{V_i\}$ as follows:

$$\mathtt{Y}_0 = \mu_0 = E(\boldsymbol{\chi}_0), \qquad \mathtt{Y}_i = E(\boldsymbol{\chi}_0 \mid V_i), \qquad i = 1, \ldots, n. \tag{6.2}$$

Clearly, $\mathtt{Y}_n = \boldsymbol{\chi}_0$ and the sequence $\{\mathtt{Y}_i\}$ is an instance of Doob's martingale [1]. Recall the following variant of the Azuma-Hoeffding inequality [1, 3], for $\{\mathtt{Y}_i\}$:

$$\mathbb{P}(\mathtt{Y}_n - \mathtt{Y}_0 \leq -a) \leq \exp\Big(\frac{-a^2}{2\sum_{i=1}^n c_i^2}\Big) \tag{6.3}$$

where $a > 0$, and c_i is a difference estimate

$$|\mathtt{Y}_i - \mathtt{Y}_{i-1}| \leq c_i. \tag{6.4}$$

Exposing a vertex (or a face containing it) changes $\boldsymbol{\beta}_0$ by at most 1 and $\boldsymbol{\beta}_1$ by at most $\boldsymbol{\beta}_1(X, \partial X) = 1 - \chi_{rel}(X, \partial X)$ thus we obtain

$$|\mathtt{Y}_i - \mathtt{Y}_{i-1}| \leq 2 + |\chi_{rel}(X, \partial X)|.$$

Let $a = \mu_0$, then

$$\mathbb{P}(\boldsymbol{\chi}_0 = 0) = \mathbb{P}(\boldsymbol{\chi}_0 \leq \mu_0 - a) = \mathbb{P}(\mathtt{Y}_n - \mathtt{Y}_0 \leq -a).$$

Using the above estimates for c_i and (6.3) yields

$$\mathbb{P}(X \subseteq |\mathsf{A}|) = \mathbb{P}(\boldsymbol{\chi}_0 = 0) \leq \exp\Big(\frac{-\mu_0^2}{2n(|\chi_{rel}(X, \partial X)| + 2)^2}\Big),$$

which completes the proof of Theorem 1.2.

Appendix A. Auxiliary proofs for Section 4

Proof of Proposition 4.2. Consider the Mayer-Vietoris sequence applied to U and V:

$$0 \to H_1(U \cap V) \xrightarrow{j_1} H_1(U) \oplus H_1(V) \to H_1(X) \to H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(X) \to 0.$$

Since $U \cap V = \partial A$ is just finitely many points, in real coefficients we have

$$0 \longrightarrow \mathbb{R}^{\beta_1(U)} \oplus \mathbb{R}^{\beta_1(V)} \xrightarrow{d_1} \mathbb{R}^{\beta_1(X)} \longrightarrow \ldots$$

From (2.7), $\chi(X) = 1 - \beta_1(X),\ \chi(U) = \beta_0(U) - \beta_1(U),\ \chi(V) = \beta_0(V) - \beta_1(V)$. Since d_1 is injective we have $\beta_1(U) + \beta_1(V) \leq \beta_1(X)$, which implies $-\beta_1(X) + \beta_1(U) \leq 0$. This proves (4.1).

Now to prove (4.2) we have two cases to consider: $\beta_0(U) > 1$ and $\beta_0(U) = 1$. First assume $\beta_0(U) > 1$. We argue by contradiction. That is, suppose $\chi(U) \leq \chi(X)$. Then $\beta_0(U) - \beta_1(U) \leq \beta_0(X) - \beta_1(X)$ so that $\beta_0(U) \leq \beta_1(U) + 1 - \beta_1(X)$. But $\beta_1(A) - \beta_1(X) \leq 0$ by the previous lemma. Therefore we obtain $\beta_0(U) \leq 1$ contrary to our assumption. Now assume $\beta_0(U) = 1$. Then $\chi(U) = 1 - \beta_1(U)$ and $\chi(X) = 1 - \beta_1(X)$ which yields $\chi(U) - \chi(X) = -\beta_1(U) + \beta_1(X) \geq 0$. Thus $\chi(U) \geq \chi(X)$. ∎

Proof of Corollary 4.3. Notice that generally X (even with $\partial X \neq \emptyset$) is homotopy equivalent to a bouquet of circles. If $|A|^c \neq \emptyset$ in X, then (since $|A|^c$ is open) we pick $p \in |A|^c$ which is not a vertex of X. Then p is in the interior of one of the edges which we denote by e. We may homotopy X away from the interior of e to a bouquet of r circles $S = \bigvee^r S^1$ in such a way that p is away from the wedge point (just collapse along the edges different from e). From Proposition 4.2,

$$\beta_1(|A|) \leq \beta_1\big(\bigvee^{r-1} S^1 \vee (S^1 - \{p\})\big) < \beta_1(S) = \beta_1(X).$$

Thus $\beta_1(|A|) < \beta_1(X)$ and therefore $\chi(X) < \chi(|A|)$, which implies the claim. □

Proof of Lemma 4.5. Observe that $X \subseteq |A'|$ to $X \subseteq |A|$. Indeed, since $|A|$ is closed if $X - |A| \neq \emptyset$ then we may choose a point in $x \in X - |A|$ such that $x \notin \partial X$, since the projection q is a homeomorphism on $X - \partial X$, we conclude that $q(x) \notin X' - |A'|$. Next, Equation (4.4) follows immediately from Corollary 4.3, the fact that A and $A_{\partial X}$ are good and the identities

$$\chi(|A'|) = \chi_{rel}(|A|, |A| \cap \partial X), \quad \chi(X') = \chi_{rel}(X, \partial X).$$

Now, thanks to (2.5) we compute

$$\chi_{rel}(X, \partial X) = \chi(X) - \#\{\partial X\},$$
$$\chi_{rel}(|A|, |A| \cap \partial X) = \chi(|A|) - \#\{|A| \cap \partial X\},$$

which yields (4.5). □

References

[1] N. Alon and J. H. Spencer. *The probabilistic method*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley and Sons Inc., Hoboken, NJ, third edition, 2008. With an appendix on the life and work of Paul Erdős.

