

Tame failures of the unique branch hypothesis and models of $\text{AD}_{\mathbb{R}} + \Theta$ is regular

Grigor Sargsyan

Department of Mathematics, Rutgers University, NJ, USA
grigor@math.rutgers.edu

Nam Trang

Department of Mathematics, UC Irvine, CA, USA
ntrang@math.uci.edu

Received 25 March 2015

Accepted 27 August 2016

Published 4 October 2016

In this paper, we show that the failure of the unique branch hypothesis (UBH) for tame iteration trees implies that in some homogenous generic extension of V there is a transitive model M containing $\text{Ord} \cup \mathbb{R}$ such that $M \models \text{AD}_{\mathbb{R}} + \Theta$ is regular. The results of this paper significantly extend earlier works from [Non-tame mice from tame failures of the unique branch hypothesis, *Canadian J. Math.* **66**(4) (2014) 903–923; Core models with more Woodin cardinals, *J. Symbolic Logic* **67**(3) (2002) 1197–1226] for tame trees.

Keywords: The Unique Branch Hypothesis; tame iteration trees; inner models; large cardinals; core model induction; hod mice.

Mathematics Subject Classification 2010: 03E15, 03E45, 03E60

0. Introduction

We establish, using the core model induction, a lower bound for certain failures of the *Unique Branch Hypothesis* (UBH), which is the statement that every iteration tree that acts on V has at most one cofinal well-founded branch. This paper is a continuation of [8], but it is self-contained.

For the rest of this paper, all trees considered are *nonoverlapping*, that is whenever E and F are extenders such that E is used before F along a branch of the tree, then $\text{lh}(E) \leq \text{crit}(F)$. Suppose there is a proper class of strong cardinals. We say κ reflects the set of strong cardinals (or κ is a strong reflecting strong) if for every λ there is an embedding $j : V \rightarrow M$ witnessing that κ is λ -strong and for

any cardinal $\mu \in [\kappa, \lambda)$, $V \models \text{“}\mu \text{ is strong”}$ if and only if $M \models \text{“}\mu \text{ is strong”}$. Now we recall the definition of *tame trees* from [8].

Definition 0.1 (Tame iteration tree). An iteration tree $\mathcal{T}$ on V is tame if for all $\alpha < \beta < \text{lh}(\mathcal{T})$ such that $\alpha = \text{pred}_{\mathcal{T}}(\beta + 1)$, $\mathcal{M}_{\alpha}^{\mathcal{T}} \models \text{“}\exists \kappa < \lambda < \text{cp}(E_{\beta}^{\mathcal{T}}) \text{ such that } \lambda \text{ is a strong cardinal and } \kappa \text{ is strong reflecting strong”}$.

As in [8], the tameness assumption is mostly for technical convenience. Most of the methods developed in [8] and in this paper can be applied in more generality but some arguments seem to be more particular to this situation.

UBH was first introduced by Martin and Steel in [2]. Towards showing UBH, Neeman, in [4], showed that a certain weakening of UBH called cUBH holds provided there are no nonbland mice.^a However, in [16], Woodin showed that in the presence of supercompact cardinals UBH can fail for tame trees. Woodin constructs alternating chains whose branches are well-founded. Extenders of such trees can be demanded to reflect the set of strong cardinals which reflect strong cardinals. Hence critical points of the branch embeddings can be demanded to be above the first strong cardinal which reflects strong cardinals. It is still an important open problem whether UBH holds for trees that use extenders that are $2^{\aleph_0}$ -closed in the models that they are chosen from.^b A positive resolution of this problem will lead to the resolution of *the inner model problem* for superstrong cardinals and beyond.^c It is worth remarking that the aforementioned form of UBH for tame trees will also lead to the resolution of the inner model problem for superstrong cardinals and beyond. Our work can be viewed as an attempt to prove UBH for tame trees by showing that its failure is strong consistency-wise.^d

We recall some material presented in [6, 8]. Recall Θ is the supremum of ordinals α such that there is a surjection from $\mathbb{R}$ onto α . Working under $\text{AD} + \text{DC}_{\mathbb{R}}$, we say that $(\theta_{\alpha} : \alpha \leq \Omega)$ is the *Solovay sequence* if: (a) θ_0 is the supremum of ordinals α such that there is an *OD* surjection from $\mathbb{R}$ onto α , (b) for $\alpha < \Omega$ (and $\theta_{\alpha} < \Theta$), $\theta_{\alpha+1}$ is the supremum of ordinals α such that for some $A \subseteq \mathbb{R}$ of Wadge rank θ_{α} , there is an OD_A surjection from $\mathbb{R}$ onto α , (c) for $\beta \leq \Omega$ limit, $\theta_{\beta} = \sup_{\alpha < \beta} \theta_{\alpha}$, and (d) $\theta_{\Omega} = \Theta$. For a set $A \subseteq \mathbb{R}$, we let θ_A be the supremum of α such that there is an OD_A surjection from $\mathbb{R}$ onto α . We may also define the Solovay sequence $(\theta_{\alpha}^{\Gamma} : \alpha \leq \Omega)$ of a pointclass Γ with sufficient closure.^e We list some important determinacy theories in increasing consistency strength: (1) AD^+ , (2) $\text{AD}^+ + \Theta > \theta_0$, (3) $\text{AD}_{\mathbb{R}}$, (4) $\text{AD}_{\mathbb{R}} + \text{DC}$, (5) $\text{AD}_{\mathbb{R}} + \Theta$ is regular.

^aWe will not use this terminology.

^bThe first extender that used in the trees Woodin constructs is not $2^{\aleph_0}$ -closed.

^cThe inner model problem, one of the main problems in inner model theory, seeks to construct canonical inner models of large cardinals (e.g., measurable cardinals, Woodin cardinals, superstrong cardinals) from appropriate large cardinal or strong combinatorial (e.g., PFA plus perhaps large cardinals) assumptions.

^dOf course, UBH for tame trees fails in the presence of supercompact cardinals by Woodin’s result. Our main theorem, Theorem 0.2, shows that it is very hard to construct such counter-examples.

^eIn particular, we demand that for $\alpha \leq \Omega$, letting $\gamma = \theta_{\alpha}^{\Gamma}$, $\wp_{\gamma}(\mathbb{R})$ is constructibly closed.

$\text{Con}((5))$ implies, among others, the consistency of $\text{MM}(\text{c})$, a significant fragment of *Martin's Maximum* (MM) and was conjectured by Woodin to be equiconsistent with a supercompact cardinal. The first author, in [5], shows that (5) is consistent relative to the existence of a Woodin cardinal which is a limit of Woodin cardinals, which is significantly weaker than a supercompact cardinal.

The following is the main theorem of the paper, which improves significantly the lower bounds obtained from [8, 11]. The paper [11] obtains (1) as a lower bound and the main result of [8] obtains (2) as a lower bound for failures of UBH for tame trees; Theorem 0.2 obtains (5) as a lower bound under the same hypothesis.

Theorem 0.2 (Main Theorem). *Suppose there is a proper class of strong cardinals and UBH fails for tame trees. Then in a set generic extension of V , there is a transitive inner model M such that $\text{Ord} \cup \mathbb{R} \subseteq M$ and $M \models \text{“AD}_{\mathbb{R}} + \Theta \text{ is regular”}$.*

We remark that there are papers in the literature that obtain “ $\text{AD}_{\mathbb{R}} + \Theta$ is regular” as a lower bound for certain theories. For instance, in [14], the second author constructed an inner model of “ $\text{AD}_{\mathbb{R}} + \Theta$ is regular” from the Proper Forcing Axiom, and in [7], the first author constructed an inner model of “ $\text{AD}_{\mathbb{R}} + \Theta$ is regular” from certain failures of covering. However, the methods developed in this paper are different from those methods developed in the two aforementioned papers in a rather significant way. In the aforementioned papers, the authors work under hypothesis that implies the failure of lower part covering. More precisely, in the aforementioned papers, equivalents of Theorem 3.4 are proved while having the luxury of knowing that $|\mathcal{P}^+| < \omega_2$ in $M[m]$ (see the beginning of Sec. 3 for definitions). Here we do not know that $|\mathcal{P}^+| < \omega_2$, yet our large cardinal assumption still allows us to get an (ω_1, ω_1) -iteration strategy with the desired properties. We anticipate that the construction of such a strategy will be useful in other similar contexts as well.

1. Preliminaries

1.1. *Stacking mice*

We recall the notions used in [8]. Fix some uncountable cardinal λ and assume ZF. Notice that any function $f : H_\lambda \rightarrow H_\lambda$ can be naturally coded by a subset of $\wp(\bigcup_{\kappa < \lambda} \wp(\kappa))$. We then let $\text{Code}_\lambda^* : H_\lambda^{H_\lambda} \rightarrow \wp(\bigcup_{\kappa < \lambda} \wp(\kappa))$ be one such coding. If $\lambda = \omega_1$ then we just write Code^* . Because for $\alpha \leq \lambda$, any (α, λ) -iteration strategy^f for a hybrid premouse^g of size $< \lambda$ is in $H_\lambda^{H_\lambda}$, we have that any such strategy is in the domain of Code_λ^* .

^fThis is an iteration strategy for stacks of less than α normal trees, each of which has length less than λ . Typically these are fine-structural n -maximal iteration trees (as defined in [3]), where n is the degree of soundness of the premouse we iterate. We will suppress this parameter throughout our paper as fine-structural iteration trees we deal with are n -maximal for an appropriate n , which will be clear in the context.

^gFor more on hybrid mice, see [5] or [10].

Suppose $\Lambda \in \text{dom}(\text{Code}_\lambda^*)$ is a strategy with hull condensation and $\mu \leq \lambda$. Recall that we say F is (μ, Λ) -mouse operator if for some $X \in H_\lambda$ and formula ϕ in the language of Λ -mice, whenever Y is such that $X \in Y$, $F(Y)$ is the minimal μ -iterable Λ -mouse satisfying $\phi[Y]$.

We then let Code_λ be Code_λ^* restricted to $F \in \text{dom}(\text{Code}_\lambda^*)$ that are defined by the following recursion:

- (1) for some $\alpha \leq \lambda$, F is an (α, λ) -iteration strategy with hull condensation for an extender mouse or a hod mouse,^h or
- (2) for some $\alpha \leq \lambda$ and for some (α, λ) -iteration strategy $\Lambda \in \text{dom}(\text{Code}_\lambda)$ with hull condensation (Λ can be $\emptyset$), F is a (λ, Λ) -mouse operator, or
- (3) for some $\alpha \leq \lambda$, for some (α, λ) -iteration strategy $\Lambda \in \text{dom}(\text{Code}_\lambda)$ with hull condensation, for some (λ, Λ) -mouse operator $G \in \text{dom}(\text{Code}_\lambda)$ and for some $\beta \leq \lambda$, F is a (β, Λ) -iteration strategy with hull condensation for some G -mouse $\mathcal{M} \in H_\lambda$.

Code_λ is the set of codes of “useful” operatorsⁱ and includes all operators that we will construct during the course of the core model induction (for an appropriate λ). When $\lambda = \omega_1$ then we just write Code instead of Code_{ω_1} . Given an $F \in \text{dom}(\text{Code}_\lambda)$ we let $\mathcal{M}_F$ be, in the case F is an iteration strategy, the structure that F iterates and, in the case F is a mouse operator, the base of the cone on which F is defined.

Let $\mathcal{P} \in H_\lambda$ be a hybrid premouse and for some $\alpha \leq \lambda$, let Σ be (α, λ) -iteration strategy with hull condensation for $\mathcal{P}$. Suppose now that $\Gamma \subseteq \wp(\bigcup_{\kappa < \lambda} \wp(\kappa))$ is such that $\text{Code}_\lambda(\Sigma) \in \Gamma$. Given a Σ -premouse $\mathcal{M}$, we say $\mathcal{M}$ is Γ -iterable if $|\mathcal{M}| < \lambda$ and $\mathcal{M}$ has a λ -iteration strategy (or (α, λ) -iteration strategy for some $\alpha \leq \lambda$) Λ such that $\text{Code}_\lambda(\Lambda) \in \Gamma$.^j We let $\text{Mice}^{\Gamma, \Sigma}$ be the set of Σ -premise that are Γ -iterable.

Definition 1.1. Given a Σ -premouse $\mathcal{M} \in H_\lambda$, we say $\mathcal{M}$ is countably α -iterable if whenever $\pi : \mathcal{N} \rightarrow \mathcal{M}$ is a countable submodel of $\mathcal{M}$, $\mathcal{N}$, as a Σ^π -mouse, is α -iterable. When $\alpha = \omega_1 + 1$ then we just say that $\mathcal{M}$ is countably iterable.

Suppose $\Gamma \subseteq \wp(\bigcup_{\kappa < \lambda} \wp(\kappa))$ for some λ . We say $\mathcal{M}$ is countably Γ -iterable if whenever π and $\mathcal{N}$ are as above, $\mathcal{N}$ is Γ -iterable, that is, $\mathcal{N}$ has a λ -iteration strategy in Γ .^k

Suppose $\mathcal{M}$ is a Σ -premouse. We then let $o(\mathcal{M}) = \text{Ord} \cap \mathcal{M}$. We also let $\mathcal{M}||\xi$ be $\mathcal{M}$ cutoff at ξ , i.e., we keep the predicate indexed at ξ . We let $\mathcal{M}|\xi$ be $\mathcal{M}||\xi$ without the last predicate. We say ξ is a *cutpoint* of $\mathcal{M}$ if there is no extender E on $\mathcal{M}$ such that $\xi \in (\text{cp}(E), \text{lh}(E)]$. We say ξ is a *strong cutpoint* if there is no E

^hIn this case as well as in cases $\alpha \geq 1$; if $\alpha = 1$, F is a strategy on normal trees of length $< \lambda$. See Sec. 1.3 for more on hod mice.

ⁱHere we regard an iteration strategy as an operator.

^jRecall that iteration strategy for a Σ -mouse must respect Σ . In particular, all Λ -iterates of $\mathcal{M}$ are Σ -premise.

^kIn the case $\Gamma \subseteq \wp(\text{HC})$, then $\mathcal{N}$ is ω_1 -iterable with a strategy Λ in Γ . We typically have in practice that Λ is the restriction of some λ -strategy for $\lambda > \omega_1$.

on $\mathcal{M}$ such that $\xi \in [\text{cp}(E), \text{lh}(E)]$. We say $\eta < o(\mathcal{M})$ is *overlapped* in $\mathcal{M}$ if η is not a cutpoint of $\mathcal{M}$. Given $\eta < o(\mathcal{M})$ we let

$$\mathcal{O}_\eta^{\mathcal{M}} = \bigcup \{ \mathcal{N} \triangleleft \mathcal{M} : \rho(\mathcal{N}) \leq \eta \text{ and } \eta \text{ is not overlapped in } \mathcal{N} \}.$$

Given a self-well-ordered¹ $a \in H_\lambda$ we define the stacks over a as follows. In the definition below, we fix a Γ as above and a strategy Σ such that $\mathcal{M}_\Sigma \in a$.

Definition 1.2.

- (1) $Lp^\Sigma(a) = \bigcup \{ \mathcal{N} : \mathcal{N} \text{ is a countably iterable sound } \Sigma\text{-mouse over } a \text{ such that } \rho(\mathcal{N}) = a \},$
- (2) $\mathcal{K}^{\lambda, \Gamma, \Sigma}(a) = \bigcup \{ \mathcal{N} : \mathcal{N} \text{ is a countably } \Gamma\text{-iterable sound } \Sigma\text{-mouse over } a \text{ such that } \rho(\mathcal{N}) = a \},$
- (3) $\mathcal{W}^{\lambda, \Gamma, \Sigma}(a) = \bigcup \{ \mathcal{N} : \mathcal{N} \text{ is a } \Gamma\text{-iterable sound } \Sigma\text{-mouse over } a \text{ such that } \rho(\mathcal{N}) = a \}.$

Remark 1.3. In the definition above, when we say “ Σ -mouse”, we really mean “ g -organized Σ -mouse” in the sense of [10]. We will suppress the term “ g -organized” in this paper as all Σ -mice considered here will be g -organized Σ -mice. The reason for considering “ g -organized Σ -mice” is because one can perform S -constructions on g -organized Σ -mice, but not on Σ -mice as defined in [9].

When $\Gamma = \wp(\bigcup_{\kappa < \lambda} \wp(\kappa))$ then we omit it from our notation. We can define the sequences $\langle Lp_\xi^\Sigma(a) : \xi < \eta \rangle$, $\langle \mathcal{K}_\xi^{\lambda, \Gamma, \Sigma}(a) : \xi < \nu \rangle$, and $\langle \mathcal{W}_\xi^{\lambda, \Gamma, \Sigma}(a) : \xi < \mu \rangle$ as usual. For Lp -operator the definition is as follows:

- (1) $Lp_0^\Sigma(a) = Lp^\Sigma(a),$
- (2) for $\xi < \eta$, if $Lp_\xi^\Sigma(a) \in H_\lambda$ then $Lp_{\xi+1}^\Sigma = Lp_+^\Sigma(Lp_\xi^\Sigma(a)),^m$
- (3) for limit $\xi < \eta$, $Lp_\xi^\Sigma = \bigcup_{\alpha < \xi} Lp_\alpha^\Sigma(a),$
- (4) η is least such that for all $\xi < \eta$, $Lp_\xi^\Sigma(a)$ is defined.

The other stacks are defined similarly.

1.2. (Γ, Σ) -Suitable premice

Again fix an uncountable cardinal λ and assume ZF. We also fix $\Sigma \in \text{dom}(\text{Code}_\lambda)$ such that Σ is a (α, λ) -iteration strategy with hull condensation and $\Gamma \subseteq \wp(\bigcup_{\kappa < \lambda} \wp(\kappa))$ such that $\text{Code}_\lambda(\Sigma) \in \Gamma$. We now import some material from [6, Subsec. 1.3]. The most important notion we need from that subsection is that of (Γ, Σ) -suitable premouse which is defined as follows.

¹That is self-well-ordered, a set a is called self-well-ordered if $\text{trc}(a \cup \{a\})$ is well-ordered in $L_1(a)$.
^m $Lp_+^\Sigma(Lp_\xi^\Sigma(a))$ is the stack of sound, countably iterable Σ -mice $\mathcal{N}$ projecting to $\leq o(Lp_\xi^\Sigma(a))$ and extends $Lp_\xi^\Sigma(a)$, having $o(Lp_\xi^\Sigma(a))$ as a strong cutpoint. Similar definitions can be made for $\mathcal{W}_+^\Sigma(\mathcal{W}_\xi^\Sigma(a))$ and $\mathcal{K}_+^\Sigma(\mathcal{K}_\xi^\Sigma(a))$.

Definition 1.4 ((Γ, Σ)-suitable premouse). A Σ -premouse $\mathcal{P}$ is (Γ, Σ) -suitable if there is a unique cardinal δ such that

- (1) $\mathcal{P} \models \text{“}\delta \text{ is the unique Woodin cardinal”}$,
- (2) $o(\mathcal{P}) = \sup_{n < \omega} (\delta^{+n})^{\mathcal{P}}$,
- (3) for every $\eta \neq \delta$, if η is a strong cutpoint of $\mathcal{P}$, then $\mathcal{W}_+^{\lambda, \Gamma, \Sigma}(\mathcal{P}|\eta) = \mathcal{P}|(\eta)^+$,
- (4) for any $\eta < o(\mathcal{P})$, if $\eta \neq \delta$, then $\mathcal{W}_+^{\lambda, \Gamma, \Sigma}(\mathcal{N}|\eta) \models \text{“}\eta \text{ is not Woodin”}$.

If $\Gamma = \wp(\bigcup_{\alpha < \lambda} \wp(\alpha))$ then we use λ instead of Γ . In particular, we use λ -suitable to mean Γ -suitable. We will do the same with all the other notions, such as fullness preservation and short tree iterability, defined in this section. Also, if Γ is fixed throughout or is clear from the context, then we simply say $\mathcal{P}$ is Σ -suitable. We let $\mathcal{P}^-$ be the structure that Σ iterates.

Suppose $\mathcal{P}$ is (Γ, Σ) -suitable. Then we let $\delta^{\mathcal{P}}$ be the δ of Definition 1.4. We then proceed as in [6, Sec. 1.3] to define (1) nice iteration tree, (2) (Γ, Σ) -short tree, (3) (Γ, Σ) -maximal tree, (4) (Γ, Σ) -correctly guided finite stack and (5) the last model of a (Γ, Σ) -correctly guided finite stack by using $\mathcal{W}^{\lambda, \Gamma, \Sigma}$ operator instead of $\mathcal{W}^{\Gamma}$ operator.

