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PFA AND GUESSING MODELS

BY

Nam Trang*

Department of Mathematics, University of California at Irvine
Irvine, CA 92717, USA
e-mail: ntrang@math.uci.edu.

ABSTRACT

This paper explores the consistency strength of The Proper Forcing Axiom (PFA) and the theory (T) which involves a variation of the Viale–Weiß guessing hull principle. We show that (T) is consistent relative to a supercompact cardinal. The main result of the paper is Theorem 0.2, which implies that the theory "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" is consistent relative to (T) and to PFA. This improves significantly the previous known best lower-bound for consistency strength for (T) and PFA, which is roughly "$\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$".

0. Introduction

Suppose $\kappa < \gamma$ are uncountable cardinals and $X \prec H_\gamma$ is such that $\kappa \leq |X|$. Let $\wp_\kappa(X) = \{a \in \wp(X) \mid |a| < \kappa\}$. We say that X is **κ-guessing** if for all $a \in X$ and for all $b \subseteq a$ such that $c \cap b \in X$ for all $c \in X \cap \wp_\kappa(X)$ then b is **X-guessed**, i.e., there is some $c \in X$ such that $c \cap X = b \cap X$. Such a b satisfying the hypothesis of the previous sentence is called **κ-approximated by X**. So a hull X is κ-guessing if whenever $a \in X$ and whenever $b \subseteq a$ is κ-approximated by X, then b is X-guessed.

In this paper, we study the strength of the following theories:

- The Proper Forcing Axiom (PFA);

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- (T): there is a cardinal $\lambda \geq 2^{\aleph_2}$ such that the set

$$\{X \prec (H_{\lambda^{++}}, \in) \mid |X| = \aleph_2, X^\omega \subseteq X, \omega_2 \subset X, \text{ and } X \text{ is } \omega_2\text{-guessing}\}$$

  is stationary.

Guessing models in [26] are ω_1-guessing in the above notations. It's not clear that the theory (T) is consistent with PFA (in contrast to Viale–Weiß principle $\mathsf{ISP}(\omega_2)$, which asserts the existence of stationary many ω_1-guessing models of size $\aleph_1$ of H_λ for all sufficiently large λ). However, it's possible that (T) is a consequence of or at least consistent with a higher analog of PFA.

The outline of the paper is as follows. In Section 1, we review some AD^+ facts that we'll be using in this paper. In Section 2, using a Mitchell-style forcing, we prove

THEOREM 0.1: *Con(*ZFC *+ there is a supercompact cardinal)* $\Rightarrow$ *Con*(T).

Of course, it is well-known that PFA is consistent relative to the existence of a supercompact cardinal. Theorem 0.4 suggests that it's reasonable to expect PFA and (T) are equiconsistent.

Recall, for an infinite cardinal λ, the principle $\square_\lambda$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that for each $\alpha < \lambda^+$,

- C_α is club in α;
- for each limit point β of C_α, $C_\beta = C_\alpha \cap \beta$;
- the order type of C_α is at most λ.

The principle $\square(\lambda)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \lambda \rangle$ such that

(1) for each $\alpha < \lambda$,
   - each C_α is club in α;
   - for each limit point β of C_α, $C_\beta = C_\alpha \cap \beta$; and

(2) there is no thread through the sequence, i.e., there is no club $E \subseteq \lambda$ such that $C_\alpha = E \cap \alpha$ for each limit point α of E.

Note that $\square_\lambda$ implies $\square(\lambda^+)$ (equivalently, $\neg\square(\lambda^+)$ implies $\neg\square_\lambda$). The main technical theorem of the paper, proved in Section 3, is the following:

THEOREM 0.2: *Supppose κ is a cardinal such that $\kappa^\omega = \kappa$. Suppose for each cardinal $\alpha \in [\kappa^+, (2^\kappa)^+]$, $\neg\square(\alpha)$. Then letting $G \subseteq Col(\omega, \kappa)$ be V-generic, in $V[G]$, there is a transitive M containing* $\mathrm{OR} \cup \mathbb{R}$ *such that $M \vDash$ "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular".*

Remark 0.3: We remark that the proof of Theorem 0.2 actually shows that we can get in V a model M containing $\mathrm{OR} \cup \mathbb{R}$ that satisfies "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular", see Remark 3.86. This is because we actually construct a hod pair $(\mathcal{P}, \Sigma)$ that generates a model of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" with the property that $(\mathcal{P}, \Sigma \restriction V) \in V$.

Hence, as a corollary, we establish the following theorem, which improves upon the conclusion of Corollary 0.2 of [1].

THEOREM 0.4: *Suppose S is one of the following theories:*

(1) PFA,
(2) (T),
(3) *there is a strongly compact cardinal.*

If S holds, then there is a transitive model M containing $\mathbb{R} \cup \mathrm{OR}$ such that $M \vDash$ "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular".[1]

Proof. We apply Theorem 0.2 and Remark 0.3. We just need to verify that S implies the hypothesis of Theorem 0.2. If S is either PFA or (T), we take $\kappa = \aleph_2$. It's well-known that the hypothesis in Theorem 0.2 regarding threadability follows from S.[2] Otherwise, take κ to be a singular, strong limit cardinal of uncountable cofinality above a strongly compact cardinal. By [15], the hypothesis of Theorem 0.2 holds at κ. ∎

In Subsection 3.1, we lay out the framework for the core model induction which allows us to construct models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" from the hypothesis of Theorem 0.2. The actual construction of such models is carried out in Subsections 3.2–3.5.

We note that the previous best known lower-bound for (T) as well as for PFA is the sharp for a proper class model with a proper class of strong cardinals and a proper class of Woodin cardinals (see [1, Corollary 0.2]), which is just a bit stronger than "$\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$" but is weaker than "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular". The method used in this paper is the core model induction method, which can be used to further improve the lower-bounds for (T) and for PFA, as opposed to the method in [1], which seems hard to generalize. In fact, it's possible to improve

[1] To the best of the author's knowledge, the result in this paper gives the best lower-bound obtained from any combinatorial principle not augmented by large cardinal assumptions.

[2] In the case S is (T), see [26] and [27].

the lowerbound consistency strength for (T) and for PFA in Theorem 0.4 to "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable" and beyond. These results involve a combination of the core model induction and techniques for constructing hod mice beyond those developed in [7] and hence will appear in a future publication.

Let LSA denote the theory "$\mathsf{AD}^+ + \Theta = \theta_{\alpha+1} + \theta_\alpha$ is the largest Suslin cardinal". LSA was first isolated by Woodin in [29] and is very recently shown to be consistent by G. Sargsyan. It is one of the strongest determinacy theories known to be consistent. We conjecture that

CONJECTURE 0.5: *Con*(T) $\Rightarrow$ *Con*(LSA) *and* *Con*(PFA) $\Rightarrow$ *Con*(LSA).

We are hopeful that methods used in this paper and their extensions can be used to settle the conjecture.

1. Basic facts about AD^+

We start with the definition of Woodin's theory of AD^+. In this paper, we identify $\mathbb{R}$ with ω^ω. We use Θ to denote the sup of ordinals α such that there is a surjection $\pi : \mathbb{R} \to \alpha$. Under AC, Θ is just the successor cardinal of the continuum. In the context of AD, Θ is shown to be the supremum of $w(A)$ [3] for $A \subseteq \mathbb{R}$. The definition of Θ relativizes to any determined pointclass (with sufficient closure properties). We denote Θ^Γ for the sup of α such that there is a surjection from $\mathbb{R}$ onto α coded by a set of reals in Γ.

Definition 1.1: AD^+ is the theory $\mathsf{ZF} + \mathsf{AD}{+}\mathsf{DC}_{\mathbb{R}}$ and

(1) for every set of reals A, there are a set of ordinals S and a formula φ such that $x \in A \Leftrightarrow L[S, x] \vDash \varphi[S, x]$. (S, φ) is called an ∞-Borel code for A;
(2) for every $\lambda < \Theta$, for every continuous $\pi : \lambda^\omega \to \omega^\omega$, for every $A \subseteq \mathbb{R}$, the set $\pi^{-1}[A]$ is determined.

AD^+ is equivalent to "AD + the set of Suslin cardinals is closed". Another, perhaps more useful, characterization of AD^+ is "$\mathsf{AD}{+}\Sigma_1$ statements reflect into Suslin co-Suslin sets" (see [21] for the precise statement).

Let $A \subseteq \mathbb{R}$; we let θ_A be the supremum of all α such that there is an $OD(A)$ surjection from $\mathbb{R}$ onto α. If Γ is a determined pointclass, and $A \in \Gamma$, we write

[3] $w(A)$ is the Wadge rank of A.

$\Gamma \restriction A$ for the set of $B \in \Gamma$ which is Wadge reducible to A. If $\alpha < \Theta^\Gamma$, we write $\Gamma \restriction \alpha$ for the set of $A \in \Gamma$ with Wadge rank strictly less than α.

Definition 1.2 (AD^+): The **Solovay sequence** is the sequence $\langle \theta_\alpha \mid \alpha \leq \lambda \rangle$ where

(1) θ_0 is the supremum of ordinals β such that there is an OD surjection from $\mathbb{R}$ onto β;
(2) if $\alpha > 0$ is limit, then $\theta_\alpha = \sup\{\theta_\beta \mid \beta < \alpha\}$;
(3) if $\alpha = \beta + 1$ and $\theta_\beta < \Theta$ (i.e., $\beta < \lambda$), fixing a set $A \subseteq \mathbb{R}$ of Wadge rank θ_β, θ_α is the sup of ordinals γ such that there is an $OD(A)$ surjection from $\mathbb{R}$ onto γ, i.e., $\theta_\alpha = \theta_A$.

Note that the definition of θ_α for $\alpha = \beta + 1$ in Definition 1.2 does not depend on the choice of A. For a pointclass Γ that satisfies AD^+ and is sufficiently closed, we can also define the Solovay sequence $\langle \theta^\Gamma_\alpha \mid \alpha \leq \lambda \rangle$ of Γ like above. For $\alpha \leq \lambda$, we say $\Gamma \restriction \theta^\Gamma_\alpha$ is a **Solovay initial segment** of Γ.

Roughly speaking, the longer the Solovay sequence is, the stronger the associated AD^+-theory is. For instance, the theory $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$ is strictly stronger than $\mathsf{AD}_\mathbb{R}$ since by [14], DC implies $\mathrm{cof}(\Theta) > \omega$ while the minimal model [4] of $\mathsf{AD}_\mathbb{R}$ satisfies $\Theta = \theta_\omega$. $\mathsf{AD}_\mathbb{R} + \Theta$ is regular is much stronger still as it implies the existence of many models of $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$. We end this section with a theorem of Woodin, which produces models with Woodin cardinals in AD^+. The theorem is important in the HOD analysis of such models.

THEOREM 1.3 (Woodin, see [4]): *Assume* AD^+. *Let* $\langle \theta_\alpha \mid \alpha \leq \Omega \rangle$ *be the Solovay sequence. Suppose* $\alpha = 0$ *or* $\alpha = \beta + 1$ *for some* $\beta < \Omega$. *Then* $\mathrm{HOD} \vDash \theta_\alpha$ *is Woodin.*

2. Upper-bound consistency strength of (T)

In this section, we prove Theorem 0.1. We follow closely the construction of Section 3 in [5].[5] We use Even and Odd to denote the classes of even ordinals

[4] From here on, whenever we talk about "models of AD^+", we always mean those M that contain $\mathrm{OR} \cup \mathbb{R}$ and satisfy AD^+.

[5] For the reader's convenience, our ω_1 will play the role of μ in Section 3 of [5], our ω_2 will play the role of κ there, and finally our α plays the same role as the α in [5].

and odd ordinals respectively. We assume the ground model V satisfies

$$\omega_1^\omega = \omega_1 + \omega_2^\omega = \omega_2 + \alpha \text{ is supercompact.}$$

Consider the following forcing iteration,

$$\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \alpha, j < \alpha \rangle,$$

with two partial orderings $\leq$ and $\leq^*$, where $\leq$ is the standard partial ordering on posets and $\leq^*$ is defined by letting $p * \dot{r} \leq^* q * \dot{s}$ if $p * \dot{r} \leq q * \dot{s}$ and $p = q$. Inductively, we ensure that the following hold:

(1) If $i < \alpha$ is even then $\mathbb{P}_i$ forces $\dot{\mathbb{Q}}_i = \mathrm{ADD}(\omega_1)$, where $\mathrm{ADD}(\omega_1)$ is the standard forcing for adding a Cohen subset of ω_1 with countable conditions.

(2) If $j < \alpha$ is odd, $\mathbb{P}_{j-1}$ forces $\langle \dot{Q}_{j-1} * \dot{Q}_j, \leq^* \rangle = \langle \mathrm{ADD}(\omega_1) * \mathrm{Col}(\omega_2, j), \leq^* \rangle$, where $\mathrm{Col}(\omega_2, j)$ is the standard forcing that collapses j to ω_2 using conditions of size at most $\aleph_1$. So for all $i < \alpha$, $\mathbb{P}_i$ forces $\dot{\mathbb{Q}}_i$ is ω_1-closed and furthermore, if i is odd, then $\mathbb{P}_{i-1}$ forces $\langle \dot{\mathbb{Q}}_{i-1} * \dot{\mathbb{Q}}_i, \leq^* \rangle$ is ω_2-strategically closed.

(3) If $i \leq \alpha$ is a limit ordinal, then $\mathbb{P}_i$ consists of all partial functions $p : i \to V$ such that $p \restriction j \in \mathbb{P}_j$ for $j < i$, $|\mathrm{dom}(p) \cap \mathrm{Even}| < \omega_1$, and $|\mathrm{dom}(p) \cap \mathrm{Odd}| < \omega_2$.

(4) For $i \leq \alpha$ and $p, q \in \mathbb{P}_i$, $q \leq p$ in $\mathbb{P}_i$ iff for all γ in the domain of p, γ is in the domain of q and $q \restriction \gamma \Vdash q(\gamma) \leq p(\gamma)$.

(5) For $i \leq \alpha$ and $p, q \in \mathbb{P}_i$, $q \leq^* p$ in $\mathbb{P}_i$ iff $q \leq p$, $\mathrm{dom}(p) \cap \mathrm{Even} = \mathrm{dom}(q) \cap \mathrm{Even}$, and for every γ in $\mathrm{dom}(p) \cap \mathrm{Even}$, $q \restriction \gamma \Vdash q(\gamma) = p(\gamma)$.

By results in Section 3 of [5], we get the following:

(a) $\langle \mathbb{P}_\beta, \leq \rangle$ is ω_1-closed for all $\beta \leq \alpha$. In particular, $\mathrm{ADD}(\omega_1)$ is the same in the ground model and in any intermediate extension.

(b) $\mathbb{P}_\alpha$ preserves ω_1, ω_2 (as well as stationary subsets of ω_1 and ω_2).

(c) $\forall \eta \leq \alpha$ such that η is inaccessible, $\mathbb{P}_\eta$ is η-cc.

(d) $\mathbb{P}_\alpha$ forces $\alpha = \omega_3$.

(e) $\forall \eta \leq \alpha$ such that η is inaccessible, letting $\mathbb{P}_\alpha = \mathbb{P}_\eta * \dot{\mathbb{Q}}$, then $\Vdash_{\mathbb{P}_\eta} \dot{\mathbb{Q}}$ satisfies the ω_2-approximation property, that is, whenever $G * H$ is V-generic for $\mathbb{P}_\eta * \dot{\mathbb{Q}}$, then if for all $x \in V[G][H]$, $x \subseteq V[G]$, it holds that if $x \cap z \in V[G]$ for all $z \in \wp_{\omega_2}(V[G])^{V[G]}$ then $x \in V[G]$.

LEMMA 2.1: *In $V[G]$, for any regular cardinal $\lambda > \omega_2$, the set*

$$\{X \prec (H_\lambda, \in) \mid |X| = \omega_2, \omega_2 \subset X, X^\omega \subseteq X, \text{ and } X \text{ is } \omega_2\text{-guessing}\}$$

is stationary. In particular, (T) *holds in $V[G]$.*

Proof. Since $\mathbb{P}_\alpha$ is ω_1-closed, $\omega_2^\omega = \omega_2$ in V, and $\omega_2^V = \omega_2^{V[G]}$,

$$(\omega_2^\omega)^{V[G]} = (\omega_2^\omega)^V = \omega_2^V = \omega_2^{V[G]}.$$

It's also clear from (a), (c), and the fact that α is an inaccessible limit of inaccessibles that $\omega_2^\omega = \omega_2$ and $2^{\omega_2} = \omega_3$ in $V[G]$.

Working in $V[G]$, fix a regular cardinal $\lambda > \omega_2$. Let

$$T = \{X \prec H_\lambda \mid |X| = \omega_2 \wedge \omega_2 \subset X \wedge X^\omega \subseteq X \wedge X \text{ is } \omega_2\text{-guessing}\}.$$

We show T is stationary. In V, let $j : V \to M$ witness that α is H_λ-supercompact. Let $G * H$ be $\mathbb{P}_{j(\alpha)}$-generic over V; j canonically lifts to $j^+ : V[G] \to M[G * H]$, where $j^+(\tau_G) = j(\tau)_{G*H}$. Let $\mathcal{F}$ be the normal filter defined from j^+, that is for all $A \in \wp_{\omega_3}(H_\lambda)^{V[G]}$,

$$A \in \mathcal{F} \Leftrightarrow \mathbb{P}_{j(\alpha)} \text{ forces over } V[G] \text{ that } j^+[H_\lambda^{V[G]}] \in j(A).$$

It's easy to check that $\mathcal{F}$ is a normal filter in $V[G]$. We now check that whenever $G*H$ is V-generic, then $H_\lambda^{V[G]} \in j^+(T)$. Fix such G, H and let $X = j^+[H_\lambda^{V[G]}]$. To simplify the notation, we also use j to denote j^+. Note that $X \prec H_{j(\lambda)}^{M[G][H]}$.

We first show in $M[G][H]$, $X^\omega \subseteq X$. Let $a \in X^\omega$ and note that $j^{-1}[a] \in V[G]$. This is because $j^{-1}[a] \subseteq V[G]$ and is a countable sequence in $V[G][H]$ and hence is in $V[G]$ since by construction, in $V[G]$, $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}$ is ω_1-closed", where $\mathbb{P}_{j(\alpha)} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}$. This easily implies that $j^{-1}[a] \in H_\lambda^{V[G]}$. Hence $j(j^{-1}[a]) = a \in X$.

Now suppose $b \subseteq z \in X$ is such that $b \subseteq X$ and whenever $d \in X \cap \wp_{\omega_2}(X)$, $d \cap b \in X$. We want to show there is a $c \in X$ such that $b \cap X = c \cap X$. To this end, note that $j^{-1}[b] \in V[G]$. This uses (e) and the assumption on b. Let $c = j(j^{-1}[b])$. Since $j^{-1}[b] \subseteq j^{-1}(z) \in H_\lambda^{V[G]}$, $j^{-1}[b] \in H_\lambda^{V[G]}$; this gives $c \in X$. It's easy then to check that $c \cap X = b \cap X$ (c need not equal b though). This completes the proof of the lemma. ∎

Proof of Theorem 0.1. Immediate from Lemma 2.1. ∎

Remark 2.2: We note that the ω_2-approximation property in (e) is crucial in the proof of Lemma 2.1. It's used to show that the hull X is ω_2-guessing.

3. Lower-bound consistency strength of (T)

In this section, we prove Theorem 0.2 and hence Theorem 0.4. The next several subsections are dedicated to setting up the core model induction, constructing hod pairs with nice properties that generate AD^+ models. Fix a V-generic $G \subseteq \mathrm{Col}(\omega, \kappa)$. Let $\lambda = 2^\kappa$. For $X \prec H_{\lambda^{++}}$ such that $|X| = \kappa$, $\kappa \subseteq X$, let $\pi_X : M_X \to H_{\lambda^{++}}$ be the uncollapse map; π_X naturally extends to a map, which we also call π_X from $M_X[G]$ to $H_{\lambda^{++}}[G]$. The core model induction will occur in $V[G]$. Our smallness assumption throughout this paper is:

> (†) : in $V[G]$, there is no model M containing all reals and ordinals such that $M \vDash$ "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular".

Among other things, (†) implies:

- There are no AD^+ models M, N such that $\mathbb{R} \cup \mathrm{OR} \subseteq M, N$ and $\wp(\mathbb{R}) \cap M \cap N$ is strictly contained in $\wp(\mathbb{R}) \cap M$ and in $\wp(\mathbb{R}) \cap N$ (see [7]). This implies that all AD^+ models constructed in the core model induction will end-extend one another.
- If M is an AD^+ model, then **Strong Mouse Capturing** (SMC) (see Footnote 26 for definition of SMC) and **Generation of Mouse Full Pointclasses** (see [7, Section 6.1]) hold in M . This fact allows us to use the hod analysis in [7] to construct hod mice.

Using the first consequence of (†), we define

Definition 3.1 (Maximal pointclass of AD^+): In $V[G]$, let

$$\Omega = \bigcup \{\wp(\mathbb{R}) \cap M \mid \mathbb{R} \cup \mathrm{OR} \subset M \wedge M \vDash \mathsf{AD}^+\}.$$

The rest of the paper is dedicated to analyzing Ω. In particular, we show that in $V[G]$,

- $\Omega \neq \emptyset$.
- Letting $\langle \theta_\alpha^\Omega \mid \alpha \leq \gamma \rangle$ be the Solovay sequence of Ω, then γ is a limit ordinal.

We will then deduce that there is indeed a model M of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular". This contradicts (†).

3.1. Framework for the core model induction. This section, consisting of several subsections, develops some terminology and framework for the core

model induction. The first subsection gives a brief summary of the theory of hod mice developed in [7]. In the next three subsections, we briefly introduce the notions of $\mathcal{F}$-premice, strategy premice, and (Θ)-g-organized $\mathcal{F}$-premice developed in [13]. For a full development of these concepts as well as proofs of lemmas stated below, the reader should consult [13]. These subsections summarize the theory and results in [13] to make the paper self-contained. The reader who wishes to see the main argument can skip them on the first read, and go back when needed. The next subsection discusses the S-constructions, which allow us to translate hybrid mice over a set a to hybrid mice over a set b where a and b are closely related. The last subsection defines core model induction operators, which are operators that we construct during the course of the core model induction in this paper.

3.1.1. *A brief introduction to hod mice.* In this paper, a hod premouse $\mathcal{P}$ is one defined as in [7]. The reader is advised to consult [7] for basic results and notations concerning hod premice and mice. Let us mention some basic first-order properties of a hod premouse $\mathcal{P}$. There are an ordinal $\lambda^{\mathcal{P}}$ and sequences $\langle(\mathcal{P}(\alpha), \Sigma^{\mathcal{P}}_\alpha) \mid \alpha < \lambda^{\mathcal{P}}\rangle$ and $\langle \delta^{\mathcal{P}}_\alpha \mid \alpha \leq \lambda^{\mathcal{P}}\rangle$ such that

(1) $\langle \delta^{\mathcal{P}}_\alpha \mid \alpha \leq \lambda^{\mathcal{P}}\rangle$ is increasing and continuous and if α is a successor ordinal then $\mathcal{P} \vDash \delta^{\mathcal{P}}_\alpha$ is Woodin;
(2) $\mathcal{P}(0) = Lp_\omega(\mathcal{P}|\delta_0)^{\mathcal{P}}$; for $\alpha < \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha+1) = (Lp^{\Sigma^{\mathcal{P}}_\alpha}_\omega(\mathcal{P}|\delta_\alpha))^{\mathcal{P}}$; for limit $\alpha \leq \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha) = (Lp^{\oplus_{\beta<\alpha}\Sigma^{\mathcal{P}}_\beta}_\omega(\mathcal{P}|\delta_\alpha))^{\mathcal{P}}$;
(3) $\mathcal{P} \vDash \Sigma^{\mathcal{P}}_\alpha$ is a $(\omega, o(\mathcal{P}), o(\mathcal{P}))$ [6]-strategy for $\mathcal{P}(\alpha)$ with hull condensation;
(4) if $\alpha < \beta < \lambda^{\mathcal{P}}$ then $\Sigma^{\mathcal{P}}_\beta$ extends $\Sigma^{\mathcal{P}}_\alpha$.

We will write $\delta^{\mathcal{P}}$ for $\delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}}$ and $\Sigma^{\mathcal{P}} = \bigoplus_{\beta<\lambda^{\mathcal{P}}} \Sigma^{\mathcal{P}}_\beta$. Note that $\mathcal{P}(0)$ is a pure extender model. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are two hod premice. Then $\mathcal{P} \trianglelefteq_{hod} \mathcal{Q}$ if there is $\alpha \leq l^{\mathcal{Q}}$ such that $\mathcal{P} = \mathcal{Q}(\alpha)$. We say then that $\mathcal{P}$ is a **hod initial segment** of $\mathcal{Q}$; $(\mathcal{P}, \Sigma)$ is a **hod pair** if $\mathcal{P}$ is a hod premouse and Σ is a strategy for $\mathcal{P}$ (acting on countable stacks of countable normal trees) such that $\Sigma^{\mathcal{P}} \subseteq \Sigma$ and this fact is preserved under Σ-iterations. Typically, we will construct hod pairs $(\mathcal{P}, \Sigma)$ such that Σ has hull condensation, branch condensation, and is Γ-fullness preserving for some pointclass Γ.

[6] This just means $\Sigma^{\mathcal{P}}_\alpha$ acts on all stacks of ω-maximal, normal trees in $\mathcal{P}$.

Suppose $(\mathcal{Q}, \Sigma)$ is a hod pair such that Σ has hull condensation. Then $\mathcal{P}$ is a $(\mathcal{Q}, \Sigma)$-hod premouse if there are ordinal $\lambda^{\mathcal{P}}$ and sequences $\langle (\mathcal{P}(\alpha), \Sigma^{\mathcal{P}}_\alpha) \mid \alpha < \lambda^{\mathcal{P}} \rangle$ and $\langle \delta^{\mathcal{P}}_\alpha \mid \alpha \leq \lambda^{\mathcal{P}} \rangle$ such that

(1) $\langle \delta^{\mathcal{P}}_\alpha \mid \alpha \leq \lambda^{\mathcal{P}} \rangle$ is increasing and continuous, and if α is a successor ordinal then $\mathcal{P} \vDash \delta^{\mathcal{P}}_\alpha$ is Woodin;
(2) $\mathcal{P}(0) = Lp^{\Sigma}_\omega(\mathcal{P}|\delta_0)^{\mathcal{P}}$ (so $\mathcal{P}(0)$ is a Σ-premouse built over $\mathcal{Q}$); for $\alpha < \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha+1) = (Lp^{\Sigma \oplus \Sigma^{\mathcal{P}}_\alpha}_\omega(\mathcal{P}|\delta_\alpha))^{\mathcal{P}}$; for limit $\alpha \leq \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha) = (Lp^{\oplus_{\beta<\alpha} \Sigma^{\mathcal{P}}_\beta}_\omega(\mathcal{P}|\delta_\alpha))^{\mathcal{P}}$;
(3) $\mathcal{P} \vDash \Sigma \cap \mathcal{P}$ is a $(\omega, o(\mathcal{P}), o(\mathcal{P}))$-strategy for $\mathcal{Q}$ with hull condensation;
(4) $\mathcal{P} \vDash \Sigma^{\mathcal{P}}_\alpha$ is a $(\omega, o(\mathcal{P}), o(\mathcal{P}))$-strategy for $\mathcal{P}(\alpha)$ with hull condensation;
(5) if $\alpha < \beta < \lambda^{\mathcal{P}}$ then $\Sigma^{\mathcal{P}}_\beta$ extends $\Sigma^{\mathcal{P}}_\alpha$.

Inside $\mathcal{P}$, the strategies $\Sigma^{\mathcal{P}}_\alpha$ act on stacks above $\mathcal{Q}$ and every Σ^P_α iterate is a Σ-premouse. Again, we write $\delta^{\mathcal{P}}$ for $\delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}}$ and $\Sigma^{\mathcal{P}} = \bigoplus_{\beta<\lambda^{\mathcal{P}}} \Sigma^{\mathcal{P}}_\beta$; $(\mathcal{P}, \Lambda)$ is a $(\mathcal{Q}, \Sigma)$-hod pair if $\mathcal{P}$ is a $(\mathcal{Q}, \Sigma)$-hod premouse and Λ is a strategy for $\mathcal{P}$ such that $\Sigma^P \subseteq \Lambda$ and this fact is preserved under Λ-iterations. The reader should consult [7] for the definition of $B(\mathcal{Q}, \Sigma)$ and $I(\mathcal{Q}, \Sigma)$. Roughly speaking, $B(\mathcal{Q}, \Sigma)$ is the collection of all hod pairs which are strict hod initial segments of a Σ-iterate of $\mathcal{Q}$ and $I(\mathcal{Q}, \Sigma)$ is the collection of all Σ-iterates of Σ. In the case $\lambda^{\mathcal{Q}}$ is limit, $\Gamma(\mathcal{Q}, \Sigma)$ is the collection of $A \subseteq \mathbb{R}$ such that A is Wadge reducible to some Ψ for which there is some $\mathcal{R}$ such that $(\mathcal{R}, \Psi) \in B(\mathcal{Q}, \Sigma)$. See [7] for the definition of $\Gamma(\mathcal{Q}, \Sigma)$ in the case $\lambda^{\mathcal{Q}}$ is a successor ordinal.

[7] constructs under AD^+ and the hypothesis that there are no models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" hod pairs that are fullness preserving, positional, commuting, and have branch condensation (see [7] for a full discussion of these notions). Such hod pairs are particularly important for our computation as they are points in the direct limit system giving rise to HOD of AD^+ models. Under AD^+, for hod pairs $(\mathcal{M}_\Sigma, \Sigma)$, if Σ is a strategy with branch condensation and $\vec{\mathcal{T}}$ is a stack on $\mathcal{M}_\Sigma$ with last model $\mathcal{N}$, $\Sigma_{\mathcal{N}, \vec{\mathcal{T}}}$ is independent of $\vec{\mathcal{T}}$. Therefore, later on we will omit the subscript $\vec{\mathcal{T}}$ from $\Sigma_{N, \vec{\mathcal{T}}}$ whenever Σ is a strategy with branch condensation and $\mathcal{M}_\Sigma$ is a hod mouse. In a core model induction, we don't quite have, at the moment $(\mathcal{M}_\Sigma, \Sigma)$ is constructed, an AD^+-model M such that $(\mathcal{M}_\Sigma, \Sigma) \in M$ but we do know that every $(\mathcal{R}, \Lambda) \in B(\mathcal{M}_\Sigma, \Sigma)$ belongs to such a model. We then can show (using our hypothesis) that $(\mathcal{M}_\Sigma, \Sigma)$ belongs to an AD^+-model.

