$$I_0$$ I 0 and combinatorics at $$ \lambda ^+$$ λ +

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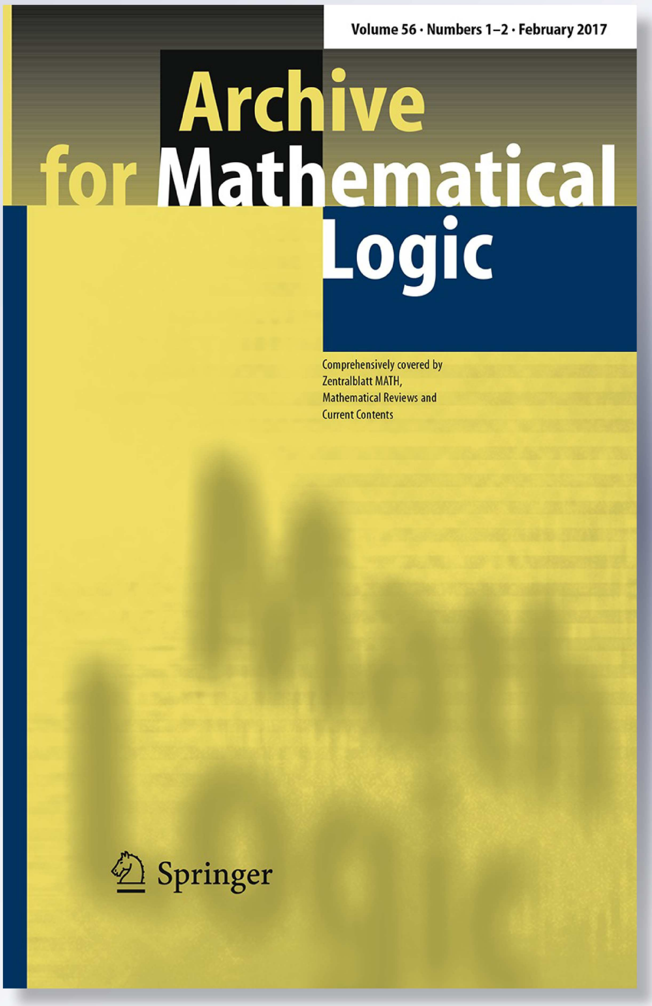

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I_0 and combinatorics at λ^+

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Abstract We investigate the compatibility of I_0 with various combinatorial principles at λ^+, which include the existence of λ^+-Aronszajn trees, square principles at λ, the existence of good scales at λ, stationary reflections for subsets of λ^+, diamond principles at λ and the singular cardinal hypothesis at λ. We also discuss whether these principles can hold in $L(V_{\lambda+1})$.

Keywords Axiom I_0 · λ^+-Aronszajn tree · Square · Weak square · Stationary reflection · Good scales · Diamond · λ-Continuum hypothsis · Generic absoluteness · λ-Goodness

Mathematics Subject Classification 03E55 · 03E35 · 03E05

1 Introduction

Axiom $I_0(\lambda)$ is the assertion that there is an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ such that $\text{crit}(j) < \lambda$. It was first proposed and studied by Woodin in the early 80's and by Laver in the 90's. For the introductory material on this axiom and its connection with other rank-into-rank axioms, we refer the readers to [14].

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Although it is stronger than the existence of supercompact cardinals in consistency strength, the statement $I_0(\lambda)$ only implies the existence of $<\lambda$-supercompact cardinals, and there are a fair number of statements that follow from supercompactness but are independent of $I_0(\lambda)$. The theme of this paper is to present some examples of this sort in the area of combinatorics at λ^+. In this context, λ is an ω-limit of very strong large cardinals, for instance, limit of $<\lambda$-supercompact cardinals.

In this paper, we consider combinatorial principles in the following list.

1. the existences of (special) λ^+-Aronszajn tree and of λ^+-Suslin tree; (see Sects. 2.1, 2.2)
2. the $\square_\lambda$ and the $\square^*_\lambda$ principles; (see Sects. 2.1, 2.2)
3. the existence of (good, very good) scales at λ^+; (see Sect. 2.3)
4. $\neg\mathrm{SR}_{\lambda^+}$, the negation of Stationary Reflection at λ^+; (see Sect. 3)
5. the $\diamondsuit_{\lambda^+}$ principle; (see Sect. 4)
6. GCH at λ; (see Sect. 4)

We are interested in the compatibility of the $I_0(\lambda)$ axiom with various φ's in the above list over the base theory $\Gamma = \mathsf{ZFC} + I_0(\lambda)$. We ask three types of questions:

- Is φ consistent with Γ?
- Is $\neg\varphi$ consistent with Γ?
- Is φ true in $L(V_{\lambda+1})$?

We categorize the results into three theorems. For the case of $\Gamma + \varphi$, we have

Theorem 1 (ZFC) *Assume $I_0(\lambda)$. Then there is a forcing poset $\mathbb{P}$ such that in its generic extension $I_0(\lambda)$ remains true, GCH holds at λ (i.e. $2^\lambda = \lambda^+$), and there is a $\square_\lambda$-sequence $\bar{D} = \langle D_\alpha \mid \alpha < \lambda^+\rangle$ and a stationary set $S \subseteq \{\alpha < \lambda^+ \mid \mathrm{cf}(\alpha) > \omega\}$ such that $S \cap \lim(D_\alpha) = \varnothing$ for all $\alpha < \lambda^+$.*

As consequences, the following statements are also true in the generic extensions:

(a) special λ^+-Aronszajn trees exist, and equivalently, $\square^*_\lambda$;
(b) λ^+-Suslin trees exist;
(c) there is a very good scale at λ^+;
(d) Stationary Reflection fails at λ^+;
(e) $\diamondsuit_{\lambda^+}$.

For the consistency of $\Gamma + \neg\varphi$, we appeal to stronger forms of I_0-type axioms. Let $I_0^\sharp(\lambda, \alpha)$ denote the following stronger form of I_0-type assertion: *There is an elementary embedding $j : L_\alpha(V^\sharp_{\lambda+1}, V_{\lambda+1}) \to L_\alpha(V^\sharp_{\lambda+1}, V_{\lambda+1})$ with $\mathrm{crit}(j) < \lambda$.* And let $I_0^\sharp(\lambda)$ denote the statement without the subscript α.

Theorem 2 (ZFC)

1. *Assume $I_0^\sharp(\lambda, \omega)$. Then there is $\bar\lambda < \lambda$ such that $I_0(\bar\lambda)$ holds and the following statements are true in V:*
   (a) there is no $\bar\lambda^+$-Aronszajn tree;
   (b) there is no scale at $\bar\lambda^+$;
   (c) Stationary Reflection holds at $\bar\lambda^+$.
   And consequently, $\square_\lambda$ and $\square^*_\lambda$ fails in V.

2. *Assume $I_0^\sharp(\lambda)$ and* GCH *holds in V_λ. Assume that Generic Absoluteness holds for $V^\sharp_{\lambda+1}$ at some α which is $V^\sharp_{\lambda+1}$-good and such that $\Theta_\lambda < \alpha < \Theta_\lambda^{V^\sharp_{\lambda+1}}$.*[1] *Then* GCH *fails first at λ, i.e. $2^\kappa = \kappa^+$ for $\kappa < \lambda$ but $2^\lambda = \lambda^+$.*

Regarding the question whether φ is true in $L(V_{\lambda+1})$, we have

Theorem 3 (ZFC) *Assume $I_0(\lambda)$. Then in $L(V_{\lambda+1})$,*

(a) there is no λ^+-Aronszajn tree;
(b) there is no scale at λ^+;
(c) both $\square_\lambda$ and $\square^*_\lambda$ fail;
(d) Stationary Reflection holds at λ;
(e) $\diamondsuit_{\lambda^+}$ fails;
(f) GCH *fails at λ;*
(g) there is no λ^+-sequence of distinct members of $V_{\lambda+1}$.