1.3. A brief introduction to HOD mice

In this paper, a hod premouse $\mathcal{P}$ is one defined as in [5].ⁿ The reader is advised to consult [5] for basic results and notations concerning hod premice and mice. Let us mention some basic first-order properties of a hod premouse $\mathcal{P}$. There are ordinal $\lambda^{\mathcal{P}}$ and sequences $\langle (\mathcal{P}(\alpha), \Sigma_{\alpha}^{\mathcal{P}}) \mid \alpha < \lambda^{\mathcal{P}} \rangle$ and $\langle \delta_{\alpha}^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}} \rangle$ such that

- (1) $\langle \delta_{\alpha}^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}} \rangle$ is increasing and continuous and if α is a successor ordinal then $\mathcal{P} \models \delta_{\alpha}^{\mathcal{P}}$ is Woodin;
- (2) $\mathcal{P}(0) = Lp_{\omega}(\mathcal{P}|\delta_0)^{\mathcal{P}}$; for $\alpha < \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha + 1) = (Lp_{\omega}^{\Sigma_{\alpha}^{\mathcal{P}}}(\mathcal{P}|\delta_{\alpha}))^{\mathcal{P}}$; for limit $\alpha \leq \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha) = (Lp_{\omega}^{\oplus_{\beta < \alpha} \Sigma_{\beta}^{\mathcal{P}}}(\mathcal{P}|\delta_{\alpha}))^{\mathcal{P}}$;
- (3) $\mathcal{P} \models \Sigma_{\alpha}^{\mathcal{P}}$ is a $(\omega, o(\mathcal{P}), o(\mathcal{P}))^o$ -strategy for $\mathcal{P}(\alpha)$ with hull condensation;
- (4) if $\alpha < \beta < \lambda^{\mathcal{P}}$ then $\Sigma_{\beta}^{\mathcal{P}}$ extends $\Sigma_{\alpha}^{\mathcal{P}}$.

We will write $\delta^{\mathcal{P}}$ for $\delta_{\lambda^{\mathcal{P}}}^{\mathcal{P}}$ and $\Sigma^{\mathcal{P}} = \bigoplus_{\beta < \lambda^{\mathcal{P}}} \Sigma_{\beta}^{\mathcal{P}}$. Note that $\mathcal{P}(0)$ is a pure extender model. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are two hod premice. Then $\mathcal{P} \leq_{\text{hod}} \mathcal{Q}$ if there is $\alpha \leq \lambda^{\mathcal{Q}}$ such that $\mathcal{P} = \mathcal{Q}(\alpha)$. We say then that $\mathcal{P}$ is a *hod initial segment* of $\mathcal{Q}$. $(\mathcal{P}, \Sigma)$ is a *hod pair* if $\mathcal{P}$ is a hod premouse and Σ is a strategy for $\mathcal{P}$ (acting on countable stacks of countable normal trees) such that $\Sigma^{\mathcal{P}} \subseteq \Sigma$ and this fact is preserved under Σ -iterations. Typically, we will construct hod pairs $(\mathcal{P}, \Sigma)$ such that Σ has hull condensation, branch condensation, and is Γ -fullness preserving for some pointclass Γ .

ⁿBy a similar remark, by “hod premice” we mean “reorganized hod premice” in the sense of [5] or “ g -organized hod premice” in the sense of [10]. Again, the reason has to do with S -constructions.

^oThis just means $\Sigma_{\alpha}^{\mathcal{P}}$ acts on all stacks of ω -maximal, normal trees in $\mathcal{P}$.

Suppose $(\mathcal{Q}, \Sigma)$ is a hod pair. $\mathcal{P}$ is a $(\mathcal{Q}, \Sigma)$ -hod premouse if there are ordinal $\lambda^{\mathcal{P}}$ and sequences $\langle (\mathcal{P}(\alpha), \Sigma_{\alpha}^{\mathcal{P}}) \mid \alpha < \lambda^{\mathcal{P}} \rangle$ and $\langle \delta_{\alpha}^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}} \rangle$ such that

- (1) $\langle \delta_{\alpha}^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}} \rangle$ is increasing and continuous and if α is a successor ordinal then $\mathcal{P} \models \delta_{\alpha}^{\mathcal{P}}$ is Woodin;
- (2) $\mathcal{P}(0) = Lp_{\omega}^{\Sigma}(\mathcal{P} \mid \delta_0^{\mathcal{P}})^{\mathcal{P}}$ (so $\mathcal{P}(0)$ is a Σ -premouse built over $\mathcal{Q}$); for $\alpha < \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha + 1) = (Lp_{\omega}^{\Sigma \oplus \Sigma_{\alpha}^{\mathcal{P}}}(\mathcal{P} \mid \delta_{\alpha}^{\mathcal{P}}))^{\mathcal{P}}$; for limit $\alpha \leq \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha) = (Lp_{\omega}^{\oplus_{\beta < \alpha} \Sigma \oplus \Sigma_{\beta}^{\mathcal{P}}}(\mathcal{P} \mid \delta_{\alpha}^{\mathcal{P}}))^{\mathcal{P}}$;
- (3) $\mathcal{P} \models \Sigma_{\alpha}^{\mathcal{P}}$ is a $(\omega, o(\mathcal{P}), o(\mathcal{P}))$ -strategy for $\mathcal{P}(\alpha)$ with hull condensation;
- (4) if $\alpha < \beta < \lambda^{\mathcal{P}}$ then $\Sigma_{\beta}^{\mathcal{P}}$ extends $\Sigma_{\alpha}^{\mathcal{P}}$.

Inside $\mathcal{P}$, the strategies $\Sigma_{\alpha}^{\mathcal{P}}$ act on stacks above $\mathcal{Q}$ and every $\Sigma_{\alpha}^{\mathcal{P}}$ iterate is a Σ -premouse. Again, we write $\delta^{\mathcal{P}}$ for $\delta_{\lambda^{\mathcal{P}}}^{\mathcal{P}}$ and $\Sigma^{\mathcal{P}} = \bigoplus_{\beta < \lambda^{\mathcal{P}}} \Sigma_{\beta}^{\mathcal{P}}$. $(\mathcal{P}, \Lambda)$ is a $(\mathcal{Q}, \Sigma)$ -hod pair if $\mathcal{P}$ is a $(\mathcal{Q}, \Sigma)$ -hod premouse and Λ is a strategy for $\mathcal{P}$ such that $\Sigma^{\mathcal{P}} \subseteq \Lambda$ and this fact is preserved under Λ -iterations. The reader should consult [5] for the definition of $B(\mathcal{Q}, \Sigma)$, and $I(\mathcal{Q}, \Sigma)$. In a core model induction, we do not quite have at the moment $(\mathcal{Q}, \Sigma)$ is constructed an AD^+ -model M such that $(\mathcal{Q}, \Sigma) \in M$ but we do know that every $(\mathcal{R}, \Lambda) \in B(\mathcal{Q}, \Sigma)$ belongs to such a model. We then can show (using our hypothesis) that $(\mathcal{Q}, \Sigma)$ belongs to an AD^+ -model.

The paper [5] constructs under AD^+ (under Strong Mouse Capturing (SMC)) hod pairs that are fullness preserving, positional, commuting, and have branch condensation. Such hod pairs are particularly important for our computation as they are points in the direct limit system giving rise to HOD of AD^+ models. For hod pairs $(\mathcal{M}_{\Sigma}, \Sigma)$, if Σ is a strategy with branch condensation and $\vec{T}$ is a stack on $\mathcal{M}_{\Sigma}$ with last model $\mathcal{N}$, $\Sigma_{\mathcal{N}, \vec{T}}$ is independent of $\vec{T}$. Therefore, later on we will omit the subscript $\vec{T}$ from $\Sigma_{\mathcal{N}, \vec{T}}$ whenever Σ is a strategy with branch condensation and $\mathcal{M}_{\Sigma}$ is a hod mouse.

1.4. HOD under AD^+

Using techniques above and the theory of hod mice developed in [5], the papers [5, 15] compute HOD (up to Θ) in AD^+ models of $V = L(\wp(\mathbb{R})) + \text{SMC}^{\mathcal{P}} + \Theta = \theta_{\alpha+1}$ for some α below “ $\text{AD}_{\mathbb{R}} + \Theta$ is regular”.

These papers show the existence of an $\mathcal{M}_{\infty}$ such that:

- (1) $\mathcal{M}_{\infty} \in \text{HOD}$.
- (2) $\mathcal{M}_{\infty}$ is a hod premouse.
- (3) $\mathcal{M}_{\infty} \mid \Theta = (V_{\Theta}^{\text{HOD}}, \vec{E}^{\mathcal{M}_{\infty} \mid \Theta}, S^{\mathcal{M}_{\infty}}, \in)$, where $S^{\mathcal{M}_{\infty} \mid \Theta}$ is the predicate for strategies of hod initial segments of $\mathcal{M}_{\infty} \mid \Theta$.

We call $\mathcal{M}_{\infty}$ the *hod limit*.

The paper [5] also computes HOD (up to Θ) in models of $V = L(\wp(\mathbb{R})) + \text{SMC} + \text{AD}_{\mathbb{R}}$ below “ $\text{AD}_{\mathbb{R}} + \Theta$ is regular” by exhibiting a hod premouse $\mathcal{M}_{\infty}$

^PSMC stands for the Strong Mouse Capturing, which says that for any hod pair $(\mathcal{P}, \Sigma)$ such that Σ has branch condensation and is fullness preserving, then $\text{MC}(\Sigma)$ holds.

satisfying (1)–(3) as above. Here $\mathcal{M}_\infty = \bigcup_{(\mathcal{Q}, \Lambda)} \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$, where $(\mathcal{Q}, \Lambda)$ is a hod pair with branch condensation and is fullness preserving and $\mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ is the direct limit of all (nondropping) Λ -iterates of $\mathcal{Q}$.

What is important for us are the notions discussed in those papers to compute HOD in the successor cases. Let $(\mathcal{P}^-, \Sigma)$ be as above and suppose also that the direct limit $\mathcal{M}_\infty(\mathcal{P}^-, \Sigma)$ agrees with HOD up to θ_α . Let

$$\mathbb{B}(\mathcal{P}^-, \Sigma) = \{B \subseteq {}^\omega\mathbb{R} \times \mathbb{R} \times \mathbb{R} \mid B \text{ is } OD, \text{ for any } (\mathcal{Q}, \Lambda) \text{ iterate of } (\mathcal{P}^-, \Sigma), \\ \text{and for any } (x, y) \in B_{(\mathcal{Q}, \Lambda)}, x \text{ codes } \mathcal{Q}\}.$$

In the above definition, we identify Λ with the set of reals $\text{Code}(\Lambda)$, and the notation $B_{(\mathcal{Q}, \Lambda)}$ refers to the section consisting of the second and third coordinates of B . We also write “ $\mathcal{P}$ is Σ -suitable” for “ $(\mathcal{P}, \Sigma)$ is a suitable pair”. For such a $\mathcal{P}$, we let $\delta^\mathcal{P}$ be the Woodin cardinal of $\mathcal{P}$ (above $\mathcal{P}^-$). If $(\mathcal{P}^-, \Sigma) = (\emptyset, \emptyset)$, then each $B \in \mathbb{B}(\emptyset, \emptyset)$ can be canonically identified with an OD set of reals and hence $\mathbb{B}(\emptyset, \emptyset)$ can be canonically identified with the collection of OD sets of reals. Suppose $B \in \mathbb{B}(\mathcal{P}^-, \Sigma)$ and $\kappa < o(\mathcal{P})$. Let $\tau_{B, \kappa}^\mathcal{P}$ be the canonical term in $\mathcal{P}$ that captures B at κ i.e., for any $g \subseteq \text{Col}(\omega, \kappa)$ generic over $\mathcal{P}$

$$B_{(\mathcal{P}^-, \Sigma)} \cap \mathcal{P}[g] = (\tau_{B, \kappa}^\mathcal{P})_g.$$

Let $\delta = \delta^\mathcal{P}$. For each $m < \omega$, let

$$\gamma_{B, m}^{\mathcal{P}, \Sigma} = \sup(\text{Hull}_1^\mathcal{P}(\mathcal{P}^- \cup \{\tau_{B, (\delta+m)^\mathcal{P}}^\mathcal{P}\}) \cap \delta), \\ H_{B, m}^{\mathcal{P}, \Sigma} = \text{Hull}_1^\mathcal{P}(\gamma_{B, m}^{\mathcal{P}, \Sigma} \cup \{\tau_{B, (\delta+m)^\mathcal{P}}^\mathcal{P}\}), \\ \gamma_B^{\mathcal{P}, \Sigma} = \sup_{m < \omega} \gamma_{B, m}^{\mathcal{P}, \Sigma},$$

and

$$H_B^{\mathcal{P}, \Sigma} = \bigcup_{m < \omega} H_{B, m}^{\mathcal{P}, \Sigma}.$$

Similar definitions can be given for $\gamma_{\vec{B}, m}^{\mathcal{P}, \Sigma}, H_{\vec{B}, m}^{\mathcal{P}, \Sigma}, \gamma_{\vec{B}}^{\mathcal{P}, \Sigma}, H_{\vec{B}}^{\mathcal{P}, \Sigma}$ for any finite sequence $\vec{B} \in \mathbb{B}(\mathcal{P}^-, \Sigma)$. One just needs to include relevant terms for each element of $\vec{B}$ in each relevant hull. The usual notions of B -iterability, strong B -iterability, and the corresponding weak iteration games $\mathcal{WG}(\mathcal{P}, \Sigma)$, $\mathcal{WG}(\mathcal{P}, \Sigma, B)$ are defined in [5, Sec. 3.1]. The papers [5, 15] show that if $(\mathcal{P}^-, \Sigma)$ is a hod pair such that

- (i) Σ is fullness preserving, commuting, positional, and has branch condensation,
- (ii) $\delta^{\mathcal{M}_\infty(\mathcal{P}^-, \Sigma)} = \theta_\alpha$ for some α ,
- (iii) $\mathcal{M}_\infty(\mathcal{P}^-, \Sigma)|\theta_\alpha = \text{HOD}|\theta_\alpha$,

then we can compute $\text{HOD}|\theta_{\alpha+1}$ as follows.

Let

$$\mathcal{F} = \{(\mathcal{P}, \Sigma, \vec{B}) \mid \vec{B} \in \mathbb{B}(\mathcal{P}^-, \Sigma)^{<\omega}, (\mathcal{P}^-, \Sigma) \text{ satisfies (i)–(iii), } \mathcal{P} \text{ is } \Sigma\text{-suitable} \\ \text{and } (\mathcal{P}, \Sigma) \text{ is strongly } \vec{B}\text{-iterable}\}.$$

The ordering on $\mathcal{F}$ is defined as follows:

$(\mathcal{P}, \Sigma, \vec{B}) \preceq (\mathcal{Q}, \Lambda, \vec{C}) \Leftrightarrow \vec{B} \subseteq \vec{C}, \exists r (r \text{ is a run of } \mathcal{WG}(\mathcal{P}, \Sigma, \vec{B}) \text{ with the last model}$

$$\mathcal{P}^* \text{ such that } (\mathcal{P}^*)^- = \mathcal{Q}^-, \Sigma_{(\mathcal{P}^*)^-} = \Lambda, \mathcal{P}^* = \mathcal{Q}|(\eta^{+\omega})^{\mathcal{Q}}$$

where $\mathcal{Q} \models \eta > o(\mathcal{Q}^-)$ is Woodin).

Suppose $(P, \Sigma, \vec{B}) \preceq (Q, \Lambda, \vec{C})$ then there is a unique map $\pi_{\vec{B}}^{(\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda)} : H_{\vec{B}}^{\mathcal{P}, \Sigma} \rightarrow H_{\vec{B}}^{\mathcal{Q}, \Lambda}$ given by strong $\vec{B}$ -iterability. $(\mathcal{F}, \preceq)$ is then directed. Let

$$\mathcal{M}_{\infty, \alpha} = \text{direct limit of } (\mathcal{F}, \preceq) \text{ under maps } \pi_{\vec{B}}^{(\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda)}.$$

Then $\mathcal{M}_{\infty, \alpha} \in \text{HOD}$ and $\mathcal{M}_{\infty, \alpha}|_{\theta_{\alpha+1}} = \text{HOD}|_{\theta_{\alpha+1}}$. Also for each $(\mathcal{P}, \Sigma, \vec{B}) \in \mathcal{F}$, let

$$\pi_{\vec{B}}^{(\mathcal{P}, \Sigma), \infty} : H_{\vec{B}}^{\mathcal{P}, \Sigma} \rightarrow \mathcal{M}_{\infty, \alpha}$$

be the natural map, and let for each such $\vec{B}$

$$H_{\vec{B}}^{\mathcal{M}_{\infty, \alpha}} = \bigcup_{(\mathcal{P}, \Sigma, \vec{B}) \in \mathcal{F}} \pi_{\vec{B}}^{(\mathcal{P}, \Sigma), \infty} [H_{\vec{B}}^{\mathcal{P}, \Sigma}],$$

and

$$\gamma_{\vec{B}}^{\mathcal{M}_{\infty, \alpha}} = \bigcup_{(\mathcal{P}, \Sigma, \vec{B}) \in \mathcal{F}} \pi_{\vec{B}}^{(\mathcal{P}, \Sigma), \infty} [\gamma_{\vec{B}}^{\mathcal{P}, \Sigma}].$$

Now suppose $f : \Theta \rightarrow \Theta$ (f could be taken from a parent ZFC universe) is such that, for each α such that $\theta_\alpha < \Theta$, $f \restriction (\theta_\alpha + 1) \in \text{HOD}$ and $\text{rng}(f \restriction (\theta_\alpha + 1)) \subset \theta_{\alpha+1}$. We call such an f *appropriate*. Fix an appropriate f and an α and let $\mathcal{F}, \mathcal{M}_{\infty, \alpha}$ be as above for α . Let $(\mathcal{P}, \Sigma, B) \in \mathcal{F}$ be such that $f \restriction (\theta_\alpha + 1) \cup \{f \restriction (\theta_\alpha + 1)\} \subset \text{rng}(\pi_{\vec{B}}^{(\mathcal{P}, \Sigma), \infty} \restriction H_{\vec{B}}^{\mathcal{P}, \Sigma})$. In particular, $\gamma_{\vec{B}}^{\mathcal{M}_{\infty, \alpha}} > f(\theta_\alpha)$. We call such a triple $(\mathcal{P}, \Sigma, B)$ *f-suitable*. We then say that a Σ -suitable $\mathcal{P}$ is (*strongly*) (f, Σ) -*iterable* if letting B_f be the *OD*-least B in $\mathbb{B}(\mathcal{P}^-, \Sigma)$ such that $(\mathcal{P}, \Sigma, B_f)$ is *f-suitable*, then $(\mathcal{P}, \Sigma)$ is (*strongly*) B -iterable. Whenever $(\mathcal{P}, \Sigma, B_f) \in \mathcal{F}$ is *f-suitable*, we also write $\gamma_f^{\mathcal{P}, \Sigma}$ for $\gamma_{B_f}^{\mathcal{P}, \Sigma}$ or simply $\gamma_f^{\mathcal{P}}$ if Σ is clear from the context.⁴

2. The Maximal Model and a Framework for the Core Model Induction

The core model induction is a method for constructing models of determinacy while working under various hypotheses. During the induction one climbs up through the Solovay hierarchy. This is a hierarchy of axioms that extend AD^+ and roughly describes how complicated the Solovay sequence is. One first defines, under a certain smallness assumption, for instance “there are no models M such that $\mathbb{R} \cup \text{Ord} \subset M$

⁴The notions of f -iterability and f -suitability here are related to those used in [13] but as far as the authors can see, they are not the same.

such that $M \models \text{AD}_{\mathbb{R}} + \Theta$ is regular”, a so-called *maximal model of AD^+* , $\mathfrak{M}$. We show $\mathfrak{M} \models \text{AD}^+$. We also show that $\mathfrak{M}$ cannot satisfy “ $\Theta = \theta_{\alpha+1}$ ” for some α and “ $\text{AD}_{\mathbb{R}} + \Theta$ is singular” as in each case, we can construct a hod pair $(\mathcal{P}, \Sigma)$ that generates $\wp(\mathbb{R}) \cap \mathfrak{M}$, but by maximality of $\mathfrak{M}$, $(\mathcal{P}, \Sigma) \in \mathfrak{M}$, which is a contradiction. This shows that there must indeed be such models M satisfying “ $\text{AD}_{\mathbb{R}} + \Theta$ is regular”.