Definition 3.2 (Hod pair below κ): $(\mathcal{P}, \Sigma)$ is a hod pair below κ if $\mathcal{P} \in V$, $|\mathcal{P}|^V \leq \kappa$, Σ is a (ω_1, ω_1) in $V[G]$ [7]-strategy with branch condensation, and is commuting, positional, and Ω-fullness preserving, and for all $(\mathcal{Q}, \Lambda) \in B(\mathcal{Q}, \Sigma)$, $\Lambda \restriction HC \in \Omega$. Furthermore, $\Sigma \restriction V \in V$.

3.1.2. *$\mathcal{F}$-premice.*

Definition 3.3: Let $\mathcal{L}_0$ be the language of set theory expanded by unary predicate symbols $\dot{E}, \dot{B}, \dot{S}$, and constant symbols $\dot{a}$, $\dot{\mathfrak{P}}$. Let $\mathcal{L}_0^- = \mathcal{L}_0 \backslash \{\dot{E}, \dot{B}\}$.

Let a be transitive. Let $\varrho : a \to \mathrm{rank}(a)$ be the rank function. We write $\hat{a} = \mathrm{trancl}(\{(a, \varrho)\})$. Let $\mathfrak{P} \in \mathcal{J}_1(\hat{a})$.

A **$\mathcal{J}$-structure over a (with parameter $\mathfrak{P}$) (for $\mathcal{L}_0$)** is a structure $\mathcal{M}$ for $\mathcal{L}_0$ such that $a^{\mathcal{M}} = a$, $(\mathfrak{P}^{\mathcal{M}} = \mathfrak{P})$, and there is $\lambda \in [1, \mathrm{Ord})$ such that $|\mathcal{M}| = \mathcal{J}_\lambda^{S^{\mathcal{M}}}(\hat{a})$. ategy with Here we also let $l(\mathcal{M})$ denote λ, the **length** of $\mathcal{M}$, and let $\hat{a}^{\mathcal{M}}$ denote $\hat{a}$.

For $\alpha \in [1, \lambda]$ let $\mathcal{M}_\alpha = \mathcal{J}_\alpha^{S^{\mathcal{M}}}(\hat{a})$. We say that $\mathcal{M}$ is **acceptable** iff for each $\alpha < \lambda$ and $\tau < o(\mathcal{M}_\alpha)$, if

$$\mathcal{P}(\tau^{<\omega} \times \hat{a}^{<\omega}) \cap \mathcal{M}_\alpha \neq \mathcal{P}(\tau^{<\omega} \times \hat{a}^{<\omega}) \cap \mathcal{M}_{\alpha+1},$$

then there is a surjection $\tau^{<\omega} \times \hat{a}^{<\omega} \to \mathcal{M}_\alpha$ in $\mathcal{M}_{\alpha+1}$.

A **$\mathcal{J}$-structure (for $\mathcal{L}_0$)** is a $\mathcal{J}$-structure over a, for some a.

As all $\mathcal{J}$-structures we consider will be for $\mathcal{L}_0$, we will omit the phrase "for $\mathcal{L}_0$". We also often omit the phrase "with parameter $\mathfrak{P}$". Note that if $\mathcal{M}$ is a $\mathcal{J}$-structure over a then $|\mathcal{M}|$ is transtive and rud-closed, $\hat{a} \in M$ and $o \cap M = \mathrm{rank}(M)$. This last point is because we construct from $\hat{a}$ instead of a.

$\mathcal{F}$-premice will be $\mathcal{J}$-structures of the following form.

Definition 3.4: A **$\mathcal{J}$-model over a (with parameter $\mathfrak{P}$)** is an acceptable $\mathcal{J}$-structure over a (with parameter $\mathfrak{P}$), of the form

$$\mathcal{M} = (M; E, B, S, a, \mathfrak{P})$$

where $\dot{E}^{\mathcal{M}} = E$, etc., and letting $\lambda = l(\mathcal{M})$, the following hold:

(1) $\mathcal{M}$ is amenable.
(2) $S = \langle S_\xi \mid \xi \in [1, \lambda) \rangle$ is a sequence of $\mathcal{J}$-models over a (with parameter $\mathfrak{P}$).

[7] Technically, this should be a (k, ω_1, ω_1)-strategy, where k is the degree of soundness of $\mathcal{P}$. But we suppress this parameter throughout our paper.

(3) For each $\xi \in [1,\lambda)$, $\dot{S}^{S_\xi} = S \restriction \xi$ and $\mathcal{M}_\xi = |S_\xi|$.
(4) Suppose $E \neq \emptyset$. Then $B = \emptyset$ and there is an extender F over $\mathcal{M}$ which is $\hat{a} \times \gamma$-complete for all $\gamma < \mathrm{crit}(F)$ and such that the premouse axioms [28, Definition 2.2.1] hold for $(\mathcal{M}, F)$, and E codes $\tilde{F} \cup \{G\}$ where: (i) $\tilde{F} \subseteq M$ is the amenable code for F (as in [23]); and (ii) if F is not type 2 then $G = \emptyset$, and otherwise G is the "longest" non-type Z proper segment of F in $\mathcal{M}$.[8]

Our notion of a "$\mathcal{J}$-model over a" is a bit different from the notion of "model with parameter a" in [11] or [28, Definition 2.1.1] in that we build into our notion some fine structure and we do not have the predicate l used in [28, Definition 2.1.1]. Note that with notation as above, if λ is a successor ordinal then $M = J(S^{\mathcal{M}}_{\lambda-1})$, and otherwise, $M = \bigcup_{\alpha<\lambda} |S_\alpha|$. The predicate $\dot{B}$ will be used to code extra information (like a (partial) branch of a tree in M).

Definition 3.5: Let $\mathcal{M}$ be a $\mathcal{J}$-model over a (with parameter $\mathfrak{P}$). Let $E^{\mathcal{M}}$ denote $\dot{E}^{\mathcal{M}}$, etc. Let $\lambda = l(\mathcal{M})$, $S^{\mathcal{M}}_0 = a$, $S^{\mathcal{M}}_\lambda = \mathcal{M}$, and $\mathcal{M}|\xi = S^{\mathcal{M}}_\xi$ for all $\xi \leq \lambda$. An **(initial) segment** of $\mathcal{M}$ is just a structure of the form $\mathcal{M}|\xi$ for some $\xi \in [1,\lambda]$. We write $\mathcal{P} \trianglelefteq \mathcal{M}$ iff $\mathcal{P}$ is a segment of $\mathcal{M}$, and $\mathcal{P} \triangleleft \mathcal{M}$ iff $\mathcal{P} \trianglelefteq \mathcal{M}$ and $\mathcal{P} \neq \mathcal{M}$. Let $\mathcal{M}||\xi$ be the structure having the same universe and predicates as $\mathcal{M}|\xi$, except that $E^{\mathcal{M}||\xi} = \emptyset$. We say that $\mathcal{M}$ is **E-active** iff $E^{\mathcal{M}} \neq \emptyset$, and **$B$-active** iff $B^{\mathcal{M}} \neq \emptyset$. **Active** means either E-active or B-active; **E-passive** means not E-active; **B-passive** means not B-active; and **passive** means not active.

Given a $\mathcal{J}$-model $\mathcal{M}_1$ over b and a $\mathcal{J}$-model $\mathcal{M}_2$ over $\mathcal{M}_1$, we write $\mathcal{M}_2 \downarrow b$ for the $\mathcal{J}$-model $\mathcal{M}$ over b, such that $\mathcal{M}$ is "$\mathcal{M}_1 \,\hat{}\, \mathcal{M}_2$". That is, $|\mathcal{M}| = |\mathcal{M}_2|$, $a^{\mathcal{M}} = b$, $E^{\mathcal{M}} = E^{\mathcal{M}_2}$, $B^{\mathcal{M}} = B^{\mathcal{M}_2}$, and $\mathcal{P} \triangleleft \mathcal{M}$ iff $\mathcal{P} \trianglelefteq \mathcal{M}_1$ or there is $\mathcal{Q} \triangleleft \mathcal{M}_2$ such that $\mathcal{P} = \mathcal{Q} \downarrow b$, when such an $\mathcal{M}$ exists. Existence depends on whether the $\mathcal{J}$-structure $\mathcal{M}$ is acceptable.

In the following, the variable i should be interpreted as follows. When $i = 0$, we ignore history, and so $\mathcal{P}$ is treated as a coarse object when determining $\mathcal{F}(0,\mathcal{P})$. When $i = 1$ we respect the history (given it exists).

[8] We use G explicitly, instead of the code $\gamma^{\mathcal{M}}$ used for G in [6, Section 2], because G does not depend on which (if there is any) wellorder of $\mathcal{M}$ we use. This ensures that certain pure mouse operators are forgetful.

Definition 3.6: An **operator $\mathcal{F}$ with domain** D is a function with domain D, such that for some cone $C = C_{\mathcal{F}}$, possibly self-wellordered (sword),[9] D is the set of pairs (i, X) such that either:

- $i = 0$ and $X \in C$, or
- $i = 1$ and X is a $\mathcal{J}$-model over $X_1 \in C$,

and for each $(i, X) \in D$, $\mathcal{F}(i, X)$ is a $\mathcal{J}$-model over X such that for each $\mathcal{P} \trianglelefteq \mathcal{F}(i, X)$, $\mathcal{P}$ is fully sound. (Note that $\mathcal{P}$ is a $\mathcal{J}$-model over X, so soundness is in this sense.)

Let $\mathcal{F}, D$ be as above. We say $\mathcal{F}$ is **forgetful** iff $\mathcal{F}(0, X) = \mathcal{F}(1, X)$ whenever $(0, X), (1, X) \in D$, and whenever X is a $\mathcal{J}$-model over X_1, and X_1 is a $\mathcal{J}$-model over $X_2 \in C$, we have $\mathcal{F}(1, X) = \mathcal{F}(1, X \downarrow X_2)$. Otherwise we say $\mathcal{F}$ is **historical**. Even when $\mathcal{F}$ is historical, we often just write $\mathcal{F}(X)$ instead of $\mathcal{F}(i, X)$ when the nature of $\mathcal{F}$ is clear from the context. We say $\mathcal{F}$ is **basic** iff for all $(i, X) \in D$ and $\mathcal{P} \trianglelefteq \mathcal{F}(i, X)$, we have $E^{\mathcal{P}} = \emptyset$. We say $\mathcal{F}$ is **projecting** iff for all $(i, X) \in D$, we have $\rho_\omega^{\mathcal{F}(i,X)} = X$.

Here are some illustrations. Strategy operators (to be explained in more detail later) are basic, and as usually defined, projecting and historical. Suppose we have an iteration strategy Σ and we want to build a $\mathcal{J}$-model $\mathcal{N}$ (over some a) that codes a fragment of Σ via its predicate $\dot{B}$. We feed Σ into $\mathcal{N}$ by always providing $b = \Sigma(\mathcal{T})$, for the $<$-$\mathcal{N}$-least tree $\mathcal{T}$ for which this information is required. So given a reasonably closed level $\mathcal{P} \trianglelefteq \mathcal{N}$, the choice of which tree $\mathcal{T}$ should be processed next will usually depend on the information regarding Σ already encoded in $\mathcal{P}$ (its history). Using an operator $\mathcal{F}$ to build $\mathcal{N}$, then $\mathcal{F}(i, \mathcal{P})$ will be a structure extending $\mathcal{P}$ and over which $b = \Sigma(\mathcal{T})$ is encoded. The variable i should be interpreted as follows. When $i = 1$, we respect the history of $\mathcal{P}$ when selecting $\mathcal{T}$. When $i = 0$ we ignore history when selecting $\mathcal{T}$. The operator $\mathcal{F}(X) = X^{\#}$ is forgetful and projecting, and not basic; here $\mathcal{F}(X) = \mathcal{F}(0, X)$.

Definition 3.7: For any P and any ordinal $\alpha \geq 1$, the operator $\mathcal{J}_\alpha^{\mathrm{m}}(\ \cdot\ ; P)$ is defined as follows.[10] For X such that $P \in \mathcal{J}_1(\hat{X})$, let $\mathcal{J}_\alpha^{\mathrm{m}}(X; P)$ be the $\mathcal{J}$-model

[9] C is a cone if there are a cardinal κ and a transitive set $a \in H_\kappa$ such that C is the set of $b \in H_\kappa$ such that $a \in L_1(b)$; a is called the base of the cone. A set a is self-wellordered if there is a well-ordering of a in $L_1(a)$. A set C is a self-wellordered cone if C is the restriction of a cone C' to its own self-wellordered elements

[10] The "m" is for "model".

$\mathcal{M}$ over X, with parameter P, such that $|\mathcal{M}| = \mathcal{J}_\alpha(\hat{X})$ and for each $\beta \in [1, \alpha]$, $\mathcal{M}|\beta$ is passive. Clearly $\mathcal{J}^{\rm m}_\alpha(\ \cdot\ ; P)$ is basic and forgetful. If $P = \emptyset$ or we wish to supress P, we just write $\mathcal{J}^{\rm m}_\alpha(\ \cdot\ )$.

Definition 3.8 (Potential $\mathcal{F}$-premouse, $\mathcal{C}_\mathcal{F}$): Let $\mathcal{F}$ be an operator with domain D of self-wellordered sets. Let $b \in C_\mathcal{F}$, so there is a well-ordering of b in $L_1[b]$. A **potential $\mathcal{F}$-premouse over** b is an acceptable $\mathcal{J}$-model $\mathcal{M}$ over b such that there is an ordinal $\iota > 0$ and an increasing, closed sequence $\langle \zeta_\alpha \rangle_{\alpha \leq \iota}$ of ordinals such that for each $\alpha \leq \iota$, we have:

(1) $0 = \zeta_0 \leq \zeta_\alpha \leq \zeta_\iota = l(\mathcal{M})$ (so $\mathcal{M}|\zeta_0 = b$ and $\mathcal{M}|\zeta_\iota = \mathcal{M}$).
(2) If $1 < \iota$ then $\mathcal{M}|\zeta_1 = \mathcal{F}(0, b)$.
(3) If $1 = \iota$ then $\mathcal{M} \trianglelefteq \mathcal{F}(0, b)$.
(4) If $1 < \alpha + 1 < \iota$ then $\mathcal{M}|\zeta_{\alpha+1} = \mathcal{F}(1, \mathcal{M}|\zeta_\alpha) \downarrow b$.
(5) If $1 < \alpha + 1 = \iota$ then $\mathcal{M} \trianglelefteq \mathcal{F}(1, \mathcal{M}|\zeta_\alpha) \downarrow b$.
(6) Suppose α is a limit. Then $\mathcal{M}|\zeta_\alpha$ is B-passive, and if E-active, then $\mathrm{crit}(E^{\mathcal{M}|\zeta_\alpha}) > \mathrm{rank}(b)$.

We say that $\mathcal{M}$ is **($\mathcal{F}$-)whole** iff ι is a limit or else, $\iota = \alpha + 1$ and $\mathcal{M} = \mathcal{F}(\mathcal{M}|\zeta_\alpha) \downarrow b$.

A **(potential) $\mathcal{F}$-premouse** is a (potential) $\mathcal{F}$-premouse over b, for some b.

Definition 3.9: Let $\mathcal{F}$ be an operator and $b \in C_\mathcal{F}$. Let $\mathcal{N}$ be a whole $\mathcal{F}$-premouse over b. A **potential continuing $\mathcal{F}$-premouse over** $\mathcal{N}$ is a $\mathcal{J}$-model $\mathcal{M}$ over $\mathcal{N}$ such that $\mathcal{M} \downarrow b$ is a potential $\mathcal{F}$-premouse over b. (Therefore $\mathcal{N}$ is a whole strong cutpoint of $\mathcal{M}$.)

We say that $\mathcal{M}$ (as above) is **whole** iff $\mathcal{M} \downarrow b$ is whole.

A **(potential) continuing $\mathcal{F}$-premouse** is a (potential) continuing $\mathcal{F}$-premouse over b, for some b.

Definition 3.10: $\mathrm{Lp}^\mathcal{F}(a)$ denotes the stack of all countably $\mathcal{F}$-iterable $\mathcal{F}$-premice $\mathcal{M}$ over a such that $\mathcal{M}$ is fully sound and projects to a.[11]

Let $\mathcal{N}$ be a whole $\mathcal{F}$-premouse over b, for $b \in C_\mathcal{F}$. Then $\mathrm{Lp}^\mathcal{F}_+(\mathcal{N})$ denotes the stack of all countably $\mathcal{F}$-iterable (above $o(\mathcal{N})$) continuing $\mathcal{F}$-premice $\mathcal{M}$ over $\mathcal{N}$ such that $\mathcal{M} \downarrow b$ is fully sound and projects to $\mathcal{N}$.

We say that $\mathcal{F}$ is **uniformly** Σ_1 iff there are Σ_1 formulas φ_1 and φ_2 in $\mathcal{L}^-_0$ such that whenever $\mathcal{M}$ is a (continuing) $\mathcal{F}$-premouse, then the set of whole

[11] Countable substructures of $\mathcal{M}$ are $(\omega, \omega_1 + 1)$-$\mathcal{F}$-iterable, i.e. all iterates are $\mathcal{F}$-premice. See [13, Section 2] for more details on $\mathcal{F}$-iterability.

proper segments of $\mathcal{M}$ is defined over $\mathcal{M}$ by φ_1 (φ_2). For such an operator $\mathcal{F}$, let $\varphi^{\mathcal{F}}_{\mathrm{wh}}$ denote the least such φ_1.

Definition 3.11 (Mouse operator): Let Y be a projecting, uniformly Σ_1 operator. A **Y-mouse operator $\mathcal{F}$ with domain D** is an operator with domain D such for each $(0,X) \in D$, $\mathcal{F}(0,X) \triangleleft \mathrm{Lp}^Y(X)$, and for each $(1,X) \in D$, $\mathcal{F}(1,X) \triangleleft \mathrm{Lp}^Y_+(X)$.[12] (So any Y-mouse operator is an operator.) A Y-mouse operator $\mathcal{F}$ is called **first order** if there are formulas φ_1 and φ_2 in the language of Y-premice such that $\mathcal{F}(0,X)$ ($\mathcal{F}(1,X)$) is the first $\mathcal{M} \triangleleft \mathrm{Lp}^Y(X)$ ($\mathrm{Lp}^Y_+(X)$) satisfying φ_1 (φ_2).

A **mouse operator** is a $\mathcal{J}^{\mathrm{m}}_1$-mouse operator.

We can then define $\mathcal{F}$-solidity, the $L^{\mathcal{F}}[\mathbb{E}]$-construction etc. as usual (see [13] for more details). We now define the kind of condensation that mouse operators need to satisfy to ensure the $L^{\mathcal{F}}[\mathbb{E}]$ converges.

Definition 3.12: Let $\mathcal{M}_1, \mathcal{M}_2$ be k-sound $\mathcal{J}$-models over a_1, a_2 and let $\pi : \mathcal{M}_1 \to \mathcal{M}_2$. Then π is **(weakly, nearly) k-good** iff $\pi \restriction a_1 = \mathrm{id}$, $\pi(a_1) = a_2$, and π is a (weak, near) k-embedding (as in [6]).

Definition 3.13: Given a $\mathcal{J}$-model $\mathcal{N}$ over a, and $\mathcal{M} \triangleleft \mathcal{N}$ such that $\mathcal{M}$ is fully sound, the **$\mathcal{M}$-drop-down sequence** of $\mathcal{N}$ is the sequence of pairs $\langle(\mathcal{Q}_n, m_n)\rangle_{n<k}$ of maximal length such that $\mathcal{Q}_0 = \mathcal{M}$ and $m_0 = \omega$ and for each $n+1 < k$:

(1) $\mathcal{M} \triangleleft \mathcal{Q}_{n+1} \trianglelefteq \mathcal{N}$ and $\mathcal{Q}_n \trianglelefteq \mathcal{Q}_{n+1}$,
(2) every proper segment of $\mathcal{Q}_{n+1}$ is fully sound,
(3) $\rho_{m_n}(\mathcal{Q}_n)$ is an a-cardinal of $\mathcal{Q}_{n+1}$,
(4) $0 < m_{n+1} < \omega$,
(5) $\mathcal{Q}_{n+1}$ is $(m_{n+1}-1)$-sound,
(6) $\rho_{m_{n+1}}(\mathcal{Q}_{n+1}) < \rho_{m_n}(\mathcal{Q}_n) \leq \rho_{m_{n+1}-1}(\mathcal{Q}_{n+1})$.

Definition 3.14: Let $\mathcal{F}$ be an operator and let C be some class of E-active $\mathcal{F}$-premice. Let b be transitive. A **(C-certified) $L^{\mathcal{F}}[\mathbb{E}, b]$-construction** is a sequence $\langle\mathcal{N}_\alpha\rangle_{\alpha\leq\lambda}$ with the following properties. We omit the phrase "over b".

We have $\mathcal{N}_0 = b$ and $\mathcal{N}_1 = \mathcal{F}(0,b)$.

Let $\alpha \in (0,\lambda]$. Then $\mathcal{N}_\alpha$ is an $\mathcal{F}$-premouse, and if α is a limit then $\mathcal{N}_\alpha$ is the lim inf of the $\mathcal{N}_\beta$ for $\beta < \alpha$. Now suppose that $\alpha < \lambda$. Then either:

[12] This restricts the usual notion defined in [11].

- $\mathcal{N}_\alpha$ is passive and is a limit of whole proper segments and $\mathcal{N}_{\alpha+1} = (\mathcal{N}_\alpha, G)$ for some extender G (with $\mathcal{N}_{\alpha+1} \in C$); or
- $\mathcal{N}_\alpha$ is ω-$\mathcal{F}$-solid. Let $\mathcal{M}_\alpha = \mathcal{C}_\omega(\mathcal{N}_\alpha)$. Let $\mathcal{M}$ be the largest whole segment of $\mathcal{M}_\alpha$. So either $\mathcal{M}_\alpha = \mathcal{M}$ or $\mathcal{M}_\alpha \downarrow \mathcal{M} \trianglelefteq \mathcal{F}_1(\mathcal{M})$. Let $\mathcal{N} \trianglelefteq \mathcal{F}_1(\mathcal{M})$ be least such that either $\mathcal{N} = \mathcal{F}_1(\mathcal{M})$ or for some $k+1 < \omega$, $(\mathcal{N} \downarrow b, k+1)$ is on the $\mathcal{M}_\alpha$-drop-down sequence of $\mathcal{N} \downarrow b$. Then $\mathcal{N}_{\alpha+1} = \mathcal{N} \downarrow b$. (Note $\mathcal{M}_\alpha \triangleleft \mathcal{N}_{\alpha+1}$.)

Definition 3.15: Let Y be an operator. We say that Y **condenses coarsely** iff for all $i \in \{0,1\}$ and $(i, \bar{X}), (i, X) \in \mathrm{dom}(Y)$, and all $\mathcal{J}$-models $\mathcal{M}^+$ over $\bar{X}$, if $\pi : \mathcal{M}^+ \to Y_i(X)$ is fully elementary, fixes the parameters in the definition of Y, then

(1) if $i = 0$ then $\mathcal{M}^+ \trianglelefteq Y_0(\bar{X})$; and
(2) if $i = 1$ and X is a sound whole Y-premouse, then $\mathcal{M}^+ \trianglelefteq Y_1(\bar{X})$.

Definition 3.16: Let Y be a projecting, uniformly Σ_1 operator. We say that Y **condenses finely** iff Y condenses coarsely and we have the following. Let $k < \omega$. Let $\mathcal{M}^*$ be a Y-premouse over a, with a largest whole proper segment $\mathcal{M}$, such that $\mathcal{M}^+ = \mathcal{M}^* \downarrow \mathcal{M}$ is sound and $\rho_{k+1}(\mathcal{M}^+) = \mathcal{M}$. Let $\mathcal{P}^*, \bar{a}, \mathcal{P}, \mathcal{P}^+$ be likewise. Let $\mathcal{N}$ be a sound whole Y-premouse over $\bar{a}$. Let $G \subseteq \mathrm{Col}(\omega, \mathcal{P} \cup \mathcal{N})$ be V-generic. Let $\mathcal{N}^+, \pi, \sigma \in V[G]$, with $\mathcal{N}^+$ a sound $\mathcal{J}$-model over $\mathcal{N}$ such that $\mathcal{N}^* = \mathcal{N}^+ \downarrow \bar{a}$ is defined (i.e., acceptable). Suppose $\pi : \mathcal{N}^* \to \mathcal{M}^*$ is such that $\pi(\mathcal{N}) = \mathcal{M}$ and either:

(1) $\mathcal{M}^*$ is k-sound and $\mathcal{N}^* = \mathcal{C}_{k+1}(\mathcal{M}^*)$; or
(2) $(\mathcal{N}^*, k+1)$ is in the $\mathcal{N}$-dropdown sequence of $\mathcal{N}^*$, and likewise $(\mathcal{P}^*, k+1), \mathcal{P}$, and either:
   (a) π is k-good, or
   (b) π is fully elementary, or
   (c) π is a weak k-embedding, $\sigma : \mathcal{P}^* \to \mathcal{N}^*$ is k-good, $\sigma(\mathcal{P}) = \mathcal{N}$ and $\pi \circ \sigma \in V$ is a near k-embedding.

Then $\mathcal{N}^+ \trianglelefteq Y_1(\mathcal{N})$.

We say that Y **almost condenses finely** iff $\mathcal{N}^+ \trianglelefteq Y_1(\mathcal{N})$ whenever the hypotheses above hold with $\mathcal{N}^+, \pi, \sigma \in V$.

In fact, the two notions above are equivalent.

LEMMA 3.17: *Let Y be an operator on a cone with base in* HC. *Suppose that Y almost condenses finely. Then Y condenses finely.*

We end this section with the following lemma (proved in Section 2 of [13]), which states that the $L^{\mathcal{F}}[\mathbb{E}]$-construction (relative to some class of background extenders) runs smoothly for a certain class of operators. In the following, if $(\mathcal{N}, G) \in C$, then G is backgrounded as in [6] or as in [16] (we additionally demand that the structure N in [16, Definition 1.1] is closed under $\mathcal{F}$).

LEMMA 3.18: *Let $\mathcal{F}$ be a projecting, uniformly Σ_1 operator which condenses finely. Suppose $\mathcal{F}$ is defined on a cone with bases in* HC. *Let $\mathbb{C} = \langle \mathcal{N}_\alpha \rangle_{\alpha \leq \lambda}$ be the (C-certified) $L^{\mathcal{F}}[\mathbb{E}, b]$-construction for $b \in C_{\mathcal{F}}$. Then* (a) *$\mathcal{N}_\lambda$ is 0-$\mathcal{F}$-solid (i.e., is an $\mathcal{F}$-premouse).*

Now suppose that $\mathcal{N}_\lambda$ is k-$\mathcal{F}$-solid.

Suppose that for a club of countable elementary $\pi : \mathcal{M} \to \mathcal{C}_k(\mathcal{N}_\lambda)$, there is an $\mathcal{F}$-putative, $(k, \omega_1, \omega_1 + 1)$-iteration strategy Σ for $\mathcal{M}$, such that every tree $\mathcal{T}$ via Σ is $(\pi, \mathbb{C})$-realizable.[13]

Then (b) *$\mathcal{N}_\lambda$ is $(k+1)$-$\mathcal{F}$-solid.*

LEMMA 3.19: *Let $Y, \mathcal{F}$ be uniformly Σ_1 operators defined on a cone over some $\mathcal{H}_\kappa$, with bases in* HC.[14] *Suppose that Y condenses finely. Suppose that $\mathcal{F}$ is a whole continuing Y-mouse operator. Then $\mathcal{F}$ condenses finely.*

The following lemma gives a stronger condensation property than fine condensation in certain circumstances. So if $\mathcal{F}$ satisfies the hypothesis of Lemma 3.20 (particularly, if $\mathcal{F}$ is one of the operators constructed in our core model induction) then the $L^{\mathcal{F}}[\mathbb{E}]$-construction converges by Lemma 3.18.

LEMMA 3.20: *Let $Y, \mathcal{F}$ be uniformly Σ_1 operators with bases in* HC. *Suppose that Y condenses finely. Suppose that $\mathcal{F}$ is a whole continuing Y-mouse operator. Then* (a) *$\mathcal{F}$ condenses finely. Moreover,* (b) *let $\mathcal{M}$ be an $\mathcal{F}$-whole $\mathcal{F}$-premouse. Let $\pi : \mathcal{N} \to \mathcal{M}$ be fully elementary with $a^{\mathcal{N}} \in C_{\mathcal{F}}$. Then $\mathcal{N}$ is an $\mathcal{F}$-whole $\mathcal{F}$-premouse. So regarding $\mathcal{F}$, the conclusion of 3.15 may be modified by replacing "$\trianglelefteq$" with "$=$".*

[13] See [13, Section 2] for a precise definition of $(\pi, \mathbb{C})$-realizability. Roughly speaking this means that models along the tree $\mathcal{T}$ are embedded into the $\mathcal{N}_\alpha$'s.

[14] We also say "operator over $\mathcal{H}_\kappa$ with bases in HC" for short.

Remark 3.21: In the context of the core model induction of this paper (and elsewhere), we often construct mouse operators $\mathcal{F}$ defined over some $\mathcal{H}_\kappa$ with base $a \notin \mathrm{HC}$. So given an $\mathcal{F}$-premouse $\mathcal{N}$, $\pi : \mathcal{N}^* \to \mathcal{N}$ elementary, and $\mathcal{N}^*$ countable, $\mathcal{N}^*$ may not be an $\mathcal{F}$ premouse. We have to make some changes for the theory above to work for these $\mathcal{F}$. For instance, in Lemma 3.18, with the notation as there, we can modify the hypothesis of the lemma in one of two ways:

(1) We can either require that $a \in \mathcal{M}$, $|\mathcal{M}| = |a|$, and the $(\pi, \mathbb{C})$-realizable strategy Σ is $(k, |a|^+, |a|^+ + 1)$-iterable.
(2) We can still require $\mathcal{M}$ is countable but the strategy Σ is a $(k, \omega_1, \omega_1+1)$-$\mathcal{F}^\pi$-strategy, where $\mathcal{F}^\pi$ is the π-pullback operator of $\mathcal{F}$.[15]

3.1.3. *Strategy premice.* We now proceed to defining Σ-premice, for an iteration strategy Σ. We first define the operator to be used to feed in Σ.