We are unable to answer the question regarding stationary reflection at λ^+ in $L(V_{\lambda+1})$, due to the lack of choice in this model. We include a scenario (see Theorem 13) where it could be true in $L(V_{\lambda+1})$, although it is unknown if that setting is even compatible with I_0.

Our discussion regarding the failure of GCH at λ in V (see Theorem 17) assumes a stronger form of Generic absoluteness. To apply it, we need to show that Gitik's one-extender-based Prikry forcing is λ-good. For that we extend the idea in [21], introduce two rank notions and develop in Sect. 5 a systematic analysis on the ranks of (finite parts of) conditions in Gitik's forcing.

Notation An $I_0(\lambda)$-embedding is an embedding that witnesses $I_0(\lambda)$. We write I_0 for the statement $\exists\lambda\, I_0(\lambda)$. For two cardinals $\kappa < \lambda$, κ regular, we write $E^\kappa_\lambda = \{\alpha < \lambda \mid \mathrm{cf}(\alpha) = \kappa\}$, and similarly write $E^{>\kappa}_\lambda$, $E^{\le\kappa}_\lambda$ to denote the obvious sets. If C is a set of ordinals, we use $\lim(C)$ to denote the set of limit ordinals of C.

2 λ^+-Aronszajn tree, good scales at λ and $\square_\lambda$

A κ-tree is a tree on κ of size κ whose every level has size $<\kappa$. A κ-*Aronszajn tree* is a κ-tree that has no cofinal branch of length κ.

2.1 There are no λ^+-Aronszajn trees and $\square_\lambda$-sequences in $L(V_{\lambda+1})$

Under ZFC, there is an ω_1-Aronszajn tree, however this is not true under the axiom of determinacy. Being more precise, assuming $\mathsf{AD}^{L(\mathbb{R})}$, there is no ω_1-Aronszajn tree in $L(\mathbb{R})$, while it may exist in V, if AC is assumed there. In this section, we show that a similar situation occurs at λ^+, assuming $I_0(\lambda)$.

Theorem 4 (ZFC) *Assume $I_0(\lambda)$. There is no λ^+-Aronszajn tree in $L(V_{\lambda+1})$.*

[1] See p. 142 for relevant definitions.

Proof The reason there is no λ^+-Aronszajn tree in $L(V_{\lambda+1})$ is the same as that of the nonexistence of ω_1-Aronszajn tree in $L(\mathbb{R})$ under $\mathsf{AD}^{L(\mathbb{R})}$. First, note that $(\lambda^+)^V = (\lambda^+)^{L(V_{\lambda+1})}$, so a λ^+-tree in $L(V_{\lambda+1})$ is also a λ^+-tree in V. We show that such a tree can not be a λ^+-Aronszajn tree.

By a theorem of Woodin (see [24, 1.B.5]), $I_0(\lambda)$ implies that

$$L(V_{\lambda+1}) \models \lambda^+ \text{ is a measurable cardinal.}$$

Assume towards a contradiction that there is a λ^+-Aronszajn tree T in $L(V_{\lambda+1})$. Let $\pi : L[T] \to M \cong \mathrm{Ult}(L[T], \mu \cap L[T])$ be the ultrapower embedding induced by a λ^+-complete measure μ on λ^+. Then $\pi(T)$ is a $\pi(\lambda^+)$-Aronszajn tree in M. Notice that as $\mathrm{crit}(\pi) = \lambda^+$ and every level of T has size $< \lambda^+$, we have $\pi(T)\restriction\lambda^+ = T$. Any node at the λ^+th level of $\pi(T)$ is a cofinal branch of $\pi``T = T$. Thus there can be no λ^+-Aronszajn tree in $L(V_{\lambda+1})$. □

The same argument gives us a similar result regarding the square principle, which is due to Jensen [13].

Definition 1 Let λ be an uncountable cardinal. A $\square_\lambda$*-sequence* is sequence $\langle C_\alpha : \alpha < \lambda^+, \alpha \in \lim(\lambda^+)\rangle$ such that for all $\alpha < \lambda^+$,

1. $C_\alpha \subseteq \alpha$ is closed and unbounded in α,
2. $\mathrm{otp}\, C_\alpha \leq \lambda$,
3. For all $\beta \in \lim(C_\alpha)$, $C_\beta = C_\alpha \cap \beta$.

We say $\square_\lambda$ *holds* if there exists a $\square_\lambda$-sequence.

Theorem 5 (ZFC) *Assume $I_0(\lambda)$. Then $L(V_{\lambda+1}) \models \neg\square_\lambda$.*

Proof Assume not, and let $\bar{C} = \langle C_\alpha : \alpha < \lambda^+, \alpha \in \lim(\lambda^+)\rangle$ be a $\square_\lambda$-sequence in $L(V_{\lambda+1})$. Let μ be a λ^+-complete ultrafilter that witnesses the measurability of λ^+ in $L(V_{\lambda+1})$. Let $\pi : L[\bar{C}] \to M \cong \mathrm{Ult}(L[\bar{C}], \mu \cap L[\bar{C}])$ be the induced elementary embedding. Then $\pi(\bar{C})$ is a $\square_{\pi(\lambda^+)}$-sequence in M. Since every C_α, $\alpha < \lambda^+$, has ordertype $\leq \lambda$ in $L[\bar{C}]$, every member of $\pi(\bar{C})$ has ordertype $\leq \pi(\lambda) = \lambda$, as $\mathrm{crit}(\pi) = \lambda^+$. Let C_{λ^+} be the λ^+th element of $\pi(\bar{C})$. So $\mathrm{otp}(C_{\lambda^+}) = \lambda$ by elementarity. But as a member of $\square_{\pi(\lambda^+)}$-sequence, C_{λ^+} is a closed unbounded subset of λ^+, hence $\mathrm{otp}(C_{\lambda^+}) = \lambda^+$. This is a contradiction! □

Remark Note that the proof only uses items 1 and 2 in the definition of the square principle, so the same argument works for weaker versions of square principles such as $\square_{\lambda,\kappa}$ ($\kappa \leq \lambda$), the approachability property at λ^+ (see [5]) etc.

Although $\square_\lambda$ implies the existence of a λ^+-Aronszajn tree (see Exercise IV.1C and the proof of Theorem IV.2.4, [8]), this does not enable us to conclude the failure of $\square_\lambda$ in $L(V_{\lambda+1})$ from Theorem 4, as the construction of a λ^+-Aronszajn tree uses λ^+-DC, which fails in $L(V_{\lambda+1})$.

2.2 λ^+-Aronszajn trees and $\square_\lambda$ in V

The two theorems above say that $I_0(\lambda)$ pushes λ^+-Aronszajn trees as well as $\square_\lambda$-sequences, if exist, out of $L(V_{\lambda+1})$, but it does not necessarily eliminate their existence in V. Next we show that given the consistency of $I_0(\lambda)$ for some λ, it is possible to produce a model with both $I_0(\lambda)$ and a λ^+-Suslin tree. A κ-*Suslin* tree is a κ-Aronszajn tree with no antichain of size κ. A κ^+-tree is *special* if it can be written as a union of κ many antichains.