Throughout the paper we work under the smallness assumption, which is assumed in V as well as in all set-generic extensions of V

(†): “there are no models M such that $\mathbb{R} \cup \text{Ord} \subset M$ such that $M \models \text{AD}_{\mathbb{R}} + \Theta$ is regular”.[†]

In this section we first recall the notion of the maximal model and some correctness results from [8]; the second part of the section sets up the framework for our core model induction.

We start by introducing universally Baire iteration strategies and mouse operators. We assume ZFC. Throughout this paper we fix a canonical method for coding sets in HC by reals. Given a real x which is a code of a set in HC, we let M_x be the structure coded by x and let $\pi_x : M_x \rightarrow N_x$ be the transitive collapse of M_x . We let WF be the set of reals which code sets in HC.

Definition 2.1 (uB operators). Suppose $\Lambda \in \text{dom}(\text{Code})$ and $\lambda \geq \omega_1$ is a cardinal. We say Λ is λ -uB if there are $< \lambda$ -complementing trees^s (T, S) witnessing that $\text{Code}(\Lambda)$ is $< \lambda$ -uB in the following stronger sense: for all $x \in WF$ and $n, m \in x$,

$$(x, n, m) \in p[T] \iff \pi_x(m) \in \Lambda(\pi_x(n)).$$

If g is a $< \lambda$ -generic then we let Λ^g be the canonical interpretation of Λ onto $V[g]$, i.e., given $a, b \in HC^{V[g]}$, $\Lambda^g(a) = b$ if and only if whenever $x \in WF^{V[g]}$ is such that $a \in N_x$ and $n \in x$ is such that $\pi_x(n) = a$ then $b = \pi_x[\{m : (x, n, m) \in (p[T])^{V[g]}\}]$.

If Λ is λ -uB for all λ then we say Λ is uB.

In the core model induction of this paper, all operators constructed are λ -uB operators (for some appropriate λ to be defined later). The definition of uB operators ensures that the “decoding” of Λ from $\text{Code}(\Lambda)$ is generically absolute.

Suppose now λ is an uncountable cardinal, g is a $< \lambda$ -generic, $a \in (H_\lambda)^V[g]$ and $\Sigma \in \text{dom}(\text{Code})$ is λ -uB. Then we define $Lp^{\Sigma, g}(a)$, $\mathcal{W}^{\lambda, \Sigma, g}(a)$ and $\mathcal{K}^{\lambda, \Sigma, g}(a)$ in $V[g]$ according to Definition 1.2. The following connects the three stacks defined above.

Proposition 2.2. *Suppose λ is an uncountable cardinal and $\Sigma \in \text{dom}(\text{Code})$ is λ -uB. For every $a \in H_\lambda^V$, $\mathcal{W}^{\lambda, \Sigma}(a) \trianglelefteq \mathcal{K}^{\lambda, \Sigma}(a) \trianglelefteq Lp^\Sigma(a)$. Moreover, for any $\eta < \lambda$ and V -generic $g \subseteq \text{Coll}(\omega, \eta)$ or $g \subseteq \text{Coll}(\omega, < \eta)$, $\mathcal{W}^{\lambda, \Sigma, g}(a) \trianglelefteq \mathcal{W}^{\lambda, \Sigma}(a)$, $\mathcal{K}^{\lambda, \Sigma, g}(a) \trianglelefteq \mathcal{K}^{\lambda, \Sigma}(a)$ and $Lp^{\Sigma, g}(a) \trianglelefteq Lp^\Sigma(a)$.*

[†]Another way of stating our smallness assumption is the statement: “there are no hod mice $\mathcal{P}$ such that $\delta^{\mathcal{P}}$ is an inaccessible limit of Woodin cardinals in $\mathcal{P}$ ”.

^sIt means that the trees project to complement in all $< \lambda$ -generic extensions.

Definition 2.3 (Hod pair below λ). Suppose that λ is an uncountable cardinal and $(\mathcal{P}, \Sigma)$ is a hod pair such that $\Sigma \in \text{dom}(\text{Code})$ is λ^+ -uB. We say $(\mathcal{P}, \Sigma)$ is a hod pair below λ if Σ has branch condensation and whenever $g \subseteq \text{Coll}(\omega, \lambda)$ is V -generic, in $V[g]$, Σ^g is ω_1 -fullness preserving.^t

Note that if $\kappa \leq \lambda$ and $(\mathcal{P}, \Sigma)$ is a hod pair below λ then $(\mathcal{P}, \Sigma)$ is a hod pair below κ . We are now in a position to introduce the maximal model of AD^+ .

Definition 2.4 (Maximal model of $\text{AD}^+ + \Theta = \theta_\Sigma$). Suppose $\mu < \lambda$ are cardinals and $g \subseteq \text{Coll}(\omega, < \mu)^u$ is V -generic. Suppose in $V[g]$, $(\mathcal{P}, \Sigma)$ is a hod pair below λ . Then we let $\mathcal{S}_{\mu,g}^{\lambda,\Sigma} = L(\mathcal{K}^{\lambda,\Sigma,g}(\mathbb{R}^{V[g]}))$. We also let

$$\mathfrak{M}_{\mu,g}^\lambda = L\left(\bigcup_{(\mathcal{P},\Sigma)} \mathcal{S}_{\mu,g}^{\lambda,\Sigma} \cap \wp(\mathbb{R})\right) \quad \text{and} \quad \Omega_{\mu,g}^\lambda = \bigcup_{(\mathcal{P},\Sigma)} \mathcal{S}_{\mu,g}^{\lambda,\Sigma} \cap \wp(\mathbb{R}),$$

where the union is over all such hod pairs $(\mathcal{P}, \Sigma)$.

Thus far strategy mice have been discussed only in situations when the underlying set was self-well-ordered. However, $\mathcal{S}_{\mu,g}^{\lambda,\Sigma}$ is a Σ -mouse over the set of reals.^v Such hybrid mice were defined in [5, Sec. 2.10] and a more detailed treatment is given in [10]. We say that $\mathcal{S}_{\mu,g}^{\lambda,\Sigma}$ is the λ - Σ -maximal model of AD^+ at μ , $\mathfrak{M}_{\mu,g}^\lambda$ is the λ -maximal model of AD^+ at μ , and $\Omega_{\mu,g}^\lambda$ is the λ -maximal point class of AD^+ at μ . Our goal is to show that (under $(\dagger)$) $\mathfrak{M}_{\mu,g}^\lambda$ is a model of “ $\text{AD}_\mathbb{R} + \Theta$ is regular”.

The next lemma connects various degrees of iterability. Below, if $\xi \in \text{Ord}$ and N is a transitive model of ZFC then we let $N_\xi = V_\xi^N$.

Lemma 2.5. *Suppose $\mu < \lambda$ are such that μ is a strong cardinal and λ is inaccessible. Let $j : V \rightarrow M$ be an embedding witnessing that μ is λ^+ -strong and let $g \subseteq \text{Coll}(\omega, < \mu)$ and $h \subseteq \text{Coll}(\omega, < j(\mu))$ be two generics such that $g = h \cap \text{Coll}(\omega, < \mu)$. Let $j^+ : V[g] \rightarrow M[h]$ be the lift of j . Let $W = V[g]$. Suppose $(\mathcal{P}, \Sigma)$ is a hod pair below μ and $a \in V_\lambda[g]$ is self-well-ordered. Then*

$$\mathcal{W}^{\lambda,\Sigma,g}(a) = \mathcal{W}^{\lambda,\Sigma,h \cap \text{Coll}(\omega, < \lambda)}(a) = \mathcal{K}^{\lambda,\Sigma,g}(a) = \mathcal{K}^{\mu,\Sigma,g}(a) = (\mathcal{W}^{j(\lambda),j(\Sigma),h}(a))^{M[h]}.$$

The proof of the previous lemma is from [8, Lemma 2.5]. The following is an easy corollary of Lemma 2.5.

Corollary 2.6. *Suppose $\mu < \kappa < \lambda$ and $j : V \rightarrow M$ are such that μ and κ are strong cardinals, λ is inaccessible, j witness that μ is λ -strong and $M \models “\kappa$ is strong cardinal”.* Let $(\mathcal{P}, \Sigma)$ be a hod pair below μ which is λ -uB. Let $g \subseteq \text{Coll}(\omega, < \kappa)$

^tRecall this means Σ^g is $\wp(\wp(\omega_1))$ -fullness preserving.

^uIn this paper, μ is typically an inaccessible cardinal.

^vActually, we need that $\mathcal{S}_{\mu,g}^{\lambda,\Sigma}$ is a Θ - g -organized Σ -mouse over $\mathbb{R}^{V[g]}$ as defined in [10]; this is a slight modification from the hierarchy of g -organized Σ -mice. This modification is needed (only for Σ -mice over $\mathbb{R}$) so that the scales analysis works out.

and $h \subseteq \text{Coll}(\omega, < j(\mu))$ be generic such that $g = h \cap \text{Coll}(\omega, < \kappa)$. Let $j^+ : V[g \cap \text{Coll}(\omega, < \mu)] \rightarrow M[h]$ be the lift of j . Then whenever $a \in V_\lambda[g]$,

$$\mathcal{W}^{\lambda, \Sigma, g}(a) = \mathcal{K}^{\kappa, \Sigma, g}(a) = \mathcal{W}^{\lambda, \Sigma, h \cap \text{Coll}(\omega, < \lambda)}(a) = (\mathcal{W}^{j(\lambda), j(\Sigma), h}(a))^{M[h]}.$$

The proof of the above is given in [8, Sec. 2], so we omit it here. Now we develop some basic notions in order to state Theorem 2.9 which we will use as a black box. Our core model induction is a typical one: we have two uncountable cardinals $\kappa < \lambda$, the core model induction operators (cmi operators) defined on bounded subsets of κ can be extended to act on bounded subsets of λ , and for any such cmi operator F acting on bounded subsets of λ , the minimal F -closed mouse with one Woodin cardinal exists and is λ -iterable.

The mouse operators that are constructed during core model induction have two additional properties: they *transfer* and *relativize well*. More precisely, fix $\Sigma \in \text{dom}(\text{Code})$ which is λ -uB. Given a Σ -mouse operator $F \in \text{dom}(\text{Code}_\lambda)$, we say

- (1) (Relativizes well) F relativizes well if there is a formula $\phi(u, v, w)$ such that whenever $X, Y \in \text{dom}(F)$ and N are such that $X \in L_1(Y)$ and N is a transitive rudimentarily closed set such that $Y, F(Y) \in N$ then $F(X) \in N$ and $F(X)$ is the unique U such that $N \models \phi[U, X, F(Y)]$.
- (2) (Transfers well) F transfers well if whenever $X, Y \in \text{dom}(F)$ are such that X is generic over $L_1(Y)$ then $F(L_1(Y)[X])$ is obtained from $F(Y)$ via S -constructions (see [5, Sec. 2.11]) and in particular, $F(L_1(Y))[X] = F(L_1(Y)[X])$.

We are now in a position to introduce the core model induction operators that we will construct in this paper.

Definition 2.7 (Core model induction operator). Suppose $|\mathbb{R}| = \kappa$, $(\mathcal{P}, \Sigma)$ is a hod pair below κ^+ . We say $F \in \text{dom}(\text{Code})$ is a Σ core model induction operator or just Σ -cmi operator if for some $\alpha \in \text{Ord}$, letting $M = \mathcal{S}_\omega^{\kappa^+, \Sigma} \parallel \alpha$, $\Gamma = \Sigma_1^M$, suppose $M \models \text{AD}^+ + \text{MC}(\Sigma)$ and one of the following holds:

- (1) F is a Σ -mouse operator which transfers and relativizes well.
- (2) For some self-well-ordered $b \in HC$ and some Σ -premouse $\mathcal{Q} \in HC^V$ over b , F is an (ω_1, ω_1) -iteration strategy (above $o(\mathcal{P})$) for a (Σ, Γ) -suitable $\mathcal{Q}$ which is Γ -fullness preserving, has branch condensation and is guided by some $\vec{A} = (A_i : i < \omega)$ such that $\vec{A} \in OD_{b, \Sigma, x}^M$ for some $x \in b$. Moreover, α ends either a weak or a strong gap in the sense of [10].
- (3) For some $H \in \text{dom}(\text{Code})$, H satisfies a or b above and for some $n < \omega$, F is $x \rightarrow \mathcal{M}_n^{\#, H}(x)$ operator or for some $b \in HC$, F is the ω_1 -iteration strategy of $\mathcal{M}_n^{\#, H}(b)^w$.

^wThis definition is taken from [8, Definition 3.1], though we omit clause (2) there as it is not necessary.

When $\Sigma = \emptyset$ then we omit it from our notation. Often times, when doing core model induction, we have two uncountable cardinals $\kappa < \lambda$ and we need to show that cmi operators in $V^{\text{Coll}(\omega, < \kappa)}$ can be extended to act on $V^{\text{Coll}(\omega, < \lambda)}$. This is a weaker notion than being λ -uB. We also need to know that for any cmi operator $F \in V^{\text{Coll}(\omega, < \kappa)}$, $\mathcal{M}_1^{\#, F}$ -exists. We make these statements more precise.

Definition 2.8 (Lifting cmi operators). Suppose $\kappa < \lambda$ are two cardinals such that κ is an inaccessible cardinal and suppose $(\mathcal{P}, \Sigma)$ is a hod pair below κ .

- (1) $\text{Lift}(\kappa, \lambda, \Sigma)$ is the statement that for every generic $g \subseteq \text{Coll}(\omega, < \kappa)$, in $V[g]$, for every Σ^g -cmi operator F there is an operator $F^* \in \text{dom}(\text{Code}_\lambda)$ such that $F = F^* \upharpoonright HC$. In this case we say F is λ -extendable. Such an F^* is necessarily unique as can be easily shown by a Skolem hull argument.^x If $\text{Lift}(\kappa, \lambda, \Sigma)$ holds, $g \subseteq \text{Coll}(\omega, < \kappa)$ is generic, and F is a Σ^g -cmi operator then we let F^λ be its extended version.
- (2) We let $\text{Proj}(\kappa, \lambda, \Sigma)^y$ be the conjunction of the following statements:
 - $\text{Lift}(\kappa, \lambda, \Sigma)$ and for every generic $g \subseteq \text{Coll}(\omega, < \kappa)$, in $V[g]$,
 - (a) for every Σ^g -cmi operator F , $\mathcal{M}_1^{\#, F}$ exists and is λ -iterable,
 - (b) for every $a \in H_{\omega_1}$, $\mathcal{K}^{\omega_1, \Sigma, g}(a) = \mathcal{W}^{\lambda, \Sigma, g}(a)$.

The following is the core model induction theorem that we will use.

Theorem 2.9. Suppose $\kappa < \lambda$ are two uncountable cardinals and suppose $(\mathcal{P}, \Sigma)$ is a hod pair below κ such that $\text{Proj}(\kappa, \lambda, \Sigma)$ holds. Then for every generic $g \subseteq \text{Coll}(\omega, < \kappa)$, $\mathcal{S}_{\kappa, g}^{\lambda, \Sigma} \models \text{AD}^+ + \theta_\Sigma = \Theta$.^z

We will not prove the theorem here as the proof of the theorem is very much like the proof of the core model induction theorems in [6] (see Theorems. 2.4 and 2.6), [9] (see Chapter 7) and [12]. To prove the theorem we have to use the scales analysis for $\mathcal{S}_{\kappa, g}^{\lambda, \Sigma}$ (see [10]). For a relevant discussion on how Theorem 2.9 is proved, see [8].

We end this section with the following useful fact on lifting strategies. Among other things it can be used to show clause (b) of $\text{Proj}(\kappa, \lambda, \Sigma)$. The following lemma is Lemma 3.5 of [8].

Lemma 2.10 (Lifting cmi operators through strongness embeddings). Suppose $\kappa < \lambda$ are such that κ is a λ -strong cardinal. Then whenever $(\mathcal{P}, \Sigma)$ is a hod pair below κ , $\text{Lift}(\kappa, \lambda, \Sigma)$ and clause (b) of $\text{Proj}(\kappa, \lambda, \Sigma)$ hold.

^xSuppose $H_0, H_1 \in \text{dom}(\text{Code}_\lambda^{V[g]})$ are two extensions of F . Working in $V[g]$, let $\pi : N \rightarrow H_{\lambda^+}[g]$ be elementary such that N is countable and $H_0, H_1 \in \text{rng}(\pi)$. Let $(\bar{H}_0, \bar{H}_1) = \pi^{-1}(H_0, H_1)$. Then it follows from the definition of being a Σ -cmi operator that $\bar{H}_0 = H_0 \upharpoonright N$ and $\bar{H}_1 = H_1 \upharpoonright N$. However, since $H_0 \upharpoonright N = F \upharpoonright N = H_1 \upharpoonright N$, we get that $N \models \bar{H}_0 = \bar{H}_1$, contradiction!

^y Proj stands for projective determinacy. The meaning is taken from clause (a).

^zTechnically speaking, the proof shows that some maximal, constructibly closed initial segment M of $\mathcal{S}_{\kappa, g}^{\lambda, \Sigma} \models \text{AD}^+ + \theta_\Sigma = \Theta$. In fact, M is the union of $\mathcal{S}$ such that for any countable transitive $\mathcal{S}^*$ such that there is an elementary map from $\mathcal{S}^*$ into $\mathcal{S}$, $\mathcal{S}^*$ has an iteration strategy projective in $\mathcal{S}$. We will, for the rest of the paper, confuse this M with $\mathcal{S}_{\kappa, g}^{\lambda, \Sigma}$; this does not change anything.

3. A Core Model Induction

Recall that we say μ reflects the set of strong cardinals (or μ is *strong reflecting strong*s) if μ is a strong cardinal and for every $\lambda > \mu$, there is an embedding $j : V \rightarrow M$ witnessing that μ is λ -strong and such that for any cardinal $\kappa \in [\mu, \lambda)$, $V \models \text{“}\kappa \text{ is strong”}$ if and only if $M \models \text{“}\kappa \text{ is strong”}$. We fix $\mu < \kappa < \lambda$ such that λ is an inaccessible cardinal, μ and κ are strong such that μ is strong reflecting strongs and κ is strong.^{aa}

Suppose $n \subseteq \text{Coll}(\omega, < j(\mu))$ is V -generic. Let $m = n \cap \text{Coll}(\omega, < \kappa)$ and $g = m \cap \text{Coll}(\omega, < \mu)$. We also let $j^+ : V[g] \rightarrow M[n]$ be the lift of j . Suppose also $\text{Proj}(\kappa, \lambda, \Psi)$ holds for all hod pairs $(\mathcal{R}, \Psi)$ below κ . We first prove (under the assumption $(\dagger)$) the following.

Theorem 3.1. *Let μ, κ, λ etc. be as above. $\mathfrak{M}_{\mu, g}^\lambda \neq \mathcal{S}_{\mu, g}^{\lambda, \Psi}$ for any hod pair $(\mathcal{S}, \Psi)$ below κ , where $\mathcal{S} \in V_\mu[g]$, $\Psi \cap V[g] \in V[g]$.*

We first restate the main theorem (Theorem 4.1) of [8] in our context. The proof of this theorem is an easy generalization of that of Theorem 4.1 of [8] combined with Theorem 2.9, so we omit it.