Definition 3.22 ($\mathfrak{B}(a, \mathcal{T}, b)$, $b^{\mathcal{N}}$): Let $a, \mathcal{P}$ be transitive, with $\mathcal{P} \in \mathcal{J}_1(\hat{a})$. Let $\lambda > 0$

$\mathcal{T}$ be an iteration tree[16] on $\mathcal{P}$, of length $\omega\lambda$, with $\mathcal{T} \restriction \beta \in a$ for all $\beta \leq \omega\lambda$. Let $b \subseteq \omega\lambda$. We define $\mathcal{N} = \mathfrak{B}(a, \mathcal{T}, b)$ recursively on $\mathrm{lh}(\mathcal{T})$, as the $\mathcal{J}$-model $\mathcal{N}$ over a, with parameter $\mathcal{P}$,[17] such that:

(1) $l(\mathcal{N}) = \lambda$,
(2) for each $\gamma \in (0, \lambda)$, $\mathcal{N}|\gamma = \mathfrak{B}(a, \mathcal{T} \restriction \omega\gamma, [0, \omega\gamma]_{\mathcal{T}})$,
(3) $B^{\mathcal{N}}$ is the set of ordinals $o(a) + \gamma$ such that $\gamma \in b$,
(4) $E^{\mathcal{N}} = \emptyset$.

We also write $b^{\mathcal{N}} = b$.

It is easy to see that every initial segment of $\mathcal{N}$ is sound, so $\mathcal{N}$ is acceptable and is indeed a $\mathcal{J}$-model (not just a $\mathcal{J}$-structure).

In the context of a Σ-premouse $\mathcal{M}$ for an iteration strategy Σ, if $\mathcal{T}$ is the $<_{\mathcal{M}}$-least tree for which $\mathcal{M}$ lacks instruction regarding $\Sigma(\mathcal{T})$, then $\mathcal{M}$ will already have been instructed regarding $\Sigma(\mathcal{T} \restriction \alpha)$ for all $\alpha < \mathrm{lh}(\mathcal{T})$. Therefore

[15] For instance, if $\mathcal{F}$ corresponds to a strategy Σ, then $\mathcal{F}^\pi$ corresponds to Σ^π, the π-pullback of Σ. If $\mathcal{F}$ is a first order mouse operator defined by (φ, a), then $\mathcal{F}^\pi$ is defined by $(\varphi, \pi^{-1}(a))$.

[16] We formally take an *iteration tree* to include the entire sequence $\langle M_\alpha^{\mathcal{T}} \rangle_{\alpha < \mathrm{lh}(\mathcal{T})}$ of models. So it is $\Sigma_0(\mathcal{T}, \mathfrak{P})$ to assert that "$\mathcal{T}$ is an iteration tree on $\mathfrak{P}$".

[17] $\mathcal{P} = M_0^{\mathcal{T}}$ is determined by $\mathcal{T}$.

if $\mathrm{lh}(\mathcal{T}) > \omega$ then $\mathfrak{B}(\mathcal{M}, \mathcal{T}, \Sigma(\mathcal{T}))$ codes redundant information (the branches already in $\mathcal{T}$) before coding $\Sigma(\mathcal{T})$. This redundancy seems to allow one to prove slightly stronger condensation properties, given that Σ has nice condensation properties (see Lemma 3.29). It also simplifies the definition.

Definition 3.23: Let Σ be a partial iteration strategy. Let C be a class of iteration trees, closed under initial segment. We say that (Σ, C) is **suitably condensing** iff for every $\mathcal{T} \in C$ such that $\mathcal{T}$ is via Σ and $\mathrm{lh}(\mathcal{T}) = \lambda + 1$ for some limit λ, either (i) Σ has hull condensation with respect to $\mathcal{T}$, or (ii) $b^{\mathcal{T}}$ does not drop and Σ has branch condensation with respect to $\mathcal{T}$, that is, any hull $\mathcal{U}^\frown c$ of $\mathcal{T}^\frown b$ is according to Σ.

When C is the class of all iteration trees according to Σ, we simply omit it from our notation.

Definition 3.24: Let φ be an $\mathcal{L}_0$-formula. Let $\mathcal{P}$ be transitive. Let $\mathcal{M}$ be a $\mathcal{J}$-model (over some a), with parameter $\mathcal{P}$. Let $\mathcal{T} \in \mathcal{M}$. We say that φ **selects** $\mathcal{T}$ **for** $\mathcal{M}$, and write $\mathcal{T} = \mathcal{T}_\varphi^{\mathcal{M}}$, iff

(a) $\mathcal{T}$ is the unique $x \in \mathcal{M}$ such that $\mathcal{M} \vDash \varphi(x)$,
(b) $\mathcal{T}$ is an iteration tree on $\mathcal{P}$ of limit length,
(c) for every $\mathcal{N} \triangleleft \mathcal{M}$, we have $\mathcal{N} \nvDash \varphi(\mathcal{T})$, and
(d) for every limit $\lambda < \mathrm{lh}(\mathcal{T})$, there is $\mathcal{N} \triangleleft \mathcal{M}$ such that $\mathcal{N} \vDash \varphi(\mathcal{T} \restriction \lambda)$.

One instance of $\phi(\mathcal{P}, \mathcal{T})$ is, in the case a is self-wellordered, the formula "$\mathcal{T}$ is the least tree on $\mathcal{P}$ that doesn't have a cofinal branch", where least is computed with respect to the canonical well-order of the model.

Definition 3.25 (Potential $\mathcal{P}$-strategy-premouse, $\Sigma^{\mathcal{M}}$): Let $\varphi \in \mathcal{L}_0$. Let $\mathcal{P}, a$ be transitive with $\mathcal{P} \in \mathcal{J}_1(\hat{a})$. A **potential $\mathcal{P}$-strategy-premouse (over a, of type φ)** is a $\mathcal{J}$-model $\mathcal{M}$ over a, with parameter $\mathcal{P}$, such that the $\mathfrak{B}$ operator is used to feed in an iteration strategy for trees on $\mathcal{P}$, using the sequence of trees naturally determined by $S^{\mathcal{M}}$ and selection by φ. We let $\Sigma^{\mathcal{M}}$ denote the partial strategy coded by the predicates $B^{\mathcal{M}|\eta}$, for $\eta \leq l(\mathcal{M})$.

In more detail, there is an increasing, closed sequence of ordinals $\langle \eta_\alpha \rangle_{\alpha \leq \iota}$ with the following properties. We will also define $\Sigma^{\mathcal{M}|\eta}$ for all $\eta \in [1, l(\mathcal{M})]$ and $\mathcal{T}_\eta = \mathcal{T}_\eta^{\mathcal{M}}$ for all $\eta \in [1, l(\mathcal{M}))$.

(1) $1 = \eta_0$ and $\mathcal{M}|1 = \mathcal{J}_1^{\mathrm{m}}(a; \mathcal{P})$ and $\Sigma^{\mathcal{M}|1} = \emptyset$.
(2) $l(\mathcal{M}) = \eta_\iota$, so $\mathcal{M}|\eta_\iota = \mathcal{M}$.

(3) Given $\eta \leq l(\mathcal{M})$ such that $B^{\mathcal{M}|\eta} = \emptyset$, we set $\Sigma^{\mathcal{M}|\eta} = \bigcup_{\eta'<\eta} \Sigma^{\mathcal{M}|\eta'}$.

Let $\eta \in [1, l(\mathcal{M})]$. Suppose there is $\gamma \in [1, \eta]$ and $\mathcal{T} \in \mathcal{M}|\gamma$ such that $\mathcal{T} = \mathcal{T}_\varphi^{\mathcal{M}|\gamma}$, and $\mathcal{T}$ is via $\Sigma^{\mathcal{M}|\eta}$, but no proper extension of $\mathcal{T}$ is via $\Sigma^{\mathcal{M}|\eta}$. Taking γ minimal such, let $\mathcal{T}_\eta = \mathcal{T}_\varphi^{\mathcal{M}|\gamma}$. Otherwise let $\mathcal{T}_\eta = \emptyset$.

(4) Let $\alpha + 1 \leq \iota$. Suppose $\mathcal{T}_{\eta_\alpha} = \emptyset$. Then $\eta_{\alpha+1} = \eta_\alpha + 1$ and $\mathcal{M}|\eta_{\alpha+1} = \mathcal{J}_1^{\mathrm{m}}(\mathcal{M}|\eta_\alpha; \mathcal{P}) \downarrow a$.

(5) Let $\alpha + 1 \leq \iota$. Suppose $\mathcal{T} = \mathcal{T}_{\eta_\alpha} \neq \emptyset$. Let $\omega\lambda = \mathrm{lh}(\mathcal{T})$. Then for some $b \subseteq \omega\lambda$, and $\mathcal{S} = \mathfrak{B}(\mathcal{M}|\eta_\alpha, \mathcal{T}, b)$, we have:
   (a) $\mathcal{M}|\eta_{\alpha+1} \trianglelefteq \mathcal{S}$.
   (b) If $\alpha + 1 < \iota$ then $\mathcal{M}|\eta_{\alpha+1} = \mathcal{S}$.
   (c) If $\mathcal{S} \trianglelefteq \mathcal{M}$ then b is a $\mathcal{T}$-cofinal branch.[18]
   (d) For $\eta \in [\eta_\alpha, l(\mathcal{M})]$ such that $\eta < l(\mathcal{S})$, $\Sigma^{\mathcal{M}|\eta} = \Sigma^{\mathcal{M}|\eta_\alpha}$.
   (e) If $\mathcal{S} \trianglelefteq \mathcal{M}$ then $\Sigma^{\mathcal{S}} = \Sigma^{\mathcal{M}|\eta_\alpha} \cup \{(\mathcal{T}, b^{\mathcal{S}})\}$.

(6) For each limit $\alpha \leq \iota$, $B^{\mathcal{M}|\eta_\alpha} = \emptyset$.

Definition 3.26 (Whole): Let $\mathcal{M}$ be a potential $\mathcal{P}$-strategy-premouse of type φ. We say $\mathcal{P}$ is **φ-whole** (or just **whole** if φ is fixed) iff for every $\eta < l(\mathcal{M})$, if $\mathcal{T}_\eta \neq \emptyset$ and $\mathcal{T}_\eta \neq \mathcal{T}_{\eta'}$ for all $\eta' < \eta$, then for some b, $\mathfrak{B}(\mathcal{M}|\eta, \mathcal{T}_\eta, b) \trianglelefteq \mathcal{M}$.[19]

Definition 3.27 (Potential Σ-premouse): Let Σ be a (partial) iteration strategy for a transitive structure $\mathcal{P}$. A **potential Σ-premouse (over a, of type φ)** is a potential $\mathcal{P}$-strategy premouse $\mathcal{M}$ (over a, of type φ) such that $\Sigma^{\mathcal{M}} \subseteq \Sigma$.[20]

Definition 3.28: Let $\mathcal{R}, \mathcal{M}$ be $\mathcal{J}$-structures for $\mathcal{L}_0$, $a = a^{\mathcal{R}}$ and $b = a^{\mathcal{M}}$. Suppose that a, b code $\mathcal{P}, \mathcal{Q}$ respectively. Let $\pi : \mathcal{R} \to \mathcal{M}$ (or

$$\pi : o(\mathcal{R}) \cup \mathcal{P} \cup \{\mathcal{P}\} \to o(\mathcal{M}) \cup \mathcal{Q} \cup \{\mathcal{Q}\}$$

respectively). Then π is a **$(\mathcal{P}, \mathcal{Q})$-weak 0-embedding** (resp., **$(\mathcal{P}, \mathcal{Q})$-very weak 0-embedding**) iff $\pi(\mathcal{P}) = (\mathcal{Q})$ and with respect to the language $\mathcal{L}_0$, π is Σ_0-elementary, and there is an $X \subseteq \mathcal{R}$ (resp., $X \subseteq o(\mathcal{R})$) such that X is cofinal in $\in^{\mathcal{R}}$ and π is Σ_1-elementary on parameters in $X \cup \mathcal{P} \cup \{\mathcal{P}\}$. If also $\mathcal{P} = \mathcal{Q}$ and

[18] We allow $\mathcal{M}_b^{\mathcal{T}}$ to be illfounded, but then $\mathcal{T}^\frown b$ is not an iteration tree, so is not continued by $\Sigma^{\mathcal{M}}$.

[19] *φ-whole* depends on φ as the definition of $\mathcal{T}_\eta$ does.

[20] If $\mathcal{M}$ is a model all of whose proper segments are potential Σ-premice, and the rules for potential $\mathcal{P}$-strategy premice require that $B^{\mathcal{M}}$ code a $\mathcal{T}$-cofinal branch, but $\Sigma(\mathcal{T})$ is not defined, then $\mathcal{M}$ is not a potential Σ-premouse, whatever its predicates are.

$\pi \restriction \mathcal{P} \cup \{\mathcal{P}\} = \mathrm{id}$, then we just say that π is a **$\mathcal{P}$-weak 0-embedding** (resp., **$\mathcal{P}$-very weak 0-embedding**).

Note that, for $(\mathcal{P}, \mathcal{Q})$-weak 0-embeddings, we can in fact take $X \subseteq o(\mathcal{R})$. The following lemma is again proved in [13, Section 3].

LEMMA 3.29: *Let $\mathcal{M}$ be a $\mathcal{P}$-strategy premouse over a, of type φ, where φ is Σ_1. Let $\mathcal{R}$ be a $\mathcal{J}$-structure for $\mathcal{L}_0$ and $a' = a^{\mathcal{R}}$, and let $\mathcal{P}'$ be a transitive structure coded by a'.*

(1) *Suppose $\pi : \mathcal{R} \to \mathcal{M}$ is a partial map such that $\pi(\mathcal{P}') = \mathcal{P}$ and either:*
   (a) *π is a $(\mathcal{P}', \mathcal{P})$-weak 0-embedding, or*
   (b) *π is a $(\mathcal{P}', \mathcal{P})$-very weak 0-embedding, and if $E^{\mathcal{R}} \neq \emptyset$ then item 4 of 3.4 holds for $E^{\mathcal{R}}$.*

   Then $\mathcal{R}$ is a $\mathcal{P}'$-strategy premouse of type φ. Moreover, if $\pi \restriction \{\mathcal{P}'\} \cup \mathcal{P}' = \mathrm{id}$ and if $\mathcal{M}$ is a Σ-premouse, where $(\Sigma, \mathrm{dom}(\Sigma^{\mathcal{M}}))$ is suitably condensing, then $\mathcal{R}$ is also a Σ-premouse.

(2) *Suppose there is $\pi : \mathcal{M} \to \mathcal{R}$ is such that $\pi(a, \mathcal{P}) = (a', \mathcal{P}')$ and either*
   (a) *π is Σ_2-elementary; or*
   (b) *π is cofinal and Σ_1-elementary, and $B^{\mathcal{M}} = \emptyset$.*

   Then $\mathcal{R}$ is a $\mathcal{P}'$-strategy premouse of type φ, and $\mathcal{R}$ is whole iff $\mathcal{M}$ is whole.

(3) *Suppose $B^{\mathcal{M}} \neq \emptyset$. Let $\mathcal{T} = \mathcal{T}^{\mathcal{M}}_{\eta}$ where $\eta < l(\mathcal{M})$ is largest such that $\mathcal{M}|\eta$ is whole. Let $b = b^{\mathcal{M}}$ and $\omega\gamma = \bigcup b$. So $\mathcal{M} \trianglelefteq \mathfrak{B}(\mathcal{M}|\eta, \mathcal{T}, b)$. Suppose there is $\pi : \mathcal{M} \to \mathcal{R}$ such that $\pi(\mathcal{P}) = \mathcal{P}'$ and π is cofinal and Σ_1-elementary. Let $\omega\gamma' = \sup \pi \text{“} \omega\gamma$.*
   (a) *$\mathcal{R}$ is a $\mathcal{P}'$-strategy premouse of type φ iff we have either (i) $\omega\gamma' = \mathrm{lh}(\pi(\mathcal{T}))$, or (ii) $\omega\gamma' < \mathrm{lh}(\pi(\mathcal{T}))$ and $b^{\mathcal{R}} = [0, \omega\gamma']_{\pi(\mathcal{T})}$.*
   (b) *If either $b^{\mathcal{M}} \in \mathcal{M}$ or π is continuous at $\mathrm{lh}(\mathcal{T})$, then $\mathcal{R}$ is a $\mathcal{P}'$-strategy premouse of type φ.*

Remark 3.30: The preceding lemma left open the possibility that $\mathcal{R}$ fails to be a $\mathcal{P}$-strategy premouse under certain circumstances (because $B^{\mathcal{R}}$ should be coding a branch that has in fact already been coded at some proper segment of $\mathcal{R}$, but codes some other branch instead). In the main circumstance we are interested in, this does not arise, for a couple of reasons. Suppose that Σ is an iteration strategy for $\mathcal{P}$ with hull condensation, $\mathcal{M}$ is a Σ-premouse, and Λ is a Σ-strategy for $\mathcal{M}$. Suppose $\pi : \mathcal{M} \to \mathcal{R}$ is a degree 0 iteration embedding

and $B^{\mathcal{M}} \neq \emptyset$ and π is discontinuous at $\mathrm{lh}(\mathcal{T})$. Then [13, Section 3] shows that $b^{\mathcal{M}} \in \mathcal{M}$. (It's not relevant whether π itself is via Λ.) It then follows from 0b of Lemma 3.29 that $\mathcal{R}$ is a Σ-mouse.

The other reason is that, supposing $\pi : \mathcal{M} \to \mathcal{R}$ is via Λ (so $\pi \restriction \mathcal{P} \cup \{\mathcal{P}\} = \mathrm{id}$), then trivially, $B^{\mathcal{R}}$ must code branches according to Σ. We can obtain such a Λ given that we can realize iterates of $\mathcal{M}$ back into a fixed Σ-premouse (with $\mathcal{P}$-weak 0-embeddings as realization maps).

Definition 3.31: Let $\mathcal{P}$ be transitive and Σ a partial iteration strategy for $\mathcal{P}$. Let $\varphi \in \mathcal{L}_0$. Let $\mathcal{F} = \mathcal{F}_{\Sigma,\varphi}$ be the operator such that:

(1) $\mathcal{F}_0(a) = \mathcal{J}_1^{\mathrm{m}}(a; \mathcal{P})$, for all transitive a such that $\mathcal{P} \in \mathcal{J}_1(\hat{a})$.
(2) Let $\mathcal{M}$ be a sound φ-whole Σ-premouse of type φ. Let $\lambda = l(\mathcal{M})$ and with notation as in 3.25, let $\mathcal{T} = \mathcal{T}_\lambda$. If $\mathcal{T} = \emptyset$ then $\mathcal{F}_1(\mathcal{M}) = \mathcal{J}_1^{\mathrm{m}}(\mathcal{M}; \mathcal{P})$. If $\mathcal{T} \neq \emptyset$ then $\mathcal{F}_1(\mathcal{M}) = \mathfrak{B}(\mathcal{M}, \mathcal{T}, b)$ where $b = \Sigma(\mathcal{T})$.

We say that $\mathcal{F}$ is a **strategy operator**.

LEMMA 3.32: *Let $\mathcal{P}$ be countable and transitive. Let φ be a formula of $\mathcal{L}_0$. Let Σ be a partial strategy for $\mathcal{P}$. Let D_φ be the class of iteration trees $\mathcal{T}$ on $\mathcal{P}$ such that for some $\mathcal{J}$-model $\mathcal{M}$, with parameter $\mathcal{P}$, we have $\mathcal{T} = \mathcal{T}_\varphi^{\mathcal{M}}$. Suppose that (Σ, D_φ) is suitably condensing. Then $\mathcal{F}_{\Sigma,\varphi}$ is uniformly Σ_1, projecting, and condenses finely.*

Definition 3.33: Let a be transitive and let $\mathcal{F}$ be an operator. We say that $\mathcal{M}_1^{\mathcal{F},\#}(a)$ **exists** iff there is a $(0, |a|, |a|^+ + 1)$-$\mathcal{F}$-iterable, non-1-small $\mathcal{F}$-premouse over a. We write $\mathcal{M}_1^{\mathcal{F},\#}(a)$ for the least such sound structure. For $\Sigma, \mathcal{P}, a, \varphi$ as in 3.31, we write $\mathcal{M}_1^{\Sigma,\varphi,\#}(a)$ for $\mathcal{M}_1^{\mathcal{F}_{\Sigma,\varphi},\#}(a)$.

Let $\mathcal{L}_0^+$ be the language $\mathcal{L}_0 \cup \{\dot{\prec}, \dot{\Sigma}\}$, where $\dot{\prec}$ is the binary relation defined by "$\dot{a}$ is self-wellordered, with ordering $\prec_{\dot{a}}$, and $\dot{\prec}$ is the canonical wellorder of the universe extending $\prec_{\dot{a}}$", and $\dot{\Sigma}$ is the partial function defined by "$\dot{\mathfrak{P}}$ is a transitive structure and the universe is a potential $\dot{\mathfrak{P}}$-strategy premouse over $\dot{a}$ and $\dot{\Sigma}$ is the associated partial putative iteration strategy for $\dot{\mathfrak{P}}$". Let $\varphi_{\mathrm{all}}(\mathcal{T})$ be the $\mathcal{L}_0$-formula "$\mathcal{T}$ is the $\dot{\prec}$-least limit length iteration tree $\mathcal{U}$ on $\dot{\mathfrak{P}}$ such that $\mathcal{U}$ is via $\dot{\Sigma}$, but no proper extension of $\mathcal{U}$ is via $\dot{\Sigma}$". Then for $\Sigma, \mathcal{P}, a$ as in 3.31, we sometimes write $\mathcal{M}_1^{\Sigma,\#}(a)$ for $\mathcal{M}_1^{\mathcal{F}_{\Sigma,\varphi_{\mathrm{all}}},\#}(a)$.

Let κ be a cardinal and suppose that $\mathfrak{M} = \mathcal{M}_1^{\mathcal{F},\#}(a)$ exists and is $(0, \kappa^+ + 1)$-iterable. We write $\Lambda_{\mathfrak{M}}$ for the unique $(0, \kappa^+ + 1)$-iteration strategy for $\mathfrak{M}$ (given that κ is fixed).

Definition 3.34: We say that $(\mathcal{F}, \Sigma, \varphi, D, a, \mathfrak{P})$ is **suitable** iff a is transitive and $\mathcal{M}_1^{\mathcal{F},\#}(a)$ exists, where either

(1) $\mathcal{F}$ is a projecting, uniformly Σ_1 operator, $C_\mathcal{F}$ is the (possibly swo'd) cone above a, D is the set of pairs $(i, X) \in \mathrm{dom}(\mathcal{F})$ such that either $i = 0$ or X is a sound whole $\mathcal{F}$-premouse, and $\Sigma = \varphi = 0$, or

(2) $\mathcal{P}, \Sigma, \varphi, D_\varphi$ are as in 3.32, $\mathcal{F} = \mathcal{F}_{\Sigma,\varphi}$, $D_\varphi \subseteq D$, D is a class of limit length iteration trees on $\mathcal{P}$, via Σ, $\Sigma(\mathcal{T})$ is defined for all $\mathcal{T} \in D$, (Σ, D) is suitably condensing and $\mathcal{P} \in \mathcal{J}_1(\hat{a})$.

We write $\mathcal{G}_\mathcal{F}$ for the function with domain $C_\mathcal{F}$, such that for all $x \in C_\mathcal{F}$, $\mathcal{G}_\mathcal{F}(x) = \Sigma(x)$ in case (ii), and in case (i), $\mathcal{G}_\mathcal{F}(0, x) = \mathcal{F}(0, x)$ and $\mathcal{G}_\mathcal{F}(1, x)$ is the least $\mathcal{R} \trianglelefteq \mathcal{F}_1(x) \downarrow a^x$ such that either $\mathcal{R} = \mathcal{F}_1(X) \downarrow a^X$ or $\mathcal{R}$ is unsound.

Lemma 3.35: *Let $\mathcal{F}$ be as in 3.34 and $\mathfrak{M} = \mathcal{M}_1^{\mathcal{F},\#}$. Then $\Lambda_\mathfrak{M}$ has branch condensation and hull condensation.*

3.1.4. *g-organized $\mathcal{F}$-premice.* Now we give an outline of the general treatment of [13] on $\mathcal{F}$-premice over an arbitrary set; following the terminology of [13], we will call these g-organized $\mathcal{F}$-premice and Θ-g-organized $\mathcal{F}$-premice. For (Θ)-g-organized $\mathcal{F}$-premice to be useful, we need to assume that the following absoluteness property holds for the operator $\mathcal{F}$. We then show that if $\mathcal{F}$ is the operator for a nice enough iteration strategy, then it does hold. We write $\mathfrak{M}$ for $\mathfrak{M}_\mathcal{F}$ and fix $a, \mathfrak{P}, \mathcal{F}, \mathcal{P}, C$ as in the previous subsection. In the following, $\delta^\mathfrak{M}$ denotes the Woodin cardinal of $\mathfrak{M}$. Again, the reader should see [13] for proofs of lemmas stated here.

Definition 3.36: Let $(\mathcal{F}, \Sigma, \varphi, C, a, \mathfrak{P})$ be suitable. We say that $\mathcal{M}_1^{\mathcal{F},\sharp}(a)$ **generically interprets** $\mathcal{F}$[21] iff, writing $\mathfrak{M} = \mathcal{M}_1^{\mathcal{F},\#}(a)$, there are formulas Φ, Ψ in $\mathcal{L}_0$ such that there is some $\gamma > \delta^\mathfrak{M}$ such that $\mathfrak{M}|\gamma \vDash \Phi$ and for any non-dropping $\Sigma_\mathfrak{M}$-iterate $\mathcal{N}$ of $\mathfrak{M}$, via a countable iteration tree $\mathcal{T}$, any $\mathcal{N}$-cardinal δ, any $\gamma \in \mathrm{Ord}$ such that $\mathcal{N}|\gamma \vDash \Phi$ & "δ is Woodin", and any g which is set-generic over $\mathcal{N}|\gamma$ (with $g \in V$), then $(\mathcal{N}|\gamma)[g]$ is closed under $\mathcal{G}_\mathcal{F}$, and $\mathcal{G}_\mathcal{F} \restriction (\mathcal{N}|\gamma)[g]$ is defined over $(\mathcal{N}|\gamma)[g]$ by Ψ. We say such a pair (Φ, Ψ) **generically determines** $(\mathcal{F}, \Sigma, \varphi, C, a)$ (or just $\mathcal{F}$).

We say an operator $\mathcal{F}$ is **nice** iff for some $\Sigma, \varphi, C, a, \mathfrak{P}$, $(\mathcal{F}, \Sigma, \varphi, C, a, \mathfrak{P})$ is suitable and $\mathcal{M}_1^{\mathcal{F},\sharp}$ generically interprets $\mathcal{F}$.

[21] In [13], this notion is called $\mathcal{F}$ determines itself on generic extensions. In this paper, "determines itself on generic extensions" will have a different meaning, as defined later.

Let $\mathcal{P} \in \mathrm{HC}$, let Σ be an iteration strategy for $\mathcal{P}$ and let C be the class of all limit length trees via Σ. Suppose $\mathcal{M}_1^{\Sigma,\#}(\mathcal{P})$ exists, (Σ, C) is suitably condensing. We say that $\mathcal{M}_1^{\Sigma,\#}(\mathcal{P})$ **generically interprets** Σ iff some (Φ, Ψ) generically determines $(\mathcal{F}_{\Sigma,\varphi_{\mathrm{all}}}, \Sigma, \varphi_{\mathrm{all}}, C, \mathcal{P})$. (Note then that the latter is suitable.)

LEMMA 3.37: *Let $\mathcal{N}, \delta$, etc. be as in 3.36, except that we allow $\mathcal{T}$ to have uncountable length, and allow g to be in a set-generic extension of V. Then $(\mathcal{N}|\gamma)[g]$ is closed under $\mathcal{G}_{\mathcal{F}}$ and letting $\mathcal{G}'$ be the interpretation of Ψ over $(\mathcal{N}|\gamma)[g]$, $\mathcal{G}' \restriction C = \mathcal{G}_{\mathcal{F}} \restriction (\mathcal{N}|\gamma)[g]$.*

We fix a nice $\mathcal{F}$, $\mathfrak{M}$, $\Lambda_{\mathfrak{M}} = \Lambda$, (Φ, Ψ) for the rest of the section. We define $\mathcal{M}_1^{\Sigma}$ from $\mathfrak{M}$ in the standard way.

See [13, Section 4] for a proof that if Σ is a strategy (of a hod mouse, a suitable mouse) with branch condensation and is fullness preserving with respect to mice in some sufficiently closed, determined pointclass Γ or if Σ is the unique strategy of a sound (Y)-mouse for some mouse operator Y that is projecting, uniformly Σ_1, $\mathcal{M}_1^{Y,\sharp}$ generically interprets Y, and condenses finely, then $M_1^{\mathcal{F},\sharp}$ generically interprets $\mathcal{F}$.

Now we are ready to define g-organized $\mathcal{F}$-premice.

Definition 3.38 (Sargsyan, [7]): Let M be a transitive structure. Let $\dot{G}$ be the name for the generic $G \subseteq \mathrm{Col}(\omega, M)$ and let $\dot{x}_{\dot{G}}$ be the canonical name for the real coding $\{(n,m) \mid G(n) \in G(m)\}$, where we identify G with $\bigcup G$. The **tree $\mathcal{T}_M$ for making M generically generic** is the iteration tree $\mathcal{T}$ on $\mathfrak{M}$ of maximal length such that:

(1) $\mathcal{T}$ is via Λ and is everywhere non-dropping.
(2) $\mathcal{T} \restriction o(M)+1$ is the tree given by linearly iterating the first total measure of $\mathfrak{M}$ and its images.
(3) Suppose $\mathrm{lh}(\mathcal{T}) \geq o(M)+2$ and let $\alpha+1 \in (o(M), \mathrm{lh}(\mathcal{T}))$. Let $\delta = \delta^{\mathcal{M}_\alpha^{\mathcal{T}}}$, the Woodin cardinal of $\mathcal{M}_\alpha^{\mathcal{T}}$, and let $\mathbb{B} = \mathbb{B}(M_\alpha^{\mathcal{T}})$ be the extender algebra of $M_\alpha^{\mathcal{T}}$ at δ. Then $E_\alpha^{\mathcal{T}}$ is the extender E with least index in $M_\alpha^{\mathcal{T}}$ such that for some condition $p \in \mathrm{Col}(\omega, M)$, $p \Vdash$"There is a $\mathbb{B}$-axiom induced by E which fails for $\dot{x}_{\dot{G}}$".