Theorem 6 (ZFC) *Assume $I_0(\lambda)$. Then it is consistent that $I_0(\lambda)$ holds and $\square_\lambda$ holds.*

Proof Let $\mathbb{P}_\lambda$ denote the standard Jensen forcing for adding a $\square_\lambda$-sequence.[2] We claim that $I_0(\lambda)$ is preserved after forcing with $\mathbb{P}_\lambda$. The point is that this forcing is $<\lambda^+$-strategically closed, therefore it adds no new subsets of λ, preserves cardinals and cofinalities up to λ^+. So it does not change $V_{\lambda+1}$ and $L(V_{\lambda+1})$, hence any $I_0(\lambda)$-embedding in V remains to witness $I_0(\lambda)$ in the generic extension. □

Jensen introduced a weak form of square principle, often denoted $\square^*_\mu$,[3] and showed that it is equivalent to the existence of a special μ^+-Aronszajn tree. So immediately we have

Corollary 1 (ZFC) *Assume $I_0(\lambda)$. Then it is consistent that $I_0(\lambda)$ holds and there is a special λ^+-Aronszajn tree.*

To produce a special λ^+-Aronszajn tree, a $\square_\lambda$-sequence seems to be a little bit overkill. Ben-David and Magidor [1] showed that, assuming ZFC, if there is a cardinal κ which is κ^+-supercompact, then it is consistent to have $\square^*_{\aleph_\omega} + \neg\square_{\aleph_\omega}$. Assume $I_0(\lambda)$, let κ be the critical point of an $I_0(\lambda)$-embedding. Then κ is κ^+-supercompact. It is unclear what we can say about $\square^*_\lambda + \neg\square_\lambda$.

Question (ZFC) Assume $I_0(\lambda)$. Is it consistent to have $\square^*_\lambda + \neg\square_\lambda$, or the probably weaker version $\exists\gamma\ (\square^*_\gamma + \neg\square_\gamma)$?

The possibility of $\square_\lambda$ (together with $I_0(\lambda)$) gives us an interesting scenario for $I_0(\lambda)$-embeddings and ultrafilters on λ^+: it is relatively consistent with ZFC + $I_0(\lambda)$ that

$$\begin{aligned}&\inf\{\mathrm{crit}(j) \mid j \text{ is an } I_0(\lambda)\text{-embedding}\}\\ &\quad \geq \sup\{\kappa \mid \exists\mu\ (\mu \text{ is an ultrafilter on } \lambda^+ \wedge \mu \text{ is } \kappa\text{-complete})\}\end{aligned}$$

Corollary 2 (ZFC) *Assume $I_0(\lambda)$ and let j be an $I_0(\lambda)$-embedding. Then it is consistent that there is no* $\mathrm{crit}(j)$*-complete ultrafilters on λ^+.*

Proof Silver and Prikry (see [15]) showed that if λ is a singular cardinal, $\lambda > \kappa$ and λ^+ carries a κ-complete ultrafilter, then $\square_\lambda$ fails. Let $\kappa = \mathrm{crit}(j)$. Then by Theorem 6, it is consistent that λ^+ carries no κ-complete ultrafilters. □

Next we prepare a theorem for showing that it is consistent to have both an $I_0(\lambda)$-embedding and a λ^+-Suslin tree.

[2] For the detail of $\mathbb{P}_\lambda$, one can read Cummings' handbook article [6, §6.6].

[3] Jensen's $\square_\mu$-principle asserts that there exists a sequence $\langle C_\alpha : \alpha < \mu^+, \alpha \text{ limit}\rangle$ such that each C_α is a nonempty set of club subsets of α, $|C_\alpha| \leq \mu$, and for all limit $\alpha < \mu^+$, all $C \in C_\alpha$ and all $\beta \in \lim(C)$, $\mathrm{otp}(C) \leq \mu$ and $C \cap \beta = C_\beta$.

Theorem 7 (ZFC) *Assume $I_0(\lambda)$. Then it is consistent that $I_0(\lambda)$ holds and there is a $\square_\lambda$-sequence $\bar{D} = \langle D_\alpha \mid \alpha < \lambda^+ \rangle$ and a $\diamondsuit_{\lambda^+}(S)$-sequence, where $S \subseteq E^{>\omega}_{\lambda^+}$ is stationary and such that $S \cap \lim(D_\alpha) = \varnothing$ for all $\alpha < \lambda^+$.*

Proof For the diamond sequence, we need to apply the forcing poset $\mathbb{P}_\lambda$ (in the proof of Theorem 6) over a ground model that satisfies GCH at λ, namely $2^\lambda = \lambda^+$. This is not difficult to achieve, as one can first force $2^\lambda = \lambda^+$ then force a square sequence, for example, using $\mathbb{Q}_\lambda = \mathrm{Coll}(\lambda^+, 2^\lambda) * \dot{\mathbb{P}}_\lambda$, where $\dot{\mathbb{P}}_\lambda$ is the $\mathrm{Coll}(\lambda^+, 2^\lambda)$-name of $\mathbb{P}_\lambda$. Note that this Levy collapse is a $<\lambda^+$-closed forcing, so this two-step iterated forcing poset does not change $V_{\lambda+1}$ and therefore the $L(V_{\lambda+1})$ of the models before and after applying $\mathbb{Q}_\lambda$ are the same, hence the same $I_0(\lambda)$-embedding in V witnesses $I_0(\lambda)$ in the generic extension.

With a little extra work, one can show that, in the $\mathbb{Q}_\lambda$-generic extension, there are a $\square_\lambda$-sequence $\bar{D} = \langle D_\alpha \mid \alpha < \lambda^+ \rangle$ and a stationary set $S \subseteq E^{>\omega}_{\lambda^+}$ such that $S \cap \lim(D_\alpha) = \varnothing$ for all $\alpha < \lambda^+$.[4] By a result of Shelah ([19], or Theorem 2.2 of [5]), if $2^{<\lambda} = \lambda$ and GCH holds at λ, then $\diamondsuit_{\lambda^+}(T)$ holds for every stationary $T \subset E^{>\omega}_{\lambda^+}$. So we also have a $\diamondsuit_{\lambda^+}(S)$-sequence in the $\mathbb{Q}_\lambda$-generic extension. □

By an argument of Jensen (see [5, §4.2]), a λ^+-Suslin tree can be constructed from a $\square_\lambda$-sequence $\bar{D}$ and a $\diamondsuit_{\lambda^+}(S)$-sequence as in Theorem 7.

Corollary 3 (ZFC) *Assume $I_0(\lambda)$. Then it is consistent that $I_0(\lambda)$ holds and there is a λ^+-Suslin tree.*

Note that in this model there are both special λ^+-Aronszajn trees and λ^+-Suslin trees. However, the notions of special Aronszajn tree and Suslin tree are mutually exclusive, it is natural to ask

QUESTION (ZFC + I_0) Is it possible to have a situation in which for some λ, $I_0(\lambda)$ holds and there are special λ^+-Aronszajn trees but not λ^+-Suslin trees, or the other way around?

Next we show that under suitable assumptions, $I_0(\lambda)$ is not compatible with the existence of λ^+-Aronszajn trees. For that we need an I_0 theorem.