Theorem 3.2. *Let μ, κ, λ etc. be as above. Suppose $(\mathcal{R}, \Psi)$ is a hod pair below κ such that $\text{Proj}(\kappa, \lambda, \Psi)$ holds. Suppose $(\mathcal{R}, \Psi) \in V_\mu[g]$. Let $\mathcal{P} = (\mathcal{M}_\infty)^{\mathcal{S}_{\mu, g}^{\lambda, \Psi}}$. Then in $M[m]$, $\mathcal{P}$ has an (ω_1, ω_1) -iteration strategy Σ such that Σ is extendable to a $(j(\mu), j(\mu))$ -strategy that is $j(\mu)$ -fullness preserving. Moreover, there is a stack $\vec{T} \in HC^{V[m]}$ on $\mathcal{P}$ according to Σ with last model $\mathcal{Q}$ such that $\pi^{\vec{T}}$ exists and in $V[m]$, $(\mathcal{Q}, \Sigma_{\mathcal{Q}, \vec{T}})$ is a hod pair below ω_1 (so in particular, $\Sigma_{\mathcal{Q}, \vec{T}}$ has branch condensation). Finally, in $V[m]$ (or equivalently in $M[m]$), $\mathcal{S}_{\kappa, m}^{\lambda, \Sigma_{\mathcal{Q}, \vec{T}}} \models \text{AD}^+ + \theta_\Psi < \Theta$.*

Proof of Theorem 3.1. This basically follows from Theorem 3.2. We outline the argument. Suppose not; then $\mathfrak{M}_{\mu, g}^\lambda = \mathcal{S}_{\mu, g}^{\lambda, \Psi}$ for some hod pair $(\mathcal{R}, \Psi) \in V[g]$ below κ , where $\mathcal{R} \in V_\mu[g]$ and $\Psi \cap V[g] \in V[g]$. Fix such a $(\mathcal{R}, \Psi)$. Applying Theorem 3.2 to $(\mathcal{R}, \Psi)$ and using elementarity of j^+ and the fact that κ is strong in M , we get that there is a hod pair $(\mathcal{Q}, \Sigma)$ below a strong cardinal $\kappa^* < \mu$ such that $(\mathcal{Q}, \Sigma) \in V_{\kappa^*}[g \cap \text{Coll}(\omega, < \kappa^*)]$ and $(\mathcal{Q}, \Sigma)$ is also a hod pair below μ such that $\Sigma \notin \mathfrak{M}_{\mu, g}^\lambda$ and $\mathcal{S}_{\mu, g}^{\lambda, \Sigma} \models \text{AD}^+$.^{bb} This contradicts the definition of $\mathfrak{M}_{\mu, g}^\lambda$. $\square$

^{aa}It should be noted that the main assumption in the definition of “tame trees” is a bit of an overkill. It would have been enough to assume that all extenders of the tree are above cardinals $\mu < \kappa$, where μ is a strong reflecting strongs and κ is strong.

^{bb}To see that Σ is fullness preserving with respect to mice in $\mathcal{S}_{\mu, g}^{\lambda, \Psi}$, using the fact that κ^* is strong, we get that Σ is λ -fullness preserving in $V[g \cap \text{Coll}(\omega, < \kappa^*)]$. Suppose $\mathcal{M}$ is a λ -iterable sound (g -organized) Ψ -mouse over $a \in HC^{V[g]}$ and $\rho_\omega(\mathcal{M}) = a$, then by S -construction, $\mathcal{M}$ is (fine structurally) equivalent to a (g -organized), sound Ψ -mouse $\mathcal{M}^*$ over some $a^* \in V_\mu[g \cap \text{Coll}(\omega, < \kappa^*)]$ and $\rho_\omega(\mathcal{M}^*) = a^*$. This observation guarantees Σ is $\mathcal{S}_{\mu, g}^{\lambda, \Psi}$ -fullness preserving.

Remark 3.3. The proof of Theorem 3.1 is one main place the hypothesis that μ reflects the set of strong cardinals is used. There are other places in this section where this assumption on μ seems to be used essentially.

The above theorem shows that the Solovay sequence of $\Omega_{\mu,g}^\lambda$ has limit length. In the following theorem, the definition of $(\mathcal{M}_\infty)^{\Omega_{\mu,g}^\lambda}$ is self-explanatory. As explained in the next section, the assumption $\text{Proj}(\kappa, \lambda, \Psi)$ in the hypothesis holds when the hypothesis of Theorem 0.2 is assumed; this follows from [8, Lemma 5.2].

Theorem 3.4. *Suppose whenever $(\mathcal{R}, \Psi)$ is a hod pair below κ then $\text{Proj}(\kappa, \lambda, \Psi)$ holds; so the Solovay sequence of $\Omega_{\mu,g}^\lambda$ has limit length. Let $\mathcal{P} = (\mathcal{M}_\infty)^{\Omega_{\mu,g}^\lambda}$ and $\mathcal{P}^+ = \mathcal{W}_\omega^{\lambda, \Sigma^-, g}(\mathcal{P})$, where Σ^- is the join of the strategies $\Sigma_{\mathcal{P}(\alpha)}$ of $\mathcal{P}(\alpha)$ for all $\alpha < \lambda^{\mathcal{P}.cc}$. Let Θ be the height of the Wadge hierarchy of $\Omega_{\mu,g}^\lambda$ (so $\Theta = o(\mathcal{P})$). If $\mathcal{P}^+ \models \Theta$ is regular, then $\Omega_{\mu,g}^\lambda$ is constructibly closed and $L(\Omega_{\mu,g}^\lambda) \models \text{“AD}_\mathbb{R} + \Theta \text{ is regular”}$. If $\mathcal{P}^+ \models \Theta$ is singular, then there is an initial segment Γ of $\Omega_{\mu,g}^\lambda$ such that $L(\Gamma) \models \text{AD}_\mathbb{R} + \Theta \text{ is regular”}$.*

Proving Theorem 3.4 is the main task of our paper. For the first conclusion, suppose $\mathcal{P}^+ \models \Theta$ is regular. Note that $L[\mathcal{P}^+] \cap \wp(\Theta) = \mathcal{P}^+ \cap \wp(\Theta)$. By standard Vopenka arguments (cf. [1]), letting $\Omega = \Omega_{\mu,g}^\lambda$, $L[\mathcal{P}^+](\Omega) \cap \wp(\mathbb{R}) = \Omega$ and $L(\Omega) \models \text{“AD}_\mathbb{R} + \Theta \text{ is regular”}$.

The rest of the section is devoted to the second conclusion. So we assume $\mathcal{P}^+ \models \Theta$ is singular. We follow arguments in [7]. Many of the main ideas of our proof come from [7]; however, in this situation, we do not know *a priori* that $|\mathcal{P}^+|^V < \mu^+$ (unlike in the situation of [7]) and this affects many of the key arguments given there. We now outline the proof of the theorem, making use of results from [7] as much as possible.

Lemma 3.5. *Suppose $\mathcal{P} \trianglelefteq \mathcal{M} \trianglelefteq \mathcal{P}^+$. Then $\rho_\omega(\mathcal{M}) \geq \Theta$.*

Proof sketch. Fix such an $\mathcal{M}$. Note that $|\mathcal{M}|^V = \mu$ since the assumption that $\mathcal{P}^+ \models \Theta$ is singular that $\Theta < \mu^+$. The methods of [7], in particular Lemma 11.8, applied to $\mathcal{M}$ show that in fact $\rho_\omega(\mathcal{M}) \geq \Theta$. $\square$

We assume throughout this section that $|\mathcal{P}^+|^V \geq \mu^+$ (so equality holds). Otherwise, [7] applies and gives Theorem 3.4. By replacing j by the ultrapower embedding via the $(\mu, j(\mu))$ -extender derived from j , we may assume $j[\mathcal{P}^+]$ is cofinal in $j(\mathcal{P}^+)$.

Lemma 3.6. *$\mathcal{P}^+ \models \text{cof}(\lambda^{\mathcal{P}})$ is measurable.*

Proof. Suppose not. Recall we set $\Sigma^- = \bigoplus_{\alpha < \lambda^{\mathcal{P}^+}} \Sigma_{\mathcal{P}(\alpha)}$. Let $\Omega = \Omega_{\mu,g}^\lambda$. We have Σ^- acts on $\mathcal{P}^+$. More precisely, whenever $\vec{T}$ (based on $\mathcal{P}$) is according to Σ^- and $\pi^{\vec{T}}$

${}^{cc}\Sigma_{\mathcal{P}(\alpha)}$ is simply the tail of a hod pair $(\mathcal{Q}, \Lambda) \in \Omega_{\mu,g}^\lambda$ where $\mathcal{M}_\infty(\mathcal{Q}, \Lambda) = \mathcal{P}(\alpha)$. Using j^+ , we can extend Λ to a unique strategy, called Λ also, acting on stacks in $j^+(\Omega_{\mu,g}^\lambda)$; so $(\mathcal{P}(\alpha), \Sigma_{\mathcal{P}(\alpha)})$ is indeed a hod pair in $j^+(\Omega_{\mu,g}^\lambda)$.

exists, then letting $\mathcal{Q} = \text{Ult}(\mathcal{P}^+, E)$, where E is the $(\text{crt}(\pi^{\vec{T}}), \text{sup}\pi^{\vec{T}}[\mathcal{P}])$ -extender derived from $\pi^{\vec{T}}$, then we can define $\sigma : \mathcal{Q} \rightarrow j(\mathcal{P}^+)$ as follows: for any $f \in \mathcal{P}^+$, any $a \in (\mathcal{Q}|\delta^{\mathcal{Q}})^{<\omega}$,

$$\sigma(i_E(f)(a)) = j(f)(\pi_{\mathcal{Q}|\delta^{\mathcal{Q}},\infty}^{\Sigma_{\vec{T},\mathcal{Q}}^-}(a)).$$

Using the fact that $\pi^{\vec{T}}$ is continuous at $\delta^{\mathcal{P}^+}$ and $j \restriction \mathcal{P} = \pi_{\mathcal{P},\infty}^{\Sigma^-}$, we get that σ is elementary, $\sigma \circ i_E = j \restriction \mathcal{P}^+$, and $\sigma \restriction \delta^{\mathcal{Q}} = \pi_{\mathcal{Q},\infty}^{\Sigma_{\vec{T},\mathcal{Q}}^-} \restriction \delta^{\mathcal{Q}}$. In particular, this implies that $\mathcal{Q}$ is well-founded.

It follows from Theorem 3.1 that $(\mathcal{P}^+, \Sigma^-) \in j^+(\Omega)$. But then letting $\mathcal{M}_\infty$ be the direct limit of all iterates of $(\mathcal{P}^+, \Sigma^-)$ in $j^+(\Omega)$ (equivalently in $M[n]$), there is an embedding $\tau : \mathcal{M}_\infty \rightarrow j(\mathcal{P}^+)$ with critical point $\delta^{\mathcal{M}_\infty}$; this is because the maps j and $\pi_{\mathcal{P},\infty}^{\Sigma^-}$ agree up to $\delta^{\mathcal{P}}$ but not at $\delta^{\mathcal{P}}$. This implies that $\mathcal{M}_\infty$ is a hod initial segment of $j(\mathcal{P}^+)$ and $\mathcal{M}_\infty \models \text{“}\delta^{\mathcal{M}_\infty} \text{ is an inaccessible limit of Woodin cardinals”}$. This contradicts our smallness assumption $(\dagger)$ and the first clause. $\square$

Definition 3.7 (Nice strategies). Suppose $\pi_{\mathcal{P}^+,\mathcal{R}} : \mathcal{P}^+ \rightarrow \mathcal{R}$, $\sigma : \mathcal{R} \rightarrow j(\mathcal{P}^+)$ are Σ_1 -elementary. Suppose $j \restriction \mathcal{P}^+ = \sigma \circ \pi_{\mathcal{P}^+,\mathcal{R}}$. Let $\alpha < \lambda^{\mathcal{R}}$. We say that an iteration strategy Σ_α for $\mathcal{R}(\alpha)$ is **nice** if and only if

- (i) Σ_α is a $j^+(\Omega_{\mu,g}^\lambda)$ -fullness preserving strategy for $\mathcal{R}(\alpha)$ with branch condensation;
- (ii) $\pi_{\mathcal{R}(\alpha),\infty}^{\Sigma_\alpha} = \sigma' \restriction \mathcal{R}(\alpha)$ for some Σ_1 elementary map $\sigma' : \mathcal{R} \rightarrow j(\mathcal{P}^+)$ such that $j \restriction \mathcal{P}^+ = \sigma' \circ \pi_{\mathcal{P}^+,\mathcal{R}}$ (so Σ_α acts on all of $\mathcal{R}$);
- (iii) if $\pi_{\mathcal{P}^+,\mathcal{R}} \in M$, then $\Sigma_\alpha \restriction M \in M$.

Now, we construct a partial strategy Σ of $\mathcal{P}^+$ in $V[n]$ with the following properties (using the terminology of [7]):

- (i) Σ extends Σ^- .
- (ii) Whenever $\vec{T} \in M_{j(\mu)} \cup M_\kappa[m]$ is a stack on $\mathcal{P}^+$, we say that $\vec{T}$ is according to Σ if:
 - (a) for all $\mathcal{R}$ a terminal node (see [7, Definition 2.1]) of $\vec{T}$, there is a map $\sigma_{\mathcal{R}} : \mathcal{R} \rightarrow j(\mathcal{P}^+)$ such that

$$j \restriction \mathcal{P}^+ = \sigma_{\mathcal{R}} \circ \pi_{\mathcal{P}^+,\mathcal{R}}^{\vec{T}}.$$

Furthermore, if $\mathcal{Q}$ and $\mathcal{R}$ are two terminal nodes and $\pi_{\mathcal{Q},\mathcal{R}}^{\vec{T}}$ exist then $\sigma_{\mathcal{Q}} = \sigma_{\mathcal{R}} \circ \pi_{\mathcal{Q},\mathcal{R}}^{\vec{T}}$.

- (b) For all terminal nodes $\mathcal{R}$, for all successor $\alpha < \lambda^{\mathcal{R}}$, letting $\sigma_{\mathcal{R}}$ be as above, there is a unique $\{j(f) : f \in \mathcal{P}^+ \wedge j(f) \text{ is appropriate}\}$ -guided,^{dd} nice

^{dd}This means $\Sigma_{\mathcal{R}(\alpha)}$ witnesses $\mathcal{R}(\alpha)$ is strongly $(j(f), \Sigma_{\mathcal{R}(\beta)})$ -iterable for all $f \in \mathcal{P}^+$ such that $j(f)$ is appropriate. Furthermore, for any correctly guided, maximal $\mathcal{T}$ according to $\Sigma_{\mathcal{R}(\alpha)}$, $\delta(\mathcal{T}) = \sup_{f \in \mathcal{P}^+} (\gamma_{j(f)}^{\mathcal{M}_{\mathcal{T}}^b})$.

- strategy $\Sigma_{\mathcal{R}(\alpha)}$ for $\mathcal{R}(\alpha)$. Furthermore, letting $\alpha = \beta + 1$, then $\Sigma_{\mathcal{R}(\alpha)}$ extends $\Sigma_{\mathcal{R}(\beta)}$.
- (c) Letting $\mathcal{R}$ be as in (b), then whenever $\vec{\mathcal{U}}$ on $\mathcal{R}$ is according to (the tail of) Σ and $\vec{\mathcal{U}}$ is based on $\mathcal{R}(\alpha)$ for some $\alpha < \lambda^{\mathcal{R}}$, then $\vec{\mathcal{U}}$ is according to $\Sigma_{\mathcal{R}(\alpha)}$.

In the above, we define Σ by inductively defining $\Sigma_{\mathcal{R}(\alpha)}$ for each $\alpha < \lambda^{\mathcal{R}}$, where $\mathcal{R}$ is a terminal node on a stack $\vec{\mathcal{T}}$ as above. First note that $\Sigma_{\mathcal{P}(\alpha)}$ is nice for each $\alpha < \lambda^{\mathcal{P}^+}$ (with clause (ii) in Definition 3.7 being witnessed by j). Now suppose $\vec{\mathcal{T}}, \mathcal{R}$ are as above. It is enough to define $\Sigma_{\mathcal{R}(\alpha)}$ for $\alpha = \beta + 1$, where by induction, we have that $\Sigma_{\mathcal{R}(\beta)}$ is nice and the supremum of the generators of $\vec{\mathcal{T}}_{\mathcal{P}^+, \mathcal{R}}$ is $\leq \delta_{\beta}^{\mathcal{R}}$.^{ee} We prove a series of lemmas that eventually leads to the construction of $\Sigma_{\mathcal{R}(\alpha)}$.

Lemma 3.8. *Let $\vec{\mathcal{T}}, \mathcal{R}, \sigma_{\mathcal{R}}, \alpha, \beta$ be as above. Then $\mathcal{R}(\alpha)$ is full in $j^+(\Omega_{\mu, g}^{\lambda})$.*

Proof. Suppose not. As before we set $\Omega = \Omega_{\mu, g}^{\lambda}$ and we have already assumed that $\Sigma_{\mathcal{R}(\beta)}$ is nice, so in particular, $\mathcal{R}(\beta)$ is $j^+(\Omega)$ -full. Let $\xi < j(\mu)$ be M -inaccessible and $(\mathcal{P}^*, \Psi) \in M_{\xi}$ be a Σ^- -hod pair in $j^+(\Omega)$ witnessing $\mathcal{R}(\alpha)$ is not full and $\lambda^{\mathcal{P}^*}$ is limit of countable cofinality (in $\mathcal{P}^*$). More precisely, there is a cutpoint ξ in $\mathcal{R}(\alpha)$ above $\mathcal{R}(\beta)$ such that in $\Gamma(\mathcal{P}^*, \Psi)$,^{ff} there is a $\Sigma_{\mathcal{R}(\beta)}$ -mouse $\mathcal{M}$ such that $\mathcal{M} \triangleleft Lp^{\Sigma_{\mathcal{R}(\beta)}}(\mathcal{R}(\alpha)|\xi) \setminus \mathcal{R}(\alpha)$. The existence of such a pair $(\mathcal{P}^*, \Psi)$ follows from the fact that $j(\mu)$ is strong in M and by Boolean comparison. Note that no levels of $\mathcal{P}^*$ extending $\mathcal{P}^+$ projects to or below Θ . This is similar to the proof of Lemma 3.5; basically, this is because if $\mathcal{P} \triangleleft \mathcal{M} \triangleleft \mathcal{P}^*$ is such that $\rho_{\omega}(\mathcal{M}) \leq \Theta$, by the proof of [7, Lemma 11.8], we get that $\mathcal{M} \triangleleft \mathcal{P}^+$. We assume that Ψ has branch condensation and is $j^+(\Omega)$ -fullness preserving.

Let $\sigma = \pi_{\mathcal{P}^+, \mathcal{R}}$ and $\sigma^+ : \mathcal{P}^* \rightarrow \mathcal{R}^*$ be the ultrapower map of $\mathcal{P}^*$ by the $(\text{crt}(\sigma), \delta^{\mathcal{R}})$ -extender derived from σ . Let $\sigma_{\mathcal{R}}^+ : \mathcal{R}^* \rightarrow j(\mathcal{P}^*)$ be defined as: for $g \in \mathcal{P}^*$, $a \in (\delta^{\mathcal{R}})^{<\omega}$,

$$\sigma_{\mathcal{R}}^+(\sigma^+(g)(a)) = j(g)(\sigma_{\mathcal{R}}(a)).$$

We have then that $\sigma_{\mathcal{R}}^+$ is elementary and $j \restriction \mathcal{P}^* = \sigma_{\mathcal{R}}^+ \circ \sigma^+$.

In V , let $\vec{\mathcal{T}}, \vec{\mathcal{R}}, \vec{\mathcal{R}}^*, \vec{\mathcal{S}}, \vec{\Sigma}, \vec{\sigma}, \vec{\sigma}^+, \vec{\sigma}_{\mathcal{R}}, \vec{\sigma}_{\mathcal{R}}^+ \in V$ be canonical $\text{Coll}(\omega, < \kappa)$ names for $\vec{\mathcal{T}}, \mathcal{R}, \mathcal{R}^*, \mathcal{R}(\alpha), \Sigma_{\beta}, \sigma, \sigma^+, \sigma_{\mathcal{R}}, \sigma_{\mathcal{R}}^+$ respectively. Let γ be a sufficiently large regular cardinal in $V[g]$ such that $V_{\gamma}[g]$ contains all relevant objects and let $\mu + 1 \subset X \prec V_{\gamma}[g]$ be of size μ and contain all relevant objects. Let $\pi : N \rightarrow X$ be the uncollapse

^{ee}If $\alpha < \lambda^{\mathcal{R}}$ is limit and $\text{cof}^{\mathcal{R}}(\alpha)$ is not measurable in $\mathcal{R}$ then we set $\Sigma_{\mathcal{R}(\alpha)}$ to be $\bigoplus_{\beta < \alpha} \Sigma_{\mathcal{R}(\beta)}$. If $\text{cof}^{\mathcal{R}}(\alpha) = \kappa$ is measurable in $\mathcal{R}$, then we let $\mathcal{S} = \text{Ult}(\mathcal{R}, E)$ where E is the total extender on κ with the least index in $\mathcal{R}$ and we define $\Sigma_{\mathcal{R}(\alpha)}$ by inductively defining $\Sigma_{\mathcal{S}(\beta)}$ for $\beta < \lambda^{\mathcal{S}}$.