Assuming that $\mathfrak{M}$ is sufficiently iterable, then $\mathcal{T}_M$ exists and has successor length.

Sargsyan noticed that one can feed in $\mathcal{F}$ into a structure $\mathcal{N}$ indirectly, by feeding in the branches for $\mathcal{T}_{\mathcal{M}}$, for various $\mathcal{M} \trianglelefteq \mathcal{N}$. The operator ${}^{\mathrm{g}}\mathcal{F}$, defined below, and used in building g-organized $\mathcal{F}$-premice, feeds in branches for such $\mathcal{T}_{\mathcal{M}}$. We will also ensure that being such a structure is first-order — other than wellfoundedness and the correctness of the branches — by allowing sufficient spacing between these branches.

In the following, we let $\mathcal{N}^{\mathcal{T}}$ denote the last model of the tree $\mathcal{T}$.

Definition 3.39: Given a formula Φ. Given a successor length, nowhere dropping tree $\mathcal{T}$ on $\mathfrak{M}$, let $P^{\Phi}(\mathcal{T})$ be the least $P \trianglelefteq \mathcal{N}^{\mathcal{T}}$ such that for some cardinal δ' of $\mathcal{N}^{\mathcal{T}}$, we have $\delta' < o(P)$ and $P \vDash \Phi +$"δ' is Woodin". Let $\lambda = \lambda^{\Phi}(\mathcal{T})$ be least such that $P^{\Phi}(\mathcal{T}) \trianglelefteq M_{\lambda}^{\mathcal{T}}$. Then δ' is a cardinal of $M_{\lambda}^{\mathcal{T}}$. Let $I^{\Phi} = I^{\Phi}(\mathcal{T})$ be the set of limit ordinals $\leq \lambda$.

We can now define the operator used for g-organization:

Definition 3.40 (${}^{\mathrm{g}}\mathcal{F}$): We define the forgetful operator ${}^{\mathrm{g}}\mathcal{F}$, for $\mathcal{F}$ such that $\mathcal{M}_1^{\mathcal{F},\sharp}$ generically interprets $\mathcal{F}$ as witnessed by a pair (Φ, Ψ). Let b be a transitive structure with $\mathfrak{M} \in \mathcal{J}_1(\hat{b})$.

We define $\mathcal{M} = {}^{\mathrm{g}}\mathcal{F}(b)$, a $\mathcal{J}$-model over b, with parameter $\mathfrak{M}$, as follows.

For each $\alpha \leq l(\mathcal{M})$, $E^{\mathcal{M}|\alpha} = \emptyset$.

Let α_0 be the least α such that $\mathcal{J}_{\alpha}(b) \vDash \mathsf{ZF}$. Then $\mathcal{M}|\alpha_0 = \mathcal{J}^{\mathrm{m}}_{\alpha_0}(b; \mathfrak{M})$.

Let $\mathcal{T} = \mathcal{T}_{\mathcal{M}|\alpha_0}$. We use the notation $P^{\Phi} = P^{\Phi}(\mathcal{T})$, $\lambda = \lambda^{\Phi}(\mathcal{T})$, etc., as in 3.39. The predicates $B^{\mathcal{M}|\gamma}$ for $\alpha_0 < \gamma \leq l(\mathcal{M})$ will be used to feed in branches for $\mathcal{T} \restriction \lambda + 1$, and therefore P^{Φ} itself, into $\mathcal{M}$. Let $\langle \xi_{\alpha} \rangle_{\alpha < \iota}$ enumerate $I^{\Phi} \cup \{0\}$.

There is a closed, increasing sequence of ordinals $\langle \eta_{\alpha} \rangle_{\alpha \leq \iota}$ and an increasing sequence of ordinals $\langle \gamma_{\alpha} \rangle_{\alpha \leq \iota}$ such that:

(1) $\eta_1 = \gamma_0 = \eta_0 = \alpha_0$.
(2) For each $\alpha < \iota$, $\eta_{\alpha} \leq \gamma_{\alpha} \leq \eta_{\alpha+1}$, and if $\alpha > 0$ then $\gamma_{\alpha} < \eta_{\alpha+1}$.
(3) $\gamma_{\iota} = l(\mathcal{M})$, so $\mathcal{M} = \mathcal{M}|\gamma_{\iota}$.
(4) Let $\alpha \in (0, \iota)$. Then γ_{α} is the least ordinal of the form $\eta_{\alpha} + \tau$ such that $\mathcal{T} \restriction \xi_{\alpha} \in \mathcal{J}_{\tau}(\mathcal{M}|\eta_{\alpha})$ and if $\alpha > \alpha_0$ then $\delta(\mathcal{T} \restriction \xi_{\alpha}) < \tau$. (We explain below why such τ exists.) And $\mathcal{M}|\gamma_{\alpha} = \mathcal{J}^{\mathrm{m}}_{\tau}(\mathcal{M}|\eta_{\alpha}; \mathfrak{M}) \downarrow b$.
(5) Let $\alpha \in (0, \iota)$. Then $\mathcal{M}|\eta_{\alpha+1} = \mathfrak{B}(\mathcal{M}|\gamma_{\alpha}, \mathcal{T} \restriction \xi_{\alpha}, \Lambda(\mathcal{T} \restriction \xi_{\alpha})) \downarrow b$.
(6) Let $\alpha < \iota$ be a limit. Then $\mathcal{M}|\eta_{\alpha}$ is passive.
(7) γ_{ι} is the least ordinal of the form $\eta_{\iota} + \tau$ such that $\mathcal{T} \restriction \lambda + 1 \in \mathcal{J}_{\eta_{\iota}+\tau}(\mathcal{M}|\eta_{\iota})$ and $\tau > o(M_{\lambda}^{\mathcal{T}})$; with this τ, $\mathcal{M} = \mathcal{J}^{\mathrm{m}}_{\tau}(\mathcal{M}|\eta_{\iota}; \mathfrak{M}) \downarrow b$

and furthermore, ${}^{g}\mathcal{F}(b)$ is acceptable and every strict segment of ${}^{g}\mathcal{F}(b)$ is sound.

Remark 3.41: It's not hard to see (cf. [13]) $\bar{\mathcal{M}} \trianglelefteq \mathcal{M} = {}^{\mathrm{g}}\mathcal{F}(b)$, the sequences $\langle \mathcal{M}|\eta_\alpha\rangle_{\alpha\leq\iota} \cap \bar{\mathcal{M}}$ and $\langle \mathcal{M}|\gamma_\alpha\rangle_{\alpha\leq\iota} \cap \bar{\mathcal{M}}$ and $\langle \mathcal{T} \restriction \alpha\rangle_{\alpha\leq\lambda+1} \cap \bar{\mathcal{M}}$ are $\Sigma_1^{\bar{\mathcal{M}}}$ in $\mathcal{L}_0^-$, uniformly in b and $\bar{\mathcal{M}}$.

Definition 3.42: Let b be transitive with $\mathfrak{M} \in \mathcal{J}_1(\hat{b})$. A **potential g-organized $\mathcal{F}$-premouse over** b is a potential ${}^{\mathrm{g}}\mathcal{F}$-premouse over b, with parameter $\mathfrak{M}$.

Lemma 3.43: *There is a formula φ_{g} in $\mathcal{L}_0$, such that for any transitive b with $\mathfrak{M} \in \mathcal{J}_1(\hat{b})$, and any $\mathcal{J}$-structure $\mathcal{M}$ over b, $\mathcal{M}$ is a potential g-organized $\mathcal{F}$-premouse over b iff $\mathcal{M}$ is a potential $\Lambda_{\mathfrak{M}}$-premouse over b, of type φ_{g}.*

Lemma 3.44: *${}^{\mathrm{g}}\mathcal{F}$ is projecting, uniformly Σ_1, basic, and condenses finely.*

Definition 3.45: Let $\mathcal{M}$ be a g-organized $\mathcal{F}$-premouse over b. We say $\mathcal{M}$ is **$\mathcal{F}$-closed** iff $\mathcal{M}$ is a limit of ${}^{\mathrm{g}}\mathcal{F}$-whole proper segments.

Because $\mathcal{M}_1^{\mathcal{F},\sharp}$ generically interprets $\mathcal{F}$, $\mathcal{F}$-closure ensures closure under $\mathcal{G}_{\mathcal{F}}$:

Lemma 3.46: *Let $\mathcal{M}$ be an $\mathcal{F}$-closed g-organized $\mathcal{F}$-premouse over b. Then $\mathcal{M}$ is closed under $\mathcal{G}_{\mathcal{F}}$. In fact, for any set generic extension $\mathcal{M}[g]$ of $\mathcal{M}$, with $g \in V$, $\mathcal{M}[g]$ is closed under $\mathcal{G}_{\mathcal{F}}$ and $\mathcal{G}_{\mathcal{F}} \restriction \mathcal{M}[g]$ is definable over $\mathcal{M}[g]$, via a formula in $\mathcal{L}_0^-$, uniformly in $\mathcal{M}, g$.*

The analysis of scales in $\mathrm{Lp}^{{}^{\mathrm{g}}\mathcal{F}}(\mathbb{R})$ runs into a problem (see [13, Remark 6.8] for an explanation). Therefore we will analyze scales in a slightly different hierarchy.

Definition 3.47: Let $X \subseteq \mathbb{R}$. We say that X is **self-scaled** iff there are scales on X and $\mathbb{R}\backslash X$ which are analytical (i.e., Σ^1_n for some $n < \omega$) in X.

Definition 3.48: Let b be transitive with $\mathfrak{M} \in \mathcal{J}_1(\hat{b})$.

Then ${}^{\mathsf{G}}\mathcal{F}(b)$ denotes the least $\mathcal{N} \trianglelefteq {}^{\mathrm{g}}\mathcal{F}(b)$ such that either $\mathcal{N} = {}^{\mathrm{g}}\mathcal{F}(b)$ or $\mathcal{J}_1(\mathcal{N}) \vDash$"$\Theta$ does not exist". (Therefore $\mathcal{J}_1^{\mathrm{m}}(b;\mathfrak{M}) \trianglelefteq {}^{\mathsf{G}}\mathcal{F}(b)$.)

We say that $\mathcal{M}$ is a **potential Θ-g-organized $\mathcal{F}$-premouse over** X iff $\mathfrak{M} \in \mathrm{HC}^{\mathcal{M}}$ and for some $X \subseteq \mathrm{HC}^{\mathcal{M}}$, $\mathcal{M}$ is a potential ${}^{\mathsf{G}}\mathcal{F}$-premouse over $(\mathrm{HC}^{\mathcal{M}}, X)$ with parameter $\mathfrak{M}$ and $\mathcal{M} \vDash$"X is self-scaled". We write $X^{\mathcal{M}} = X$.

In our application to core model induction, we will be most interested in the cases that either $X = \emptyset$ or $X = \mathcal{F} \restriction \mathrm{HC}^{\mathcal{M}}$. Clearly Θ-g-organized $\mathcal{F}$-premousehood is not first order. Certain aspects of the definition, however, are:

Definition 3.49: Let "I am a Θ-g-organized premouse over X" be the $\mathcal{L}_0$ formula ψ such that for all $\mathcal{J}$-structures $\mathcal{M}$ and $X \in \mathcal{M}$ we have $\mathcal{M} \models \psi(X)$ iff (i) $X \subseteq \mathrm{HC}^{\mathcal{M}}$; (ii) $\mathcal{M}$ is a $\mathcal{J}$-model over $(\mathrm{HC}^{\mathcal{M}}, X)$; (iii) $\mathcal{M}|1 \models$"X is self-scaled"; (iv) every proper segment of $\mathcal{M}$ is sound; and (v) for every $\mathcal{N} \trianglelefteq \mathcal{M}$:

(1) if $\mathcal{N} \models$"Θ exists" then $\mathcal{N} \downarrow (\mathcal{N}|\Theta^{\mathcal{N}})$ is a $\mathfrak{P}^{\mathcal{N}}$-strategy premouse of type φ_{g};
(2) if $\mathcal{N} \models$"Θ does not exist" then $\mathcal{N}$ is passive.

LEMMA 3.50: *Let $\mathcal{M}$ be a $\mathcal{J}$-structure and $X \in \mathcal{M}$. Then the following are equivalent:* (i) *$\mathcal{M}$ is a Θ-g-organized $\mathcal{F}$-premouse over X;* (ii) *$\mathcal{M} \models$"I am a Θ-g-organized premouse over X" and $\mathfrak{P}^{\mathcal{M}} = \mathfrak{M}$ and $\Sigma^{\mathcal{M}} \subseteq \Lambda_{\mathfrak{M}}$;* (iii) *$\mathcal{M}|1$ is a Θ-g-organized premouse over X and every proper segment of $\mathcal{M}$ is sound and for every $\mathcal{N} \trianglelefteq \mathcal{M}$,*

(1) *if $\mathcal{N} \models$"Θ exists" then $\mathcal{N} \downarrow (\mathcal{N}|\Theta^{\mathcal{N}})$ is a g-organized $\mathcal{F}$-premouse;*
(2) *if $\mathcal{N} \models$"Θ does not exist" then $\mathcal{N}$ is passive.*

LEMMA 3.51: *$\mathsf{G}\mathcal{F}$ is basic and condenses finely.*

Definition 3.52: Suppose $\mathcal{F}$ is a nice operator and is an iteration strategy and $X \subseteq \mathbb{R}$ is self-scaled. We define $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, X)$ as the stack of all Θ-g-organized $\mathcal{F}$-mice $\mathcal{N}$ over (H_{ω_1}, X) (with parameter $\mathfrak{M}$). We also say (Θ-g-organized) $\mathcal{F}$-premouse **over** $\mathbb{R}$ to in fact mean over H_{ω_1}.

Remark 3.53: It's not hard to see that for any such X as in Definition 3.52, $\wp(\mathbb{R}) \cap \mathrm{Lp}^{\mathsf{g}\mathcal{F}}(\mathbb{R}, X) = \wp(\mathbb{R}) \cap \mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, X)$. Suppose $\mathcal{M}$ is an initial segment of the first hierarchy and $\mathcal{M}$ is E-active. Note that $\mathcal{M} \models$ "Θ exists" and $\mathcal{M}|\Theta$ is $\mathcal{F}$-closed. By induction below $\mathcal{M}|\Theta^{\mathcal{M}}$, $\mathcal{M}|\Theta^{\mathcal{M}}$ can be rearranged into an initial segment $\mathcal{N}'$ of the second hierarchy. Above $\Theta^{\mathcal{M}}$, we simply copy the E- and B-sequence from $\mathcal{M}$ over to obtain an $\mathcal{N} \triangleleft \mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, X)$ extending $\mathcal{N}'$.

In core model induction applications, we often have a pair $(\mathcal{P}, \Sigma)$ where $\mathcal{P}$ is a hod premouse and Σ is $\mathcal{P}$'s strategy with branch condensation and is fullness preserving (relative to mice in some pointclass) or $\mathcal{P}$ is a sound (hybrid)

premouse projecting to some countable set a and Σ is the unique (normal) ω_1+1-strategy for $\mathcal{P}$. Let $\mathcal{F}$ be the operator corresponding to Σ (using the formula $\varphi_{\rm all}$) and suppose $\mathcal{M}_1^{\mathcal{F},\sharp}$ exists. [13, Lemma 4.8] shows that $\mathcal{F}$ condenses finely and $\mathcal{M}_1^{\mathcal{F},\sharp}$ generically interprets $\mathcal{F}$. Also, the core model induction will give us that $\mathcal{F}\restriction\mathbb{R}$ is self-scaled.[22] Thus, we can define $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R},\mathcal{F}\restriction\mathbb{R})$ as above (assuming sufficient iterability of $\mathcal{M}_1^{\mathcal{F},\sharp}$). A core model induction is then used to prove that $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R},\mathcal{F}\restriction\mathbb{R})\models\mathsf{AD}^+$. What's needed to prove this is the scales analysis of $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R},\mathcal{F}\restriction\mathbb{R})$, from the optimal hypothesis (similar to those used by Steel; see [19] and [20]).[23] This is carried out in [13]; we will not go into details here, though we simply note that for the scales analysis to go through under optimal hypotheses, we need to work with the Θ-g-organized hierarchy, instead of the g-organized hiearchy.

3.1.5. *Brief remarks on S-constructions.* Suppose $\mathcal{F}$ is a nice operator (with parameter $\mathfrak{P}$) and suppose $\mathcal{M}$ is a $\mathcal{G}$-mouse (over some transitive a), where $\mathcal{G}$ is either ${}^{\mathrm{g}}\mathcal{F}$ or ${}^{\mathsf{G}}\mathcal{F}$. Suppose δ is a cutpoint of $\mathcal{M}$ and suppose $\mathcal{N}$ is a transitive structure such that $\delta\subseteq\mathcal{N}\subseteq\mathcal{M}|\delta$, $\mathfrak{P}\in\mathcal{N}$. Suppose $\mathbb{P}\in\mathcal{J}_\omega[\mathcal{N}]$ is such that $\mathcal{M}|\delta$ is $\mathbb{P}$-generic over $\mathcal{J}_\omega[\mathcal{N}]$ and suppose whenever $\mathcal{Q}$ is a $\mathcal{G}$-mouse over $\mathcal{N}$ such that $H^{\mathcal{Q}}_\delta=\mathcal{N}$ then $\mathcal{M}|\delta$ is $\mathbb{P}$-generic over $\mathcal{Q}$. Then the S-constructions (or P-constructions) from [12] give a $\mathcal{G}$-mouse $\mathcal{R}$ over $\mathcal{N}$ such that $\mathcal{R}[\mathcal{M}|\delta]=\mathcal{M}$. The S-constructions give the sequence $(\mathcal{R}_\alpha:\delta<\alpha\leq\lambda)$ of $\mathcal{G}$-premice over $\mathcal{N}$, where

(i) $\mathcal{R}_{\delta+1}=\mathcal{J}^{\mathrm{m}}_\omega(\mathcal{N})$;

(ii) if α is limit, then let $\mathcal{R}^*_\alpha=\bigcup_{\beta<\alpha}\mathcal{R}_\beta$. If $\mathcal{M}|\alpha$ is passive, then let $\mathcal{R}_\alpha=\mathcal{R}^*_\alpha$. So $\mathcal{R}_\alpha$ is passive. If $B^{\mathcal{M}|\alpha}\neq\emptyset$, then let

$$\mathcal{R}_\alpha=(|\mathcal{R}^*_\alpha|;\emptyset,B^{\mathcal{M}|\alpha},\bigcup_{\beta<\alpha}S^{\mathcal{R}_\beta},\mathcal{N},\mathfrak{P}).$$

[22] We abuse notation here, and will continue to do so in the future. Technically, we should write $\mathcal{F}\restriction\mathrm{HC}$.

[23] Suppose $\mathcal{P}=\mathcal{M}_1^\sharp$ and Σ is $\mathcal{P}$'s unique iteration strategy. Let $\mathcal{F}$ be the operator corresponding to Σ. Suppose $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R},\mathcal{F}\restriction\mathbb{R})\models\mathsf{AD}^+ + \mathsf{MC}$. Then in fact $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R})\cap\wp(\mathbb{R})=\mathrm{Lp}(\mathbb{R})\cap\wp(\mathbb{R})$. This is because in $L(\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R},\mathcal{F}\restriction\mathbb{R}))$, $L(\wp(\mathbb{R}))\models\mathsf{AD}^+ + \Theta=\theta_0+\mathsf{MC}$ and hence by [9], in $L(\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R},\mathcal{F}\restriction\mathbb{R}))$, $\wp(\mathbb{R})\subseteq\mathrm{Lp}(\mathbb{R})$. Therefore, even though the hierarchies $\mathrm{Lp}(\mathbb{R})$ and $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R},\mathcal{F}\restriction\mathbb{R})$ are different, as far as sets of reals are concerned, we don't lose any information by analyzing the scales pattern in $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R},\mathcal{F}\restriction\mathbb{R})$ instead of that in $\mathrm{Lp}(\mathbb{R})$.

Suppose $E^{\mathcal{M}|\alpha} \neq \emptyset$; let $E^* = E^{\mathcal{M}|\alpha} \cap |\mathcal{R}^*_\alpha|$, then we let

$$\mathcal{R}_\alpha = (|\mathcal{R}^*_\alpha|; E^*, \emptyset, \bigcup_{\beta<\alpha} S^{\mathcal{R}_\beta}, \mathcal{N}, \mathfrak{P}).$$

By the hypothesis, we have $\mathcal{R}_\alpha[\mathcal{M}|\delta] = \mathcal{M}|\alpha$.

(iii) Suppose we have already constructed $\mathcal{R}_\alpha$ and (by the hypothesis) maintain that $\mathcal{R}_\alpha[\mathcal{M}|\delta] = \mathcal{M}|\alpha$. Then $\mathcal{R}_{\alpha+1} = \mathcal{J}^{\mathrm{m}}_\omega(\mathcal{R}_\alpha)$.

(iv) λ is such that $\mathcal{R}_\lambda[\mathcal{M}|\delta] = \mathcal{M}$. We set $\mathcal{R}_\lambda = \mathcal{R}$.

We note that the full construction from [12] does not require that δ is a cutpoint of $\mathcal{M}$ but we don't need the full power of the S-constructions in our paper. Also, the fact that $\mathcal{M}$ is g-organized (or Θ-g-organized) is important for our constructions above because it allows us to get past levels $\mathcal{M}|\alpha$ for which $B^{\mathcal{M}|\alpha} \neq \emptyset$. Because of this fact, in this paper, hod mice are reorganized into the g-organized hierarchy, that is if $\mathcal{P}$ is a hod mouse then $\mathcal{P}(\alpha+1)$ is a g-organized $\Sigma_{\mathcal{P}(\alpha)}$-premouse for all $\alpha < \lambda^{\mathcal{P}}$. The S-constructions are also important in many other contexts. One such context is the local HOD analysis of levels of $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$, which features in the scales analysis of $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$ (cf. [13]).

3.1.6. *Core model induction operators.* To analyze Ω, we adapt the framework for the core model induction developed above and the scales analysis in [13], [20], and [19]. We are now in a position to introduce the core model induction operators that we will need in this paper. These are particular kinds of (hybrid) mouse operators that are constructed during the course of the core model induction. These operators can be shown to satisfy the sort of condensation described above and determine themselves on generic extensions.

Suppose $\mathcal{F}$ is a nice operator and Γ is an inductive-like pointclass that is determined. Let $\mathfrak{M} = \mathcal{M}_1^{\mathcal{F},\sharp}$. $\mathrm{Lp}^{g\mathcal{F}}(x)$ is defined as in the previous section. We write $\mathrm{Lp}^{g\mathcal{F},\Gamma}(x)$ for the stack of sound, projecting to x $g\mathcal{F}$-premice $\mathcal{M}$ over x such that every countable, transitive $\mathcal{M}^*$ embeddable into $\mathcal{M}$ has an ω_1-$g\mathcal{F}$-iteration strategy in Γ.

Definition 3.54: Let $t \in \mathrm{HC}$ with $\mathfrak{M} \in \mathcal{J}_1(t)$. Let $1 \leq k < \omega$. A premouse $\mathcal{N}$ over t is **$\mathcal{F}$-Γ-k-suitable** (or just **k-suitable** if Γ and $\mathcal{F}$ are clear from the context) iff there is a strictly increasing sequence $\langle \delta_i \rangle_{i<k}$ such that:

(1) $\forall \delta \in \mathcal{N}$, $\mathcal{N} \vDash$"δ is Woodin" if and only if $\exists i < k(\delta = \delta_i)$.

(2) $o(\mathcal{N}) = \sup_{i<\omega}(\delta_{k-1}^{+i})^{\mathcal{N}}$.

(3) If $\mathcal{N}|\eta$ is a ${}^{\mathrm{g}}\mathcal{F}$-whole strong cutpoint of $\mathcal{N}$ then $\mathcal{N}|(\eta^+)^{\mathcal{N}} = \mathrm{Lp}^{{}^{g}\mathcal{F},\Gamma}(\mathcal{N}|\eta)$.[24]

(4) Let $\xi < o(\mathcal{N})$, where $\mathcal{N} \vDash$"ξ is not Woodin". Then $C_\Gamma(\mathcal{N}|\xi) \vDash$"$\xi$ is not Woodin".

We write $\delta_i^{\mathcal{N}} = \delta_i$; also let $\delta_{-1}^{\mathcal{N}} = 0$ and $\delta_k^{\mathcal{N}} = o(\mathcal{N})$.

Definition 3.55 (relativizes well): Let $\mathcal{F}$ be a Y-mouse operator for some operator Y. We say that $\mathcal{F}$ **relativizes well** if there is a formula $\phi(x, y, z)$ such that for any $a, b \in \mathrm{dom}(\mathcal{F})$ such that $a \in L_1(b)$ and have the same cardinality, whenever N is a transitive model of ZFC^- such that N is closed under Y, $\mathcal{F}(b) \in N$, then $\mathcal{F}(a) \in N$ and is the unique $x \in N$ such that $N \vDash \phi[x, a, \mathcal{F}(b)]$.

Definition 3.56 (determines itself on generic extensions): Suppose $\mathcal{F}$ is a Y-mouse operator for some operator Y. We say that $\mathcal{F}$ **determines itself on generic extensions** if there is a formula $\phi(x, y, z)$, a parameter a such that for almost all transitive structures N of ZFC^- such that $\omega_1 \subset N$, N contains a and is closed under $\mathcal{F}$, for any generic extension $N[g]$ of N in V, $\mathcal{F} \cap N[g] \in N[g]$ and is definable over $N[g]$ via (ϕ, a), i.e., for any $x \in N[g] \cap \mathrm{dom}(\mathcal{F})$, $\mathcal{F}(a) = b$ if and only if b is the unique $c \in N[g]$ such that $N[g] \vDash \phi[x, c, a]$.[25]

The following definition gives examples of "nice model operators". This is not a standard definition and is given here for convenience more than anything. These are the kind of model operators that the core model induction in this paper deals with. We by no means claim that these operators are all the useful model operators that one might consider. Recall we fixed a V-generic $G \subseteq Col(\omega, \kappa)$.

Definition 3.57 (Core model induction operators): Suppose $(\mathcal{P}, \Sigma)$ is a hod pair below κ; assume furthermore that Σ is a (λ^+, λ^+)-strategy. Let $\mathcal{F} = \mathcal{F}_{\Sigma,\varphi_{\mathrm{all}}}$ (note that $\mathcal{F}$, ${}^{g}\mathcal{F}$ are basic, projecting, uniformly Σ_1, and condenses finely). Assume $\mathcal{F} \restriction \mathbb{R}$ is self-scaled. We say J is a Σ core model induction operator or just a Σ-cmi operator if in $V[G]$, one of the following holds:

[24] Literally we should write "$\mathcal{N}|(\eta^+)^{\mathcal{N}} = \mathrm{Lp}^{\Gamma}(\mathcal{N}|\eta) \downarrow t$", but we will be lax about this from now on.

[25] By "almost all", we mean for all such N with the properties listed above and N satisfies some additional property. In practice, this additional property is: N is closed under $\mathcal{M}_1^{\mathcal{F},\sharp}$.

(1) J is a projecting, uniformly Σ_1, first order $\mathcal{F}$-mouse operator (or ${}^{\mathrm{g}}\mathcal{F}$-mouse operator) defined on a cone of $(H_{\omega_1})^{V[G]}$ above some $a \in (H_{\omega_1})^{V[G]}$. Furthermore, J relativizes well.

(2) For some $\alpha \in \mathrm{OR}$ such that α ends either a weak or a strong gap in the sense of [19] and [13], letting $M = \mathrm{Lp}^{{}^{\mathrm{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})||\alpha$ and $\Gamma = (\Sigma_1)^M$, $M \vDash \mathsf{AD}^+ + \mathsf{MC}(\Sigma)$.[26] For some transitive $b \in H^{V[G]}_{\omega_1}$ and some g-organized $\mathcal{F}$-premouse $\mathcal{Q}$ over b, $J = \mathcal{F}_\Lambda$, where Λ is an (ω_1, ω_1)-iteration strategy for a 1-suitable (or more fully $\mathcal{F}$-Γ-1-suitable) $\mathcal{Q}$ which is Γ-fullness preserving, has branch condensation and is guided by some self-justifying-system (sjs) $\vec{A} = (A_i : i < \omega)$ such that $\vec{A} \in OD^M_{b,\Sigma,x}$ for some real x and $\vec{A}$ seals the gap that ends at α.[27]

Remark 3.58: 1) The Σ-cmi operators J we construct in this paper also determine themselves on generic extensions. If J is defined as in (1) and determines itself on generic extensions, then so does the "next operator" $\mathcal{M}_1^{J,\sharp}$. If J is defined as in (2), then [13] shows that $M_1^{J,\sharp}$ generically interprets J; from this, the proof of Lemma 3.46 (see [13, Lemma 4.21]) shows that J determines itself on generic extensions.

2) Suppose J is defined on a cone over $(H_{\omega_1})^{V[G]}$ above some transitive $a \in H^V_{\kappa^+}$ and $J \restriction V \in V$. During the course of construction, we show that knowing J on V is sufficient to determine J on $V[G]$. During the course of the core model induction, we'll be first constructing these Σ-cmi operators J's on $H^V_{\kappa^+}$ (above some a); then we show how to extend J to $HC^{V[G]}$; we then lift J to $H^V_{\lambda^+}$, which then extends J to $H^{V[G]}_{\lambda^+}$.

3) By results in [7], under (†), if $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation, then Σ has hull condensation. The same is true for $(\mathcal{Q}, J)$ in Definition 3.57. This implies that Σ (J) is suitably condensing.

3.2. Getting $M_1^{J,\sharp}$ and lifting. We assume the hypothesis of Theorem 0.2. We fix a V-generic $G \subseteq Col(\omega, \kappa)$ and recall that we let $\lambda = 2^\kappa$. Suppose $(\mathcal{P}^*, \Sigma)$ is a hod pair below κ such that Σ is an (κ^+, κ^+)-strategy in $V[G]$ and

[26] $\mathsf{MC}(\Sigma)$ stands for the Mouse Capturing relative to Σ which says that for $x, y \in \mathbb{R}$, x is $\mathrm{OD}(\Sigma, y)$ (or equivalently x is $\mathrm{OD}(\mathcal{F}, y)$) iff x is in some ${}^{\mathrm{g}}\mathcal{F}$-mouse over y. SMC is the statement that for every hod pair $(\mathcal{P}, \Sigma)$ such that Σ is fullness preserving and has branch condensation, then $\mathsf{MC}(\Sigma)$ holds.