Theorem (Cramer [4]) *Assume $I_0^\sharp(\lambda, \omega)$. Then I_0 holds unboundedly often below λ, i.e. for any $\beta < \lambda$, $I_0(\bar{\lambda})$ holds at some $\bar{\lambda}$ such that $\beta < \bar{\lambda} < \lambda$.*

The theorem we state here is stronger than the original version ([4, Theorem 3.9]), which states only that I_0 holds below λ. The key points are the following two basic facts in I_0 analysis: (1) For any $\beta < \lambda$, there is an $I_0(\lambda)$ embedding k such that $\beta < \mathrm{crit}(k) < \lambda$; (2) By Martin's lemma, every $I_0(\lambda)$ embedding k has square roots with critical points arbitrarily close to $\mathrm{crit}(k)$. Given any $\beta < \lambda$, one can run Cramer's proof (of Theorem 3.3 and 3.9, [4]) with only inverse limits of $I_0(\lambda)$ embeddings with critical points above β, the $\bar{\lambda}_J$'s for such inverse limits are all above β, and thus one can get $I_0(\bar{\lambda})$ for some $\bar{\lambda}$ such that $\beta < \bar{\lambda} < \lambda$.

By a result of Shelah (see [7, Fact 2.10]), if there is a supercompact κ and λ is a cardinal such that $\mathrm{cf}(\lambda) < \kappa < \lambda$, then $\square^*_\lambda$ fails (in fact, the proof just needs κ

[4] The argument for the existence of such $\bar{D}$ and S can be found in [5], the paragraph prior to 4.2.

to be λ^+-supercompact). Let j be any $I_0(\lambda)$-embedding and $\kappa = \mathrm{crit}(j)$. Then κ is $<\lambda$-supercompact. Under $I_0^\sharp(\lambda,\omega)$, by Cramer's theorem (the version above), there is a $\bar\lambda$ such that $\kappa < \bar\lambda < \lambda$ and $I_0(\bar\lambda)$. Then κ is $\bar\lambda^+$-supercompact, so we have $\square^*_{\bar\lambda}$ fails and that there is no special $\bar\lambda^+$-Aronszajn tree. The elimination of the adjective "special" follows from an examination of Cramer's argument.

Theorem 8 (ZFC) *Assume $I_0^\sharp(\lambda,\omega)$. Then there is a $\bar\lambda < \lambda$ such that $I_0(\bar\lambda)$ holds and there is no $\bar\lambda^+$-Aronszajn tree.*

Proof In [17], Magidor and Shelah show that if λ is a singular limit of strongly compact cardinals, then λ^+ carries no Aronszajn trees. For our purpose, it suffices to have λ being a limit of λ^+-strongly compact cardinals. Let $\bar\lambda$ be as in the original version of Cramer's theorem (i.e. the case of $\beta = -1$ in the version quoted above). Then $I_0(\bar\lambda)$ holds. In Cramer's proof of his Theorem, this $\bar\lambda$ is obtained via an inverse limit $(J, \mathbf{j})$ such that $\bar\lambda = \lambda_J$. Let $\mathbf{j} = \langle j_n : n < \omega\rangle$, then $\bar\lambda = \lim_{n<\omega} \mathrm{crit}(j_n)$. Here each j_n is an $I_0(\lambda)$ embedding, thus each $\mathrm{crit}(j_n)$ is a $<\lambda$-strongly compact. Therefore $\bar\lambda$ is a limit of $\bar\lambda^+$-strongly compact cardinals. Then by Magidor-Shelah's theorem, there is no $\bar\lambda^+$-Aronszajn tree. □

We have shown that

Corollary 4 (ZFC) *Let $\varphi(\lambda)$ be one of the following statements.*

1. *there is a $\square_\lambda$-sequence.*
2. *there is a $\square^*_\lambda$-sequence, or equivalently, there[3.] is a special λ^+-Aronszajn tree.*
4. *there is a λ^+-Suslin tree.*
5. *there is a λ^+-Aronszajn tree.*

Then

(a) *Assume $I_0(\lambda)$. There is a model in which $I_0(\lambda) + \varphi(\lambda)$ holds.*
(b) *Assume $I_0^\sharp(\lambda,\omega)$. Then there is a $\bar\lambda < \lambda$ such that $I_0(\bar\lambda) + \neg\varphi(\bar\lambda)$ holds.*

Contrast Corollary 4 with Solovay's theorem (see [22,23]) regarding the incompatibility of square principle with supercompact cardinals, more precisely: If $\kappa \le \lambda$ and κ is λ^+-supercompact, then $\square_\lambda$ fails.

2.3 Good scales at λ

Next we discuss good scales at λ. We are going to show that there is no (very) good scale at λ in $L(V_{\lambda+1})$ and to add the assertion of its existence to the list in Corollary 4. In this paper, as λ is a singular cardinal of countable cofinality, we consider only the set $\prod_{i<\omega}\kappa_i$, where $\bar\kappa = \langle\kappa_i : i < \omega\rangle$ is a sequence of regular cardinals such that $\lambda = \sup_{i<\omega}\kappa_i$, and the ideal I on ω that consists of all finite subsets of ω. Given $f, g \in \prod_i \kappa_i$, $f <_I g$ if and only if $\omega\backslash\{i \mid f(i) < g(i)\} \in I$. *A scale of length α in* $\prod_i\kappa_i/I$ is a $<_I$-increasing sequence $\langle f_i : i < \alpha\rangle$ in $\prod_i\kappa_i$ which is cofinal in $\prod_i\kappa_i$ under the relation $<_I$. *A scale for* λ is a pair $(\bar\kappa, \bar f)$, where $\bar f$ is a scale of length λ^+ in $\prod_i\kappa_i/I$. As λ is singular, a basic fact of PCF theory is that, there exists a scale for λ.

Definition 2 1. Suppose $(\bar{\kappa}, \bar{f})$ is a scale for λ. A point $\alpha < \lambda^+$ is *good for* $(\bar{\kappa}, \bar{f})$ iff there is an $A \subset \alpha$ unbounded in α and $i < \omega$ such that

$$\forall \alpha, \beta \in A \forall j > i\, (\alpha < \beta \to f_\alpha(j) < f_\beta(j)).$$

2. Let $\langle g_i : i < \beta \rangle$ be a $<_I$-increasing sequence in $\prod_i \kappa_i$ and $g \in \prod_i \kappa_i$. g is an *exact upper bound (eub)* for $\langle g_i : i < \beta \rangle$ if $g_i <_I g$ for every $i < \beta$ and for any $h \in \prod_i \kappa_i$, $h <_I g \Rightarrow h \leq_I g_i$ for some $i < \beta$.

By Shelah's PCF theory, the set of good points in a scale for λ is a stationary subset of λ^+. This set is determined by the sequence $\bar{\kappa}$ modulo the nonstationary ideal on λ^+.

Definition 3 A scale $(\bar{\kappa}, \bar{f})$ for λ is *good* if except a nonstationary subset of λ^+ every point of uncountable cofinality is good for $\bar{f}$.

A scale $(\bar{\kappa}, \bar{f})$ for λ is *very good* if for every limit $\alpha < \lambda^+$ such that $\mathrm{cf}(\alpha) > \omega$, there is a $C \subseteq \alpha$ club in α and an integer $m < \omega$ such that for all $n > m$, $\langle f_\beta(n) : \beta \in C \rangle$ is strictly increasing.