^{ff}This is the set of $A \subseteq \mathbb{R}$ such that $A <_w \Lambda_{\mathcal{S}(\alpha)}$ for some iterate $(\mathcal{S}, \Lambda)$ of $(\mathcal{P}^*, \Psi)$ and $\alpha < \lambda^{\mathcal{S}}$. The existence of $(\mathcal{P}^*, \Psi)$ follows from the fact that Ω 's Solovay sequence is of limit order type.

map. Let $\bar{m} \in V[g]$ be $\text{Coll}(\omega, \pi^{-1}(\kappa))$ -generic over N . For any $a \in X$, let $\bar{a} = \pi^{-1}(a)$.

Let $\mathcal{M} = \mathcal{M}_{\omega}^{\sharp, \Psi}$ and let Π be $\mathcal{M}$'s $j(\mu)$ -strategy in $M[n]$ (the $\mathcal{M}$ before is behind us now). We assume also that $(\mathcal{M}, \Pi \restriction V[g]) \in X$. Let $\mathcal{N}$ be an iterate (below the first Woodin cardinal of $\mathcal{M}$) such that $H =_{\text{def}} H_{\xi}^M$ is generically generic^{gg} over $\mathcal{N}$ for the extender algebra $\mathbb{B}_{\delta}^{\mathcal{N}}$, where δ is the first Woodin cardinal of $\mathcal{N}$. Then in $\mathcal{N}[H][m]$, the following conditions holds:

$$D(\mathcal{N}[H][m]) \models \text{“in } L(\Gamma(\mathcal{P}^*, \Psi), \mathbb{R}), \mathcal{R}(\alpha) \text{ is not full”},$$

where $D(\mathcal{N}[H][m])$ is the derived model of $\mathcal{N}[H][m]$. A similar fact holds of $\bar{\mathcal{N}}$ inside $N[\bar{m}]$. In fact, letting $\mathcal{M} = \bar{\mathcal{N}}[\bar{H}]$, then inside $\mathcal{M}$, letting λ be the sup of $\mathcal{M}$'s Woodin cardinals:

$$\begin{aligned} \emptyset \Vdash_{\text{Col}(\omega, < \bar{\kappa})} \Vdash_{\text{Col}(\omega, < \lambda)} \text{ in the derived model, } L(\Gamma(\bar{\mathcal{P}}^*, \bar{\Psi})) \\ \text{witnesses that } \dot{\mathcal{S}} \text{ is not full.} \end{aligned} \quad (3.1)$$

Note that

$$\bar{\Psi} = \Psi^{\pi} \restriction N \quad \text{and} \quad \bar{\Pi} = \Pi^{\pi} \restriction N, \quad (3.2)$$

and

$$\pi \restriction \pi^{-1}(j(\mathcal{P}^*)) \circ (\sigma_{\bar{\mathcal{R}}}^{\bar{\tau}})_{\bar{m}} =_{\text{def}} \tau \in M[g], \quad (3.3)$$

and

$$\Psi^{\pi} = j(\Psi)^{j \circ \pi} = j(\Psi)^{\tau \circ (\sigma^{\bar{\tau}})_{\bar{m}}} \quad \text{is Wadge reducible to } \Lambda =_{\text{def}} j(\Psi)^{\tau}. \quad (3.4)$$

Combining (3.1), (3.2), and (3.4), letting $\mathcal{W} = \bar{\mathcal{R}}^*_{\bar{m}}$ and $\mathcal{S} = \bar{\mathcal{S}}$, we get (in $\Omega_{\kappa, m}^{\lambda}$),

$$\text{in } L(\Gamma(\mathcal{W}, \Lambda), \mathbb{R}), \quad \mathcal{S} \text{ is not full.} \quad (3.5)$$

This means that if we perform an $\mathbb{R}^{V[m]}$ -genericity iteration via Λ , then letting $\mathcal{W}^*$ be the iterate, inside $D(\mathcal{W}^*)$, we have

$$\mathcal{S} \text{ is not full.} \quad (3.6)$$

This contradicts results in [5] on internal fullness of hod mice. $\square$

Definition 3.9. For $f \subseteq \delta^{\mathcal{P}}$ and $f \in \mathcal{P}^+$. We say an $\mathcal{M} \triangleleft \mathcal{P}^+$ is f -**nice** if $\rho_{\omega}(\mathcal{M}) = \Theta$, $f \in \mathcal{M}$, $\mathcal{M} \models \Theta$ is the largest cardinal, and $j \restriction \mathcal{M}$ is cofinal in $j(\mathcal{M})$.

Fix an appropriate $f \in \mathcal{P}^+$. Let $\mathcal{M} \triangleleft \mathcal{P}^+$ be f -nice; note that the set of f -nice $\mathcal{M}$'s is unbounded in $\mathcal{P}^+$. We construct a strategy Σ_f witnessing $\mathcal{R}(\alpha)$ is strongly $(j(f), \Sigma_{\mathcal{R}(\beta)})$ -iterable. First, we construct a realizable strategy for $\mathcal{R}(\alpha)$. Let $\tau_{\mathcal{M}} = j \restriction \mathcal{M}$. Note that $\tau_{\mathcal{M}} \in M$ and by f -niceness of $\mathcal{M}$, $\pi_{\mathcal{P}^+, \mathcal{R}} \restriction \mathcal{M}$ is cofinal in $\pi_{\mathcal{P}^+, \mathcal{R}}(\mathcal{M})$. By absoluteness, $\Sigma_{\mathcal{R}(\beta)} \in M[n]$, and the fact that $\pi_{\mathcal{P}^+, \mathcal{R}}(\mathcal{M})$ is countable in $M[n]$, there is in $M[n]$ an elementary $\sigma_{\mathcal{M}} : \pi_{\mathcal{P}^+, \mathcal{R}}(\mathcal{M}) \rightarrow j(\mathcal{M})$

^{gg}See [5, Sec. 2.10] for the definition.

such that

- $\sigma_{\mathcal{M}} \circ \pi_{\mathcal{P}^+, \mathcal{R}} \restriction \mathcal{M} = \tau_{\mathcal{M}}$.
- $\sigma_{\mathcal{M}} \restriction \mathcal{R}(\beta) = \pi_{\mathcal{R}(\beta), \infty}^{\Sigma_{\mathcal{R}(\beta)}}$.

Let $\Sigma' = j(\mathcal{P})_{\mathcal{R}(\alpha)}^{\sigma_{\mathcal{M}}}$ be the $\Sigma_{\mathcal{M}}$ -pullback of $\mathcal{R}(\alpha)$. By constructions in [7, Sec. 11], whenever $\mathcal{S}$ is a nondropping Σ' -iterate of $\mathcal{R}(\alpha)$, then there is an embedding $\sigma_{\mathcal{S}} : \mathcal{S} \rightarrow \sigma_{\mathcal{M}}(\mathcal{R}(\alpha))$ such that $\sigma_{\mathcal{S}} \circ \pi_{\mathcal{R}(\alpha), \mathcal{S}}^{\Sigma'} = \sigma_{\mathcal{M}} \restriction \mathcal{R}(\alpha)$.

Remark 3.10. The above construction, though stated as a definition for a strategy of $\mathcal{R}(\alpha)$ in $V[m]$ as part of defining a partial strategy Σ for $\mathcal{P}^+$, indeed gives an inductive definition of a strategy $\Lambda_{\mathcal{M}} \in M[n]$ for $\mathcal{M}$ for stacks in $M_{j(\mu)}[n]$; the reason is because $\tau_{\mathcal{M}} \in M$. Furthermore by construction, given any $\Lambda_{\mathcal{M}}$ -iterate $\mathcal{S}$ of $\mathcal{M}$, there is some $\sigma : \mathcal{S} \rightarrow j(\mathcal{M})$ such that $\sigma \circ \pi_{\mathcal{M}, \mathcal{S}} = j \restriction \mathcal{M}$.

Lemma 3.11. *All nondropping Σ' -iterates of $\mathcal{R}(\alpha)$ are $j^+(\Omega_{\mu, g}^\lambda)$ -full. Furthermore, Σ' has branch condensation and is positional and commuting.*

Proof sketch. The proof is almost the same as that of Lemma 3.8. We only outline the main changes. Let $\mathcal{S}$ be a nondropping Σ' -iterate of $\mathcal{R}(\alpha)$ and suppose $\mathcal{S}$ is not full. Let $(\mathcal{P}^*, \Psi)$ be as in the proof of Lemma 3.8 witnessing this. Let E be the $(\text{crt}(\pi_{\mathcal{P}^+, \mathcal{R}}), \delta^{\mathcal{R}})$ -extender derived from $\pi_{\mathcal{P}^+, \mathcal{R}}$. Let $\mathcal{Q} = \text{Ult}(\mathcal{P}^*, E)$, $\mathcal{N}' = i_E^{\mathcal{Q}}(\mathcal{M})$, and $\mathcal{N} = \text{Ult}(\mathcal{M}, E)$.

Claim 3.12. $\mathcal{N}' = \mathcal{N} = \pi_{\mathcal{P}^+, \mathcal{R}}(\mathcal{M})$ and $i_E^{\mathcal{Q}} \restriction \mathcal{M} = i_E^{\mathcal{M}} = \pi_{\mathcal{P}^+, \mathcal{R}} \restriction \mathcal{M}$.

Proof. We just prove $\mathcal{N} = \pi_{\mathcal{P}^+, \mathcal{R}}(\mathcal{M})$ and $i_E^{\mathcal{M}} = \pi_{\mathcal{P}^+, \mathcal{R}} \restriction \mathcal{M}$. By definition of E and the choice of $\mathcal{M}$, there is a factor map $l : \mathcal{N} \rightarrow \pi_{\mathcal{P}^+, \mathcal{R}}(\mathcal{M})$ such that $\text{crt}(l) \geq \delta^{\mathcal{R}}$ and l is cofinal in $\pi_{\mathcal{P}^+, \mathcal{R}}(\mathcal{M})$. Note that both $\mathcal{N}$ and $\pi_{\mathcal{P}^+, \mathcal{R}}(\mathcal{M})$ satisfy $\delta^{\mathcal{R}}$ is the largest cardinal. It means that l is the identity. Similarly, $\mathcal{N}' = \pi_{\mathcal{P}^+, \mathcal{R}}(\mathcal{M})$ and $i_E^{\mathcal{Q}} = \pi_{\mathcal{P}^+, \mathcal{R}} \restriction \mathcal{M}$. $\square$

Now as in the proof of Lemma 3.8, $\pi =_{\text{def}} \pi_{\mathcal{P}^+, \mathcal{R}}$ lifts to $\pi^+ : \mathcal{P}^* \rightarrow \mathcal{Q}$ and there is a map $\sigma_{\mathcal{Q}} : \mathcal{Q} \rightarrow j(\mathcal{P}^*)$ extending $\sigma_{\mathcal{M}}$ (this uses the claim) such that $\sigma_{\mathcal{Q}} \circ \pi^+ = j \restriction \mathcal{P}^*$.

Now, $\tau =_{\text{def}} \pi_{\mathcal{R}(\alpha), \mathcal{S}}^{\Sigma'}$ can be extended to $\tau^+ : \mathcal{Q} \rightarrow \mathcal{S}^+$ (τ^+ is simply the ultrapower map by the $(\text{crt}(\tau), \delta^{\mathcal{S}})$ -extender derived from τ) and there is a map $\sigma_{\mathcal{S}^+} : \mathcal{S}^+ \rightarrow j(\mathcal{P}^*)$ such that $\sigma_{\mathcal{Q}} = \sigma_{\mathcal{S}^+} \circ \tau^+$. The rest of the proof is just like that of Lemma 3.8.

That Σ' has branch condensation, is positional and commuting follows from [5, Lemma 3.26] and the fact that $\text{cof}(\delta^{\mathcal{P}})$ is measurable in $\mathcal{P}^+$ since Σ' can be taken to be the pullback of some hod pair $(\mathcal{R}, \Lambda)$ in $j^+(\Omega_{\mu, g}^\lambda)$ such that $\lambda^{\mathcal{R}}$ is limit and Λ has branch condensation and is fullness preserving. $\square$

Let $f \subseteq \delta^{\mathcal{P}}$ and $f \in \mathcal{P}^+$; let $\mathcal{M}$ be f -nice, and $\tau_{\mathcal{M}} = j \restriction \mathcal{M}$. Again, note that $\tau_{\mathcal{M}} \in M$. We now define the notion of $(f, \mathcal{M})$ -condensation. Suppose in $M[n]$, $\mathcal{S}$ is

a hod premouse such that $\mathcal{S}$ is $\tau_{\mathcal{M}}$ -realizable, that is, there are maps $\pi : \mathcal{M} \rightarrow \mathcal{S}$ and $\tau_{\mathcal{S}} : \mathcal{S} \rightarrow j(\mathcal{M})$ in $M[n]$ such that $\tau_{\mathcal{M}} = \tau_{\mathcal{S}} \circ \pi$. Letting $\Sigma_{\tau_{\mathcal{S}}} = j(\mathcal{P})^{\tau_{\mathcal{S}}}$, we define the set $A_{f,\mathcal{M},\tau_{\mathcal{S}}}$ as follows: for any Σ_1 formula ϕ , for any $s \in (\mathcal{S}|\delta^{\mathcal{S}})^{<\omega}$,

$$(\phi, s) \in A_{f,\mathcal{M},\tau_{\mathcal{S}}} \Leftrightarrow j(\mathcal{M}) \models \phi[\pi_{\Sigma_{\tau_{\mathcal{S}}(\gamma),\infty}}^{\Sigma_{\tau_{\mathcal{S}}}}(s), \tau_{\mathcal{M}}(f)],$$

where γ is such that $s \in \mathcal{S}(\gamma)$. We also let

$$T_{f,\mathcal{M}} = \{(\phi, s) \mid \phi \text{ is a } \Sigma_1 \text{ formula, } s \in (\mathcal{P})^{<\omega}, \text{ and } \mathcal{M} \models \phi[s, f]\}.$$

In the following definition, we reuse the notions just defined.

Definition 3.13. Let $\mathcal{M}$ be f -nice and $\tau_{\mathcal{M}} = j \upharpoonright \mathcal{M}$. Suppose $\pi : \mathcal{M} \rightarrow \mathcal{S}$ and $\tau_{\mathcal{S}} : \mathcal{S} \rightarrow j(\mathcal{M})$ are such that $\tau_{\mathcal{M}} = \tau_{\mathcal{S}} \circ \pi$. We say $\tau_{\mathcal{S}}$ **has** $(f, \mathcal{M})$ -**condensation** (in $M[n]$) if whenever $\mathcal{W}$ is $\tau_{\mathcal{S}}$ -realizable as witnessed by $(\pi^*, \tau_{\mathcal{W}})$, then $\pi^*(\pi(T_{f,\mathcal{M}})) = A_{f,\mathcal{M},\tau_{\mathcal{W}}}$.

The following theorem and its proof is from [7], but here we apply it to $\mathcal{M}$.

Theorem 3.14 (($f, \mathcal{M}$)-condensation lemma). *Let $\mathcal{M}$ be f -nice and $\tau_{\mathcal{M}} = j \upharpoonright \mathcal{M}$. Then $\tau_{\mathcal{M}}$ has $(f, \mathcal{M})$ -condensation.*

Proof. Working in $M[n]$, let $\mu \leq \nu < j(\mu)$ be such that ν is M -inaccessible. Let $\mathcal{R}_{\nu}$ be the direct limit of all hod pairs $(\mathcal{W}, \Sigma)$ such that $\mathcal{W} \in M[n \cap \text{Coll}(\omega, < \nu)]$, Σ is $j^+(\Omega_{\mu,g}^{\lambda})$ -fullness preserving, positional, commuting, and has branch condensation. Let $Y_{\nu} = \bigcup_{\alpha < \lambda^{\mathcal{R}_{\nu}}} \pi_{\mathcal{R}_{\nu}(\alpha),\infty}^{\Sigma_{\mathcal{R}_{\nu}(\alpha)}}[\mathcal{R}_{\nu}(\alpha)]$. Let $X \subset j(\mathcal{M})$ be countable in $M[n]$.^{hh} Let $\mathcal{R}_{\nu}^*$ be the transitive collapse of $H_1^{j(\mathcal{M})}(X \cup Y_{\nu})$ and σ_{ν} be the uncollapse map. We say that ν is X -good if $\sigma_{\nu} \upharpoonright \delta^{\mathcal{R}_{\nu}^*} = \bigcup_{\alpha < \lambda^{\mathcal{R}_{\nu}}} \pi_{\mathcal{R}_{\nu}(\alpha),\infty}^{\Sigma_{\mathcal{R}_{\nu}(\alpha)}}$. The proof of [7, Lemma 11.9] shows that there are cofinally many $\nu < j(\mu)$ that are X -good for any such X . When $X = \tau_{\mathcal{M}}[\mathcal{M}]$, and ν is X -good, we say ν is a *good point*.

For a good point ν , we can define an iteration strategy Λ_{ν} (for stacks in $M_{j(\mu)}[n]$) for $\mathcal{R}_{\nu}^*$ the same way $\Lambda_{\mathcal{M}}$ was defined in Remark 3.10, but using σ_{ν} instead of j . Λ_{ν} has the following properties:

- Whenever $\mathcal{S}$ is a nondropping Λ_{ν} -iterate of $\mathcal{R}_{\nu}^*$, $\mathcal{S}|\delta^{\mathcal{S}}$ is $j^+(\Omega_{\mu,g}^{\lambda})$ -full.ⁱⁱ Furthermore, for each $\alpha < \lambda^{\mathcal{S}}$, $(\Lambda_{\nu})_{\mathcal{S}(\alpha)}$ has branch condensation.
- Letting $\mathcal{S}$ be as above, there is a map $\sigma : \mathcal{S} \rightarrow j(\mathcal{M})$ such that $\sigma \circ \pi_{\mathcal{R}_{\nu}^*,\mathcal{S}} = \sigma_{\nu}$.

Let $\mathcal{M}_{\nu}$ be the direct limit of all Λ_{ν} -iterates in $j^+(\Omega_{\mu,g}^{\lambda})$ and m_{ν} be the σ_{ν} -realization map given by the construction of Λ_{ν} .

As in the proof of [7, Lemma 11.15], it suffices to show:

$$\text{there is a } \tau_{\mathcal{M}}[\mathcal{M}]\text{-good } \nu \text{ such that } \sigma_{\nu} \text{ has } (f, \mathcal{M})\text{-condensation.} \quad (3.7)$$

^{hh} $\tau_{\mathcal{M}}[\mathcal{M}]$ is an example of such an X .

ⁱⁱ $\mathcal{S}$ is not full at the top, so we cannot demand more than this.

The proof of this now is just that of [7, Lemma 11.15] using remarks in Lemma 3.11 and the fact that $\tau_{\mathcal{M}} \in M$ (this replaces the hypothesis $|\mathcal{P}^+|^V < \mu^+$ used in [7, Lemma 11.9]). We outline the proof here for the reader's convenience.

Suppose 3.7 fails. We can then find a sequence $(\mathcal{Q}_i, \pi_i, \tau_i, k_i, \psi_i, \nu_i : i < \omega) \in M[n]$ such that

- (1) $\nu_0 = \mu$, $\mathcal{R}_0 = \mathcal{M}$, and $(\nu_i : i < \omega)$ is an increasing sequence of good points,
- (2) for $i < \omega$, $\mathcal{Q}_i$ is σ_{ν_i} -realizable as witness by (π_i, τ_i) and $k_i : \mathcal{Q}_i \rightarrow \mathcal{R}_{\nu_{i+1}}^* =_{\text{def}} \mathcal{R}_{i+1}$ is given by $k_i = \sigma_{\nu_i}^{-1} \circ \tau_i$,
- (3) for $i < \omega$, $\sigma_{\nu_i}[\mathcal{R}_i] \subseteq \text{rng}(\sigma_{\nu_{i+1}})$, $\psi_i = \sigma_{\nu_{i+1}}^{-1} \circ \sigma_{\nu_i}$ and for $i < m$, letting $\psi_{i,m} = \sigma_{\nu_m}^{-1} \circ \sigma_{\nu_i}$ and $f_i = \psi_{0,i}(f)$, $\pi_i(T_{f_i, \mathcal{R}_i}) \neq A_{f_i, \mathcal{Q}_i, \tau_i}$ (i.e., $(\mathcal{Q}_i, \pi_i, \tau_i)$ witnesses that σ_{ν_i} does not have $(f_i, \mathcal{R}_i)$ -condensation).