[27] This implies that $\vec{A}$ is Wadge cofinal in $\mathbf{Env}(\mathbf{\Gamma})$, where $\Gamma = \Sigma_1^M$. Note that $\mathbf{Env}(\mathbf{\Gamma}) = \wp(\mathbb{R})^{\mathbf{M}}$ if α ends a weak gap and $\mathbf{Env}(\mathbf{\Gamma}) = \wp(\mathbb{R})^{\mathrm{Lp}^{\mathbf{\Sigma}}(\mathbb{R})|(\alpha+\mathbf{1})}$ if α ends a strong gap.

$\Sigma \restriction V \in V$ (or $(\mathcal{P}^*, \Sigma) = (\emptyset, \emptyset)$). Suppose J is a Σ-cmi-operator. As part of the induction, we assume J is defined on a cone in $H^{V[G]}_{\lambda^+}$ above some $x \in H^V_{\kappa^+}$ and $J \restriction V \in V$.[28] We first show $\mathcal{M}_1^{J,\sharp}(a)$ exists (and is (κ^+, κ^+)-iterable) for $a \in H^V_{\kappa^+} \cap \mathrm{dom}(J)$. We then show that $\mathcal{M}_1^{J,\sharp}(a)$ is defined on $H^V_{\lambda^+}$ and $\mathcal{M}_1^{J,\sharp}$ is (λ^+, λ^+)-iterable for all $a \in H^V_{\lambda^+}$. Finally, we get that $\mathcal{M}_1^{J,\sharp}$ is a Σ-cmi operator defined on a cone in $H^{V[G]}_{\lambda^+}$.

Let $\mathcal{F} = \mathcal{F}_{\Sigma,\varphi_{\mathrm{all}}}$. Let A code $\mathcal{P}^*$ and $\mathrm{Lp}^{\mathcal{F}}_*(A)$ be the union of all $\mathcal{N}$ such that $\mathcal{N}$ is ω-sound above A, $\mathcal{N}$ is a countably iterable Σ-premouse over A and $\rho_\omega(\mathcal{N}) \leq \sup(A)$. This means whenever $\pi : \mathcal{N}^* \to \mathcal{N}$ is elementary, $\mathcal{N}^*$ is countable, transitive, then $\mathcal{N}^*$ is $(\omega, \omega_1 + 1)$ iterable via a unique strategy Λ such that whenever $\mathcal{M}$ is a Λ-iterate of $\mathcal{N}^*$, then $\mathcal{M}$ is a Σ^π-premouse. As a matter of notations, in V, for A a bounded subset of $(\lambda^+)^V$, we set

$$\mathrm{Lp}^{\Sigma}_1(A) = \mathrm{Lp}^{\mathcal{F}}_*(A).$$

Suppose $\mathrm{Lp}^{\Sigma}_\alpha(A)$ has been defined for $\alpha < \lambda^+$,

$$\mathrm{Lp}^{\Sigma}_{\alpha+1}(A) = \mathrm{Lp}^{\mathcal{F}}_{*,+}(\mathrm{Lp}^{\Sigma}_\alpha(A)),^{29}$$

amd for $\xi < \kappa^+$ limit,

$$\mathrm{Lp}^{\Sigma}_{\alpha+1}(A) = \bigcup_{\alpha<\xi} \mathrm{Lp}^{\Sigma}_\alpha(A).$$

We define $\mathrm{Lp}^{\Sigma}_\xi(A)$ and $\mathrm{Lp}^{\mathfrak{C}\Sigma}_\lambda(A)$ similarly for $\xi \leq \lambda^+$, in the presence of $\mathcal{M}_1^{\Sigma,\sharp}$. We also write $\mathrm{Lp}^{\Sigma}(A)$ for $\mathrm{Lp}^{\mathcal{F}}_*(A)$ and similarly for $\mathrm{Lp}^{\mathfrak{g}\Sigma}(A)$ and $\mathrm{Lp}^{\mathfrak{C}\Sigma}(A)$. We work in V for a while.

LEMMA 3.59: *Let A be a subset of λ coding $\mathcal{P}^*$. Then $Lp^{\Sigma}_{\lambda^+}(A) \vDash \lambda^+$ exists. Similarly, $Lp^{\mathfrak{g}\Sigma}_{\lambda^+}(A)$, $Lp^{\mathfrak{C}\Sigma}_{\lambda^+}(A) \vDash \lambda^+$ exists.*

Proof. Suppose not. This easily implies that we can construct over $\mathrm{Lp}^{\Sigma}_{\lambda^+}(A)$ a $\square_\lambda$-sequence.[30] This contradicts $\neg\square_\lambda$ in V. ∎

The following gives the main consequence of the failures of squares assumption. It allows us to run covering arguments later.

[28] We note the specific requirement that the cone over which J is defined is above some $x \in V$. These are the Σ-cmi-operators that we will propagate in our core model induction. We will not deal with all Σ-cmi-operators.

[29] $\mathrm{Lp}^{\mathcal{F}}_{*,+}(\mathrm{Lp}^{\Sigma}_\alpha(A))$ is defined similarly to $\mathrm{Lp}^{\mathcal{F}}_*$ but here we stack continuing, $\mathcal{F}$-sound $\mathcal{F}$-premice. One can show by induction on α that $Lp^{\mathcal{F}}_{*,+}(Lp^{\sigma}_\alpha(A)) = Lp^{\mathcal{F}}_*(Lp^{\Sigma}_\alpha(A))$.

[30] Squares hold in $\mathrm{Lp}^{\Sigma}_{\kappa^+}(A)$ because Σ has hull and branch condensation.

LEMMA 3.60: *Let A, Σ be as in 3.59. Let $M \in \{Lp^{\Sigma}_{\lambda^+}(A), Lp^{g\Sigma}_{\lambda^+}(A), Lp^{\mathsf{G}\Sigma}_{\lambda^+}(A)\}$ and $\gamma = (\lambda^+)^M$. Then $cof(\gamma) \leq \kappa$.*

Proof. First note that $\gamma < \lambda^+$ by Lemma 3.59. Now suppose $\mathrm{cof}(\gamma) = \xi$ for some regular cardinal $\xi \in [\kappa^+, \lambda]$. Let $f : \xi \to \gamma$ be cofinal and continuous. Using f and $\square_\lambda$ in M, by a standard argument (see [10]) we can construct a non-threadable sequence of length ξ.[31] This contradicts $\neg\square(\xi)$. ∎

Let S be the set of $X \prec H_{\lambda^{++}}$ such that $\kappa \subseteq X$, $|X| = \kappa$, $X^\omega \subseteq X$, and X is cofinal in the ordinal height of $\mathrm{Lp}^{\Sigma}(B)$, $\mathrm{Lp}^{\mathsf{G}\Sigma}(B)$ and $J, (\mathcal{P}^* \cup \{\mathcal{P}^*\}, \Sigma) \in X$.[32] So S is stationary. As before, we let $\pi_X : M_X \to H_{\lambda^{++}}$ be the uncollapsed map and λ_X be the critical point of π_X. We first prove some lemmas about "lifting" operators. In the following, when we write "$\mathrm{Lp}^{\mathsf{G}\mathcal{F}}$", we implicitly assume $\mathcal{M}_1^{\mathcal{F},\sharp}$ exists and is (λ^+, λ^+)-iterable. We will prove this at the end of the section.

LEMMA 3.61: *Suppose $A^* \subseteq \lambda$. Suppose $X \in S$ such that $A^* \in X$ and X is cofinal in $Lp^{\Sigma}(A^*)$ (there are stationary many such X because $cof(o(Lp^{\Sigma}(A^*))) \leq \kappa$ by 3.60). Let $\pi_X(A) = A^*$. Then $Lp^{\Sigma}(A) \subseteq M_X$. The same conclusion holds if we replace $Lp^{\Sigma}(A)$ by $Lp^{\mathsf{G}\Sigma}(A)$ or $Lp^{g\Sigma}(A)$.*

Proof. We just prove the first clause. Suppose not. Then let $\mathcal{M} \lhd \mathrm{Lp}^{\Sigma}(A)$ be the least counterexample. Let E be the (λ_X, λ)-extender derived from π_X. Let $\mathcal{N} = \mathrm{Ult}(\mathcal{M}, E)$. Then any countable transitive $\mathcal{N}^*$ embeddable into $\mathcal{N}$ (via σ) is embeddable into $\mathcal{M}$ (via τ) such that $i_E \circ \tau = \sigma$ by countable completeness of E. So $\mathcal{N}^*$ is $\omega_1 + 1$ Σ^σ-iterable because $\mathcal{M} \lhd \mathrm{Lp}^{\Sigma}(A)$, $\sigma^{-1}(\mathcal{P}^*) = \tau^{-1}(\mathcal{P}^*)$, and $\sigma \restriction \sigma^{-1}(\mathcal{P}^*) = \tau \restriction \sigma^{-1}(\mathcal{P}^*)$. So $\mathcal{N} \lhd \mathrm{Lp}^{\Sigma}(A^*)$. But since π_X is cofinal in $\mathrm{Lp}^{\Sigma}(A^*)$, $\mathcal{N} \notin \mathrm{Lp}^{\Sigma}(A^*)$. Contradiction. ∎

LEMMA 3.62: 1) *If H is defined by (ψ, a) on $H^{V[G]}_{\omega_1}$ (as in clause 1 of 3.57) with $a \in V$ and $H \restriction V \in V$, then H can be extended to an operator H^+ defined by (ψ, a) on H_{λ^+}. Furthermore, H^+ relativizes well.*

2) *If $(\mathcal{Q}, F)$ and Γ are as in clause 2 of Definition 3.57, where F plays the role of Λ there with $(\mathcal{Q}, F \restriction V) \in V$, then F can be extended to a (λ^+, λ^+)-strategy that has branch condensation. Furthermore, there is a unique such extension.*

[31] A thread will allows us to construct a Σ-mouse projecting to A and extends $M|\gamma$ but not in M. This is a contradiction to the definition of M.

[32] This means $(\mathcal{P}^*, \Sigma \restriction V) \in X$ and $\Sigma \in X[G]$ but we will abuse notation here.

Proof. To prove 1), first let A^* be a bounded subset of λ^+ (in the cone above a) and let $X \in S$ such that $A^* \in X$ and X is cofinal in $\mathrm{Lp}^{\Sigma}(A^*)$. Let $\pi_X(A) = A^*$. We assume H is an $\mathcal{F}$-mouse operator. By Lemma 3.61, $H(A) \in M_X$ and hence we can define $H^+(A^*) = \pi_X(H(A))$ (as the first level $\mathcal{M} \triangleleft \mathrm{Lp}^{\Sigma}(A^*)$ that satisfies $\psi[A^*, a]$). This defines H^+ on all bounded subsets of κ^+. The same proof works for H being a ${}^{\mathrm{g}}\mathcal{F}$-mouse operator. We can then define H^+ on all of H_{κ^+} using the fact that H relativizes well and $|H_\kappa| = \kappa$. It's easy to see then that H^+ also relativizes well.

We first prove the "uniqueness" clause of 2). Suppose F_1 and F_2 are two extensions of F and let $\mathcal{T}$ be according to both F_1 and F_2. Let $b_1 = F_1(\mathcal{T})$ and $b_2 = F_2(\mathcal{T})$. If $b_1 \neq b_2$ then $\mathrm{cof}(lh(\mathcal{T})) = \omega$. So letting $\mathcal{T}^*$ be a hull of $\mathcal{T}$ such that $|\mathcal{T}^*| \leq \omega_2$ and letting $\pi : \mathcal{T}^* \to \mathcal{T}$ be the hull embedding, then $\{b_1, b_2\} \in \mathrm{rng}(\pi)$. Then $\pi^{-1}[b_1] = F(\mathcal{T}^*) \neq \pi^{-1}[b_2] = F(\mathcal{T}^*)$. Contradiction.

To show existence, let $F_{\kappa^+} = F$. Inductively for each $\kappa^+ \leq \xi < \lambda^+$ such that ξ is a limit ordinal, we define a strategy F_ξ extending F_α for $\alpha < \xi$ and F_ξ acts on trees of length ξ. For $X \prec Y \prec H_{\lambda^{++}}$, let $\pi_{X,Y} = \pi_Y^{-1} \circ \pi_X$. Let $\mathcal{T}$ be a tree of length ξ such that for every limit $\xi^* < \xi$, $\mathcal{T} \restriction \xi^*$ is according to F_{ξ^*}. We want to define $F_\xi(\mathcal{T})$.

For $X \in S$ such that X is cofinal in $\mathrm{Lp}^{\mathrm{g}\Sigma}_{*,+}(\mathcal{M}(\mathcal{T}))$ (such an X exists by the proof of 3.60 again),[33] let $(\mathcal{T}_X, \xi_X) = \pi_X^{-1}(\mathcal{T}, \xi)$ and $b_X = F(\mathcal{T}_X)$. Let c_X be the downward closure of $\pi_X[b_X]$ and $c_{X,Y}$ be the downward closure of $\pi_{X,Y}[b_X]$.

Claim: *For all $\gamma < \xi$, either $\forall^* X \in S$ [34] $\gamma \in c_X$ or $\forall^* X \in S$ $\gamma \notin c_X$.*

Proof. The proof is similar to that of Lemma 2.5 in [17] so we only sketch it here. Suppose for contradiction that there are stationarily many $X \in S$ such that $\gamma \in c_X$ and there are stationarily many $Y \in S$ such that $\gamma \notin c_Y$. Suppose first $\mathrm{cof}(\xi) \in [\omega_1, \kappa]$. Note that $\mathrm{crt}(\pi_X)$, $\mathrm{crt}(\pi_Y) > \kappa$. It's easy then to see that $\pi_X[b_X]$ is cofinal in ξ and $\pi_Y[c_Y]$ is cofinal in ξ. Hence $c_X = c_Y$. Contradiction.

Now suppose $\mathrm{cof}(\xi) = \omega$. Fix a surjection $f : \lambda \twoheadrightarrow \xi$. $\forall^* X \in S$ $(f, \xi) \in X$ so let $(f_X, \xi_X) = \pi_X^{-1}(f, \xi)$. For each such X, let α_X be least such that $f_X[\alpha_X] \cap b_X$ is cofinal in ξ_X. By Fodor's lemma,

$$\exists \alpha \exists U \ (U \text{ is stationary} \wedge \forall X \in U \ \alpha_X = \alpha).$$

[33] Recall $\mathrm{Lp}^{\mathrm{g}\Sigma}_{*,+}(\mathcal{M}(\mathcal{T}))$ is just $\mathrm{Lp}^{\mathrm{g}\mathcal{F}}_{*,+}(\mathcal{M}(\mathcal{T}))$.

[34] This means the set of such X is $C \cap S$ for some club C.

By symmetry and by thinning out U, we may assume

$$X \in U \Rightarrow \pi_X^{-1}(\gamma) \in b_X.$$

Fix $Y \in S$ such that $\gamma \notin c_Y$ and $\alpha < \lambda_Y$. Since U is stationary, there is some $X \in U$ such that $Y \prec X$, which implies

$$\pi_{Y,X}[f_Y[\alpha]] = f_X[\alpha]$$

is cofinal in b_X and hence $\mathcal{T}_Y^\frown \pi_{Y,X}^{-1}[b_X]$ is a hull of $\mathcal{T}_X$. Since F condenses well, $\pi_{Y,X}^{-1}[b_X] = b_Y$. This contradicts the fact that $\pi_X^{-1}(\gamma) \in b_X$ but $\pi_Y^{-1}(\gamma) \notin b_Y$.

Finally, suppose $\mathrm{cof}(\xi) \geq \kappa^+$. The case $\mathcal{T}_X$ is maximal is proved exactly as in Lemma 1.25 of [17]. Suppose $\mathcal{T}_X$ is short and is according to F. Note that $\mathrm{lh}(\mathcal{T}_X)$ has uncountable cofinality (in V). We claim that $\forall^* X \in S\ b_X = F(\mathcal{T}_X) \in M_X$. Given the claim we get that for any two such $X \prec Y$ satisfying the claim, $\pi_{X,Y}(b_X)$ is cofinal in $\mathcal{T}_Y$ and hence $\pi_{X,Y}(b_X) = b_Y$. This gives $c_{X,Y}$ is an initial segment of b_Y, which is what we want to prove.

Now to see $\forall^* X \in S$, $b_X = F(\mathcal{T}_X) \in M_X$. We first remind the reader $\mathcal{Q}(\mathcal{T}_X)$ is the least $\mathcal{Q} \trianglelefteq \mathrm{Lp}_+^{\mathrm{g}\Sigma,\Gamma}(\mathcal{M}(\mathcal{T}_X))$ that defines the failure of Woodinness of $\delta(\mathcal{T}_X)$. Since $\delta(\mathcal{T})$ has uncountable cofinality (in V and in $V[G]$), by a standard interpolation argument, whenever $\mathcal{M}_0, \mathcal{M}_1 \in \mathrm{Lp}_+^{\mathrm{g}\Sigma,\Gamma}(\mathcal{M}(\mathcal{T}_X))$ then we have either $\mathcal{M}_0 \trianglelefteq \mathcal{M}_1$ or $\mathcal{M}_1 \trianglelefteq \mathcal{M}_0$. So the "leastness" of $\mathcal{Q}(\mathcal{T}_X)$ is justified in this case. By the same proof as that of Lemma 3.61 and the fact that X is cofinal in $\mathrm{Lp}_{*,+}^{\mathrm{g}\Sigma}(\mathcal{M}(\mathcal{T}))$ and $\mathrm{Lp}_+^{\mathrm{g}\Sigma,\Gamma}(\mathcal{M}(\mathcal{T}_X)) \trianglelefteq \mathrm{Lp}_{*,+}^{\mathrm{g}\Sigma}(\mathcal{M}(\mathcal{T}))$, we get $\mathcal{Q}(\mathcal{T}_X) \in M_X$.

Now $F(\mathcal{T}_X) = b_X$ is the unique branch b such that $\mathcal{Q}(b, \mathcal{T}_X)$ exists and $(\mathcal{Q}(b, \mathcal{T}_X)^*, \mathcal{M}(\mathcal{T}_X))$ [35] is $\mathcal{Q}(\mathcal{T}_X)$. The uniqueness of b_X follows from a standard comparison argument. By an absoluteness argument and the fact that $\mathcal{Q}(\mathcal{T}_X) \in M_X$, $b_X \in M_X$. We're done. ∎

Using the claim, we can just define

$$\gamma \in F_\xi(\mathcal{T}) \Leftrightarrow \forall^* X \in S\ \gamma \in c_X.$$

It's easy to verify that with this definition, the unique extension of F to a (κ^+, κ^+) strategy has branch condensation. This completes the proof sketch of the lemma. ∎

[35] See [18] for more on $*$-translations. $(\mathcal{Q}(b, \mathcal{T}_X)^*, \mathcal{M}(\mathcal{T}_X))$ is fine-structurally equivalent to $\mathcal{Q}(b, \mathcal{T}_X)$ but itself is a ${}^{\mathrm{G}}\Sigma$-premouse over $\mathcal{M}(\mathcal{T}_X)$.

LEMMA 3.63: (1) *If H is a defined by (ψ, a) on $H^{V[G]}_{\omega_1}$ as in clause 1 of 3.57 with $a \in H^V_{\kappa^+}$, then H can be extended to a first order mouse operator H^+ defined by (ψ, a) on $H^{V[G]}_{\lambda^+}$. Furthermore, H^+ relativizes well and if H determines itself on generic extensions then so does H^+.*

(2) *If $(\mathcal{Q}, \Lambda)$ and Γ are as in clause 2 of Definition 3.57, where F plays the role of J there and $(\mathcal{Q}, \Lambda \restriction V) \in V$, then Λ can be extended to a unique (λ^+, λ^+)-strategy that has branch condensation in $V[G]$.*

Proof. For (1), let $b \in H^{V[G]}_{\lambda^+}$ and let $\tau \in H^V_{\lambda^+}$ be a nice $\mathrm{Col}(\omega, \kappa)$-name for b ($\mathrm{Col}(\omega, \kappa)$ is κ^+-cc so such a name exists by the choice of λ).[36] Assume H is a Σ-mouse operator (the other case is proved similarly). Let $X \in S$ be such that $\mathcal{P}^* \cup \{\mathcal{P}^*\}, \Sigma, b, \tau, H(\tau) \in X[G]$; here we use Lemma 3.62 to get that $H(\tau)$ is defined. Let $(\bar{b}, \bar{\tau}) = \pi_X^{-1}(b, \tau)$. Then $\pi_X^{-1}(H(\tau)) = H(\bar{\tau}) \in M_X$ by condensation of H. Since H relativizes well, $H(\bar{b}) \in M_X[G]$. This means we can define $H^+(b)$ to be $\pi_X(H(\bar{b}))$. We need to see that $H^+(b)$ is countably Σ-iterable in $V[G]$. So let $\pi : \mathcal{N} \to H^+(b)$ with $\mathcal{N}$ countable transitive in $V[G]$ and $\pi(b^*) = b$. Let $X \subset Y \in S$ be such that $\mathrm{ran}(\pi) \subseteq \mathrm{ran}(\pi_Y)$; then $H(\pi_Y^{-1}(b)) \in M_Y[G]$ and there is an embedding from $\mathcal{N}$ into $H(\pi_Y^{-1}(b))$, so $\mathcal{N}$ has an $(\omega_1, \omega_1 + 1)$-Σ-iteration strategy. The definition doesn't depend on the choice of X and it's easy to see that H^+ satisfies the conclusion.

For (2), let $M \in H^{V[G]}_{\lambda^+}$ be transitive and $\tau \in H^V_{\lambda^+}$ be a $\mathrm{Col}(\omega, \kappa)$-term for M. We define the extension Λ^+ of Λ as follows (it's easy to see that there is at most one such extension). In $N = L^{\Lambda^*}_{\lambda^+}[A, \mathfrak{M}]$, where $A \subseteq \lambda$ codes $tr.cl.(\tau)$ and a well-ordering of $tr.cl.(\tau)$, Λ^* is the unique (λ^+, λ^+)-Λ-strategy for $\mathfrak{M} = \mathcal{M}_1^{\Lambda, \sharp}$ in V; Λ^* exists by Lemma 3.62.

Let $\mathcal{T}_{tr.cl.(\tau)}$ be according to Λ^* and be defined as in Definition 3.38. Note that $(\lambda^+)^N < (\lambda^+)^V$ by $\neg\square_\lambda$ and the fact that $\square_\lambda$ holds in N, so $\mathcal{T}_{tr.cl.(\tau)} \in N$ and has length less than $o(N) = \lambda^+$. Let $\mathcal{R}$ be the last model of $\mathcal{T}_{tr.cl(\tau)}$ and note that by the construction of $\mathcal{T}_{tr.cl.(\tau)}$, M is generic over $\mathcal{R}$. Let $\mathcal{U} \in M$ be a tree according to Λ^+ of limit length, then set $\Lambda^+(\mathcal{U}) = b$ where b is given by (the proof of) [13, Lemma 4.8] by interpreting Λ over generic extensions of $\mathcal{R}$. By a simple reflection argument, it's easy to see that $\Lambda^+(\mathcal{U})$ doesn't depend

[36] In particular, a nice $\mathrm{Col}(\omega, \kappa)$-name for a real can be considered a subset of κ and hence a nice $\mathrm{Col}(\omega, \kappa)$-name for $\mathbb{R}^{V[G]}$ is an element of $H^V_{\lambda^+}$.

on M.[37] This completes the construction of Λ^+. It's easy to see that F^+ has branch condensation. ∎

Let J be as above. We now proceed to construct $\mathcal{M}_1^{J,\sharp}$. We denote $\mathcal{M}_0^{J,\sharp}(x)$ for the least E-active, sound J-mouse over x.

LEMMA 3.64: *For every A bounded in λ^+, $\mathcal{M}_0^{J,\sharp}(A)$ exists.*

Proof. By Lemma 3.62, it's enough to show that if A is a bounded subset of κ^+, then $\mathcal{M}_0^{J,\sharp}(A)$ exists. Fix such an A and let $X \in S$ such that $\sup(A) \cup \{A, \sup(A), J\} \subseteq X$ and X is cofinal in the $L^J[A]$-successor of κ^+, which has cofinality at most κ. Hence $\pi_X(A) = A$. Let

$$\mu = \{B \subseteq \lambda_X \mid \lambda_X \in \pi_X(B) \wedge B \in L^J[A]\ ^{38}\}.$$

CLAIM 3.65: *μ is a countably complete $L^J[A]$-ultrafilter,*

Proof. Let $Q = L^J[A]$ and $P = \pi_X^{-1}(Q)$. Let $\eta = \kappa^+$ and $\xi = \pi_X^{-1}((\eta^+)^Q)$. Let $\kappa_0 = \lambda_X$. Then $\xi = (\kappa_0^+)^Q$. This is because X is cofinal in $(\eta^+)^Q$. So μ is indeed total over $L^J[A]$. Using the fact that $X^\omega \subset X$, we get that μ is countably complete. ∎

We need to know that when iterating $L^J[A]$ by μ and its images, the iterates are $L^J[A]$. This follows from a well-known argument by Kunen. The point is that iterates of $L^J[A]$ by μ and its images can be realized back into $L^J[A]$ and hence since J condenses well, the ultrapowers are $L^J[A]$. We outline the proof here for the reader's convenience (see [22, Theorem 28] for a similar argument).

Let $\mu_0 = \mu$, $\xi_0 = \xi$, and $\mathcal{M}_0 = (L^J_\xi[A], \mu_0)$. By the usual Kunen's argument, $\mathcal{M}_0$ is an amenable structure. By induction on $\alpha < \kappa^+$, we define:

(1) $\mathcal{M}_\alpha$, the α-th iterate of $\mathcal{M}_0$ by μ_0 and its images,
(2) maps $\pi^*_{\beta,\alpha} : \mathcal{M}_\beta \to \mathcal{M}_\alpha$ for $\beta < \alpha$,
(3) maps $\pi_{\beta,\alpha} : L^J[A] \to L^J[A]$ extending $\pi^*_{\beta,\alpha}$,
(4) maps $\tau_\alpha : L^J[A] \to L^J[A]$ such that $\forall \beta < \alpha$, $\tau_\beta = \tau_\alpha \circ \pi_{\beta,\alpha}$.

[37] Let M, M^* be such that $\mathcal{U} \in M \cap M^*$; let τ, τ^* be nice $\mathrm{Col}(\omega, \kappa)$-terms for M, M^* respectively. In $V[G]$, let $X[G]$ contain all relevant objects and $X \in S$. Let $\bar{a} = \pi_X^{-1}(a)$ for all $a \in X[G]$. Then letting b_0, b_1 be the branches of $\bar{\mathcal{U}}$ given by applying [13, Lemma 4.8] in $L^{\Lambda^*}[tr.cl.(\bar{\tau}), <_1, \mathfrak{M}], L^{\Lambda^*}[tr.cl.(\bar{\tau^*}), <_2, \mathfrak{M}]$ (built inside $M_X[G]$), where $<_1$ is a well-ordering of $\bar{\tau}$ and $<_2$ is a well-ordering of $\bar{\tau^*}$. Then $b_0 = b_1$ as both are according to Λ, since $(\mathfrak{M}, \Lambda^*)$ generically interprets Λ in $V[G]$.

[38] We only build $L^J[A]$ up to λ^+.

For $\alpha = 0$, let $\pi^*_{0,1}, \pi_{0,1}$ be the μ_0-ultrapower maps and let E be the (κ_0, ω_3^V)-extender derived from π_X and let

$$\tau_0 : L^J[A] \to \mathrm{Ult}(L^J[A], E).$$

It's easy to see that:

- τ_0 has a stationary set of fixed points.
- $\mathrm{Ult}(L^J[A], E) = L^J[A]$ (similarly, $\mathrm{Ult}(L^J[A], \mu_0) = L^J[A]$). Let $Y \prec H_{\lambda^{++}}$ be countable containing all relevant objects and $\pi : M \to Y$ be the uncollapse map and for each $a \in Y$, let $a^* = \pi^{-1}(a)$. Using countable completeness of E, it is easy to check that in M, $\mathrm{Ult}(L^{J^*}[A^*], E^*)$ realizes into $L^J[A]$ and is in fact $L^{J^*}[A^*]$.

If α is limit, let $\mathcal{M}_\alpha$ be the direct limit of the system $(\mathcal{M}_\beta, \pi^*_{\gamma,\beta})_{\gamma<\beta<\alpha}$, $\tau_\alpha = \lim_{\beta<\alpha} \tau_\beta$, and $\pi^*_{\beta,\alpha}, \pi_{\beta,\alpha}$ be natural direct limit maps.

Suppose $\alpha = \beta + 1$ and $\mathcal{M}_\beta = (L^J_{\xi_\beta}[A], \mu_\beta)$ is an amenable structure, $\kappa_\beta = \mathrm{crt}(\mu_\beta) = \pi_{0,\beta}(\kappa_0)$, and $\mu_\beta = \pi_{0,\beta}[\mu_0]$. Let $\pi^*_{\beta,\alpha}, \pi_{\beta,\alpha}$ be μ_β-ultrapower maps. For any $f \in L^J[A]$,

$$\tau_\alpha(\pi_{\beta,\alpha}(f)(\kappa_\beta)) = \tau_\beta(f)(\kappa_\beta).$$

We need to check that τ_α is elementary. This is equivalent to checking μ_β is derived from τ_β, i.e.,

$$C \in \mu_\beta \Leftrightarrow \kappa_\beta \in \tau_\beta(C). \tag{3.1}$$

To see (3.1), let $\nu < \mu_0$ and $W = \mu_0 \cap L^J[A]$, $f : \kappa_0 \to \wp(\kappa_0) \cap L^J_\nu[A]$. Let c be a finite set of fixed points of τ_0 and such that

$$\forall \xi < \kappa_0 \ f(\xi) = \tau^{L^J[A]}[c](\xi) \cap \kappa_0.$$

So

$$L^J[A] \vDash \forall \xi < \kappa_0 \ (\tau[c](\xi) \cap \kappa_0 \in W \Leftrightarrow \kappa_0 \in \tau[c](\xi)). \tag{3.2}$$

This fact is preserved by $\pi_{0,\beta}$ and gives (3.1).

One can also show by induction that $\mathrm{crt}(\tau_\gamma) = \kappa_\gamma$ for all $\gamma \leq \beta$. This is because κ_γ is the only generator of μ_γ.