Theorem 9 (ZFC) *Assume $I_0(\lambda)$. There is no (good, very good) scale at λ in $L(V_{\lambda+1})$.*

Proof It suffices to show that there is no scale at λ in $L(V_{\lambda+1})$. Suppose otherwise and let $(\bar{\kappa}, \bar{f})$ be a scale for λ in $L(V_{\lambda+1})$. Let μ be a λ^+-complete ultrafilter that witnesses the measurability of λ^+ in $L(V_{\lambda+1})$. Let

$$\pi : L[\bar{\kappa}, \bar{f}] \to M \cong \mathrm{Ult}(L[\bar{\kappa}, \bar{f}], \mu \cap L[\bar{\kappa}, \bar{f}])$$

be the induced elementary embedding. Since $L[\bar{\kappa}, \bar{f}] \models \forall \alpha < \beta\, (f_\alpha <_I f_\beta)$, by elementarity, $f_\alpha <_I \pi(\bar{f})(\lambda^+)$ in M, for every $\alpha < \lambda^+$. Since $<_I$ is absolute, that is also true in $L(V_{\lambda+1})$. But then $\bar{f}$ is not a scale in $L(V_{\lambda+1})$. Contradiction! □

Similar to the situation of $\square_\lambda$, we have

Theorem 10 *1. Assume $I_0(\lambda)$. Then there is a model of* ZFC $+ I_0(\lambda)$*, in which there is a (very) good scale at λ.*

1. Assume $I_0^\sharp(\lambda, \omega)$. Then there is a $\bar{\lambda} < \lambda$ such that $I_0(\bar{\lambda})$ holds and there is no good scale at $\bar{\lambda}$.

Proof 1 follows from Corollary 4-1 and a theorem of Cummings, Foreman and Magidor (see [7, Theorem 3.1]): If λ is singular and $\kappa < \lambda$, then $\square_{\lambda,\kappa}$[5] implies that there is a very good scale at λ. $\square_\lambda$ implies $\square_{\lambda,\kappa}$, therefore in the model obtained by adding a $\square_\lambda$-sequence, there is a very good scale at λ.

For 2, we need a theorem of Shelah (see [20], or [5, Theorem 18.1]): If there is a κ such that $\mathrm{cf}(\lambda) < \kappa < \lambda$ and κ is λ^+-supercompact, then there is no good scale at λ. By the discussion in the paragraph following Cramer's Theorem on p. 136, one can arrange $I_0(\bar{\lambda})$ for some $\bar{\lambda} > \kappa = \mathrm{crit}(j)$, but κ is $<\lambda$-supercompact, in particular $\bar{\lambda}^+$-supercompact, therefore, there is no good scale at $\bar{\lambda}$. □

[5] The definition of $\square_{\lambda,\kappa}$ is irrelevant to our proof, we refer the reader to Cummings [5] for details.

Corollary 5 (ZFC) *The assertion that "there is a (very) good scale at λ" can be added to the list in Corollary 4.*

3 Stationary reflection at λ^+

Let κ be an uncountable regular cardinal. Let S be a stationary subset of κ. S *reflects at* α if $\alpha < \kappa$, $\mathrm{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary in α. *Stationary Reflection Principle for* T, where $T \subseteq \kappa$ is stationary, says that for every stationary $S \subseteq T$, S reflects at some $\alpha < \kappa$.

In this section, we show that I_0 is compatible with either side of the Stationary Reflection Principle. Let SR_{λ^+} denote the Stationary Reflection Principle for λ^+.

Theorem 11 (ZFC) *Assume $I_0^\sharp(\lambda, \omega)$. Then there is a $\bar\lambda < \lambda$ such that $I_0(\bar\lambda)$ holds and $\mathrm{SR}_{\bar\lambda^+}$ is true.*

Proof As before (see p. 136, after Cramer's Theorem), this hypothesis yields κ, $\bar\lambda$ such that $\kappa < \bar\lambda < \lambda$ and κ is $\bar\lambda^+$-supercompact. Then it follows from the standard argument that the Stationary Reflection Principle for $\bar\lambda^+$ is true: Fix a stationary $S \subseteq \bar\lambda^+$. Let $\pi : V \to M$ be an embedding witnessing the $\bar\lambda^+$-supercompactness of κ. We claim that

Claim π"S is a stationary subset of $\gamma = \sup \pi$"$S = \sup \pi$"$\bar\lambda^+$ in M.

Let C be a closed and unbounded subset of γ in M. Since π"$\bar\lambda^+$ is κ-closed, i.e. closed under supremum of $< \kappa$-sequences, π"$\bar\lambda^+ \cap C$ is a κ-closed and unbounded subset of γ. Pull it back, $D = \pi^{-1}$"(π"$\bar\lambda^+ \cap C$) is a κ-closed and unbounded subset of λ^+. Then we have $S \cap D \neq \varnothing$. And then π"$S \cap C \neq \varnothing$. Thus π"S is stationary in γ.

Since π"$S \subseteq \pi(S) \cap \gamma$, we have

$$M \models \exists \gamma < \pi(\bar\lambda^+)\ (\pi(S) \text{ reflects at } \gamma).$$

By elementarity, $V \models S$ reflects at some $\alpha < \bar\lambda^+$. □

It is well known that $\Box_\kappa$ implies that the Stationary Reflection Principle fails for every stationary $T \subseteq \kappa^+$ (see [7, Theorem 1]). So one can obtain the failure of $\mathrm{SR}_{\bar\lambda^+}$ by forcing a square sequence. As discussed in the proof of Theorem 7, that forcing is $<\lambda^+$-strategically closed, it preserves $I_0(\lambda)$, therefore we have both $I_0(\lambda)$ and $\neg\mathrm{SR}_{\lambda^+}$ in the generic extension. One can also force directly a non-reflecting stationary subset of λ^+. One can find such a forcing in Cummings' handbook article (see [6, §6.5]). That forcing is λ^+-strategically closed, therefore adds no new subsets of λ. Thus in $V[G]$, we also have both $I_0(\lambda)$ and $\neg\mathrm{SR}_{\lambda^+}$.

Theorem 12 (ZFC) *Assume $I_0(\lambda)$ is consistent. Then so is $I_0(\lambda) + \neg\mathrm{SR}_{\lambda^+}$.*

Corollary 6 (ZFC) *The assertion SR_{λ^+} can be added to the list in Corollary 4.*

The question left is that

– Assuming $I_0(\lambda)$, is it true that $L(V_{\lambda+1}) \models \mathrm{SR}_{\lambda^+}$?

Our first attempt is to try the trick we did in the proofs for the nonexistence of λ^+-Aronszajn tree (see Theorem 4) and of $\square_\lambda$-sequences (see Theorem 5) in $L(V_{\lambda+1})$. However, the SR_{λ^+} case is subtle. Its negation is the following statement

$$\exists S \notin \mathscr{I}_{\lambda^+} \forall\alpha \in E^{>\omega}_{\lambda^+} \exists C_\alpha (C_\alpha \text{ is club in } \alpha \wedge S \cap \alpha \cap C_\alpha = \varnothing).$$

Here $\mathscr{I}_{\lambda^+}$ denote the nonstationary ideal on λ^+ and $E^{>\omega}_{\lambda^+}$ denote the set of ordinals $<\lambda^+$ with uncountable cofinalities. For each such α, let $\mathcal{C}_\alpha$ be the collection of clubs C in α such that $S \cap C \cap \alpha = \emptyset$. We would like to take the ultrapower of the structure $L[\langle \mathcal{C}_\alpha : \alpha < \lambda^+\rangle, S]$ by a measure on λ^+. The problem is that Łos theorem fails for the ultrapower. In particular, we are not able to show that, letting i be the ultrapower map and $\langle \mathcal{D}_\beta : \beta < i(\lambda^+)\rangle = i(\langle \mathcal{C}_\alpha : \alpha < \lambda^+\rangle)$, for each $\beta < i(\lambda^+)$, $\mathcal{D}_\beta \neq \emptyset$. Also, since λ^+-DC fails in $L(V_{\lambda+1})$, we are unable to choose, for each $\alpha < \lambda^+$, a $C_\alpha \in \mathcal{C}_\alpha$ and consider the ZFC model $L[\langle C_\alpha : \alpha < \lambda^+\rangle, S]$.