Let now ν be a good point such that $\sup_{i < \omega} \nu_i < \nu < j(\mu)$ and letting $X = \bigcup_{i < \omega} (\tau_i[\mathcal{Q}_i] \cup \sigma_{\nu_i}[\mathcal{R}_{\nu_i}])$, $X \subseteq \text{rng}(\sigma_\nu)$. Let $(\mathcal{S}^*, \Phi^*) \in j^+(\Omega_{\mu, g}^\lambda)$ be a hod pair below $j(\mu)$ such that $\mathcal{M}_\nu \triangleleft \mathcal{M}_\infty(\mathcal{S}^*, \Phi^*)^{\text{jj}}$ and $\lambda^{\mathcal{S}}$ is limit with cofinality ω in $\mathcal{S}^*$. Let $B = m_\nu^{-1}(j(f))$. Let now $\sigma_i = m_\nu^{-1} \circ \sigma_{\nu_i}$ and $\tau_i^* = m_\nu^{-1} \circ \tau_i$. Notice now that we can define the notion of $(f_i, \mathcal{R}_i)$ -condensation also for the embeddings σ_i . We leave it to the reader to fill in the definition. Now notice that we have that

- (4) for $i < \omega$, $\mathcal{Q}_i$ is σ_i -realizable as witness by (π_i, τ_i^*) and $k_i : \mathcal{Q}_i \rightarrow \mathcal{R}_{i+1}$ is given by $k_i = \sigma_i^{-1} \circ \tau_i^*$,
- (5) $(\mathcal{Q}_i, \pi_i, \tau_i^*)$ witnesses that σ_i does not have $(f_i, \mathcal{R}_i)$ -condensation.

The importance of this move is that the badness of $(\mathcal{Q}_i, \mathcal{R}_i, \pi_i, \tau_i^*, k_i, \psi_i, \sigma_i : i < \omega)$ can now be witnessed in the derived model of $\mathcal{S}^*$ as computed by Φ^* . More precisely, letting $\Sigma_i = \bigoplus_{\alpha < \lambda^{\mathcal{R}_i}} \Lambda_{\nu_i}(\alpha)$ and $\Psi_i = (\tau_i\text{-pullback of } j(\Sigma_i)^h)$,

(1): in $M[n]$, letting $N = D(\mathcal{S}^*, \Phi^*) = L(\Gamma(\mathcal{S}^*, \Phi^*), \mathbb{R})$, in N , there is a formula $\theta(u, v)$ and a finite set of ordinals t such that for every i , $(\phi, s) \in T_{f_i, \mathcal{R}_i}$ if and only if $\theta[\pi_{\mathcal{R}_i(\alpha)}^{\Sigma_i}(s), t]$ where α is the least such that $s \in [\delta_\alpha^{\mathcal{R}_i}]^{<\omega}$. However, in N , for each i , there is a pair $(\phi_i, s_i) \in T_{\mathcal{Q}_i, \pi_i(f_i)}$ such that $\neg \theta[\pi_{\mathcal{Q}_i(\alpha)}^{\Psi_i}(s_i), t]$ where α is the least such that $s \in [\delta_\alpha^{\mathcal{Q}_i}]^{<\omega}$.

Suppose K is a transitive model of AD^+ and $b = ((\mathcal{M}_i, \Sigma_i), \mathcal{N}_i, \gamma_i, l_i, \xi_i, C : i < \omega) \in K$ is such that $(\mathcal{M}_i, \mathcal{N}_i, \gamma_i, l_i, \xi_i, C : i < \omega) \in HC^K$. Suppose $\theta(u, v)$ is a formula and t is a finite sequence of ordinals. We write $K \models "(b, \theta(u, v), t) \text{ is bad}"$ if in K , letting $K^* = L(\{D \subseteq \mathbb{R} : w(D) \leq t(0)\})$ then $b \in K^*$ and in K^*

- (6) for every $i < \omega$, $\mathcal{M}_i$ is a hod premouse such that $\lambda^{\mathcal{M}_i}$ is limit and Σ_i is an ω_1 -iteration strategy for $\mathcal{M}_i|\delta^{\mathcal{M}_i}$ with the property that for every $\alpha < \lambda^{\mathcal{M}_i}$, $(\Sigma_i)_{\mathcal{M}_i(\alpha)}$ has branch condensation and is fullness preserving,
- (7) for every i , $\xi_i : \mathcal{M}_i \rightarrow \mathcal{M}_{i+1}$,
- (8) for every i , $\mathcal{N}_i$ is a ξ_i -realizable as witnessed by (γ_i, l_i) ,

^{jj}The direct limit is taken inside $j^+(\Omega_{\mu, g}^\lambda)$.

- (9) for every $\alpha < \lambda^{\mathcal{N}_i}$, letting $\Psi_i = (l_i\text{-pullback of } \Sigma_i)$, $(\Psi_i)_{\mathcal{N}_i(\alpha)}$ has branch condensation and is fullness preserving,
- (10) $C \in M_0 \cap \wp(\delta^{\mathcal{M}_i})$ and letting $C_0 = C$ and $C_{i+1} = \xi_i(C_i)$, for every i ,

$$(\phi, s) \in T_{C_i, \mathcal{M}_i} \quad \text{if and only if } \theta[\pi_{\mathcal{M}_i(\alpha), \infty}^{\Sigma_i}(s), t],$$

where α is least such that $s \in [\delta_\alpha^{\mathcal{M}_i}]^{<\omega}$ but for every i , there is $(\phi_i, s_i) \in T_{\gamma_i(C_i), \mathcal{N}_i}$ such that $\neg\theta[\pi_{\mathcal{N}_i(\alpha), \infty}^{\Psi_i}(s), t]$ where α is least such that $s_i \in [\delta_\alpha^{\mathcal{N}_i}]^{<\omega}$.

In $M[n]$, let $(\mathcal{W}^*, \Pi^*)$ be a $(\mathcal{P}^+, \Sigma^-)$ -hod pair such that $\mathcal{W}^* \in M_\gamma[g]$ for some M -cardinal $\gamma < j(\mu)$ but greater than ν , Π^* is a $(j(\lambda), j(\lambda))$ -strategy that is $j^+(\Omega_{\mu, g}^\lambda)$ -fullness preserving, $\Pi^* \cap j^+(\Omega_{\mu, g}^\lambda) \in (\Omega_{\mu, g}^\lambda)$, and $\Gamma(\mathcal{W}^*, \Pi^*) = \Gamma(\mathcal{S}^*, \Phi^*)$ in $j^+(\Omega_{\mu, g}^\lambda)$. Let $b = ((\mathcal{R}_i, \Sigma_i), \mathcal{Q}_i, \pi_i, k_i, \psi_i, f_i : i < \omega)$. We can then rewrite (1) in terms of $(\mathcal{W}^*, \Pi^*)$ and get that

(2): in $M[h]$, letting $N = D(\mathcal{W}^*, \Pi^*) = L(\Gamma(\mathcal{W}^*, \Pi^*), \mathbb{R})$, in N , there is a formula $\theta(u, v)$ and a finite set of ordinals t such that $(b, \theta(u, v), t)$ is bad.

Let then $\mathcal{N}^* = \mathcal{M}_\omega^{\#, \Pi^*, \oplus_{i < \omega} \Sigma_i}$. Let $\mathcal{N}$ be an iterate of $\mathcal{N}^*$ via the canonical iteration strategy of $\mathcal{N}^*$ such that H_γ^M is generically generic over the extender algebra of $\mathcal{N}$ at its bottom Woodin cardinal. We can now witness (3) inside $\mathcal{N}[H_\gamma^M][h_\gamma]$ as follows:

(3): $D(\mathcal{N}[H_\gamma^M][h_\gamma]) \models$ “letting $N = D(\mathcal{W}^*, \Pi^*) = L(\Gamma(\mathcal{W}^*, \Pi^*), \mathbb{R})$, in N , there is a formula $\theta(u, v)$ and a finite set of ordinals t such that $(b, \theta(u, v), t)$ is bad”.

We will get a contradiction using (3). Notice that the sequence $a = (\mathcal{R}_i, \psi_i, \Sigma_i, f_i : i < \omega) \in M$. However, the sequence $(\mathcal{Q}_i, \pi_i, k_i : i < \omega)$ may not be in M . Let then $d \in M^{\text{Coll}(\omega, < \gamma)}$ be a name for $(\mathcal{Q}_i, \pi_i, k_i : i < \omega)$. Let $\zeta = (j(\mu)^+)^M$, and let $\pi : P[g] \rightarrow (H_\zeta^M)[g]$ be such that $P \in V$, $\text{cp}(\pi) > \mu$, $|P|^V = \mu$, and all relevant objects are contained in $\text{rng}(\pi)$. Let $\mathcal{M} = \pi^{-1}(\mathcal{N})$, $e = \pi^{-1}(a)$ and $c = \pi^{-1}(d)$. Let for $i < \omega$, $e(i) = (\mathcal{K}_i, \xi_i, \bar{\Sigma}_i, g_i : i < \omega)$ and $(\mathcal{W}, \Pi) = \pi^{-1}(\mathcal{W}^*, \Pi^*)$. Also we let $\bar{\gamma} = \pi^{-1}(\gamma)$. By elementarity, (3) gives that

(4): whenever $\bar{m} \subseteq \text{Coll}(\omega, < \pi^{-1}(j(\mu)))$ is $P[g]$ -generic then in $P[g][\bar{m}]$, letting $\bar{k} = \bar{m} \cap \text{Coll}(\omega, < \bar{\gamma})$ $d = d_{g * \bar{k}}$, for $i < \omega$, $d(i) = (\mathcal{S}_i, \gamma_i, l_i)$ and $g = ((\mathcal{K}_i, \bar{\Sigma}_i), \mathcal{S}_i, \gamma_i, l_i, \xi_i, g_i : i < \omega)$, $D(\mathcal{M}[H_\gamma^P][g * \bar{k}]) \models$ “letting $N = D(\mathcal{W}, \Pi) = L(\Gamma(\mathcal{W}, \Pi), \mathbb{R})$, in N , there is a formula $\theta(u, v)$ and a finite set of ordinals t such that $(g, \theta(u, v), t)$ is bad”.

Using genericity iterations we can completely internalize (5) to $\mathcal{M}^* = \mathcal{M}[H_\gamma^P]$ and get that

(5): in $\mathcal{M}^*$, there is a name $d^* \in (\mathcal{M}^*)^{\text{Coll}(\omega, < \bar{\gamma})}$ such that whenever $\bar{k} \subseteq \text{Coll}(\omega, < \bar{\gamma})$ is $\mathcal{M}^*$ -generic then letting $d = d^*_{\bar{k}}$, for $i < \omega$, $d(i) = (\mathcal{S}_i, \gamma_i, l_i)$ and $g = ((\mathcal{K}_i, \bar{\Sigma}_i), \mathcal{S}_i, \gamma_i, l_i, \xi_i, g_i : i < \omega)$, $D(\mathcal{M}^*[\bar{k}]) \models$ “letting $N = D(\mathcal{W}, \Pi) = L(\Gamma(\mathcal{W}, \Pi), \mathbb{R})$, in N , there is a formula $\theta(u, v)$ and a finite set of ordinals t such that $(g, \theta(u, v), t)$ is bad”.

Work now in $M[n]$. Notice that for every i , $\bar{\Sigma}_i = ((\pi\text{-pullback of } \Sigma_i)) \restriction P$ and $\Pi = ((\pi\text{-pullback of } \Pi^*)) \restriction P$. In what follows, we abuse our notation and let for every i , $\bar{\Sigma}_i = (\pi\text{-pullback of } \Sigma_i)$ and $\Pi = (\pi\text{-pullback of } \Pi^*)$ in all $M[n]$. It then follows that in $M[n]$, $\mathcal{M}$ is a $\Pi \oplus (\bigoplus_{i < \omega} \bar{\Sigma}_i)$ -mouse. Let now $C = D(\mathcal{W}, \Pi)$. It is easy to see that (5) gives $(\mathcal{S}_i, \gamma_i, l_i : i < \omega)$ such that if $g = ((\mathcal{K}_i, \bar{\Sigma}_i), \mathcal{S}_i, \gamma_i, l_i, \xi_i, g_i : i < \omega)$

(6): in C , there exist a formula $\theta(u, v)$ and a finite set of ordinals t such that $(g, \theta(u, v), t)$ is bad.

Fix then $\theta(u, v)$ and t as in (6). Let E_i be the $(\delta^{\mathcal{K}_i}, \delta^{\mathcal{K}_{i+1}})$ -extender derived from ξ_i and F_i be $(\delta^{\mathcal{K}_i}, \delta^{\mathcal{S}_i})$ -extender derived from γ_i . Let $\mathcal{K}_0^+ = \mathcal{W}$, $\mathcal{S}_i^+ = \text{Ult}(\mathcal{K}_i, F_i)$ and $\mathcal{K}_{i+1}^+ = \text{Ult}(\mathcal{K}_i^+, E_i)$. Let $p_i = \sigma_{\nu_i} \circ (\pi \restriction \mathcal{K}_i)$. Then we have that p_i , γ_i , ξ_i and l_i extend to $p_i^+ : \mathcal{K}_i^+ \rightarrow j(\mathcal{W})$, $\gamma_i^+ : \mathcal{K}_i^+ \rightarrow \mathcal{S}_i^+$, $\xi_i^+ : \mathcal{K}_i^+ \rightarrow \mathcal{K}_{i+1}^+$ and $l_i^+ : \mathcal{S}_i^+ \rightarrow \mathcal{K}_{i+1}^+$ such that $p_i^+ = p_{i+1}^+ \circ \xi_i^+$ and $\xi_i^+ = l_i^+ \circ \gamma_i^+$.

By a standard argument (e.g., see [15, Lemma 4.3]), we can simultaneously iterate $(\mathcal{K}_i^+, \mathcal{S}_i^+ : i < \omega)$ using strategies $\Pi_i = (p_i^+\text{-pullback of } \pi(\Pi))$ and $\Omega_i = (l_i^+ \circ p_i^+\text{-pullback of } \pi(\Pi))$ to make $\mathbb{R}^{M[n]}$ generic. Such genericity iterations have been used by many authors. The details of such genericity iterations are spelled out in [6, Definition 1.35]. The outcome of this iteration is a sequence of models $(\mathcal{K}_{i,\omega}, \mathcal{S}_{i,\omega} : i < \omega)$ and embeddings $(\xi_{i,\omega}, \gamma_{i,\omega}, l_{i,\omega} : i < \omega)$ with the property that $\xi_{i,\omega} : \mathcal{K}_{i,\omega} \rightarrow \mathcal{K}_{i+1,\omega}$, $\gamma_{i,\omega} : \mathcal{K}_{i,\omega} \rightarrow \mathcal{S}_{i,\omega}$, $l_{i,\omega} : \mathcal{S}_{i,\omega} \rightarrow \mathcal{K}_{i+1,\omega}$ and for every $i < \omega$, $\xi_{i,\omega} = l_{i,\omega} \circ \xi_{i,\omega}$. Moreover, the iterations $\mathcal{K}_i^+\text{-to-}\mathcal{K}_{i,\omega}$ and $\mathcal{S}_i^+\text{-to-}\mathcal{S}_{i,\omega}$ are above respectively $\delta^{\mathcal{K}_i}$ and $\delta^{\mathcal{S}_i}$. Let then $C_i = D(\mathcal{K}_{i,\omega})$ and $D_i = D(\mathcal{S}_{i,\omega})$. One important remark is that for every $i < \omega$, $\mathcal{K}_{i,\omega}$ is a $\bar{\Sigma}_i$ -hod premouse and $\mathcal{S}_{i,\omega}$ is a Ψ_i -premouse where $\Psi_i = (l_i\text{-pullback of } \bar{\Sigma}_i)$. Another important remark is that $C_i \subseteq D_i \subseteq C_{i+1}$. The most important remark, however, is that the construction of the sequences $(\mathcal{K}_{i,\omega}, \mathcal{S}_{i,\omega} : i < \omega)$ and $(\xi_{i,\omega}, \gamma_{i,\omega}, l_{i,\omega} : i < \omega)$ guarantees that the direct limit of $\mathcal{K}_{i,\omega}$ under $\xi_{i,\omega}$ is well-founded. Let then n be such that for every $m \geq n$, $\xi_{m,\omega}(t) = t$. It then follows from (6) and the fact that for every $i < \omega$, $C \subseteq C_i$ and $C \subseteq D_i$ that

(7): for every $i < \omega$, in C_i , for every (ϕ, s) such that ϕ is a formula and $s \in [\delta^{\mathcal{K}_i}]^{<\omega}$, $\mathcal{K}_i \models \phi[s, B_i]$ if and only if $\theta[\pi_{\mathcal{K}_i(\alpha), \infty}^{\bar{\Sigma}_i}(s), t]$ where $\alpha < \lambda^{\mathcal{K}_i}$ is least such that $s \in [\delta_\alpha^{\mathcal{K}_i}]^{<\omega}$.

(8): for every i , in D_i , there is a formula ϕ and $s \in [\delta^{\mathcal{S}_i}]^{<\omega}$ such that $\mathcal{S}_i \models \phi[s, \gamma_i(B_i)]$ and $\neg \theta[\pi_{\mathcal{S}_i(\alpha), \infty}^{\Psi_i}(s), t]$ where $\alpha < \lambda^{\mathcal{S}_i}$ is least such that $s \in [\delta_\alpha^{\mathcal{S}_i}]^{<\omega}$.

It follows from elementarity of $\gamma_{i,\omega}$, (7) and the fact that if $i \geq n$ then $\gamma_{i,\omega}(t) = t$ that

(9): for every $i \geq n$, in D_i , for every (ϕ, s) such that ϕ is a formula and $s \in [\delta^{\mathcal{S}_i}]^{<\omega}$ and $\mathcal{S}_i \models \phi[s, \gamma_i(B_i)]$ if and only if $\theta[\pi_{\mathcal{S}_i(\alpha), \infty}^{\Psi_i}(s), t]$ where $\alpha < \lambda^{\mathcal{S}_i}$ is least such that $s \in [\delta_\alpha^{\mathcal{S}_i}]^{<\omega}$.

Clearly (8) and (9) contradict one another. This completes the proof of the theorem. $\square$

Lemma 3.11 and Theorem 3.14 immediately give us the following corollary.

Lemma 3.15. *Let $f, \mathcal{M}, \sigma_{\mathcal{M}}, \mathcal{R}, \Sigma'$ be defined prior to Remark 3.10. Let $\sigma'_{\mathcal{M}} : \pi_{\mathcal{P}^+, \mathcal{R}}(\mathcal{M}) \rightarrow j(\mathcal{M})$ be defined by: $\sigma'_{\mathcal{M}}(\pi_{\mathcal{P}^+, \mathcal{R}}(g)(a)) = \tau_{\mathcal{M}}(g)(\pi_{\mathcal{R}(\alpha), \infty}^{\Sigma'}(a))$ for all $g \in \mathcal{M}$ and $a \in \mathcal{R}(\alpha)^{<\omega}$. Then $\sigma'_{\mathcal{M}} \in M[n]$ is Σ_1 -elementary, and*

$$\pi_{\mathcal{R}(\alpha), \infty}^{\Sigma'} \upharpoonright H_f^{\mathcal{R}(\alpha)} = \sigma'_{\mathcal{M}} \upharpoonright H_f^{\mathcal{R}(\alpha)}. \quad (3.8)$$

Furthermore, Σ' is commuting, positional, witnesses $\mathcal{R}(\alpha)$ is strongly $(\Sigma_{\mathcal{R}(\beta)}, j(f))$ -iterable and has branch condensation.

Proof. First, Σ_1 -elementarity of $\sigma'_{\mathcal{M}}$ follows from Theorem 3.14 and the fact that $\mathcal{M}$ is g -suitable for every $g \subseteq \delta^{\mathcal{P}}$ and $g \in \mathcal{M}$.