So $\mathcal{M}_0^{J,\sharp}$ exists (and is (λ^+, λ^+)-iterable by Lemma 3.62). ∎

LEMMA 3.66: *Suppose A is a bounded subset of λ^+. Then $\mathcal{M}_1^{J,\sharp}(A)$ exists and is (λ^+, λ^+)-iterable.*

Proof. It suffices to show $\mathcal{M}_1^{J,\sharp}(a)$ exists for a a bounded subset of κ^+ (with a coding x). Fix such an a and suppose not. Then the Jensen-Steel core model (cf. [2]) $K^J(a)$ exists.[39] Let $\gamma \geq \kappa^+$ be a successor cardinal in $K^J(a)$. Since $\lambda^+ = \mathrm{o}(K^J(a)) > \kappa^+$ and is a limit of cardinals in $K^J(a)$ (by the proof of 3.59), we can take $\gamma < \lambda^+$. Weak covering (cf. [2, Theorem 1.1 (5)]) gives us

$$\mathrm{cof}(\gamma) \geq |\gamma| \geq \kappa^+. \tag{3.3}$$

Let $\vec{C}$ be a $\square$-sequence in $K^J(a)$ witnessing $\square(\gamma)$. By a standard argument, one can construct from $\vec{C}$ a sequence witnessing $\square(\mathrm{cof}(\gamma))$; but $\mathrm{cof}(\gamma) \geq \kappa^+$ by (3.3) and $\neg\square(\mathrm{cof}(\gamma))$. Contradiction. ∎

Lemmas 3.62, 3.63, and 3.66 allow us to extend $\mathcal{M}_1^{J,\sharp}$ to $H_{\lambda^+}^{V[G]}$.

LEMMA 3.67: *Suppose J is defined on a cone above some $a \in H_{\kappa^+}^V$ in $H_{\omega_1}^{V[G]}$ as in case 1 of 3.57. Then for every $b \in H_{\lambda^+}^{V[G]}$ coding a, $\mathcal{M}_1^{J,\sharp}(b)$ is defined (and is (λ^+, λ^+)-iterable in $V[G]$). Otherwise, letting $(\mathcal{Q}, \Lambda), x, \Gamma$ be as in 2 of Definition 3.57 and $J = \mathcal{F}_{\Lambda,\varphi_{\mathrm{all}}}$, then $\mathcal{M}_1^{J,\sharp}(a)$ is defined for all $a \in H_{\lambda^+}^{V[G]}$ coding $x, \mathcal{Q}$. Furthermore, these operators determine themselves on generic extensions if J does.*

3.3. THE CORE MODEL INDUCTION THEOREM. Let $(\mathcal{P}^*, \Sigma), \mathcal{F}$ be as in the previous section. When an ${}^g\mathcal{F}$-premouse $\mathcal{P}$ is 1-$\mathcal{F}$-Γ-suitable, we simply say $\mathcal{P}$ is Γ-suitable if $\mathcal{F}$ is clear from the context. Recall that under AD, if X is any set then θ_X is the least ordinal which isn't a surjective image of $\mathbb{R}$ via an OD_X function.

The following is an outline of the proof of the core model induction theorem. We will follow the standard convention and use upper-case Greek letters Γ, Ω etc. to denote lightface pointclasses, bold upper-case Greek letters $\mathbf{\Gamma}, \mathbf{\Omega}$ etc. to denote boldface pointclasses. Given a point class $\mathbf{\Gamma}$, we let $\hat{\mathbf{\Gamma}}$ denote the dual pointclass of $\mathbf{\Gamma}$ and $\mathbf{\Delta_\Gamma}$ denote the pointclass $\mathbf{\Gamma} \cap \hat{\mathbf{\Gamma}}$. For more on the envelope $\mathbf{Env}(\mathbf{\Gamma})$, the notion of C_Γ and other relevant descriptive set theoretic notions, see [28].

We refer the reader to [13] for the scales analysis in $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$ that we use in the proof of Theorem 3.72. We recall some notions which are obvious generalizations of those in [11] and [28]. The following definitions refer to $V[G]$.

[39] By our assumption and the fact that J condenses finely, $K^{c,J}(a)$ (constructed up to λ^+) converges and is (λ^+, λ^+)-iterable. See Lemma 3.18.

Definition 3.68: We say that the coarse mouse witness condition $W_\gamma^{*,{}^{\mathrm{g}}\mathcal{F}}$ holds if, whenever $U \subseteq \mathbb{R}$ and both U and its complement have scales in $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\gamma$, then for all $k < \omega$ and $x \in \mathbb{R}$ there is a coarse (k, U)-Woodin ${}^{\mathrm{g}}\mathcal{F}$-mouse[40] containing x with an $(\omega_1 + 1)$-iteration ${}^{\mathrm{g}}\mathcal{F}$-strategy whose restriction to H_{ω_1} is in $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\gamma$.

Remark 3.69: By the proof of [11, Lemma 3.3.5], $W_\gamma^{*,{}^{\mathrm{g}}\mathcal{F}}$ implies $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\gamma \vDash \mathsf{AD}$.

Definition 3.70: An ordinal γ is a critical ordinal in $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$ if there is some $U \subseteq \mathbb{R}$ such that U and $\mathbb{R}\backslash U$ have scales in $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|(\gamma + 1)$ but not in $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\gamma$. In other words, γ is critical in $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$ just in case $W_{\gamma+1}^{*,{}^{\mathrm{g}}\mathcal{F}}$ does not follow trivially from $W_\gamma^{*,{}^{\mathrm{g}}\mathcal{F}}$.

Definition 3.71: Let $\mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$ be the initial segment of $\mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$ that is the union of all $\mathcal{M} \lhd \mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$ such that every countable $\mathcal{M}^*$ embeddable into $\mathcal{M}$ has an iteration strategy in $\mathcal{M}$.

We will prove in the next theorem that $\mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R}) \vDash \mathsf{AD}^+$; in fact, this is the maximal model of $\mathsf{AD}^+ + \Theta = \theta_\Sigma$ in light of [18, Theorem 17.1]. We note that

$$\wp(\mathbb{R}) \cap L(\mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})) = \wp(\mathbb{R}) \cap \mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$$

but don't know if $\mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R}) = \mathrm{Lp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$ in general.

THEOREM 3.72: *Assume the hypothesis of Theorem 0.2 and* (†). *Suppose* $(\mathcal{P}, \Sigma)$ *is a hod pair below* κ *and* Σ *is a* (λ^+, λ^+)*-strategy in* $V[G]$ *with branch condensation. Let* $\mathcal{F}$ *be the corresponding operator (i.e.,* $\mathcal{F} = \mathcal{F}_{\Sigma,\varphi_{\mathrm{all}}}$*). Suppose* $\mathcal{F} \restriction \mathbb{R}$ *is self-scaled. Then in* $V[G]$, $\mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R}) \vDash \mathsf{AD}^+ + \theta_\Sigma = \Theta$. *Hence,* $\wp(\mathbb{R}) \cap \mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R}) \subseteq \Omega$.

Proof. As shown in subsection 3.2, our hypothesis implies that for every Σ-cmi operator H, $\mathcal{M}_1^{H,\sharp}$ exists and can be extended to $H_{\lambda^+}^{V[G]}$; furthermore, these operators determine themselves on generic extensions. We will use this fact and refer the reader to subsection 3.2 for the proof. Working in $V[G]$, let α be the strict supremum of the ordinals γ such that

[40] This is the same as the usual notion of a (k, U)-Woodin mouse, except that we demand the mouse is closed under ${}^{\mathrm{g}}\mathcal{F}$.

(1) the coarse mouse witness condition $W^{*,{}^{\mathrm{g}}\mathcal{F}}_{\gamma+1}$ holds[41];
(2) γ is a critical ordinal in $\mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$ (i.e., $\gamma+1$ begins a gap in $\mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$).

Using the fact that $\mathcal{M}_1^{H,\sharp}$ exists for every Σ-cmi operator H, it's easily seen that α is a limit ordinal. By essentially the same proof, with obvious modifications, as that in [17], we can advance past inadmissible gaps and admissible gaps in $\mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\alpha$. In each case, say $[\gamma, \xi]$ is a gap in $\mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\alpha$ and $W^{*,{}^{\mathrm{g}}\mathcal{F}}_{\gamma}$, the proof in [17, Sections 1.4, 1.5] and the scales analysis in [13][42] allow us to construct a nice operator F on a cone above some x in $HC^{V[G]}$ such that $x \in H^V_{\omega_3}$, $F \restriction V \in V$. By the previous subsection, we can extend F to a nice operator on $H^{V[G]}_{\kappa^+}$ (also called F) with $F \restriction V \in V$. Again, by the previous subsection, we can construct a sequence of nice operators $(F_n : n < \omega)$, where $F_0 = F$, $F_{n+1} = \mathcal{M}_1^{F_n,\sharp}$, and these operators witness $W^{*,{}^{\mathrm{g}}\mathcal{F}}_{\xi+1}$ (or $W^{*,{}^{\mathrm{g}}\mathcal{F}}_{\xi+2}$ if the gap is strong).

Hence, the (lightface) pointclass $\Gamma = \Sigma_1^{\mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F}\restriction\mathbb{R})|\alpha}$ is inductive-like and $\mathbf{\Delta_\Gamma} = \wp(\mathbb{R}) \cap \mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\alpha$. Since Γ is inductive-like and $\mathbf{\Delta_\Gamma}$ is determined, $\mathbf{Env}(\mathbf{\Gamma})$ is determined by Theorem 3.2.4 of [28]. Since whenever γ is a critical ordinal in $\mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$ and $W^{*,{}^{\mathrm{g}}\mathcal{F}}_{\gamma+1}$ holds then AD holds in $\mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|(\gamma+1)$, we have that AD holds in $\mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\alpha$.

Now we claim that $\mathbf{Env}(\mathbf{\Gamma}) = \wp(\mathbb{R}) \cap \mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$. This implies $\mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R}) \vDash \mathsf{AD}^+ + \Theta = \theta_\Sigma$ as desired. We first show $\mathbf{Env}(\mathbf{\Gamma}) \subseteq \wp(\mathbb{R}) \cap \mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$. Let $A \in \mathbf{Env}(\mathbf{\Gamma})$, say $A \in Env(\Gamma)(x)$ for some $x \in \mathbb{R}$. By definition of Env, for each countable $\sigma \subseteq \mathbb{R}$, $A \cap \sigma = A' \cap \sigma$ for some A' that is Δ_1-definable over $\mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\alpha$ from x and some ordinal parameter. In V, let τ be the canonical name for x and let $X \prec H_{\lambda^{++}}$ be such that $|X|^V = \kappa$, $\kappa \subseteq X$, $X^\omega \subseteq X$, X is cofinal in the ordinal height of

[41] This is defined similarly to $W^*_{\gamma+1}$ but relativized to the operator ${}^{\mathrm{g}}\mathcal{F}$. Similarly, we can also define the fine-structural mouse witness condition $W^{\mathrm{g}\mathcal{F}}_{\gamma+1}$.

[42] If γ is such that $M = \mathrm{sLp}^{{}^{\mathsf{G}}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\gamma$ is inadmissible, then M is passive. Then [13] gives us that $\mathbf{\Sigma}^M_1$ has the scales property assuming $M \vDash \mathsf{AD}$. This is the main reason why we analyze scales in Θ-g-organized premice; if M were g-organized, it could be that $\dot{B}^M \neq \emptyset$ and the argument in [13] does not seem to give us the scales property of $\mathbf{\Sigma}^M_1$ from AD^{M}.

$\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$.[43] We also assume $\mathcal{P} \cup \{\mathcal{P}, \tau\} \subseteq X$, $\Sigma \cap M_X[G] \in M_X[G]$, and $A \in \mathrm{ran}(\pi_X)$. Let $\sigma = \mathbb{R} \cap M_X[G]$. By $\mathsf{MC}(\Sigma)$ in $\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\alpha$, $A \cap \sigma \in \mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\sigma, \mathcal{F} \restriction \sigma)$ (this is because $\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\sigma, \mathcal{F} \restriction \sigma)$ and $\mathrm{sLp}^{\mathsf{g}\mathcal{F}}(\sigma, \mathcal{F} \restriction \sigma)$ have the same $\wp(\sigma)$, cf. [13, Section 5]; also, in applying $\mathsf{MC}(\Sigma)$, we need that definability is done without referencing the extender sequence and we can do this since we are inside $\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\sigma, \mathcal{F} \restriction \sigma)$, where the self-iterability condition helps us define the extender sequence). As shown in Lemma 3.75, $\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\sigma, \mathcal{F} \restriction \sigma) = (\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\sigma, \mathcal{F} \restriction \sigma))^{M_X[G]}$. Hence

$$M_X[G] \vDash A \cap \sigma \in \mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\sigma, \mathcal{F} \restriction \sigma).$$

By elementarity, the fact that $\pi_X(\mathcal{P}) = \mathcal{P}$, and $\pi_X(\Sigma \cap M_X[G]) = \Sigma$, we get $A \in \mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$.

Now assume toward a contradiction that $\mathbf{Env}(\mathbf{\Gamma}) \subsetneq \wp(\mathbb{R}) \cap \mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$. Hence $\alpha < \Theta^{\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})}$. Let β^* be the end of the gap starting at α in $\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$. Let $\beta = \beta^*$ if the gap is weak and $\beta = \beta^* + 1$ if the gap is strong. Note that

$$\alpha \leq \beta,$$
$$\wp(\mathbb{R})^{\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\beta} = \mathbf{Env}(\mathbf{\Gamma})^{\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})} \subseteq^{44} \mathbf{Env}(\mathbf{\Gamma}) \subsetneq \wp(\mathbb{R}) \cap \mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R}).$$

Hence $\beta < \Theta^{\mathrm{sLp}^{G_{\mathcal{F}}}(\mathbb{R})}$ and $\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\beta$ projects to $\mathbb{R}$. Furthermore, $\mathrm{Lp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\beta \vDash \mathsf{AD}{+}\Gamma\text{-}\mathsf{MC}(\Sigma)$, where $\Gamma\text{-}\mathsf{MC}(\Sigma)$ is the statement: for any countable transitive a, $(\mathrm{Lp}^{\mathsf{g}\mathcal{F}}(a))^{\Gamma} \cap \wp(a) = C_{\Gamma}(a)$. Now $\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\beta \vDash \Gamma\text{-}\mathsf{MC}(\Sigma)$ is clear; if $\beta = \beta^*$, $\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\beta \vDash \mathsf{AD}$ by the fact that $[\alpha, \beta^*]$ is a Σ_1-gap; otherwise, $\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\beta \vDash \mathsf{AD}$ by the Kechris–Woodin transfer theorem (see [3]). Since $\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\beta$ projects to $\mathbb{R}$, every countable sequence from $\mathbf{Env}(\mathbf{\Gamma})^{\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})}$ is in $\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|(\beta + 1)$. The scales analysis of [13], Theorem 4.3.2 and Corollary 4.3.4 of [28] together imply that there is a self-justifying-system $\mathcal{A} = \{A_i \mid i < \omega\} \subseteq \mathbf{Env}(\mathbf{\Gamma})^{\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})}$ containing a universal Γ set. By a theorem of Woodin and the fact that $\Sigma \cap V \in V$,

[43] Let ϵ be the name of $\mathbb{R}^{V[G]}$. Note that the ordinal height of $\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$ is the ordinal height of $\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\epsilon, \mathcal{F} \restriction \epsilon)$ and the latter is in V. This has cofinality at most ω_2 since the construction in [24] gives a coherent, nonthreadable sequence of length $\mathrm{o}(\mathrm{sLp}^{\mathsf{G}\mathcal{F}}(\epsilon, \mathcal{F} \restriction \epsilon))$.

[44] We get equality in this case but we don't need this fact.

we can get a pair $(\mathcal{N}, \Lambda)$ such that $\mathcal{N} \in V$, $|\mathcal{N}|^V \leq \kappa$, $\mathcal{N}$ is a Γ-suitable ${}^{\mathrm{g}}\mathcal{F}$-premouse, and Λ is the strategy for $\mathcal{N}$ guided by $\mathcal{A}$.[45] Arguments in the last subsection allow us to then lift Λ to an (λ^+, λ^+)-strategy that condenses well in $V[G]$.

Using the hypothesis of Theorem 0.2, we can get a sequence of nice operators $(F_n : n < \omega)$ where each F_n is in $\mathrm{sLp}^{\mathsf{G}^{\mathcal{F}}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|(\beta+1)$. Namely, let $F_0 = \mathcal{F}_\Lambda$, and let $F_{n+1} = \mathcal{M}_1^{F_n,\sharp}$ be the F_n-Woodin Σ-cmi operator. Each F_n is first defined on a cone in $H^V_{\omega_3}$; then using the lemmas in the previous subsection, we can extend F_n to $H^{V[G]}_{\kappa^+}$ and furthermore, each F_n is nice (i.e., condenses and relativizes well and determines itself on generic extensions). These operators are all projective in $\mathcal{A}$ and are cofinal in the projective-like hierarchy containing $\mathcal{A}$, or equivalently in the Levy hierarchy of sets of reals definable from parameters over $\mathrm{sLp}^{\mathsf{G}^{\mathcal{F}}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})|\beta$. Together these model operators can be used to establish the coarse mouse witness condition $W^{*,{}^{\mathrm{g}}\mathcal{F}}_{\beta+1}$. Therefore $\beta < \alpha$ by the definition of α, which is a contradiction. ∎

3.4. Beyond "$\mathrm{AD}^+ + \Theta = \Theta_\Sigma$". Let $(\mathcal{P}^*, \Sigma), \mathcal{F}$ be as in Section 3.2. In this section, we prove

Theorem 3.73: *Let $G \subseteq Col(\omega, \kappa)$ be V-generic. Then in $V[G]$, there is a model M such that* $\mathrm{OR} \cup \mathbb{R} \subseteq M$ *and* $M \vDash$ "$\mathsf{AD}^+ + \Theta > \theta_\Sigma$".

The rest of this subsection is devoted to proving Theorem 3.73. We assume $(\mathcal{P}^*, \Sigma) = (\emptyset, \emptyset)$ (the proof of the general case just involves more notations; in particular, for the general case, we work in the hierarchy $\mathrm{Lp}^{\mathsf{G}^{\mathcal{F}}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$ instead of in $\mathrm{Lp}(\mathbb{R})$). Suppose the conclusion of the theorem fails. By the results of Subsection 3.2 and Theorem 3.72, in $V[G]$

$$\mathrm{sLp}(\mathbb{R}) \vDash \mathsf{AD}^+ + \Theta = \theta_0.$$ [46]

Working in $V[G]$, let $\mathbf{\Omega_0} = \wp(\mathbb{R}) \cap \mathrm{sLp}(\mathbb{R})$, $\theta = \Theta^{\Omega_0}$, and $\Gamma = (\Sigma_1)^{\mathrm{sLp}(\mathbb{R})}$.

[45] We get first a pair $(\mathcal{N}^*, \Lambda^*) \in V[G]$ with $\mathcal{N}^*$ being Γ-suitable, Λ^* is an (ω_1, ω_1)-strategy in $V[G]$ guided by $\mathcal{A}$. By boolean comparisons, we can obtain such a pair $(\mathcal{N}, \Lambda)$. The details are given in [17] and [11].

[46] We could have also worked with the hierarchy $\mathrm{sLp}^{\mathsf{G}^{\mathcal{F}}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R})$ where $\mathcal{F}$ is associated with the canonical strategy of $\mathcal{M}_1^\sharp$. As mentioned before, these hierarchies construct the same sets of reals.

In V, let $\tau \in H_{\lambda^+}$ be a canonical name for $\mathbb{R}^{V[G]}$. Let S be as in Section 3.2. For $X \in S$ such that X is cofinal in $o(\mathrm{Lp}(\mathbb{R}))$ and $o(\mathrm{sLp}(\mathbb{R}))$,[47] let $\mathbb{R}_X = \pi_X^{-1}(\mathbb{R})$, $\mathbf{\Omega}_X = \pi_X^{-1}(\Omega_0)$, $\Gamma_X = \pi_X^{-1}(\Gamma)$, and $\theta_X = \pi_X^{-1}(\theta)$; we note that $\theta = \Theta^{\mathbf{\Omega_0}} = o(\mathrm{sLp}(\mathbb{R})^{V[G]})$ is the supremum of the Wadge ranks of sets in $\mathbf{\Omega_0}$. Also, let $\eta = (\underset{\sim}{\delta}^2_1)^{\mathrm{sLp}(\mathbb{R})}$ and $\eta_X = \pi_X^{-1}(\eta)$; so η is the largest Suslin cardinal of $\mathrm{sLp}(\mathbb{R})$. Let T be the tree of scales for a universal Γ-set and $T_X = \pi_X^{-1}(T)$. As usual, T is a tree on $\omega \times \eta$. For $s \in \omega^{<\omega}$, we let $T_s = \{t \mid (s,t) \in T\}$.

Following [28], we define

Definition 3.74: Let Γ, T, η be as above.

(1) $\wp^{\mathbf{\Gamma}}(\eta)$ is the σ-algebra consisting of subsets $Y \subseteq \eta$ such that $Y \in L[T,z]$ for some real z.
(2) $\mathrm{meas}^{\mathbf{\Gamma}}(\eta)$ is the set of countably complete measures on $\wp^{\mathbf{\Gamma}}(\eta)$.
(3) Using the canonical bijection $\eta \to \eta^{<\omega}$, we can define $\wp^{\mathbf{\Gamma}}(\eta^n)$, $\mathrm{meas}^{\mathbf{\Gamma}}(\eta^n)$, $\wp^{\mathbf{\Gamma}}(\eta^{<\omega})$, $\mathrm{meas}^{\mathbf{\Gamma}}(\eta^{<\omega})$ in a similar fashion.

Lemma 3.75: *Suppose $X \in S$. Then in $V[G]$,*

$$Lp(\mathbb{R}_X)^{\Omega_0} \subseteq Lp(\mathbb{R}_X)^{M_X[G]} = Lp(\mathbb{R}_X).$$

This implies

$$meas^{\mathbf{\Gamma_X}}(\eta_X^{<\omega}) \subseteq (meas^{\mathbf{\Gamma_X}}(\eta_X^{<\omega}))^{M_X[G]}.$$

Proof. We first prove in $V[G]$,

$$\mathrm{Lp}(\mathbb{R}_X)^{M_X[G]} = \mathrm{Lp}(\mathbb{R}_X).$$

If $\mathcal{M} \lhd \mathrm{Lp}(\mathbb{R}_X)^{M_X[G]}$ is a sound mouse that projects to $\mathbb{R}_X$, then $\mathcal{M}$ is embeddable into a level of $\mathrm{Lp}(\mathbb{R})^{V[G]}$. So $\mathcal{M} \lhd \mathrm{Lp}(\mathbb{R}_X)$. To see the converse, let $\mathcal{M} \lhd \mathrm{Lp}(\mathbb{R}_X)$, then letting $\mathcal{M}^* \lhd \mathrm{Lp}(\tau_X)$ be the S-translation of $\mathcal{M}$, then by Lemma 3.61, $\mathcal{M}^* \in M_X$; so $\mathcal{M} \in M_X[G]$. Now in $M_X[G]$, let $\pi : \mathcal{N} \to \mathcal{M}$ be an elementary embedding and $\mathcal{N}$ is countable, transitive; then in $V[G]$, $\mathcal{N}$ is iterable via a unique iteration strategy; so in $M_X[G]$, $\mathcal{N}$ is iterable via a unique iteration strategy. This means $\mathcal{M} \lhd \mathrm{Lp}(\mathbb{R}_X)^{M_X[G]}$.

To see the first inclusion, note that $\mathrm{Lp}(\mathbb{R}_X)^{\mathbf{\Omega_0}}$ can easily be computed from $\mathrm{Lp}(\tau_X)$ (using G- and the S-constructions) and $\mathrm{Lp}(\tau_X) \in M_X$ by Lemma 3.61.

[47] One can construct a coherent sequence of length θ in $\mathrm{sLp}(\mathbb{R}^{V[G]})$ as in [24]. Our hypothesis and the properties of the sequence then imply that $\mathrm{cof}(o(\mathrm{sLp}(\mathbb{R})) \leq \kappa$. Similarly, one can show $\mathrm{cof}(o(\mathrm{Lp}(\mathbb{R})) \leq \kappa$.

This gives $\mathrm{Lp}^{\mathbf{\Omega_0}}(\mathbb{R}_X) \subseteq \mathrm{Lp}(\mathbb{R}_X)^{M_X[G]}$ because given any $\mathcal{M} \triangleleft \mathrm{Lp}(\mathbb{R}_X)^{\mathbf{\Omega_0}}$, letting $\mathcal{M}^*$ be a premouse over τ_X which results from the S-constructions of $\mathcal{M}$, $\mathcal{M}^*$ has a unique strategy Λ in $\mathrm{Lp}(\mathbb{R})^{V[G]}$ because $\mathcal{M}$ does. So $\Lambda \restriction V \in V$ by homogeneity. By the proof of Lemma 3.62, Λ can be uniquely extended in V to an (λ^+, λ^+)-strategy (also called Λ).[48] The previous subsection also shows that $\mathcal{M}_1^{\Lambda,\sharp}$ exists and is (κ^+, κ^+)-iterable in V. By Lemma 3.63, Λ can be extended to a (λ^+, λ^+) strategy in $V[G]$, but this means $\mathcal{M} \triangleleft \mathrm{Lp}(\mathbb{R}_X)$ in $V[G]$.

We just prove $meas^{\mathbf{\Gamma x}}(\eta_X) \subseteq (meas^{\mathbf{\Gamma x}}(\eta_X))^{M_X[G]}$ as the proof of the general case is an easy generalization of the proof of the special case. We need to define "code sets" for measures in $meas^{\mathbf{\Gamma x}}(\eta_X)$. Fix a map from $\mathbb{R}_X$ onto $\wp(\eta_X)^{\Gamma_X}$: $x \mapsto Y_x$ in $M_X[G]$ such that the relation $\{(x, \alpha) \mid \alpha \in Y_x\} \in \Sigma_1^{\mathrm{sLp}(\mathbb{R}_X)}$. We then define the code set C_μ for each $\mu \in meas^{\mathbf{\Gamma x}}(\eta_X)$ as $x \in C_\mu \Leftrightarrow Y_x \in \mu$. For each such μ, C_μ is easily seen to be $OD^{\Omega_0}(\mathbb{R}_X)$ (as each such measure is principal, being a countably complete measure on a countable set in $V[G]$), and so $C_\mu \in \mathrm{Lp}(\mathbb{R}_X)^{M_X[G]}$ by MC in Ω_0 and by the first part. So $\mu \in M_X[G]$ and is countably complete there. This proves the second inclusion. ∎

For X as in the lemma, we can choose a set $C_X \in M_X$ of canonical names for measures in $[meas^{\mathbf{\Gamma x}}(\eta_X^{<\omega})]^{M_X[G]}$. Since $M_X^\omega \subset M_X$, M_X contains all ω-sequences of its terms for code sets of measures in $[meas^{\mathbf{\Gamma x}}(\eta_X^{<\omega})]^{M_X[G]}$.

Now let $\sigma = \pi_X[[meas^{\mathbf{\Gamma x}}(\eta_X^{<\omega})]^{M_X[G]}] \subseteq meas^{\mathbf{\Gamma}}(\eta^{<\omega})$; σ is a countable set of measures in $V[G]$. For $\mu \in meas^{\mathbf{\Gamma}}(\eta^{<\omega}) \cap \sigma$, let $\overline{\mu}$ be such that $\pi_X(\overline{\mu}) = \mu$. Suppose μ concentrates on η^n and let $\langle \mu_i \mid i \leq n \rangle$ be the projections of μ (that is $A \in \mu_i \Leftrightarrow \{s \in \eta^n \mid s \restriction i \in A\} \in \mu$). Note that μ_0 is the trivial measure. Define $\langle \overline{\mu}_i \mid i \leq n \rangle$ similarly for $\overline{\mu}$. Let $G_T^{\sigma,\mu}$ be the game defined in Definition 4.1.2 of [28]. For the reader's convenience, we give the definition of $G_T^{\sigma,\mu}$. I starts by playing $m_0, \ldots, m_n$, s_n, h_n; II responds by playing a measure μ_{n+1}. From the second move on, I plays m_i, s_i, h_i and II plays a measure μ_{i+1} for all $i > n$.[49]

Rules for I:

- $m_k < \omega$ for all $k < \omega$.
- $T_{(m_0,\ldots,m_{n-1})} \in \mu = \mu_n$
- $s_i \in j_{\mu_i}(T_{(m_0,\ldots,m_{i-1})})$, in particular $s_i \in j_{\mu_i}(\eta)^{i+1}$ for all $i \geq n$.
- $s_n \supsetneq [\mathrm{id}]_{\mu_n}$.

[48] Note that every tree $\mathcal{T}$ according to Λ is short and guided by $\mathcal{Q}$-structures.

[49] The game can be defined over V using the forcing relation and C_X.

- $j_{\mu_i,\mu_{i+1}}(s_i) \subsetneq s_{i+1}$ for all $i \geq n$.
- $h_i \in \mathrm{OR}$ for all $i \geq n$.
- $j_{\mu_i,\mu_{i+1}}(h_i) > h_{i+1}$ for all $i \geq n$.

Rules for II:

- $\mu_{i+1} \in \sigma$ is a measure on η^{i+1} projecting to μ_i for all $i \geq n$.
- μ_{i+1} concentrates on $T_{(m_0,\ldots,m_i)} \subset \eta^{i+1}$.

The first player that violates one of these rules loses, and if both players follow the rules for all ω moves, then I wins. The game is closed, so is determined.

LEMMA 3.76: *Player II has a winning strategy for $G_T^{\sigma,\mu}$ for all $\mu \in \sigma$.*

Proof. Suppose for contradiction that I has a winning strategy in $G_T^{\sigma,\mu}$, that is if both players follow all the rules of the game, then I can continue playing for ω moves. Suppose I plays integers $m_0, \ldots, m_n$ such that $T_{(m_0,\ldots,m_{n-1})} \in \mu = \mu_n$, an $s_n \in j_{\mu_n}(\eta)^{n+1}$ such that $s_n \in j_{\mu_n}(T_{(m_0,\ldots,m_n)})$ such that $[id]_{\mu_n} \subsetneq s_n$, and some $h_n \in \mathrm{OR}$ on his first move. II then responds with $\mu_{n+1} = \pi_X(\overline{\mu}_{n+1})$, where

$$A \in \overline{\mu}_{n+1} \Leftrightarrow s_n \in j_{\mu_n}(\pi_X(A)).$$

We have that $\overline{\mu}_{n+1} \in meas^{\mathbf{\Gamma x}}(\eta_X^{<\omega}) \in M_X[G]$ (so μ_{n+1} is defined). Similarly, suppose for $i > n$, I has played (m_i, s_i, h_i) such that

- $m_i \in \omega$,
- $s_i \in j_{\mu_i}(T_{(m_0,\ldots,m_i)})$,
- $j_{\mu_j,\mu_{j+1}}(s_j) \subsetneq s_{j+1}$ for $j < i$,
- $h_i \in \mathrm{OR}$,
- $j_{\mu_j,\mu_{j+1}}(h_j) > h_{j+1}$ for all $j < i$.