We also considered the function $\alpha \mapsto \alpha\backslash S$. Since S reflects nowhere, for each $\alpha \in E^{>\omega}_{\lambda^+}$, $\alpha\backslash S$ contains a closed unbounded subset of α. If $E^{<\omega}_{\lambda^+} \in \mu$, then $\lambda^+\backslash S = [\alpha \mapsto \alpha\backslash S]_\mu$. By elementarity, $\lambda^+\backslash S$ contains a closed unbounded subset of λ^+. S is a stationary subset of λ^+ in V and thus stationary in $M \cong \mathrm{Ult}(L[S], \mu \cap L[S])$, so $S \cap (\lambda^+\backslash S) \neq \varnothing$. This would be a contradiction! But unfortunately μ concentrates on $E^{\omega}_{\lambda^+}$, this argument does not work.

We will obtain stationary reflection in $L(V_{\lambda+1})$ from a slightly stronger principle, which unfortunately is not yet known to be consistent relative to $I_0(\lambda)$.

Theorem 13 (ZFC) *Assume $L(V_{\lambda+1}) \models \lambda^+$ is $V_{\lambda+1}$-supercompact.*[6] *Then* SR_{λ^+} *holds in* $L(V_{\lambda+1})$.

Proof Work in $L(V_{\lambda+1})$. Fix a measure μ witnessing that λ^+ is $V_{\lambda+1}$-supercompact. For each $\sigma \in \mathscr{P}_{\lambda^+}(V_{\lambda+1})$, let $M_\sigma = \mathsf{HOD}_{\sigma\cup\{\sigma\}}$ and let $M = \prod_\sigma M_\sigma/\mu$ be the μ-ultraproduct of the structures M_σ's.

Claim Łos theorem holds for this ultraproduct.

Proof of Claim The proof is by induction on the complexity of formulas. It's enough to show the following. Suppose $\varphi(x, y)$ is a formula such that the claim holds for φ and f is a function such that $\{\sigma \mid M_\sigma \models \exists x\varphi[x, f(\sigma)]\} \in \mu$. We show that $M \models \exists x\varphi[x, [f]_\mu]$.

Let $g(\sigma) = \{x \in \sigma \mid (\exists y \in \mathsf{OD}(x))(M_\sigma \models \varphi[y, f(\sigma)])\}$. Then $\{\sigma \mid g(\sigma)$ is a non-empty subset of $\sigma\} \in \mu$. By normality of μ, there is a fixed x such that $\{\sigma : x \in g(\sigma)\} \in \mu$. Hence we can define $h(\sigma)$ to be the least y in $\mathsf{OD}(x)$ such that $M_\sigma \models \varphi[y, f(\sigma)]$. It's easy to see then that $M \models \varphi[[h]_\mu, [f]_\mu]$.

[6] This means there is a fine, normal, λ^+-complete measure μ on $\mathscr{P}_{\lambda^+}(V_{\lambda+1})$. Fineness and completeness have standard meanings. In the context where full AC does not hold, normality is defined as follows: suppose $F : \mathscr{P}_{\lambda^+}(V_{\lambda+1}) \to \mathscr{P}_{\lambda^+}(V_{\lambda+1})$ is such that $\{\sigma : F(\sigma) \subseteq \sigma \wedge F(\sigma) \neq \emptyset\} \in \mu$, then there is some x such that $\{\sigma : x \in F(\sigma)\} \in \mu$.

For each x, let c_x be the constant function $f : \mathscr{P}_{\lambda^+}(V_{\lambda+1}) \to \{x\}$. By λ^+-completeness, it is easy to see that for each $\alpha < \lambda^+$, $\alpha = [c_\alpha]_\mu$. Also for each set x, there is some $a \in V_{\lambda+1}$ such that x is $\mathsf{OD}(a)$. In particular, if x is a set of ordinals, by fineness of μ, $\{\sigma \mid x \in M_\sigma\} \in \mu$. Also if $A \subseteq V_{\lambda+1}$, then $A \in \mathsf{HOD}[\tau]$ for some $\tau \in V_{\lambda+1}$, and by the fineness of μ, we have $\{\sigma \mid A \cap \sigma \in M_\sigma\} \in \mu$; also by the normality of μ, $A = [\sigma \mapsto A \cap \sigma]_\mu$. By Łos theorem, these imply that $A \in M$. In particular, $V_{\lambda+1} \in M$.

Now let $S \subseteq \lambda^+$ be stationary and $S^* = [c_S]_\mu$. By the previous paragraph, in M, $S^* \cap \lambda^+ = S$ (note that $(\lambda^+)^M = \lambda^+$ because $V_{\lambda+1} \in M$) and hence $S^* \cap \lambda^+$ is stationary in M. By Łos,

$$\{\sigma \mid \exists \alpha < \lambda^+ \; M_\sigma \vDash S \cap \alpha \text{ is stationary}\} \in \mu.$$

By normality of μ, there is some $\alpha < \lambda^+$ such that

$$\{\sigma \mid M_\sigma \vDash S \cap \alpha \text{ is stationary}\} \in \mu.$$

Now we claim that $S \cap \alpha$ is stationary. Let $C \cap \alpha$ be club in α. By the discussion above, $\{\sigma \mid C \in M_\sigma\} \in \mu$. Fix σ such that $C \in M_\sigma$ and $M_\sigma \vDash$ "$S \cap \alpha$ is stationary". Now in M_σ, C is club in α, so $C \cap S \cap \alpha \neq \emptyset$. This shows $S \cap \alpha$ is stationary. □

Remark The proof above works also if we are in a model M of the form $L(V_{\lambda+1})[\mu]$ and $M \vDash \mu$ is a normal, fine, λ^+-complete measure on $\mathscr{P}_{\lambda^+}(V_{\lambda+1})$. We are optimistic that such a model can be constructed from $I_0(\lambda)$ or from its strengthenings.

4 Diamond and GCH at λ

First of all, assuming I_0, no matter whether $\diamondsuit_{\lambda^+}$ is true or not in the universe, diamond sequence can not exist in $L(V_{\lambda+1})$.

Theorem 14 (ZFC) *Assume $I_0(\lambda)$. Then in $L(V_{\lambda+1})$, $2^\lambda \neq \lambda^+$ and $\diamondsuit_{\lambda^+}$ fails.*

Proof It is a ZF theorem that $\diamondsuit_{\lambda^+}$ yields an injective function from $\mathscr{P}(\lambda)$ into λ^+. The inverse of this injective function gives a λ^+-sequence of distinct subsets of λ. So we have $L(V_{\lambda+1}) \vDash \diamondsuit_{\lambda^+} \to (2^\lambda = \lambda^+)$. But $2^\lambda = \lambda^+$ implies that $V_{\lambda+1}$ is wellorderable in $L(V_{\lambda+1})$, this contradicts the fact that $L(V_{\lambda+1}) \vDash \neg\mathsf{AC}$. □

This proof utilizes the fact that GCH at λ leads to the violation of the fact that $L(V_{\lambda+1})$ is not a full choice model. Here we give another proof, which shows that both $\diamondsuit_{\lambda^+}$ and GCH at λ violates a weaker statement in $L(V_{\lambda+1})$. It is the following analog of the AD-fact that there is no ω_1-sequence of distinct reals.