[2] S. Athreya, R. Roy, and A. Sarkar. On the coverage of space by random sets. *Adv. in Appl. Probab.*, 36(1):1–18, 2004.

[3] K. Azuma. Weighted sums of certain dependent random variables. *Tôhoku Math. J. (2)*, 19:357–367, 1967.

[4] B. Bollobás. *Random graphs*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1985.

[5] B. Bollobás, S. Janson, and O. Riordan. On covering by translates of a set. *Random Structures Algorithms*, 38(1-2):33–67, 2011.

[6] K. Borsuk. On the imbedding of systems of compacta in simplicial complexes. *Fund. Math.*, 35:217–234, 1948.

[7] P. Bürgisser, F. Cucker, and M. Lotz. Coverage processes on spheres and condition numbers for linear programming. *Ann. Probab.*, 38(2):570–604, 2010.

[8] V. de Silva and R. Ghrist. Coordinate-free coverage in sensor networks with controlled boundaries via homology. *Intl. J. Robotics Research*, 25(12):1205–1222, 2006.

[9] V. de Silva and R. Ghrist. Coverage in sensor networks via persistent homology. *Algebr. Geom. Topol.*, 7:339–358, 2007.

[10] V. de Silva and R. Ghrist. Homological sensor networks. *Notices Amer. Math. Soc.*, 54(1):10–17, 2007.
[11] J. Eckhoff. Helly, Radon, and Carathéodory type theorems. In *Handbook of convex geometry, Vol. A, B*, pages 389–448. North-Holland, Amsterdam, 1993.
[12] M. E. A. El-Mikkawy. Explicit inverse of a generalized Vandermonde matrix. *Appl. Math. Comput.*, 146(2-3):643–651, 2003.
[13] W. Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley and Sons Inc., New York, 1971.
[14] L. Flatto. A limit theorem for random coverings of a circle. *Israel J. Math.*, 15:167–184, 1973.
[15] L. Flatto and D. J. Newman. Random coverings. *Acta Math.*, 138(3–4):241–264, 1977.
[16] E. N. Gilbert. The probability of covering a sphere with N circular caps. *Biometrika*, 52:323–330, 1965.
[17] P. Hall. *Introduction to the theory of coverage processes.* Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley and Sons Inc., New York, 1988.
[18] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.
[19] J. C. Hausmann. On the Vietoris-Rips complexes and a cohomology theory for metric spaces. In *Prospects in topology (Princeton, NJ, 1994)*, volume 138 of *Ann. of Math. Stud.*, pages 175–188. Princeton Univ. Press, Princeton, NJ, 1995.
[20] M. Kahle. Topology of random clique complexes. *Discrete Math.*, 309(6):1658–1671, 2009.
[21] M. Kahle and E. Meckes. Limit theorems for Betti numbers of random simplicial complexes. *Homology Homotopy Appl.*, 15(1):343–374, 2013.
[22] A. Kisielewicz. A solution of Dedekind's problem on the number of isotone Boolean functions. *J. Reine Angew. Math.*, 386:139–144, 1988.
[23] E. L. Lehmann. *Elements of large-sample theory.* Springer Texts in Statistics. Springer-Verlag, New York, 1999.
[24] L. Lovász. *Combinatorial problems and exercises.* North-Holland Publishing Co., Amsterdam, second edition, 1993.
[25] C. Meyer. *Matrix analysis and applied linear algebra.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000.
[26] R. E. Miles. The asymptotic values of certain coverage probabilities. *Biometrika*, 56:661–680, 1969.
[27] I. Molchanov and V. Scherbakov. Coverage of the whole space. *Adv. in Appl. Probab.*, 35(4):898–912, 2003.
[28] P. A. P. Moran and S. Fazekas de St Groth. Random circles on a sphere. *Biometrika*, 49:389–396, 1962.
[29] J. Pullen. On the complete coverage probability of polyhedra. *Ph.D. thesis (Tulane University)*, 2012.
[30] A. F. Siegel. Asymptotic coverage distributions on the circle. *Ann. Probab.*, 7(4):651–661, 1979.
[31] A. F. Siegel and L. Holst. Covering the circle with random arcs of random sizes. *J. Appl. Probab.*, 19(2):373–381, 1982.
[32] W. L. Stevens. Solution to a geometrical problem in probability. *Ann. Eugenics*, 9:315–320, 1939.
[33] A. H. Wallace. *Algebraic topology: Homology and cohomology.* W. A. Benjamin, Inc., New York, 1970.
[34] M. C. Wendl. Random covering of multiple one-dimensional domains with an application to DNA sequencing. *SIAM J. Appl. Math.*, 68(3):890–905, 2007/08.
[35] W. A. Whitworth. DCC exercises in choice and chance. *Dover, (Reprint of 1897 edition)*, 1965.

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: rako@tulane.edu

Department of Mathematics, Mercer University, Macon, GA 31207
E-mail address: pullen_j@mercer.edu