II then responds with $\mu_{i+1} = \pi_X(\overline{\mu}_{i+1})$, where

$$A \in \overline{\mu}_{i+1} \Leftrightarrow s_i \in j_{\mu_i}(\pi_X(A)).$$

Again, μ_{i+1} makes sense since $\bar{\mu}_{i+1} \in M_X[G]$.

After ω many moves, the players play a real $x = (m_0, m_1, \ldots)$, a tower of measures $(\mu_i \mid i < \omega)$, a sequence of ordinals $(h_i \mid n \leq i < \omega)$ witnessing the tower $(\mu_i \mid n \leq i < \omega)$ is illfounded, and the sequence $(s_i \mid n \leq i < \omega)$. By closure of M_X and the fact that we can find a canonical name for each $\bar{\mu}_i$ in M_X,[50] the sequence $(\bar{\mu}_i \mid i < \omega) \in M_X[G]$ (and so $(\mu_i \mid i < \omega) = \pi_X(\bar{\mu}_i \mid i < \omega)$).

[50] In fact, the definition of $\bar{\mu}_i$ only depends on s_i and not on G. Furthermore, (the codeset of) $\bar{\mu}_i$ is also OD in $M_X[G]$ from $T_X = \pi_X^{-1}(T)$ (so the $\bar{\mu}_i$'s have symmetric names in

In $M_X[G]$, the tower $(\bar{\mu}_i \mid i < \omega)$ is illfounded. By Lemma 3.5.9 of [28], there is a tree $\bar{W} \in L[T_X, x]$ for some $x \in \mathbb{R}^{V[G]}$ on $\omega \times \eta_X$ such that the $\bar{\mu}_i$'s concentrate on $\bar{W}$ and the function $\bar{h}(i) = [\mathrm{rk}_{\bar{W}}]_{\bar{\mu}_i}$ is a pointwise minimal witness to the illfoundedness of $(\bar{\mu}_i \mid i < \omega)$. Let $h = \pi_X(\bar{h})$ and $W = \pi_X(\bar{W})$; since $\bar{\mu}_i$'s concentrate on $\bar{W}$, $s_i \in j_{\mu_i}(W)$ for all i.

Let $h'(i) = \mathrm{rk}_{j_{\mu_i}(W)}(s_i)$. We have

(1) $j_{\mu_i,\mu_{i+1}}(h'(i)) = j_{\mu_i,\mu_{i+1}}(\mathrm{rk}_{j_{\mu_i}(W)}(s_i)) > h'(i+1)$ for all i as $j_{\mu_i,\mu_{i+1}}(s_i) \subsetneq s_{i+1}$;

(2) $h'(n) = \mathrm{rk}_{j_{\mu_n}(W)}(s_n) < \mathrm{rk}_{j_{\mu_n}(W)}([id]_{\mu_n}) = h(n)$ because $[id]_{\mu_n} \subsetneq s_n$.

So h' witnesses $(\mu_i \mid i < \omega)$ is illfounded and $h'(n) < h(n)$. Contradiction. ∎

Lemma 3.76 easily implies that for each $\mu \in meas^{\mathbf{\Gamma}}(\eta^{<\omega})$, there is a countable set of measures $\sigma \subset meas^{\mathbf{\Gamma}}(\eta^{<\omega})$ that stabilizes μ (in the sense of [28, Section 4]). By a simple argument using DC and the fact that if σ stabilizes μ then any $\sigma' \supseteq \sigma$ stabilizes μ, we get a countable $\tau \subset meas^{\Gamma}(\eta^{<\omega})$ that stabilizes every $\mu \in \tau$.

Knowing this, [28, Sections 4.1, 4.3] constructs a self-justifying system $\mathcal{A}$ for $\mathbf{Env}(\mathbf{\Gamma}) = \mathbf{\Omega_0}$ in $V[G]$. Using the argument in [11, Section 5.5], we can then find a pair $(\mathcal{N}, \Psi)$ such that $\mathcal{N} \in V$, $|\mathcal{N}|^V \leq \kappa$, $\mathcal{N}$ is Γ-suitable, and Ψ is a $(\omega_1^{V[G]}, \omega_1^{V[G]})$-strategy for $\mathcal{N}$ such that Ψ is Γ-fullness preserving and has branch condensation (and hence hull condensation by results in [7]); furthermore, $\Psi \restriction V \in V$. In fact Ψ is guided by $\mathcal{A}$ and hence $\Psi \notin \Omega_0$. By the lemmas in Section 3.2, we can then extend Ψ to a (λ^+, λ^+)-strategy in V and further to an (λ^+, λ^+) strategy in $V[G]$ (also called Ψ) that has branch condensation. Furthermore, results of the previous section allow us to construct operators $x \mapsto \mathcal{M}_n^{\Psi,\sharp}(x)$ for all $n < \omega$. This means $PD(\Psi)$ holds and since Ψ is guided by a self-justifying system, we can conclude by standard methods that the operator $\mathcal{F} = \mathcal{F}_{\Psi,\varphi_{\mathrm{all}}}$ is self-scaled. This allows us to run a core model induction as before to show in $V[G]$

$$\mathrm{sLp}^{\subset\mathcal{F}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R}) \vDash \text{“}L(\wp(\mathbb{R})) \vDash \mathsf{AD}^+ + \Theta = \theta_1''.\ ^{51}$$

M_X); hence we can think of the game $G_T^{\sigma,\mu}$ as being defined in V where player II plays finite sequences of ordinals in $X[G]$, which are π_X-images of the sequences of ordinals that define the $\bar{\mu}_i$'s in $M_X[G]$.

51 Of course, what we showed in the previous section also shows $\mathcal{M}_1^{\Psi,\sharp}$ exists and is (λ^+, λ^+)-iterable.

The above construction works in general and allows us to show that "$\mathrm{sLp}^{\mathsf{G}^{\mathcal{F}}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R}) \vDash \mathsf{AD}^+ + \Theta = \theta_{\alpha+1}$" for any hod pair $(\mathcal{P}^*, \Sigma) \in \Omega$ below κ and Σ is Ω-fullness preserving, and Σ is projectively equivalent to a set of Wadge rank θ_α, where $\mathcal{F} = \mathcal{F}_{\Sigma,\varphi_{\mathrm{all}}}$. In other words, we have shown that the Solovay sequence of Ω is of limit length.

3.5. Getting $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. In the previous subsections, we show that the core model induction cannot stop at successor stages, i.e., in $V[G]$, $L(\Omega, \mathbb{R})$ cannot satisfy $\mathsf{AD}^+ + \Theta = \theta_{\alpha+1}$ for some $\alpha \geq -1$.[52] This means the Solovay sequence of Ω is of limit order type. In this subsection, we show that there is some Solovay initial segment Ω^* of Ω such that $L(\Omega^*, \mathbb{R}) \vDash$ "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular". This contradicts (†). So we get after all that there is a model of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular".

Let $\langle \theta^\Omega_\alpha \mid \alpha \leq \lambda \rangle$ be the Solovay sequence of Ω. We write θ_β for θ^Ω_β and Θ for Θ^Ω and let $\alpha = \mathrm{cof}^{V[G]}(\Theta)$. Note that λ is limit and for each $\beta < \lambda$, $L(\Omega \restriction \theta_\beta, \mathbb{R}) \cap \wp(\mathbb{R}) = \Omega \restriction \theta_\beta$. Note also that $\Theta < \kappa^{++}$ since otherwise, we've already reached a model of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" by the following lemma.

Lemma 3.77: *Suppose $\Theta = \kappa^{++}$. Then in $V[G]$, $\Omega = \wp(\mathbb{R}) \cap L(\Omega, \mathbb{R})$. Consequently, $L(\Omega, \mathbb{R}) \vDash$ "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular".*

Proof. Suppose not. Let α be the least such that $\rho_\omega(L_\alpha(\Omega, \mathbb{R})) = \mathbb{R}$. Hence $\alpha \geq \Theta$ by our assumption. Let $f : \alpha \times \Omega \twoheadrightarrow L_\alpha(\Omega, \mathbb{R})$ be a surjection that is definable over $L_\alpha(\Omega, \mathbb{R})$ (from parameters).

We first define a sequence $\langle H_i \mid i < \omega \rangle$ as follows. Let $H_0 = \mathbb{R}$. By induction, suppose H_n is defined and there is a surjection from $\mathbb{R} \twoheadrightarrow H_n$. Suppose (ψ, a) is such that $a \in H_n$ and $L_\alpha(\Gamma, \mathbb{R}) \vDash \exists x \psi[x, a]$. Let $(\gamma_{a,\psi}, \beta_{a,\psi})$ be the $<_{lex}$-least pair such that there is a $B \in \Gamma$ with Wadge rank $\beta_{a,\psi}$ such that

$$L_\alpha(\Omega, \mathbb{R}) \vDash \psi[f(\gamma_{a,\psi}, B), a].$$

Let then $H_{n+1} = H_n \cup \{f(\gamma_{a,\psi}, B) \mid L_\alpha(\Omega, \mathbb{R}) \vDash \exists x \psi[x, a] \wedge w(B) = \beta_{a,\psi} \wedge a \in H_n\}$. It's easy to see that there is a surjection from $\mathbb{R} \twoheadrightarrow H_{n+1}$. This uses the fact that $\Theta^\Omega = \Theta$ is regular, which implies $\sup\{\beta_{a,\psi} \mid a \in H_n \wedge L_\alpha(\Omega, \mathbb{R}) \vDash$

[52] Ω can also be characterized as the set of all $A \subseteq \mathbb{R}$ such that A is Wadge reducible to a Σ-cmi-operator J that determines itself on generic extensions, for some hod pair $(\mathcal{P}, \Sigma)$ below ω_2.

$\exists x\psi[x,a]\} < \Theta$. Let $H = \bigcup_n H_n$. By construction, $H \prec L_\alpha(\Omega,\mathbb{R})$. Finally, let M be the transitive collapse of H.

Say $M = L_\beta(\Omega^*,\mathbb{R})$. By construction, $\Omega^* = \Omega \restriction \theta_\gamma$ for some γ such that $\theta_\gamma < \Theta$. But then $\rho_\omega(L_\beta(\Omega^*,\mathbb{R})) = \mathbb{R}$. This contradicts that Ω^* is constructibly closed. This gives $\Omega = \wp(\mathbb{R}) \cap L(\Omega,\mathbb{R})$ and in fact, $L(\Omega,\mathbb{R}) \vDash \mathsf{AD}_\mathbb{R} + \Theta$ is regular. ∎

Now let $\mathcal{H}$ be the direct limit of all hod pairs $(\mathcal{Q},\Lambda) \in \Omega$ such that Λ is Ω-fullness preserving and has branch condensation. For each $X \in S$ such that $\{\mathcal{H}\} \subseteq X$, $\Omega \in X[G]$, let $(\Omega_X,\mathcal{H}_X,\Theta_X,\alpha_X) = \pi_X^{-1}(\Omega,\mathcal{H},\Theta,\alpha)$. For each $\beta < \lambda'_X$, where λ'_X is the order-type of the closure of the set of Woodin cardinals in $\mathcal{H}_X$, let $\Sigma_{X,\beta}$ be the canonical strategy for $\mathcal{H}_X(\beta)$, which is the tail of a hod pair $(\mathcal{Q},\Lambda)$ below κ (in $M_X[G]$) and $\mathcal{H}_X(\beta)$ is the direct limit of all Λ-iterates in $M_X[G]$. The fact that $\mathcal{H}$ ($\mathcal{H}_X$, respectively) is the direct limit of hod mice in Ω follows from our smallness assumption (†) and the remarks after it. Then $(H_X(\beta),\Sigma_{X,\beta})$ is a hod pair below κ. Let $\Sigma_X^- = \bigoplus_{\beta<\lambda_X} \Sigma_{X,\beta}$ and

$$\mathcal{H}_X^+ = [\mathrm{Lp}^{\mathrm{g}\Sigma_X^-}(\mathcal{H}_X)]^{\Omega}.^{53}$$

Finally, let $\mathcal{H}^+$ be the union of all $\mathcal{M}$ such that $\mathcal{M}$ is sound, $\mathcal{H} \lhd \mathcal{M}$, $\rho_\omega(\mathcal{M}) \leq o(\mathcal{H})$, and whenever $\mathcal{M} \in X \in S$, then $\pi_X^{-1}(\mathcal{M}) \lhd \mathcal{H}_X^+$.

Lemma 3.78: 1) $\forall^* X \in S\ \mathcal{H}_X^+ \in M_X$ *and* $\pi_X(\mathcal{H}_X^+) = \mathcal{H}^+$.
2) *Let* X *be as in* 1). *Then no levels of* $\mathcal{H}_X^+$ *project across* Θ_X.

Proof. To prove 1), note that $\mathrm{cof}^V(o(\mathcal{H}^+)) \leq \kappa$. To see this, first note that $o(\mathcal{H}^+) < \kappa^{++}$; this follows from $\neg\square_{\kappa^+}$. We can then rule out $\mathrm{cof}^V(o(\mathcal{H}^+)) = \kappa^+$ using $\neg\square(\kappa^+)$ (since otherwise, the $\square$-sequence constructed in $\mathcal{H}^+$ of length $o(\mathcal{H}^+)$ gives rise to an nonthreadable coherent sequence witnessing $\square(\kappa^+)$, contradicting $\neg\square(\kappa^+)$). This means, $\forall^* X \in S$, the range of π_X is cofinal in $o(\mathcal{H}^+)$. This implies

$$\forall^* X \in S, \mathcal{H}_X^+ \trianglelefteq \pi_X^{-1}(\mathcal{H}^+) \in X.$$

Otherwise, fix such an X and let $\mathcal{M}_X \lhd \mathcal{H}_X^+$ be such that $\mathcal{M}_X \notin \pi_X^{-1}(\mathcal{H}^+)$, $o(\mathcal{M}_X)$ has cofinality ω, $\rho_1(\mathcal{M}_X) = \Theta_X$ and let $\mathcal{M} = \mathrm{Ult}_0(\mathcal{M}_X, E_X)$ where E_X is the (λ_X,Θ)-extender derived from π_X. Since π_X is cofinal in $o(\mathcal{H}^+)$,

[53] Recall that our convention is: $\mathcal{H}_X(\alpha+1)$ is a g-organized $\Sigma_{\mathcal{H}_X(\alpha)} = \Sigma_{X,\alpha}$-mouse for each $\alpha < \lambda_X$. In general, hod mice in this paper are g-organized.

$\mathcal{M} \notin \mathcal{H}^+$. But whenever $Y \in S$ is such that $\mathcal{M} \in Y$, it's easy to see that $\pi_Y^{-1}(\mathcal{M}) \in \mathcal{H}_Y^+$.[54] So $\mathcal{M} \in \mathcal{H}^+$ after all. Contradiction.

Suppose equality fails. By pressing down, there is some $\mathcal{M} \triangleleft \mathcal{H}^+$, some stationary set $T \subseteq S$ such that for $X \in T$, $\mathcal{M} \notin \pi_X(\mathcal{H}_X^+)$. But $\forall^* X\ \pi_X^{-1}(\mathcal{M}) \triangleleft \mathcal{H}_X^+$ by definition of $\mathcal{H}^+$. Contradiction. This completes the proof of 1).

To prove 2), suppose for contradiction that there is a $\mathcal{P} \triangleleft \mathcal{H}_X^+$ such that $\rho_\omega(\mathcal{P}) < \Theta_X$. Let $\mathcal{P}$ be the least such. By 1), $\mathcal{P} \in M_X$. Let $\beta < \lambda'_X$ be least such that $\rho_\omega(\mathcal{P}) \leq \delta_\beta^{\mathcal{P}}$ and $\delta_\beta > \mathrm{cof}^{\mathcal{P}}(\lambda^{\mathcal{P}})$. $\mathcal{P}$ can be considered a hod premouse over $(\mathcal{H}_X(\beta), \Sigma_{X,\beta})$. Using π_X, we can define a strategy Λ for $\mathcal{P}$ such that Λ acts on stacks above $\delta_\beta^{\mathcal{P}}$ and extends $\bigoplus_{\alpha<\lambda_X} \Sigma_{X,\alpha}$ (the strategy is simply $\bigoplus_{\alpha<\lambda_X} \Sigma_{X,\alpha}$ for stacks based on $\mathcal{H}_X$ (above $\delta_\beta^{\mathcal{P}}$), but the point is that it also acts on all of $\mathcal{P}$ because of π_X). By a core model induction similar to the previous subsections using the fact that Λ has branch condensation and noting that Λ can be extended to $H_{\kappa^+}^{V[G]}$, we can show $L^{\mathsf{G}\mathcal{F}}(\mathbb{R}) \vDash \mathsf{AD}^+$, where $\mathcal{F} = \mathcal{F}_{\Lambda,\varphi_{\mathrm{all}}}$, and hence $L(\Lambda \restriction \mathrm{HC}, \mathbb{R}) \vDash \mathsf{AD}^+$. This implies $Code(\Lambda \restriction \mathrm{HC}) \in \Omega$ by definition of Ω.

In Ω, let $\mathcal{F}$ be the direct limit system of $\Sigma_{X,\beta}$ hod pairs $(\mathcal{Q}, \Psi)$ Dodd–Jensen equivalent to $(\mathcal{P}, \Lambda)$. $\mathcal{F}$ can be characterized as the direct limit system of $\Sigma_{X,\beta}$ hod pairs $(\mathcal{Q}, \Psi)$ in Ω such that Ψ is $\Gamma(\mathcal{P}, \Lambda)$-fullness preserving and has branch condensation and $\Gamma(\mathcal{Q}, \Psi) = \Gamma(\mathcal{P}, \Lambda)$. $\mathcal{F}$ only depends on $\Sigma_{X,\beta}$ and the Wadge rank of $\Gamma(\mathcal{P}, \Lambda)$ and hence is $OD_{\Sigma_{X,\beta}}^{L(\mathbb{R},C)}$ for some $C \in \Omega$.

Fix such a C and note that $L(\mathbb{R}, C) \vDash \mathsf{AD}^+ + \mathsf{SMC}$. Let $A \subseteq \delta_\beta^{\mathcal{P}}$ witness $\rho_\omega(\mathcal{P}) \leq \delta_\beta^{\mathcal{P}}$, that is, there is a formula ϕ such that for all $\alpha \in \delta_\beta^{\mathcal{P}}$,

$$\alpha \in A \Leftrightarrow \mathcal{P} \vDash \phi[\alpha, p],$$

where p is the standard parameter of $\mathcal{P}$. Now A is $OD_{\Sigma_{X,\beta}}$ in $L(\mathbb{R}, C)$; this is because letting $\mathcal{M}_\infty$ be the direct limit of $\mathcal{F}$ under iteration maps, then in $L(\mathbb{R}, C)$, $\mathcal{M}_\infty \in HOD_{\Sigma_{X,\beta}}$ and A witnesses that $\rho_\omega(\mathcal{M}_\infty) \leq \delta_\beta^{\mathcal{P}}$. By SMC in $L(\mathbb{R}, C)$ and the fact that $\mathcal{H}_X(\beta+1)$ is Ω-full, we get that $A \in \mathcal{P}$. This is a contradiction. ∎

[54] Note that $\pi_Y^{-1}(\mathcal{M}) = \mathrm{Ult}_0(\mathcal{M}_X, E_{X,Y})$ where $E_{X,Y}$ is derived from $\pi_{X,Y}$ the same way E_X is derived from π_X. Let $Z \prec H_{\omega_6}$ be countable and contain all relevant objects and $\pi : M \to Z$ be the uncollapse map. Write a^* for $\pi^{-1}(a)$ for $a \in Z$. Then it's easy to see using countable completeness of $E_{X,Y}$ that $(\pi_Y^{-1}(\mathcal{M}))^*$ is embeddable into $\mathcal{M}_X$, which in turns gives $(\pi_Y^{-1}(\mathcal{M}))^* \triangleleft ((\mathcal{H}_Y)^+)^*$.

Now let X be as in Lemma 3.78. Using the embedding π_X and the construction in [8, Section 11], we obtain a strategy Σ_X for $\mathcal{H}^+_X$ such that

(1) Σ_X extends Σ^-_X.
(2) for any Σ_X-iterate $\mathcal{P}$ of $\mathcal{H}^+_X$ via a stack $\vec{\mathcal{T}}$ such that $i^{\vec{\mathcal{T}}}$ exists, there is an embedding $\sigma : \mathcal{P} \to \mathcal{H}^+$ such that $\pi_X = \sigma \circ i^{\vec{\mathcal{T}}}$. Furthermore, letting $\Sigma_{\mathcal{P}}$ be the $\vec{\mathcal{T}}$-tail of Σ_X, for all $\alpha < \lambda^{\mathcal{P}}$, $\Sigma_{\mathcal{P}(\alpha)} \in \Omega$ has branch condensation.
(3) Σ_X is $\Gamma(\mathcal{H}^+_X, \Sigma_X)$-fullness preserving.

Remark 3.79: the construction in [8] is nontrivial in the case that $\mathcal{H}^+_X \vDash \mathrm{cof}(\Theta_X)$ is measurable; otherwise, as mentioned in the proof of Lemma 3.78, Σ_X is simply Σ^-_X but, because of π_X, it acts on all of $\mathcal{H}^+_X$.

We claim that $\Sigma_X \in \Omega$. Let $(\mathcal{Q}, \Lambda)$ be a Σ_X-iterate of $\mathcal{H}^+_X$ such that

a) $\mathcal{Q} \in V$, $|\mathcal{Q}|^V \leq \kappa$;
b) $\Lambda \restriction V \in V$;
c) Λ has branch condensation.

c) follows from results in [7]. a) and b) can be ensured using boolean comparisons (see [7]). Using a), b), c), and arguments in previous subsections, we get that in $V[G]$,

$$L^{\mathsf{G}^{\mathcal{F}}}(\mathbb{R}, \mathcal{F} \restriction \mathbb{R}) \vDash \mathsf{AD}^+,$$

where $\mathcal{F} = \mathcal{F}_{\Lambda, \varphi_{\mathrm{all}}}$. This means $\Lambda \in \Omega$, and hence $\Sigma \in \Omega$.

LEMMA 3.80: $\forall^* X \in S\ \Sigma_X$ *is* Ω*-fullness preserving* .

Proof. Suppose not. Let $\vec{\mathcal{T}}_X$ be according to Σ_X with end model $\mathcal{Q}_X$ such that $\mathcal{Q}_X$ is not Ω-full. This means there is a strong cut point γ such that letting $\alpha \leq \lambda^{\mathcal{Q}_X}$ be the largest such that $\delta^{\mathcal{Q}_X}_\alpha \leq \gamma$, then in Ω, there is a mouse $\mathcal{M} \lhd \mathrm{Lp}^{\mathrm{g}\Sigma_{\mathcal{Q}_X(\alpha)}}(\mathcal{Q}_X|\gamma)$[55] such that $\mathcal{M} \notin \mathcal{Q}_X$. Let $k : \mathcal{Q}_X \to \mathcal{H}^+$ be such that $\pi_X = k \circ i^{\vec{\mathcal{T}}_X}$. We use i to denote $i^{\vec{\mathcal{T}}_X}$ from now on.

Let $(\mathcal{P}_X, \Sigma_{\mathcal{P}_X}) \in V$ be a Σ^-_X-hod pair such that:

- $\Gamma(\mathcal{P}_X, \Sigma_{\mathcal{P}_X}) \vDash \mathcal{Q}$ is not full as witnessed by $\mathcal{M}$.
- $\Sigma_{\mathcal{P}_X} \in \Omega$ is fullness preserving and has branch condensation.
- $\lambda^{\mathcal{P}_X}$ is limit and $\mathrm{cof}^{\mathcal{P}_X}(\lambda^{\mathcal{P}_X})$ is not measurable in $\mathcal{P}_X$.

Such a pair $(\mathcal{P}_X, \Sigma_{\mathcal{P}_X})$ exists by boolean comparisons.

[55] The case where $\gamma = \delta_\alpha$ and $\mathcal{M} \lhd \mathrm{Lp}^{\mathsf{G}_{\oplus_{\beta<\alpha}\Sigma_{\mathcal{Q}_X(\beta)}}}(\mathcal{Q}|\gamma)$ is similar.

By arguments similar to that used in 3.78, for almost all $X \in S$, no levels of $\mathcal{P}_X$ project across $\mathcal{H}_X$ and in fact, $o(\mathcal{H}_X^+)$ is a cardinal of $\mathcal{P}_X$. The second clause follows from the following argument. Suppose not and let $\mathcal{N}_X \lhd \mathcal{P}_X$ be least such that $\rho_\omega(\mathcal{N}_X) = \Theta_X$ for stationary many $X \in S$. By minimality of $\mathcal{N}_X$ and an argument similar to that in Lemma 3.78, we may assume for stationary many $X \in S$, $\mathcal{N}_X \in M_X$. Fix such an X. Let $f : \kappa^* \to \Theta_X$ be an increasing and cofinal map in $\mathcal{H}_X^+$, where $\kappa^* = \mathrm{cof}^{\mathcal{H}_X^+}(\Theta_X)$. We can construe $\mathcal{N}_X$ as a sequence $g = \langle \mathcal{N}_\alpha \mid \alpha < \kappa^* \rangle$, where $\mathcal{N}_\alpha = \mathcal{N}_X \cap \delta_{f(\alpha)}^{\mathcal{H}_X^+}$. Note that $\mathcal{N}_\alpha \in \mathcal{H}_X^+$ for each $\alpha < \kappa^*$. Now let $\mathcal{R}_0 = \mathrm{Ult}_0(\mathcal{H}_X^+, \mu)$, $\mathcal{R}_1 = \mathrm{Ult}_\omega(\mathcal{N}_X, \mu)$, where $\mu \in \mathcal{H}_X^+$ is the (extender on the sequence of $\mathcal{H}_X^+$ coding a) measure on κ^* with Mitchell order 0. Let $i_0 : \mathcal{H}_X^+ \to \mathcal{R}_0$, $i_1 : \mathcal{N}_X \to \mathcal{R}_1$ be the ultrapower maps. Letting $\delta = \delta_{\lambda^{\mathcal{H}_\sigma^+}} = \Theta_X$, it's easy to see that $i_0 \restriction (\delta + 1) = i_1 \restriction (\delta + 1)$ and $\wp(\delta)^{\mathcal{R}_0} = \wp(\delta)^{\mathcal{R}_1}$. This means $\langle i_1(\mathcal{N}_\alpha) \mid \alpha < \kappa^* \rangle \in \wp(\delta)^{\mathcal{R}_0}$. By fullness of $\mathcal{H}_X^+$ in Ω,[56] $\langle i_1(\mathcal{N}_\alpha) \mid \alpha < \kappa^* \rangle \in \mathcal{H}_X^+$. Using i_0, $\langle i_1(\mathcal{N}_\alpha) \mid \alpha < \kappa^* \rangle \in \mathcal{H}_X^+$, and the fact that $i_0 \restriction \mathcal{H}_X^+ | \Theta_X = i_1 \restriction \mathcal{N}_X | \Theta_X \in \mathcal{H}_X^+$, we can get $\mathcal{N}_X \in \mathcal{H}_X^+$ as follows. For any $\alpha, \beta < \Theta_X$, $\alpha \in \mathcal{N}_\beta$ if and only if $i_0(\alpha) \in i_1(\mathcal{N}_\beta) = i_0(\mathcal{N}_\beta)$. Since $\mathcal{H}_X^+$ can compute the right hand side of the equivalence, it can compute the sequence $\langle \mathcal{N}_\alpha \mid \alpha < \kappa^* \rangle$. Contradiction.

In other words, $\mathcal{P}_X$ thinks $\mathcal{H}_X^+$ is full. For here on, let $\mathcal{P} = \mathcal{P}_X$, $\Sigma_{\mathcal{P}} = \Sigma_{\mathcal{P}_X}$, $(\vec{\mathcal{T}}_X, \mathcal{Q}_X) = (\vec{\mathcal{T}}, \mathcal{Q})$. Let

$$\pi_X^* : \mathcal{P} \to \mathcal{H}^{++}$$

be the ultrapower map by the $(\mathrm{crt}(\pi_X), \Theta)$-extender E_{π_X} induced by π_X. Note that π_X^* extends $\pi_X \restriction \mathcal{H}_X^+$ and $\mathcal{H}^{++}$ is wellfounded since X is closed under ω-sequences. Let

$$i^* : \mathcal{P} \to \mathcal{R}$$

be the ultrapower map by the $(\mathrm{crt}(i), \delta^{\mathcal{Q}})$-extender induced by i. Note that $\mathcal{Q} \lhd \mathcal{R}$ and $\mathcal{R}$ is wellfounded since there is a natural map

$$k^* : \mathcal{R} \to \mathcal{H}^{++}$$

extending k and $\pi_X^* = k^* \circ i^*$. Without loss of generality, we may assume $\mathcal{M}$'s unique strategy $\Sigma_{\mathcal{M}} \leq_w \Sigma_{\mathcal{P}}$. Also, let $(\dot{\mathcal{Q}}, \dot{\vec{\mathcal{T}}})$ be the canonical $Col(\omega, \omega_2)$-names for $(\mathcal{Q}, \vec{\mathcal{T}})$. Let K be the transitive closure of $H_{\omega_2} \cup (\dot{\mathcal{Q}}, \dot{\vec{\mathcal{T}}})$.

[56] Any $A \subset \delta$ in $\mathcal{R}_0$ is $OD^{\Omega}_{\Sigma_X^-}$ (as in the proof of Lemma 3.78, this means $\mathrm{OD}^{L(\mathbb{R},C)}_{\Sigma_X^-}$ for some $C \in \Omega$) and so by Strong Mouse Capturing (SMC, see [7]), $A \in \mathcal{H}_X^+$.