Theorem 15 (ZFC) *Assume $I_0(\lambda)$. Then there is no λ^+-sequence of distinct members of $\mathscr{P}(\lambda)$ in $L(V_{\lambda+1})$.*

Proof The key point again is that λ^+ is measurable in $L(V_{\lambda+1})$. Suppose $X = \langle x_\alpha : \alpha < \lambda^+\rangle$ is a sequence of distinct subsets of λ. Let

$$\pi : L[X] \to M \cong \mathrm{Ult}(L[X], \mu \cap L[X])$$

be the ultrapower embedding induced by a λ^+-complete measure μ on λ^+. Then in M, $\pi(X)$ is a $\pi(\lambda^+)$-sequence of distinct subsets of λ. Every member of $\pi(X)$ is represented by a function $\lambda^+ \to \{x_\alpha \mid \alpha < \lambda^+\}$ in V, in particular, let $[f]$ be the λ^+th element of $\pi(X)$.

Claim f is constant on a measure one subset $A \subset \lambda^+$.

Proof of Claim For each $\beta < \lambda$, there is a unique $i_\beta \in \{0, 1\}$ such that

$$A_\beta^{i_\beta} = \{\alpha < \lambda^+ \mid f(\alpha)(\beta) = i_\beta\}$$

is a measure one subset of λ^+. By λ^+-completeness, the set $A = \bigcap\{A_\beta^{i_\beta} \mid \beta < \lambda\}$ has measure one. Therefore for every $\alpha \in A$, $f(\alpha)(\beta) = i_\beta$.

This means that $[f]$ equals to x_α for some $\alpha < \lambda^+$, contradicting to the assumption that members of $\pi(X)$ are all distinct. □

Remark $\mathscr{P}(\lambda)$ is the above theorem can be replaced by $V_{\lambda+1}$ (using a well ordering of V_λ of length λ), but not by $H(\lambda^+)$ (as $\lambda^+ \subset H(\lambda^+)$ gives a counter example).

This theorem effectively rules out $2^\lambda \geq \lambda^+$ in $L(V_{\lambda+1})$, thus gives a more direct reason why $\diamondsuit_{\lambda^+}$ and GCH at λ fail in $L(V_{\lambda+1})$.

As we have discussed earlier (see the proof of Theorem 7), one can easily obtain $\diamondsuit_{\lambda^+}$ by forcing $2^\lambda = \lambda^+$ (using Levy collapse $\mathrm{Coll}(\lambda^+, 2^\lambda)$) without adding bounded subsets of λ, therefore preserves $2^{<\lambda} = \lambda$ and $I_0(\lambda)$. Thus we have

Theorem 16 (ZFC) *Assume I_0 is consistent. Then the following are consistent*

1. $\exists\lambda(I_0(\lambda) + \diamondsuit_{\lambda^+})$,
2. $\exists\lambda(I_0(\lambda) + 2^\lambda = \lambda^+)$.

Regarding GCH, Dimonte–Friedman (see [9, Corollary 3.9]) sketches an argument that it is relatively consistent with I_0 that GCH fails, in particular at λ. However, there are flaws in that argument. We will remark on this after proving our next theorem. Here we show the compatibility of $I_0(\lambda)$ with the first failure of GCH at λ, and consequently with $\neg\diamondsuit_{\lambda^+}$, from a stronger form of I_0-type axiom and a strong generic absoluteness assumption. A few definitions.

Definition 4 Suppose $X \subseteq V_{\lambda+1}$.

1. Let $\Theta_\lambda^X =_{\mathrm{def}} \{\alpha \mid L(X, V_{\lambda+1}) \models$ there is a surjective $\pi : V_{\lambda+1} \to \alpha\}$.
2. An ordinal $\alpha < \Theta_\lambda^X$ is *X-good* if every element of $L_\alpha(X, V_{\lambda+1})$ is definable in $L_\alpha(X, V_{\lambda+1})$ from an element in $V_{\lambda+1} \cup \{X\}$.

Definition 5 Assume $j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$ is a proper elementary embedding and $\mathrm{crit}(j) < \lambda$. Let $(M_\omega, j_{0,\omega})$ be the ω-iterate of $(L(X, V_{\lambda+1}), j)$. Suppose $\alpha < \Theta_\lambda^X$ and α is X-good. We say that *Generic Absoluteness holds for X at α* if the following proposition holds:

Suppose $\mathbb{P} \in j_{0,\omega}(V_\lambda)$, $G \in V$ *is an* M_ω*-generic filter for* $\mathbb{P}$, *and* $\mathrm{cof}(\lambda) = \omega$ *in* M_ω. *Then there exist* $\alpha' \le \alpha$ *and* $X' \subseteq V_{\lambda+1}$ *such that*

$$L_{\alpha'}(X', M_\omega[G] \cap V_{\lambda+1}) \prec L_\alpha(X, V_{\lambda+1}).$$

The details of the definition of "proper" I_0 embedding is not important here, the key point is that if an I_0 embedding is proper then it is iterable (see [25, Lemma 17, p. 136]). We refer the readers to Woodin's monograph [25] for relevant terminology and basic I_0 theory. Recent works by S. Cramer [2,3] suggest that the Generic Absoluteness hypothesis in the following theorem is redundant, but at the moment, we do not see how to make do without it.

Theorem 17 (ZFC) *Assume* $I_0^\sharp(\lambda)$ *and* GCH *holds in* V_λ. *Assume that Generic Absoluteness holds for* $V_{\lambda+1}^\sharp$ *at some* α *which is* $V_{\lambda+1}^\sharp$*-good and such that* $\Theta_\lambda < \alpha < \Theta_\lambda^{V_{\lambda+1}^\sharp}$. *Then* GCH *fails first at* λ, *i.e.* $2^\kappa = \kappa^+$ *for all* $\kappa < \lambda$ *but* $2^\lambda = \lambda^+$. *As a consequence,* $\diamondsuit_{\lambda^+}$ *fails.*

Proof Let M_ω be the ω-iterate of $L(V_{\lambda+1}^\sharp, V_{\lambda+1})$ by j. Then by elementarity, $\lambda = j_{0,\omega}(\mathrm{crit}(j))$ is $<j_{0,\omega}(\lambda)$-strong in M_ω and GCH holds in $j_{0,\omega}(V_\lambda)$. Pick an $\eta \in [\lambda^{++}, j_{0,\omega}(\lambda))$. Let $\mathbb{P} = \mathbb{P}_{\lambda,\eta}$ be Gitik's one-extender-based Prikry forcing (with a single extender) that changes the cofinality of λ to ω and adds η many cofinal ω-sequence in λ (see [11]). The key is to show that $\mathbb{P}$ is λ-good in M_ω, as this implies that there are M_ω-generic filters in V (see [21, Proposition 3.20] or [25, p. 405]). The next section is devoted to verifying this matter.