By changing $\mathcal{M}$ if necessary, we can assume that $\rho_1(\mathcal{M}) = \delta^{\mathcal{P}}$ and there is some $h \in \mathcal{M}$ such that $\text{rng}(h \upharpoonright \mathcal{P}) \subseteq \delta^{\mathcal{P}}$ and $f, \tau_{B_f, \kappa}^{\mathcal{R}(\alpha)}, \gamma_f^{\mathcal{R}(\alpha)}$ are Σ_1 computable in $\pi_{\mathcal{P}^+, \mathcal{R}}(\mathcal{M})$ from $\pi_{\mathcal{P}^+, \mathcal{R}}(h)$ for all $\kappa \in \{((\delta_{\alpha}^{\mathcal{R}})^{+n})^{\mathcal{R}} \mid n < \omega\}$. Then applying Theorem 3.14 to $(h, \mathcal{M})$, we get (3.8). The second clause follows from Lemma 3.11. $\square$

Working in $V[n]$, we fix an enumeration $\langle g_k \mid k < \omega \rangle$ and $\langle f_k = j(g_k) \mid k < \omega \rangle$ of $\{f \mid f \in \mathcal{P}^+ \wedge j(f) \text{ is appropriate}\}$ and $\{j(f) \mid f \in \mathcal{P}^+ \wedge j(f) \text{ is appropriate}\}$ respectively, so that whenever $H_{f_k}^{\mathcal{R}(\alpha)} \subseteq H_{f_l}^{\mathcal{R}(\alpha)}$ then $k \leq l$. Note that for any k , there is some $l \geq k$ such that $H_{f_k}^{\mathcal{R}(\alpha)} \subseteq H_{f_l}^{\mathcal{R}(\alpha)}$.

For each l and f_l -suitable $\mathcal{M}$, fix strategy $\Lambda_l \in M[n]$ for $\mathcal{R}(\alpha)$ extending $\Sigma_{\mathcal{R}(\beta)}$, map $\tau'_{\mathcal{M}, l}$ satisfying Lemma 3.15 for f_l . We also demand for $l \leq k$ such that $H_{f_l}^{\mathcal{R}(\alpha)} \subseteq H_{f_k}^{\mathcal{R}(\alpha)}$,

$$\pi^{\Lambda_k} \upharpoonright H_{f_l}^{\mathcal{R}(\alpha)} = \pi^{\Lambda_l} \upharpoonright H_{f_l}^{\mathcal{R}(\alpha)}. \quad (3.9)$$

Again, let $\Omega = \Omega_{\mu, g}^{\lambda}$. We plan to construct strategy $\Sigma_{\mathcal{R}(\alpha)}$ by taking “limit” of the Λ_l ’s as follows. For simplicity, suppose $\mathcal{T} \in M_{j(\mu)}[n]$ is a normal, correctly guided, maximal tree on $\mathcal{R}(\alpha)^{\text{kk}}$ using extenders above $\mathcal{R}(\beta)$. Let $\mathcal{M}(\mathcal{T})^+$ be the

^{kk}If $\mathcal{T}$ is short, then there is a unique branch b given by the $\mathcal{Q}$ -structure.

end model of the tree $\mathcal{T}$; more precisely, $\mathcal{M}^+(\mathcal{T})$ is defined to be $\text{Lp}^{\Sigma_{\mathcal{R}(\beta)}, \Omega}(\mathcal{M}(\mathcal{T}))$.¹¹ For each n , let $b_n = \Lambda_n(\mathcal{T})$. We let $\Sigma_{\mathcal{R}(\alpha)}(\mathcal{T}) = b$, where

$$\xi \in b \Leftrightarrow \exists k \forall l \geq k (\xi \in b_l). \quad (3.10)$$

Let $\mathcal{H}$ be the transitive collapse of $\bigcup_n H_{f_n}^{\mathcal{M}(\mathcal{T})^+}$ and τ be the uncollapse map. We want to show that

- (i) b is cofinal in $\mathcal{T}$;
- (ii) $\mathcal{H} = \mathcal{M}(\mathcal{T})^+$ and τ is the identity. This gives that $\Sigma_{\mathcal{R}(\alpha)}$ is fullness preserving;
- (iii) $\Sigma_{\mathcal{R}(\alpha)}$ acts on all of $\mathcal{R}$ and is $\sigma'_{\mathcal{R}}$ -realizable for some Σ_1 elementary embedding $\sigma'_{\mathcal{R}} : \mathcal{R} \rightarrow j(\mathcal{P}^+)$ such that $\sigma'_{\mathcal{R}} \circ \pi_{\mathcal{P}^+, \mathcal{R}} = j \upharpoonright \mathcal{P}^+$;
- (iv) $\Sigma_{\mathcal{R}(\alpha)}$ has branch condensation and is guided by $\{j(f) \mid f \in \mathcal{P}^+ \wedge j(f) \text{ is appropriate}\}$.

Lemma 3.16. $\mathcal{H} = \mathcal{M}(\mathcal{T})^+$ and τ is the identity.

Proof. Let $\pi = \bigcup_l \pi_{\mathcal{M}(\mathcal{T})^+, \infty}^{\Lambda_l} \upharpoonright H_{f_l}^{\mathcal{M}(\mathcal{T})^+}$ and $k = \pi \circ \tau$. Note that

$$\mathcal{R}(\alpha) = \bigcup_l H_{f_l}^{\mathcal{R}(\alpha)}.$$

This is because every $x \in \mathcal{R}(\alpha)$ has the form $\pi_{\mathcal{P}^+, \mathcal{R}}(g_l \upharpoonright (\delta_{\beta}^{\mathcal{R}} + 1))(a)$ for some $l < \omega$ and some $a \in (\mathcal{R} \upharpoonright \delta_{\beta}^{\mathcal{R}})^{<\omega}$ and $\mathcal{R}(\beta) \cup \{\pi_{\mathcal{P}^+, \mathcal{R}}(g_k \upharpoonright \delta_{\beta}^{\mathcal{R}} + 1)\} \subset H_{f_k}^{\mathcal{R}(\alpha)}$ for all k .^{mm} It means that there is a Σ_1 map $i : \mathcal{R}(\alpha) \rightarrow \mathcal{H}$. Furthermore, letting E be the $(\text{crt}(i), \delta^{\mathcal{H}})$ -extender derived from i , then E gives the ultrapower map $i^+ : \mathcal{R} \rightarrow \mathcal{H}^+ =_{\text{def}} \text{Ult}(\mathcal{R}, E)$ extending i . Letting $k^+(i^+(g)(a)) = \sigma_{\mathcal{R}}(g)(k(a))$, we have that: $k^+ \circ i^+ = \sigma_{\mathcal{R}}$.

Now we can use the proof of Lemma 3.8 to conclude that $\mathcal{H}$ is full. If τ is not the identity, then letting $\gamma = \text{crt}(\tau)$, we have: γ is Woodin in $\mathcal{H}$, $\tau(\gamma) = \delta(\mathcal{T})$ is Woodin in $\mathcal{M}(\mathcal{T})^+$, and $\mathcal{H} \triangleleft \mathcal{M}(\mathcal{T})^+$. Since $\mathcal{H}$ is full, γ is Woodin in $\mathcal{M}(\mathcal{T})^+$. This contradicts the fact that there are no Woodin cardinals in $\mathcal{M}(\mathcal{T})^+$ between $\delta_{\beta}^{\mathcal{R}}$ and $\delta(\mathcal{T})$. $\square$

Lemma 3.16 proves (ii); furthermore, it implies that $\sup \gamma_{f_n}^{\mathcal{M}(\mathcal{T})^+} = \delta(\mathcal{T})$. This means that b is cofinal in $\mathcal{T}$, hence proves (i) (see [9, Theorem 5.4.14] for an argument). We also get that $\Sigma_{\mathcal{R}(\alpha)}$ is guided by $\{f_n \mid n < \omega\}$ (and is the unique such strategy). At this point, we do not know that $\Sigma_{\mathcal{R}(\alpha)} \in M[n]$ and has branch condensation. The following is the main technical lemma.

¹¹Recall that maximal trees always have the last model; regardless of whether there is a cofinal branch.

^{mm} $\pi_{\mathcal{P}^+, \mathcal{R}}(g_k \upharpoonright \delta_{\beta}^{\mathcal{R}} + 1) \subset H_{f_k}^{\mathcal{R}(\alpha)}$ holds by elementarity of $\pi_{\mathcal{P}^+, \mathcal{R}}$ and the fact that the corresponding containment holds in $\mathcal{P}^+$.

Lemma 3.17 (Notations as above). *The following conditions hold:*

- (1) $\Sigma_{\mathcal{R}(\alpha)}$ acts on all of $\mathcal{R}$ and whenever $i : \mathcal{R} \rightarrow \mathcal{S}$ is according to $\Sigma_{\mathcal{R}(\alpha)}$, there are embeddings $\sigma'_{\mathcal{R}} : \mathcal{R} \rightarrow j(\mathcal{P}^+)$ such that $j \upharpoonright \mathcal{P}^+ = \sigma'_{\mathcal{R}} \circ \pi_{\mathcal{P}^+, \mathcal{R}}$ and $\tau : \mathcal{S} \rightarrow j(\mathcal{P}^+)$ such that $\sigma'_{\mathcal{R}} = \tau \circ i$ and $\tau \upharpoonright \mathcal{S}(i(\alpha)) = \pi_{\mathcal{S}(i(\alpha)), \infty}^{\Phi}$, where Φ is the i -tail of $\Sigma_{\mathcal{R}(\alpha)}$.
- (2) $\Sigma_{\mathcal{R}(\alpha)}$ has branch condensation.
- (3) If $\pi_{\mathcal{P}^+, \mathcal{R}} \in M$, then $\Sigma_{\mathcal{R}(\alpha)}$ is in $j^+(\Omega_{\mu, g}^{\lambda})$.

Proof. For (1), the map $\sigma'_{\mathcal{R}}$ is defined as follows: for any $x \in \mathcal{R}$, letting $x = \pi_{\mathcal{P}^+, \mathcal{R}}(g_l)(a)$ for some $l < \omega$ and $a \in (\delta_{\alpha}^{\mathcal{R}})^{<\omega}$, then letting $k \geq l$ be such that $a \in H_{f_k}^{\mathcal{R}(\alpha)}$

$$\sigma'_{\mathcal{R}}(x) = j(g_l)(\pi_{\mathcal{R}(\alpha), \infty}^{\Lambda_k}(a)) = f_l(\pi_{\mathcal{R}(\alpha), \infty}^{\Lambda_k}(a)).$$

The map is well-defined by line (3.9). Using Theorem 3.14, we can show that $\sigma'_{\mathcal{R}}$ is Σ_1 elementary as follows. Suppose φ is Σ_1 and x, g_l, a are as above (we may increase l and assume $x = \pi_{\mathcal{P}^+, \mathcal{R}}(g_l)(a)$ and $a \in H_{f_l}^{\mathcal{R}(\alpha)}$). Suppose

$$j(\mathcal{P}^+) \models \varphi[f_l(\pi_{\mathcal{R}(\alpha), \infty}^{\Lambda_l}(a))] \Leftrightarrow \mathcal{R} \not\models \varphi[x].$$

Since φ is Σ_1 and $j[\mathcal{P}^+]$ is cofinal in $j(\mathcal{P}^+)$, we can find some $\mathcal{M} \triangleleft \mathcal{P}^+$ such that the above is equivalent to

$$j(\mathcal{M}) \models \varphi[f_l(\pi_{\mathcal{R}(\alpha), \infty}^{\Lambda_l}(a))] \Leftrightarrow \pi_{\mathcal{P}^+, \mathcal{R}}(\mathcal{M}) \not\models \varphi[x].$$

This contradicts Theorem 3.14 applied to $\mathcal{M}, f_l$. Theorem 3.14 also gives that $j \upharpoonright \mathcal{P}^+ = \sigma'_{\mathcal{R}} \circ \pi_{\mathcal{P}^+, \mathcal{R}}$ and

$$\sigma'_{\mathcal{R}} \upharpoonright \mathcal{R}(\alpha) = \pi_{\mathcal{R}(\alpha), \infty}^{\Sigma_{\mathcal{R}(\alpha)}}. \quad (3.11)$$

By Lemma 3.16, $\mathcal{S}(i(\alpha)) = \bigcup_l H_{f_l}^{\mathcal{S}(i(\alpha))}$. Now define $\tau : \mathcal{S} \rightarrow j(\mathcal{P}^+)$ as follows. Let $\vec{\mathcal{U}}$ be the stack giving rise to i and $\Lambda = (\Sigma_{\mathcal{S}(i(\beta))})_{\vec{\mathcal{U}}}$. For $x \in \mathcal{S}$, say $x = i(g)(a)$ for some $g \in \mathcal{R}$ and $a \in \mathcal{S}(i(\alpha))^{<\omega}$, and say $g = \pi_{\mathcal{P}^+, \mathcal{R}}(g_l)(b)$ for some $b \in \mathcal{R}(\beta)^{<\omega}$, we let

$$\tau(x) = f_l(\pi_{\mathcal{S}(i(\beta)), \infty}^{\Lambda}(i(b)))(\pi_{\mathcal{S}(i(\alpha)), \infty}^{\Phi}(a)).$$

Using line (3.11), we get that τ is Σ_1 elementary and $\tau \circ i = \sigma'_{\mathcal{R}}$. This proves (1). The following claim proves (2).

Claim 3.18. $\Sigma_{\mathcal{R}(\alpha)}$ has branch condensation.

Proof. Suppose not. Then there are a (nondropping) stack $\vec{\mathcal{W}}$ with last model $\mathcal{S} \in M[n]$ and a normal tree $\mathcal{T}$ of limit length based on a window $(\delta_{\beta^*}^{\mathcal{S}}, \delta_{\alpha^*}^{\mathcal{S}})$ such that

- (1) $\beta^* = \pi^{\vec{\mathcal{W}}}(\beta)$, $\alpha^* = \pi^{\vec{\mathcal{W}}}(\alpha) = \beta^* + 1$.
- (2) $\vec{\mathcal{W}}$ and $\mathcal{T}$ are according to $\Sigma_{\mathcal{R}(\alpha)}$.
- (3) Generators of $\vec{\mathcal{W}}$ are below $\delta_{\beta^*}^{\mathcal{S}}$.

- (4) $\Sigma_{\mathcal{S}(\beta^*), \vec{\mathcal{W}}}$ has branch condensation.
 (5) Let $b = \Sigma_{\mathcal{S}, \vec{\mathcal{W}}}(\mathcal{T})$. There are a cofinal branch $c \neq b$, an iteration map $i : \mathcal{R} \rightarrow \mathcal{Y}$ according to $\Sigma_{\mathcal{R}(\alpha)}$, and a $\sigma : \mathcal{M}_c^{\mathcal{T}} \rightarrow \mathcal{Y}$ such that $\sigma \circ \pi_c^{\mathcal{T}} \circ \pi^{\vec{\mathcal{W}}} = i$.

We proceed to obtain a contradiction. Let $\tau : \mathcal{Y} \rightarrow j(\mathcal{P}^+)$ come from the construction of $\Sigma_{\mathcal{R}(\alpha)}$ and $\sigma_{\mathcal{S}} : \mathcal{S} \rightarrow j(\mathcal{P}^+)$ be the realization map. By arguments above and the fact that $\mathcal{M}_c^{\mathcal{T}}$ realizes into $j(\mathcal{P}^+)$ via $\tau \circ i$ and $\tau \circ i$ factors into $\sigma_{\mathcal{S}}$, there is a strategy Λ such that $\pi_{\mathcal{M}_c^{\mathcal{T}}(\alpha^*), \infty}^{\Lambda} = \psi \upharpoonright \mathcal{M}_c^{\mathcal{T}}(\alpha^*)$ for some ψ such that $j \upharpoonright \mathcal{P}^+ = \psi \circ \pi_c^{\mathcal{T}} \circ \pi^{\vec{\mathcal{W}}}$ and Λ witnesses that $\mathcal{M}_c^{\mathcal{T}}(\alpha^*)$ is strongly $(\mathcal{S}(\beta^*), f_k)$ -iterable for all $k < \omega$. This means $\Lambda_{\mathcal{M}_c^{\mathcal{T}}(\alpha^*)} = \Sigma_{\mathcal{M}_b^{\mathcal{T}}(\alpha^*), \vec{\mathcal{W}} \sim \mathcal{T} \sim b}$.

Let $\Psi = \Sigma_{\mathcal{M}_b^{\mathcal{T}}(\alpha^*), \vec{\mathcal{W}} \sim \mathcal{T} \sim b}$. Let $\phi : \mathcal{M}_b^{\mathcal{T}} \rightarrow j(\mathcal{P}^+)$ be the realization map. Note that

$$\phi \upharpoonright \mathcal{M}_b^{\mathcal{T}}(\alpha^*) = \psi \upharpoonright \mathcal{M}_c^{\mathcal{T}}(\alpha^*) = \pi_{\mathcal{M}_c^{\mathcal{T}}(\alpha^*), \infty}^{\Lambda} = \pi_{\mathcal{M}_b^{\mathcal{T}}(\alpha^*), \infty}^{\Psi}.$$

Now we aim to show $b = c$, which contradicts our assumption.

By assumptions on $\mathcal{W}$, we have

$$\delta_{\alpha^*}^{\mathcal{S}} = \sup(A), \quad \text{where}$$

$$A = \{\gamma < \delta_{\alpha^*}^{\mathcal{S}} \mid \exists g \in \mathcal{P}^+ \exists b \in (\delta_{\beta^*}^{\mathcal{R}})^{<\omega} \exists a \in (\delta_{\beta^*}^{\mathcal{S}})^{<\omega} \gamma = \pi^{\mathcal{W}}(\pi_{\mathcal{P}^+, \mathcal{R}}(g)(b))(a)\}.$$

For each $\gamma = \pi^{\mathcal{W}}(\pi_{\mathcal{P}^+, \mathcal{R}}(g)(b))(a) \in A$,

$$\psi \circ \pi_c^{\mathcal{T}}(\gamma) = j(g)(\pi^{\Psi_{\mathcal{M}_b^{\mathcal{T}}(\alpha^*)}}(\pi^{\vec{\mathcal{W}}}(b), a)), \quad (3.12)$$

and

$$\phi \circ \pi_b^{\mathcal{T}}(\gamma) = j(g)(\pi^{\Psi_{\mathcal{M}_b^{\mathcal{T}}(\alpha^*)}}(\pi^{\vec{\mathcal{W}}}(b), a)). \quad (3.13)$$

Since $\phi \upharpoonright \mathcal{M}_b^{\mathcal{T}}(\alpha^*) = \psi \upharpoonright \mathcal{M}_c^{\mathcal{T}}(\alpha^*)$, we get that

$$\pi_c^{\mathcal{T}}(\gamma) = \pi_b^{\mathcal{T}}(\gamma). \quad (3.14)$$

Equations (3.12), (3.13), and (3.14) imply $\text{rng}(\pi_b^{\mathcal{T}}) \cap \text{rng}(\pi_c^{\mathcal{T}})$ is cofinal in $\delta(\mathcal{T})$. So $b = c$, which is a contradiction. $\square$

Now we show that $\Sigma_{\mathcal{R}(\alpha)} \cap M \in M$ in the case $\pi_{\mathcal{P}^+, \mathcal{R}} \in M$. In this case, $\mathcal{R} \in M$. Note also that the construction of $\Sigma_{\mathcal{R}(\alpha)}$ does not depend on the enumeration of the set $\{j(f) \mid f \in \mathcal{P}^+\}$ in $V[n]$; it only depends on the set itself. So in fact, $\Sigma_{\mathcal{R}(\alpha)} \upharpoonright V \in V$. The following claim is another place the fact that κ is strong in M is used.

Claim 3.19. $\Sigma_{\mathcal{R}(\alpha)} \upharpoonright M \in M$.

Proof. Suppose first $\mathcal{R} \in M_{\kappa}$. As noted before, $\Sigma_{\mathcal{R}(\alpha)} \cap M_{\kappa} \in M$ (since $M_{\kappa+1} = V_{\kappa+1}$). Using the fact that κ is strong in M , we can lift $\Sigma_{\mathcal{R}(\alpha)} \cap M_{\kappa}$ to a strategy Λ in $M_{j(\mu)}$ acting on trees in $M_{j(\mu)}$, and Λ has branch condensation and is $j^+(\Omega_{\mu, g}^{\lambda})$ -fullness preserving. We claim that $\Sigma_{\mathcal{R}(\alpha)} \cap M = \Lambda \in M$. Suppose not.