Let $\mathcal{W} = \mathcal{M}_\omega^{\Sigma_\mathcal{P},\sharp}$ and Λ be the unique strategy of $\mathcal{W}$. Let $\mathcal{W}^*$ be a Λ-iterate of $\mathcal{W}$ below its first Woodin cardinal that makes K-generically generic. Then in $\mathcal{W}^*[K]$, the derived model $D(\mathcal{W}^*[K])$ satisfies

$$L(\Gamma(\mathcal{P}, \Sigma_\mathcal{P}), \mathbb{R}) \vDash \dot{\mathcal{Q}} \text{ is not full.}^{57}$$

So the above fact is forced over $\mathcal{W}^*[K]$ for $\dot{\mathcal{Q}}$.

Let $H \prec H_{\lambda^{+++}}$ be countable such that all relevant objects are in H. Let $\pi : M \to H$ invert the transitive collapse and for all $a \in H$, let $\overline{a} = \pi^{-1}(a)$. By the countable completeness of E_{π_X} there is a map $\overline{\pi} : \overline{\mathcal{R}} \to \mathcal{H}_X^{++}$ such that

$$\pi \restriction \overline{\mathcal{P}} = \overline{\pi} \circ \overline{i^*}.^{58}$$

Let $\Sigma_{\overline{\mathcal{P}}}$ be the π-pullback of $\Sigma_\mathcal{P}$ and $\Sigma_{\overline{\mathcal{R}}}$ be the $\overline{\pi}$-pullback of $\Sigma_\mathcal{P}$. Note that $\Sigma_{\overline{\mathcal{P}}}$ extends $\pi^{-1}(\Sigma_\mathcal{P})$ and $\Sigma_{\overline{\mathcal{P}}}$ is also the $\overline{i^*}$-pullback of $\Sigma_{\overline{\mathcal{R}}}$; so in particular, $\Sigma_{\bar{\mathcal{P}}} \leq_w \Sigma_{\bar{\mathcal{R}}}$. We also confuse $\bar{\Lambda}$ with the π-pullback of Λ. Hence $\Gamma(\overline{\mathcal{P}}, \Sigma_{\overline{\mathcal{P}}})$ witnesses that $\overline{\mathcal{Q}}$ is not full and this fact is forced over $\bar{\mathcal{W}}^*[\bar{K}]$ for the name $\dot{\mathcal{Q}}$. This means if we further iterate $\bar{\mathcal{W}}^*$ to $\mathcal{Y}$ such that $\mathbb{R}^{V[G]}$ can be realized as the symmetric reals over $\mathcal{Y}$, then in the derived model $D(\mathcal{Y})$,

$$L(\Gamma(\bar{\mathcal{P}}, \Sigma_{\bar{\mathcal{P}}})) \vDash \bar{\mathcal{Q}} \text{ is not full.} \tag{3.4}$$

In the above, we have used the fact that the interpretation of the UB-code of the strategy for $\bar{\mathcal{P}}$ in $\mathcal{Y}$ to its derived model is $\Sigma_{\bar{\mathcal{P}}} \restriction \mathbb{R}^{V[G]}$; this key fact is proved in [7, Theorem 3.26].

Now we iterate $\overline{\mathcal{R}}$ to $\mathcal{S}$ via $\Sigma_{\bar{\mathcal{R}}}$ to realize $\mathbb{R}^{V[G]}$ as the symmetric reals for the collapse $Col(\omega, < \delta^\mathcal{S})$, where $\delta^\mathcal{S}$ is the sup of $\mathcal{S}$'s Woodin cardinals. By (3.4) and the fact that $\Sigma_{\bar{\mathcal{P}}} \leq_w \Sigma_{\bar{\mathcal{R}}}$, we get that in the derived model $D(\mathcal{S})$,

$$\bar{\mathcal{Q}} \text{ is not full as witnessed by } \bar{\mathcal{M}}.$$

So $\Sigma_{\bar{\mathcal{M}}}$ is $\mathrm{OD}_{\Sigma_{\overline{\mathcal{Q}}}}$ in $D(\mathcal{S})$ and hence $\overline{\mathcal{M}} \in \overline{\mathcal{R}}$. This contradicts internal fullness of $\bar{\mathcal{Q}}$ in $\bar{\mathcal{R}}$. ∎

We continue with a key definition, due to G. Sargsyan. This definition is first formulated in [8] and we reformulate it a bit to fit our situation.

Definition 3.81 (Sargsyan): Suppose $X \in S$ and $A \in \mathcal{H}_X^+ \cap \wp(\Theta_X)$. We say that π_X has A**-condensation** if whenever $\mathcal{Q}$ is such that there are elementary embeddings $\upsilon : \mathcal{H}_X^+ \to \mathcal{Q}$, $\tau : \mathcal{Q} \to \mathcal{H}^+$ such that $\mathcal{Q}$ is countable in $V[G]$ and

$\pi_X = \tau \circ \upsilon$, then $\upsilon(T_{\mathcal{H}^+_X,A}) = T_{\mathcal{Q},\tau,A}$, where

$$T_{\mathcal{H}^+_X,A} = \{(\phi, s) \mid s \in [\Theta_X]^{<\omega} \wedge \mathcal{H}^+_X \vDash \phi[s, A]\},$$

and

$$T_{\mathcal{Q},\tau,A} = \{(\phi, s) \mid s \in [\delta^{\mathcal{Q}}_\alpha]^{<\omega} \text{ for some } \alpha < \lambda_{\mathcal{Q}} \wedge \mathcal{H}^+ \vDash \phi[i^{\Sigma^{\tau,-}_{\mathcal{Q}}}_{\mathcal{Q}(\alpha),\infty}(s), \pi_X(A)]\},$$

where $\Sigma^\tau_{\mathcal{Q}}$ is the τ-pullback strategy and $\Sigma^{\tau,-}_{\mathcal{Q}} = \bigoplus_{\alpha<\lambda^{\mathcal{Q}}} \Sigma^\tau_{\mathcal{Q}(\alpha)}$. We say π_X has **condensation** if it has A-condensation for every $A \in \mathcal{H}^+_X \cap \wp(\Theta_X)$.

The following is the key lemma (cf. [8, Section 11]).

LEMMA 3.82: $\forall^* X \in S$ π_X *has condensation.*

Proof. Suppose not. Let T be the set of counterexamples. Hence T is stationary. For each $X \in T$, let A_X be the $\lesssim_X$-least such that π_X fails to have A_X-condensation, where $\lesssim_X$ is the canonical well-ordering of $\mathcal{H}^+_X$. Recall that if $(\mathcal{P}, \Sigma)$ is a hod pair such that $\delta^{\mathcal{P}}$ has measurable cofinality then $\Sigma^- = \bigoplus_{\alpha<\lambda^{\mathcal{P}}} \Sigma_{\mathcal{P}(\alpha)}$. We say that a tuple $\{\langle \mathcal{P}_i, \mathcal{Q}_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega\rangle, \mathcal{M}_{\infty,Y}\}$ is a **bad tuple** if

(1) $Y \in S$;
(2) $\mathcal{P}_i = \mathcal{H}^+_{X_i}$ for all i, where $X_i \in T$;
(3) for all $i < j$, $X_i \prec X_j \prec Y$;
(4) $\mathcal{M}_{\infty,Y}$ is the direct limit of iterates $(\mathcal{Q}, \Lambda)$ of $(\mathcal{H}^+_Y, \Sigma_Y)$ such that Λ has branch condensation;
(5) for all i, $\xi_i : \mathcal{P}_i \to \mathcal{Q}_i$, $\sigma_i : \mathcal{Q}_i \to \mathcal{M}_{\infty,Y}$, $\tau_i : \mathcal{P}_{i+1} \to \mathcal{M}_{\infty,Y}$, and $\pi_i : \mathcal{Q}_i \to \mathcal{P}_{i+1}$;
(6) for all i, $\tau_i = \sigma_i \circ \xi_i$, $\sigma_i = \tau_{i+1} \circ \pi_i$, and $\pi_{X_i,X_{i+1}} \restriction \mathcal{P}_i =_{\text{def}} \phi_{i,i+1} = \pi_i \circ \xi_i$;
(7) $\phi_{i,i+1}(A_{X_i}) = A_{X_{i+1}}$;
(8) for all i, $\xi_i(T_{\mathcal{P}_i,A_{X_i}}) \neq T_{\mathcal{Q}_i,\sigma_i,A_{X_i}}$.

In 8, $T_{\mathcal{Q}_i,\sigma_i,A_{X_i}}$ is computed relative to $\mathcal{M}_{\infty,Y}$, that is

$$T_{\mathcal{Q}_i,\sigma_i,A_{X_i}} =$$
$$\{(\phi, s) \mid s \in [\delta^{\mathcal{Q}_i}_\alpha]^{<\omega} \text{ for some } \alpha < \lambda^{\mathcal{Q}_i} \wedge \mathcal{M}_{\infty,Y} \vDash \phi[i^{\Sigma^{\sigma_i,-}_{\mathcal{Q}_i}}_{\mathcal{Q}_i(\alpha),\infty}(s), \tau_i(A_{X_i})]\}.$$

CLAIM: *There is a bad tuple.*

Proof. For brevity, we first construct a bad tuple

$$\{\langle \mathcal{P}_i, \mathcal{Q}_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega\rangle, \mathcal{H}^+\}$$

with $\mathcal{H}^+$ playing the role of $\mathcal{M}_{\infty,Y}$. We then simply choose a sufficiently large $Y \in S$ and let $i_Y : \mathcal{H}^+_Y \to \mathcal{M}_{\infty,Y}$ be the direct limit map, $m_Y : \mathcal{M}_{\infty,Y} \to \mathcal{H}^+$ be the natural factor map, i.e., $m_Y \circ i_Y = \pi_Y$. It's easy to see that for all sufficiently large Y, the tuple $\{\langle \mathcal{P}_i, \mathcal{Q}_i, m_Y^{-1} \circ \tau_i, m_Y^{-1} \circ \xi_i, m_Y^{-1} \circ \pi_i, m_Y^{-1} \circ \sigma_i \mid i < \omega\rangle, \mathcal{M}_{\infty,Y}\}$ is a bad tuple.

The key point is 6. Let $A^*_X = \pi_X(A_X)$ for all $X \in T$. By Fodor's lemma, there is an A such that $\exists^* X \in T\ A^*_X = A$. So there is an increasing and cofinal sequence $\{X_\alpha \mid \alpha < \omega_3\} \subseteq T$ such that for $\alpha < \beta$, $\pi_{X_\alpha, X_\beta}(A_{X_\alpha}) = A_{X_\beta} = \pi^{-1}_{X_\beta}(A)$. This easily implies the existence of such a tuple

$$\{\langle \mathcal{P}_i, \mathcal{Q}_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega\rangle, \mathcal{H}^+\}. \qquad \blacksquare$$

Fix a bad tuple $\mathcal{A} = \{\langle \mathcal{P}_i, \mathcal{Q}_i, \tau_i, \xi_i, \pi_i, \sigma_i \mid i < \omega\rangle, \mathcal{M}_{\infty,Y}\}$. Let $(\mathcal{P}^+_0, \Pi)$ be a ${}^{\mathrm{g}}\Sigma^-_{\mathcal{P}_0}$-hod pair such that

$$\Gamma(\mathcal{P}^+_0, \Pi) \vDash \mathcal{A} \text{ is a bad tuple.}$$

We may also assume $(\mathcal{P}^+_0, \Pi \restriction V) \in V$, $\lambda^{\mathcal{P}^+_0}$ is limit of nonmeasurable cofinality in $\mathcal{P}^+_0$ and there is some $\alpha < \lambda^{\mathcal{P}^+_0}$ such that $\Sigma_Y \leq_w \Pi_{\mathcal{P}^+_0(\alpha)}$. This type of reflection is possible because we replace $\mathcal{H}^+$ by $\mathcal{M}_{\infty,Y}$. Let $\mathcal{W} = \mathcal{M}^{\sharp,\Sigma_Y,\Pi,\oplus_{n<\omega}\Sigma_{X_n}}_\omega$ and Λ be the unique strategy of $\mathcal{W}$. If $\mathcal{Z}$ is the result of iterating $\mathcal{W}$ via Λ to make $\mathbb{R}^{V[G]}$ generic, then letting h be $\mathcal{Z}$-generic for the Levy collapse of the sup of $\mathcal{Z}$'s Woodin cardinals to ω such that $\mathbb{R}^{V[G]}$ is the symmetric reals of $\mathcal{Z}[h]$, then in $\mathcal{Z}(\mathbb{R}^{V[G]})$,

$$\Gamma(\mathcal{P}^+_0, \Pi) \vDash \mathcal{A} \text{ is a bad tuple.}$$

Now we define by induction $\xi^+_i : \mathcal{P}^+_i \to \mathcal{Q}^+_i$, $\pi^+_i : \mathcal{Q}^+_i \to \mathcal{P}^+_{i+1}$, $\phi^+_{i,i+1} : \mathcal{P}^+_i \to \mathcal{P}^+_{i+1}$ as follows. $\phi^+_{0,1} : \mathcal{P}^+_0 \to \mathcal{P}^+_1$ is the ultrapower map by the $(\mathrm{crt}(\pi_{X_0,X_1}), \Theta_{X_1})$-extender derived from π_{X_0,X_1}. Note that $\phi^+_{0,1}$ extends $\phi_{0,1}$. Let $\xi^+_0 : \mathcal{P}^+_0 \to \mathcal{Q}^+_0$ extend ξ_0 be the ultrapower map by the $(\mathrm{crt}(\xi_0), \delta^{\mathcal{Q}_0})$-extender derived from ξ_0. Finally let $\pi^+_0 = (\phi^+_{0,1})^{-1} \circ \xi^+_0$. The maps $\xi^+_i, \pi^+_i, \phi^+_{i,i+1}$ are defined similarly. Let also $\mathcal{M}_Y = \mathrm{Ult}(\mathcal{P}^+_0, E)$, where E is the (λ_X, Θ_Y)-extender derived from $\pi_{X,Y}$. There are maps $\epsilon_{2i} : \mathcal{P}^+_i \to \mathcal{M}_Y$, $\epsilon_{2i+1} : \mathcal{Q}^+_i \to \mathcal{M}_Y$ for all i such that $\epsilon_{2i} = \epsilon_{2i+1} \circ \xi^+_i$ and $\epsilon_{2i+1} = \epsilon_{2i+2} \circ \pi^+_i$. When $i = 0$, ϵ_0 is simply i_E. Letting $\Sigma_i = \Sigma^-_{\mathcal{P}_i}$ and $\Psi = \Sigma^-_{\mathcal{Q}_i}$, $A_i = A_{X_i}$, there is a finite sequence of ordinals t and a formula $\theta(u, v)$ such that in $\Gamma(\mathcal{P}^+_0, \Pi)$

(9) for every $i < \omega$, $(\phi, s) \in T_{\mathcal{P}_i, A_i} \Leftrightarrow \theta[i^{\Sigma_i}_{\mathcal{P}_i(\alpha),\infty}, t]$, where α is least such that $s \in [\delta^{\mathcal{P}_i}_\alpha]^{<\omega}$;

(10) for every i, there is $(\phi_i, s_i) \in T_{\mathcal{Q}_i, \xi_i(A_i)}$ such that $\neg\theta[i^{\Psi_i}_{\mathcal{Q}_i(\alpha)}(s_i), t]$ where α is least such that $s_i \in [\delta^{\mathcal{Q}_i}_\alpha]^{<\omega}$.

The pair (θ, t) essentially defines a Wadge-initial segment of $\Gamma(\mathcal{P}_0^+, \Pi)$ that can define the pair $(\mathcal{M}_{\infty,Y}, A)$, where $\tau_i(A_i) = A$ for some (any) i.

Now let $X \prec H_{\lambda^{+++}}$ be countable that contains all relevant objects and $\pi : M \to X$ invert the transitive collapse. For $a \in X$, let $\overline{a} = \pi^{-1}(a)$. By countable completeness of the extender E, there is a map $\pi^* : \overline{\mathcal{M}_Y} \to \mathcal{P}_0$ such that $\pi \restriction \overline{\mathcal{M}_Y} = \epsilon_0 \circ \pi^*$. Let $\overline{\Pi_i}$ be the $\pi^* \circ \overline{\epsilon_i}$-pullback of Π. Note that in $V[G]$, $\overline{\Sigma_Y} \leq_w \overline{\Pi_0} \leq_w \overline{\Pi_1} \leq_w \cdots \leq_w \Pi^{\pi^*}$.

Let $\dot{\mathcal{A}} \in (H_{\bar{\kappa}})^M$ be the canonical name for $\bar{\mathcal{A}}$. It's easy to see (using the assumption on $\mathcal{W}$) that if $\mathcal{W}^*$ is a result of iterating $\bar{\mathcal{W}}$ via $\bar{\Lambda}$ (we confuse $\bar{\Lambda}$ with the π-pullback of Λ; they coincide on M) in M below the first Woodin of $\bar{\mathcal{W}}$ to make H-generically generic, where H is the transitive closure of $H^M_{\omega_2} \cup \dot{A}$, then in $\mathcal{W}^*[H]$, the derived model of $\mathcal{W}^*[H]$ at the sup of $\mathcal{W}^*$'s Woodin cardinals satisfies

$$L(\bar{\mathcal{P}}_0, \mathbb{R}) \vDash \dot{\mathcal{A}} \text{ is a bad tuple.}$$

Now we stretch this fact out to $V[G]$ by iterating $\mathcal{W}^*$ to $\mathcal{W}^{**}$ to make $\mathbb{R}^{V[G]}$-generic. In $\mathcal{W}^{**}(\mathbb{R}^{V[G]})$, letting $i : \mathcal{W}^* \to \mathcal{W}^{**}$ be the iteration map, then

$$\Gamma(\bar{\mathcal{P}}_0^+, \bar{\Pi}) \vDash i(\bar{\mathcal{A}})^{59} \text{ is a bad tuple.}$$

By a similar argument as in Theorem 3.1.25 of [25], we can use the strategies $\overline{\Pi_i}^+$'s to simultanously execute a $\mathbb{R}^{V[G]}$-genericity iteration. The last branch of the iteration tree is wellfounded. The process yields a sequence of models $\langle \overline{\mathcal{P}^+_{i,\omega}}, \overline{\mathcal{Q}^+_{i,\omega}} \mid i < \omega \rangle$ and maps $\overline{\xi^+_{i,\omega}} : \overline{\mathcal{P}^+_{i,\omega}} \to \overline{\mathcal{Q}^+_{i,\omega}}$, $\overline{\pi^+_{i,\omega}} : \overline{\mathcal{Q}^+_{i,\omega}} \to \overline{\mathcal{P}^+_{i+1,\omega}}$, and $\overline{\phi^+_{i,i+1,\omega}} = \overline{\pi^+_{i,\omega}} \circ \overline{\pi^+_{i,\omega}}$. Furthermore, each $\overline{\mathcal{P}^+_{i,\omega}}, \overline{\mathcal{Q}^+_{i,\omega}}$ embeds into a Π^{π^*}-iterate of $\overline{\mathcal{M}_Y}$ and hence the direct limit $\mathcal{P}_\infty$ of $(\overline{\mathcal{P}^+_{i,\omega}}, \overline{\mathcal{Q}^+_{j,\omega}} \mid i, j < \omega)$ under maps $\overline{\pi^+_{i,\omega}}$'s and $\overline{\xi^+_{i,\omega}}$'s is wellfounded. We note that $\overline{\mathcal{P}^+_{i,\omega}}$ is a ${}^{\mathrm{g}}\Sigma^\pi_i$-premouse and $\overline{\mathcal{Q}^+_{i,\omega}}$ is a ${}^{\mathrm{g}}\Psi^\pi_i$-premouse because the genericity iterations are above $\overline{\mathcal{P}_i}$ and $\overline{\mathcal{Q}_i}$ for all i and by [7, Theorem 3.26], the interpretation of the strategy of $\bar{\mathcal{P}}_i$ ($\bar{\mathcal{Q}}_i$, respectively) in the derived model of $\overline{\mathcal{P}^+_{i,\omega}}$ ($\overline{\mathcal{P}^+_{i,\omega}}$, respectively) is ${}^{\mathrm{g}}\Sigma^\pi_i$ (${}^{\mathrm{g}}\Psi^\pi_i$, respectively). Let C_i be the derived model of $\overline{\mathcal{P}^+_{i,\omega}}$, D_i be the derived model of $\overline{\mathcal{Q}^+_{i,\omega}}$ (at the sup of the Woodin cardinals of each model); then $\mathbb{R}^{V[G]} = \mathbb{R}^{C_i} = \mathbb{R}^{D_i}$. Furthermore, $C_i \cap \wp(\mathbb{R}) \subseteq D_i \cap \wp(\mathbb{R}) \subseteq C_{i+1} \cap \wp(\mathbb{R})$ for all i.

9, 10 and the construction above give us that there is a $t \in [\mathrm{OR}]^{<\omega}$, a formula $\theta(u,v)$ such that

(11) for each i, in C_i, for every (ϕ, s) such that $s \in \delta^{\overline{\mathcal{P}_i}}$, $(\phi, s) \in T_{\overline{\mathcal{P}_i}, \overline{A_i}} \Leftrightarrow \theta[i^{\overline{\Sigma_i}}_{\overline{\mathcal{P}_i}(\alpha),\infty}(s), t]$ where α is least such that $s \in [\delta^{\overline{\mathcal{P}_i}}_\alpha]^{<\omega}$.

Let n be such that for all $i \geq n$, $\overline{\xi^+_{i,\omega}}(t) = t$. Such an n exists because the direct limit $\mathcal{P}_\infty$ is wellfounded as we can arrange that $\mathcal{P}_\infty$ is embeddable into a Π^{π^*}-iterate of $\bar{\mathcal{M}}_Y$. By elementarity of $\overline{\xi^+_{i,\omega}}$ and the fact that $\overline{\xi^+_{i,\omega}} \restriction \mathcal{P}_i = \overline{\xi_i}$,

(12) for all $i \geq n$, in D_i, for every (ϕ, s) such that $s \in \delta^{\overline{\mathcal{Q}_i}}$, $(\phi, s) \in T_{\overline{\mathcal{Q}_i}, \overline{\xi_i}(\overline{A_i})} \Leftrightarrow \theta[i^{\overline{\Psi_i}}_{\overline{\mathcal{Q}_i}(\alpha),\infty}(s), t]$ where α is least such that $s \in [\delta^{\overline{\mathcal{Q}_i}}_\alpha]^{<\omega}$.

However, using 10, we get

(13) for every i, in D_i, there is a formula ϕ_i and some $s_i \in [\delta^{\overline{\mathcal{Q}_i}}]^{<\omega}$ such that $(\phi_i, s_i) \in T^{\overline{\mathcal{Q}_i}, \overline{\xi_i}(\overline{A_i})}$ but $\neg\phi[i^{\overline{\Psi_i}}_{\overline{\mathcal{Q}_i}(\alpha),\infty}(s_i), t]$ where α is least such that $s \in [\delta^{\overline{\mathcal{Q}_i}}_\alpha]^{<\omega}$.

Clearly 12 and 13 give us a contradiction. This completes the proof of the lemma. ∎

Remark 3.83: The main ideas of the proof above originate from [8, Lemma 11.15]. The main difference is in the situation of [8, Lemma 11.15]; there is an elementary embedding j acting on all of V, so roughly speaking, the iterability of the $\mathcal{P}_i^+$'s is justified by embedding them into $j(\mathcal{P}_0^+)$. Here we don't have such a j; we use pressing down arguments, countable closure of hulls $X \in S$ and reflection arguments instead.

Fix an X satisfying the conclusion of Lemma 3.82. Suppose $(\mathcal{Q}, \vec{\mathcal{T}}) \in I(\mathcal{H}^+_X, \Sigma_X)$ is such that $i^{\vec{\mathcal{T}}} : \mathcal{H}^+_X \to \mathcal{Q}$ exists. Let $\gamma^{\vec{\mathcal{T}}}$ be the sup of the generators of $\vec{\mathcal{T}}$. For each $x \in \mathcal{Q}$, say $x = i^{\vec{\mathcal{T}}}(f)(s)$ for $f \in \mathcal{H}^+_X$ and $s \in [\delta^{\mathcal{Q}}_\alpha]^{<\omega}$, where $\delta^{\mathcal{Q}}_\alpha \leq \gamma^{\vec{\mathcal{T}}}$ is least such, then let $\tau_{\mathcal{Q}}(x) = \pi_X(f)(i^{\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}}_{\mathcal{Q}(\alpha),\infty}(s))$.

Remark 3.84: By Lemma 3.82 and [7, Theorem 3.26], $\tau_{\mathcal{Q}}$ is elementary and $\tau_{\mathcal{Q}} \restriction \delta^{\mathcal{Q}} = i^{\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}}_{\mathcal{Q}|\delta^{\mathcal{Q}},\infty} \restriction \delta^{\mathcal{Q}} = i^{\Lambda}_{\mathcal{Q}|\delta^{\mathcal{Q}},\infty} \restriction \delta^{\mathcal{Q}}$, where Λ is the $\tau_{\mathcal{Q}}$-pullback strategy of $\mathcal{Q}|\delta^{\mathcal{Q}}$.

LEMMA 3.85: *Fix an X satisfying the conclusion of Lemma 3.82. Suppose $(\mathcal{Q}, \vec{\mathcal{T}}) \in I(\mathcal{H}^+_X, \Sigma_X)$ and $(\mathcal{R}, \vec{\mathcal{U}}) \in I(\mathcal{Q}, \Sigma_{\mathcal{Q},\vec{\mathcal{T}}})$ are such that $i^{\vec{\mathcal{T}}}, i^{\vec{\mathcal{U}}}$ exist and $\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}$ and $\Sigma_{\mathcal{R},\vec{\mathcal{U}}}$ have branch condensation. Then $\tau_{\mathcal{Q}} = \tau_{\mathcal{R}} \circ i^{\vec{\mathcal{U}}}$.*

Proof. Let $x \in \mathcal{Q}$. There are some $f \in \mathcal{H}^+_X$ and $s \in [\gamma^{\vec{\mathcal{T}}}]^{<\omega}$ such that

$$x = i^{\vec{\mathcal{T}}}(f)(s).$$

So

$$\tau_{\mathcal{Q}}(x) = \pi_X(f)(i^{\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}}_{\mathcal{Q},\infty}(s)).$$

On the other hand,

$$\begin{aligned}\tau_{\mathcal{R}} \circ i^{\vec{\mathcal{U}}}(x) &= \tau_{\mathcal{R}} \circ i^{\vec{\mathcal{U}}}(i^{\vec{\mathcal{T}}}(f)(s)) = \pi_X(f)(i^{\Sigma_{\mathcal{R},\vec{\mathcal{U}}}}_{\mathcal{R},\infty} \circ i^{\vec{\mathcal{U}}} \circ i^{\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}}(s)) \\ &= \pi_X(f)(i^{\Sigma_{\mathcal{Q},\vec{\mathcal{T}}}}_{\mathcal{Q},\infty}(s)) = \tau_{\mathcal{Q}}(x). \qquad \blacksquare\end{aligned}$$

Let $\mathcal{M}_\infty(\mathcal{H}^+_X, \Sigma_X)$ be the direct limit of all hod pairs $(\mathcal{Q}, \Lambda) \in I(\mathcal{H}^+_X, \Sigma_X)$ such that Λ has branch condensation. The lemma implies that the map $\sigma : \mathcal{M}_\infty(\mathcal{H}^+_X, \Sigma_X) \to \mathcal{H}^+$ defined as

> $\sigma(x) = y$ iff whenever $(\mathcal{R}, \Lambda) \in I(\mathcal{H}^+_X, \Sigma_X)$ is such that Λ has branch condensation, and $i^\Lambda_{\mathcal{R},\infty}(x^*) = x$ for some x^*, then $y = \tau_{\mathcal{R}}(x^*)$

is elementary and $\mathrm{crt}(\sigma) = \delta =_{\mathrm{def}} \delta^{\mathcal{M}_\infty(\mathcal{H}^+_X, \Sigma_X)}$. This implies that $\mathcal{M}_\infty(\mathcal{H}^+_X, \Sigma_X) \vDash$ "δ is regular". Let $(\mathcal{Q}, \Lambda) \in I(\mathcal{H}^+_X, \Sigma_X)$ be such that Λ has branch condensation. By a similar argument as those used before, we get $\Lambda \in \Omega$ and in fact since $\mathcal{Q} \vDash$ "$\delta^{\mathcal{Q}}$ is regular", we easily get that $N = L(\Gamma(\mathcal{Q}, \Lambda), \mathbb{R}) \vDash$ "Θ is regular" (note that Θ^N is the image of $\delta^{\mathcal{Q}}$ under the direct limit map into the direct limit of all Λ-iterates). This contradicts the assumption that there is no model M satisfying "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular". Such an M has to exist after all. This finishes this subsection and the proof of Theorem 0.2.

Remark 3.86: In the above, there are $(\mathcal{Q}, \Lambda) \in V$ that are in $I(\mathcal{H}^+_X, \Sigma_X)$ (this is via a standard boolean comparison argument, cf. [7]). By taking a countable hull, we can find a countable hod pair $(\mathcal{Q}^*, \Lambda^*)$ that generates in V a model of "$\mathsf{AD}_{\mathbb{R}}$+Θ is regular" by an $\mathbb{R}$-genericity iteration argument using the fact that Λ^* has branch condensation and is κ-universally Baire.

4. Question and open problems

We conjecture the following (the proof of which will settle Conjecture 0.5).

Conjecture 4.1: *Suppose κ is a cardinal such that $\kappa^\omega = \kappa$. Let $\lambda = 2^\kappa$. Suppose for every cardinal $\alpha \in [\kappa^+, \lambda^+]$, $\neg\square(\alpha)$. Then in $V^{Col(\omega,\kappa)}$, there are models M containing $\mathbb{R} \cup \mathrm{OR}$ such that $M \vDash \mathsf{LSA}$.*

We're hopeful that the conjecture has a positive answer. This is because we believe it's possible to construct hod mice generating models of LSA from the hypothesis of Conjecture 4.1.

We end the paper with the following technical questions, whose solutions seem to require new core model induction techniques for working with hulls that are not closed under countable sequences. Note that in the most interesting cases (e.g., under PFA) ω_1-guessing models of size $\aleph_1$ cannot be closed under ω-sequences.

Question 4.2: Let $\kappa = 2^{\aleph_2}$. Can one construct a model of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" from the existence of stationary many ω_1-guessing models $X \prec H_{\kappa^{++}}$ such that $|X| = \aleph_1$?

Question 4.3: Let $\kappa = 2^{\aleph_2}$. Can one construct a model of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" from $\neg\square(\alpha)$ for all $\alpha \in [\omega_2, \kappa^+]$?

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