Then by a standard fact, $\text{cof}^V(\delta_\alpha^\mathcal{R}) = \omega$.ⁿⁿ Let $\langle x_n \mid n < \omega \rangle$ be cofinal in $\delta_\alpha^\mathcal{R}$ and say, $x_n = \pi_{\mathcal{P}^+, \mathcal{R}}(h_n)(a_n)$ for some $h_n \in \mathcal{P}^+$ and $a_n \in \mathcal{R}(\beta)$.

From our assumption, $((\delta^{\mathcal{P}})^+)^{\mathcal{P}^+} = \mu^+$. Hence, there is some $\mathcal{N} \triangleleft \mathcal{P}^+$ such that $\rho_\omega(\mathcal{N}) = \delta^{\mathcal{P}}$, $\{h_n \mid n < \omega\} \subseteq \mathcal{N}$ and in $\mathcal{P}^+$, $\text{cof}(o(\mathcal{N})) = \omega$. Let $\mathcal{N}$ be the least that contains some such h_n 's and let p be the standard parameter of $\mathcal{N}$ so for each n , h_n is definable in $\mathcal{N}$ from p and some $a \in \mathcal{P}$. Note that we can assume $\pi_{\mathcal{P}^+, \mathcal{R}}[\delta^{\mathcal{P}}] \subseteq \mathcal{R}(\beta)$. This means $\sup(A) = \delta_\alpha^\mathcal{R}$, where

$$A = \{g(x) \mid g \in \pi_{\mathcal{P}^+, \mathcal{R}}(\mathcal{N}) \wedge x \in \mathcal{R}(\beta)\}.$$

By the choice of $\mathcal{N}$ and the fact that A can be computed in $\mathcal{R}$, $\delta_\alpha^\mathcal{R}$ is singular in $\mathcal{R}$, which is a contradiction.

Now assume $\mathcal{R} \in M_{j(\mu)} \setminus M_\kappa$. Note that whenever $\mathcal{W} \in V_\kappa[m] = M_\kappa[m]$ is such that there is an embedding $\pi : \mathcal{W} \rightarrow \mathcal{R}$ with $\pi(\gamma) = \alpha$ and there is an embedding $\pi^* : \mathcal{P}^+ \rightarrow \mathcal{W}$ such that $\pi_{\mathcal{P}^+, \mathcal{R}} = \pi \circ \pi^*$, then $\mathcal{W}(\gamma)$ has a nice strategy as witnessed by some realization map $i : \mathcal{W} \rightarrow j(\mathcal{P}^+)$ such that the strategy restricted to $M_\kappa[m]$ has branch condensation. Let $h : M \rightarrow N$ witness κ be $j(\mu)^+$ -strong in M and we may assume $h = i_F^M$ for some extender F ; h can be extended to a map, which we also call h , from $M[n]$ to $N[p]$ for some V -generic $p \subseteq \text{Col}(\omega, < h(\kappa))$ such that $p \restriction \text{Col}(\omega, < j(\mu)) = n$. Then by absoluteness, in $N[n]$, there is an embedding π from $\mathcal{R}$ into $h(\mathcal{R})$ such that $h(\pi_{\mathcal{P}^+, \mathcal{R}}) = \pi \circ \pi_{\mathcal{P}^+, \mathcal{R}}$. And hence there is a strategy $\Lambda \in M[n]$ of $\mathcal{R}$ with branch condensation acting on stacks in $M_{j(\mu)}[n] = N_{j(\mu)[n]}$ based on $\mathcal{R}(\alpha)$ such that $\Lambda \restriction M \in M$.^{oo} By the same argument as above, we get that $\Lambda \restriction M = \Sigma_{\mathcal{R}(\alpha)} \restriction M$. $\square$

We now prove (3). Let Λ be as in the proof of Claim 3.19; so $\Lambda \in M[n]$, Λ has branch condensation and is $j^+(\Omega_{\mu, g}^\lambda)$ -fullness preserving. It is enough to prove $\Lambda = \Sigma_{\mathcal{R}(\alpha)}$ so that $\Sigma_{\mathcal{R}(\alpha)} \in M[n]$; (3) then follows from $\text{Proj}(j(\kappa), j(\lambda), \Sigma_{\mathcal{R}(\alpha)})$. The equality follows from an argument similar to the proof of Claim 3.18 and the fact that for any $k : \mathcal{R} \rightarrow \mathcal{M}$ in $M[n]$ according to Λ and $\Sigma_{\mathcal{R}(\alpha)}$, there is an $i : \mathcal{R} \rightarrow \mathcal{S}$ according to $\Lambda \restriction M = \Sigma_{\mathcal{R}(\alpha)} \restriction M$ such that there is some map $\sigma : \mathcal{M} \rightarrow \mathcal{S}$ such that $\sigma \in M[n]$ and $\sigma \circ k = i$.^{pp} This completes the proof of the lemma. $\square$

We have finished the construction of a partial strategy Σ acting on trees in $M_{j(\mu)} \cup M_\kappa[m]$. During the course of the construction, we also showed that

ⁿⁿThere is a tree $\mathcal{U} \in M$ of limit length with $\Sigma_{\mathcal{R}(\alpha)}(\mathcal{T}) = b \neq c = \Lambda(\mathcal{T})$. Hence $\text{cof}^V(\delta(\mathcal{U})) = \omega$. Since $\delta_\alpha^\mathcal{R}$ is mapped cofinally into $\delta(\mathcal{U})$ by either branch embedding, $\text{cof}^V(\delta_\alpha^\mathcal{R}) = \omega$.

^{oo}From the point of view of $\text{Ult}(V, F)[n]$, the strategy Λ is $\{i_F^V(j(f)) \mid f \in \mathcal{P}^+\}$ -guided, acts on trees in $i_F^V(M)[n]$ and does not depend on any generic enumeration of the set $\{i_F^V(j(f)) \mid f \in \mathcal{P}^+\}$, so if in addition $\mathcal{W} \in M_\kappa$, then Λ 's restriction to $N_{j(\mu)} = M_{j(\mu)} = i_F^V(M)_{j(\mu)}$ is in M even though we first collapsed $\mathcal{R}$ to ω in N to find an embedding from $\mathcal{R}$ into $\pi(\mathcal{R})$.

^{pp}Suppose $\vec{\mathcal{W}} \hat{\sim} \mathcal{T}$ is according to both strategies, $\vec{\mathcal{W}}$ is on $\mathcal{R}(\beta)$ and $\mathcal{T}$ on $\mathcal{M}^{\vec{\mathcal{W}}}$ is such that $b = \Lambda_{\mathcal{M}\mathcal{T}}(\mathcal{T}) \neq c = (\Sigma_{\mathcal{R}(\alpha)})_{\mathcal{M}\mathcal{T}}(\mathcal{T})$. Then there is some iteration $i : \mathcal{R} \rightarrow \mathcal{S}$ according to $\Lambda \restriction M = \Sigma_{\mathcal{R}(\alpha)} \restriction M$ such that there are iteration maps $\sigma_0 : \mathcal{M}_b^{\vec{\mathcal{T}}} \rightarrow \mathcal{S}$ according to the tail of Λ and $\sigma_1 : \mathcal{M}_c^{\vec{\mathcal{T}}} \rightarrow \mathcal{S}$ according to the tail of $\Sigma_{\mathcal{R}(\alpha)}$. We can then run the proof of Claim 3.18 to get that $b = c$.

nondropping iterates $\mathcal{R}$ of Σ are $j^+(\Omega_{\mu,g}^\lambda)$ -full and there is a Σ_1 elementary embedding $\sigma_{\mathcal{R}} : \mathcal{R} \rightarrow j(\mathcal{P}^+)$ such that $\sigma_{\mathcal{R}} \circ \pi_{\mathcal{P}^+, \mathcal{R}} = j \restriction \mathcal{P}^+$, and for each $\alpha < \lambda^{\mathcal{R}}$, $\sigma_{\mathcal{R}} \restriction \mathcal{R} \restriction \delta_{\alpha}^{\mathcal{R}}$ is the iteration map according to the $\sigma_{\mathcal{R}}$ -nice strategy Σ_{α} constructed above. Next we show branch condensation of Σ .

Lemma 3.20. *Σ has branch condensation.*

Proof. Suppose not. Then as in [5], we can find a “minimal place” where branch condensation fails. More precisely, there are a (nondropping) stack $\vec{\mathcal{W}}$ with last model $\mathcal{R}$ and a normal tree $\mathcal{T}$ of limit length based on a window $(\delta_{\alpha}^{\mathcal{R}}, \delta_{\alpha+1}^{\mathcal{R}})$ such that we have the following.

- (1) $\vec{\mathcal{W}}$ and $\mathcal{T}$ are according to Σ .
- (2) Generators of $\vec{\mathcal{W}}$ are below $\delta_{\alpha}^{\mathcal{R}}$.
- (3) $\Sigma_{\mathcal{R}(\alpha), \vec{\mathcal{W}}}$ has branch condensation.
- (4) Let $b = \Sigma_{\mathcal{R}, \vec{\mathcal{W}}}(\mathcal{T})$. There are a cofinal branch $c \neq b$, an iteration map $i : \mathcal{P}^+ \rightarrow \mathcal{S}$ according to Σ , and a $\sigma : \mathcal{M}_c^{\mathcal{T}} \rightarrow \mathcal{S}$ such that $\sigma \circ \pi_c^{\mathcal{T}} \circ \pi^{\vec{\mathcal{W}}} = i$.

The rest of the proof is just as in the proof of Claim 3.18. $\square$

Let $\Lambda = \Sigma \restriction M_{\kappa}[m] \in M[m]$. Using the fact that κ is strong in M , we extend Λ to a strategy $\Lambda^+ \in M[n]$ such that Λ^+ acts on all stacks in $M_{j(\mu)}[n]$, has branch condensation, and is $j(\Omega_{\mu,g}^\lambda)$ -fullness preserving. Furthermore, since $\Sigma \cap M_{\kappa} \in M$, we also get that $\Lambda^+ \restriction M \in M$.

Lemma 3.21. *$\Sigma \restriction M \subseteq \Lambda^+ \restriction M \in M$.*

Proof. Suppose $\Sigma \restriction M \not\subseteq \Lambda^+ \restriction M$. Then there are in M a (nondropping) stack $\vec{\mathcal{W}}$ with last model $\mathcal{R}$ and a normal tree $\mathcal{T}$ of limit length based on a window $(\delta_{\alpha}^{\mathcal{R}}, \delta_{\alpha+1}^{\mathcal{R}})$ such that we have the following:

- (1) $\vec{\mathcal{W}}$ and $\mathcal{T}$ are according to both strategies.
- (2) Generators of $\vec{\mathcal{W}}$ are below $\delta_{\alpha}^{\mathcal{R}}$.
- (3) Let $b = \Sigma_{\mathcal{R}, \vec{\mathcal{W}}}(\mathcal{T})$ and $c = \Lambda_{\mathcal{R}, \vec{\mathcal{W}}}^+(\mathcal{T})$. Then $b \neq c$.

This means in V as well as in M , $\text{cof}(\delta_{\alpha+1}^{\mathcal{R}}) = \omega$. The rest of the proof is just as in the proof of Claim 3.19. $\square$

Let $\Psi = \Lambda^+ \restriction M = \Sigma \restriction M$. By generic comparison using the fact that Λ^+ has branch condensation and is $j^+(\Omega_{\mu,g}^\lambda)$ -fullness preserving (see [5]), we get the following:

For any Λ^+ -iterate $\mathcal{R}$ of $\mathcal{P}^+$, letting $i : \mathcal{P}^+ \rightarrow \mathcal{R}$ be the iteration map, there is a Ψ -iterate $\mathcal{S}$ of $\mathcal{P}^+$ such that, letting $h : \mathcal{P}^+ \rightarrow \mathcal{S}$ be the iteration map, there is a map $\sigma : \mathcal{R} \rightarrow \mathcal{S}$ such that $h = \sigma \circ i$. In fact, σ is a Λ^+ -iteration map.

Using the above paragraph and the properties of Σ , letting $\mathcal{M}_\infty(\mathcal{P}^+, \Lambda^+)$ be the direct limit of (all countable) Λ^+ -iterates of $\mathcal{P}^+$ in $M[n]$, then there is a Σ_1 elementary map $\tau : \mathcal{M}_\infty(\mathcal{P}^+, \Lambda^+) \rightarrow \mathcal{P}^+$ such that $\tau \circ \pi_{\mathcal{P}^+, \infty}^{\Lambda^+} = j \restriction \mathcal{P}^+$ and $\text{crt}(\tau) = \delta^{\mathcal{M}_\infty(\mathcal{P}^+, \Lambda^+)}$.

The map τ is defined as follows: for any $x \in \mathcal{M}_\infty(\mathcal{P}^+, \Lambda^+)$, let $\mathcal{R} \in M$ be a Ψ -iterate of $\mathcal{P}^+$ such that there is some $y \in \mathcal{R}$ such that $\pi_{\mathcal{R}, \infty}^{\Psi_{\mathcal{R}}}(y) = x$. Now by construction of Ψ , there is a map $\tau_{\mathcal{R}} : \mathcal{R} \rightarrow j(\mathcal{P}^+)$ such that $j \restriction \mathcal{P}^+ = \tau_{\mathcal{R}} \circ \pi_{\mathcal{P}^+, \mathcal{R}}^{\Psi}$ and $\tau_{\mathcal{R}} \restriction \delta^{\mathcal{R}}$ agrees with the iteration map by Ψ . We then let $\tau(x) = \tau_{\mathcal{R}}(y)$.

Lemma 3.22. *$(\mathcal{P}^+, \Sigma)$ is a hod pair below κ in $V[m]$ and $(\mathcal{P}^+, \Sigma \restriction HC^{M[n]}) \in j^+(\Omega_{\mu, g}^\lambda)$.*

Proof. The second clause follows from the first clause and $\text{Proj}(j(\kappa), j(\lambda), \Sigma)$ in M . It suffices to prove the first clause. Let $l = m \cap \text{Coll}(\omega, \nu)$ where $\mu < \nu < \kappa$ and $\mathcal{P}^+$ is countable in $V^{\text{Coll}(\omega, \nu)}$ and let $a = \{f \mid f \in \mathcal{P}^+ \wedge f \text{ is appropriate}\}$ and $b = \{j(f) \mid f \in \mathcal{P}^+ \wedge f \text{ is appropriate}\}$. Let $\phi(\mathcal{P}^+, a, b, \vec{T})$ be the formula stating “ $\vec{T}$ is a stack on $\mathcal{P}^+$ according to Σ ”. Fix a regular ξ much larger than $j(\mu)$ and let $X \prec V_\xi[l]$ be countable in $V[l]$ and X contains all relevant objects and $o(\mathcal{P}^+) \subset X$. Let $\pi : N \rightarrow X$ be the uncollapse map and $(\bar{b}, \bar{\kappa}) = \pi^{-1}(b, \kappa)$. Let $\bar{m} \in V[l]$ be N -generic for a poset in $H_{\bar{\kappa}}^N$. Let $\vec{T} \in M[\bar{m}]$. Then we claim that

$$M[\bar{m}] \models \phi[\mathcal{P}^+, a, \bar{b}, \vec{T}] \Leftrightarrow V[l] \models \phi[\mathcal{P}^+, a, b, \vec{T}]. \quad (3.15)$$

Suppose $\vec{T} = \langle \mathcal{T}_\alpha, \mathcal{M}_\alpha \mid \alpha \leq \eta \rangle$ and suppose $\mathcal{T}_\eta$ is based on $\mathcal{M}_\eta(\gamma + 1)$ for some γ . Suppose by induction, (3.15) holds for $\vec{T} \restriction \alpha$ for all $\alpha < \eta$. We now show (3.15) holds for $\mathcal{T}_\eta$. But (3.15) holds for $\mathcal{T}_\eta$ because $\pi[\bar{b}] = b$ and so $\mathcal{T}_\eta$ is $\bar{b}$ -guided in $M[\bar{m}]$ if and only if $\mathcal{T}_\eta$ is b -guided in $V[l]$. $\square$

The above definition of τ and Lemma 3.22 imply that the direct limit $\mathcal{M}_\infty(\mathcal{P}^+, \Lambda^+)$ is in $j^+(\Omega_{\mu, g}^\lambda)^{\text{qq}}$ and $\mathcal{M}_\infty(\mathcal{P}^+, \Lambda^+) \models \delta^{\mathcal{M}_\infty(\mathcal{P}^+, \Lambda^+)}$ is regular. By elementarity, $\mathcal{P}^+ \models \delta^{\mathcal{P}}$ is regular. This contradicts our assumption.

The argument above shows that if $\mathcal{P}^+ \models \delta^{\mathcal{P}}$ is singular, then there must be a model of “ $\text{AD}_{\mathbb{R}} + \Theta$ is regular”.^{rr}

4. The Strength of Failure of UBH for Tame Trees

In this section, we prove Theorem 0.2. Here, we fix a tame tree U on V . We let $\mu < \kappa < \lambda$ be as in the previous section; the assumption that U is tame allows us to choose μ, κ, λ below the critical point of all extenders used in U . The proof of [8, Lemma 5.2] shows that whenever $(\mathcal{R}, \Psi)$ is a hod pair below κ , then $\text{Proj}(\kappa, \lambda, \Psi)$; as mentioned in the previous section, this is where we use the arrangement on

^{qq}This means that there is some α such that $\mathcal{M}_\infty(\mathcal{P}^+, \Lambda^+) \in L(j^+(\Omega_{\mu, g}^\lambda \restriction \alpha))$. Recall that there are cofinally many α such that $j^+(\Omega_{\mu, g}^\lambda \restriction \alpha)$ is constructibly closed.

^{rr}Here we use $(\dagger)$ to invoke the analysis in [5].

μ, κ, λ , though technically, we just need the strength of μ, κ to be a bit past $\delta(U)$. The hypothesis of Theorems 3.2 and 3.4 is satisfied. These theorems in turn imply the conclusion of Theorem 0.2.

Acknowledgments

We would like to thank the referee for discovering various mistakes and typos in the previous draft and for many helpful suggestions that significantly improve the exposition. The first author would like to thank the NSF for its generous support through Grant No. DMS-1352034. The second author would like to thank the NSF for its generous support through Grant No. DMS-1565808.

References

- [1] A. Caicedo, P. Larson, G. Sargsyan, J. R. Schindler, J. R. Steel and M. Zeman, Square in $\mathbb{P}_{\max}$ extensions, to appear in *Israel J. Math.*, available at math.uci.edu/~mzeman.
- [2] D. A. Martin and J. R. Steel, Iteration trees, *J. Amer. Math. Soc.* **7**(1) (1994) 1–73.
- [3] W. J. Mitchell and J. R. Steel, *Fine Structure and Iteration Trees*, Lecture Notes in Logic, Vol. 3 (Springer, Berlin, 1994).
- [4] I. Neeman, Inner models in the region of a Woodin limit of Woodin cardinals, *Ann. Pure Appl. Logic* **116**(1–3) (2002) 67–155.
- [5] G. Sargsyan, A tale of hybrid mice (2011), available at <http://math.rutgers.edu/~gs481/>.
- [6] G. Sargsyan, Non-tame mouse from the failure of square at a singular strong limit cardinal, *J. Math. Logic* (2014).
- [7] G. Sargsyan, Covering with universally Baire operators, *Adv. Math.* **268** (2015) 603–665.
- [8] G. Sargsyan and N. Trang, Non-tame mice from tame failures of the unique branch hypothesis, *Canadian J. Math.* **66**(4) (2014) 903–923.
- [9] R. Schindler and J. R. Steel, The core model induction, available at math.berkeley.edu/~steel.
- [10] F. Schlutzenberg and N. Trang, Scales in hybrid mice over $\mathbb{R}$, arXiv: 1210.7258v4 (2016).
- [11] J. R. Steel, Core models with more Woodin cardinals, *J. Symbolic Logic* **67**(3) (2002) 1197–1226.
- [12] J. R. Steel, PFA implies $\text{AD}^{\text{L}(\mathbb{R})}$, *J. Symbolic Logic* **70**(4) (2005) 1255–1296.
- [13] J. R. Steel and H. W. Woodin, HOD as a core model, (2012).
- [14] N. Trang, PFA and guessing models, to appear on *the Israel Journal of Mathematics* (2015), available at math.uci.edu/~ntrang.
- [15] N. Trang, HOD in natural models of AD^+ , *Ann. Pure Appl. Logic* **165**(10) (2014) 1533–1556.
- [16] W. H. Woodin, Suitable extender models I, *J. Math. Logic* **1**(2) (2010) 101–339.