Let $G \subseteq \mathbb{P}$ be an M_ω-generic filter in V. Then $2^\lambda = \eta$ holds in $M_\omega[G]$. As $\Theta_\lambda < \alpha$, $j \restriction L_{\Theta_\lambda}(V_{\lambda+1}) \in L_\alpha(V_{\lambda+1}^\sharp, V_{\lambda+1})$.[7] By Generic Absoluteness for $V_{\lambda+1}^\sharp$ at α, there is an $\alpha' \le \alpha$ and an $X' \subseteq V_{\lambda+1}$ such that

$$L_{\alpha'}(X', M_\omega[G] \cap V_{\lambda+1}) \prec L_\alpha(V_{\lambda+1}^\sharp, V_{\lambda+1}).$$

By the definability of sharp, $X' = (M_\omega[G] \cap V_{\lambda+1})^\sharp$. Since $j \restriction L_{\Theta_\lambda}(V_{\lambda+1})$ is in $L_\alpha(V_{\lambda+1}^\sharp, V_{\lambda+1})$, there is a

$$j' \in L_{\alpha'}((M_\omega[G] \cap V_{\lambda+1})^\sharp, M_\omega[G] \cap V_{\lambda+1})$$

such that $\mathrm{dom}(j') = L_{\Theta'}(M_\omega[G] \cap V_{\lambda+1})$, where Θ' is the Θ_λ computed in $L(M_\omega[G] \cap V_{\lambda+1})$, and such that the $L(M_\omega[G] \cap V_{\lambda+1})$-ultrafilter $\mu_{j'}$ given by $X \in \mu_{j'}$ iff $j' \restriction V_\lambda \in j'(X)$ induces an elementary embedding of $L(M_\omega[G] \cap V_{\lambda+1})$ into itself. This gives us $I_0(\lambda)$ in $M_\omega[G]$.

There is a little wrinkle: it is not clear that $M_\omega[G]$ is a choice model. Notice that as the ω-iterate of $L(V_{\lambda+1})$, $M_\omega = L(j_{0,\omega}(V_{\lambda+1})) = L((V_{\lambda_\omega+1})^M)$, where $\lambda_\omega = j_{0,\omega}(\lambda)$. By elementarity, M_ω satisfies $<\lambda_\omega^+$-DC, so $M_\omega[G]$ has a well ordering of its V_{λ_ω} (not $V_{\lambda_\omega+1}$!). Denote that well ordering as Λ. Note that in $M_\omega[G]$, the $I_0(\lambda)$-embedding and

[7] See [16, Theorem 3(ii)].

the witness for the first failure of GCH (at λ) are both in V_{λ_ω}, so in $L(V_{\lambda_\omega}, \Lambda)^{M_\omega[G]}$, $I_0(\lambda)$ and the first failure of GCH (at λ) remains, and in addition AC holds. □

A few remarks

1. The GCH assumption in the theorem is not essential. Suppose $j : L(V^\sharp_{\lambda+1}, V_{\lambda+1}) \to L(V^\sharp_{\lambda+1}, V_{\lambda+1})$ is a proper elementary embedding with $\mathrm{crit}(j) < \lambda$. Relativize Dimonte–Friedman argument (see [9]) for $L(V_{\lambda+1})$, then there is a poset $\mathbb{P}$ (backward Easton forcing up to λ) such that in its generic extension $V[H]$, j can be lifted to $L(V^\sharp_{\lambda+1}, V_{\lambda+1})[H]$ and GCH holds in V_λ. According to Dimonte–Friedman [9], this poset is above ω, so we have

$$L(V^\sharp_{\lambda+1}, V_{\lambda+1})[H] = L(V[H]^\sharp_{\lambda+1}, V[H]_{\lambda+1}).$$

Moreover, this poset is λ^+-c.c. and is definable in

$$N = L_{\alpha'}((M_\omega[G] \cap V_{\lambda+1})^\sharp, M_\omega[G] \cap V_{\lambda+1}).$$

Notice that N and V agree on V_λ and the elementary embedding witnessing Generic Absoluteness for $V^\sharp_{\lambda+1}$ (at α), let us call it π, has critical point $\geq (\lambda^+)^N$. Thus π can be lifted to a $\bar{\pi} : N[H_0] \to L_\alpha(V[H]^\sharp_{\lambda+1}, V[H]_{\lambda+1})$. Again

$$N[H_0] = L_{\alpha'}((M_\omega[G][H_0] \cap V[H]_{\lambda+1})^\sharp, M_\omega[G][H_0] \cap V[H]_{\lambda+1}).$$

Therefore the generic absoluteness assumption is also preserved by $\mathbb{P}$.

2. We pointed out earlier that there are some issues with the argument Dimonte–Friedman sketched for the compatibility of I_0 with the failure of GCH at λ (see [9, Corollary 3.9]). To be more specific, one is that it is not clear why $j \restriction L_\alpha(V_{\lambda+1})$ falls in the range of π, and then it would make no sense to talk about $\pi^{-1}(j \restriction L_\alpha(V_{\lambda+1}))$. The second issue is more serious: the hypothesis of their corollary, that generic absoluteness holds for all $\alpha < \Theta$, is not enough to ensure that $\pi^{-1}(j \restriction L_\alpha(V_{\lambda+1})), \alpha < \Theta$, can be pieced together to form j^*. It is unclear why (the union of) the sequence $\langle \pi^{-1}(j \restriction L_\alpha(V_{\lambda+1})) : \alpha < \Theta \rangle$ is in the domain of π. The anonymous reviewer points out that even if one tries to repair the first issue by taking an elementary embedding k such $k \restriction L_\alpha(V_{\lambda+1})$ is in the range, using elementarity, the problem of how to piece together all the k's remains.

3. However, the current status of generic absolutness is only up to $L_\delta(V_{\lambda+1})$, where δ is least such that $L_\delta(V_{\lambda+1}) \prec L(V_{\lambda+1})$, which is due to Cramer [2]. It is not clear at this point if generic absoluteness assumption in the hypothesis of our theorem follows from the existence of an elementary embedding $j : L(V^\sharp_{\lambda+1}, V_{\lambda+1}) \to L(V^\sharp_{\lambda+1}, V_{\lambda+1})$ with $\mathrm{crit}(j) < \lambda$.

4. After we proved the λ-goodness of Gitik's forcing (see Sect. 5), we were pointed out that one could use Merimovich's $\mathbb{P}_E$ (see §3 of [18]) instead of $\mathbb{P}$ in the above proof, and the λ-goodness of $\mathbb{P}_E$ follows easily from Lemma 3.25 of [18]. However, we stick to our choice here, the purpose is two-folded. One is that we have found no

written account of the proof of the analog of Lemma 3.25 of [18] for Gitik's forcing;[8] and two we would like to promote the rank analysis for the Prikry-type forcings. Some simple applications of the rank analysis can be found in §3.4 of [21].

5 The one-extender-based Prikry forcing is λ-good

5.1 Preliminaries on λ-good forcings

In order to apply the Generic Absoluteness Theorem, we need to ensure that their generics exist in V. For that, we use a notion of λ-goodness for posets due to Woodin [25].

Definition 6 Let λ be an infinite cardinal. We say a partially ordered set $\mathbb{P}$ is λ-*good* (in V) if it adds no bounded subsets of λ and for every generic filter G and for every $A \subset \text{Ord}$ in $V[G]$ and of size $< \lambda$, there is a non-$\subset$-decreasing ω-sequence $\langle A_i : i < \omega\rangle$ such that $A = \bigcup_i A_i$ and each A_i, $i < \omega$, is in V.

Below is a relativized version of Proposition 3.8 of [21], which asserts that generics for forcings that are λ-good in the ωth iterate exist in V.

Proposition *Assume that $j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$ is a proper elementary embedding with critical point $< \lambda$. Let $(M_\omega, j_{0,\omega})$ be the ω-iterate of $(L(X, V_{\lambda+1}), j)$. Suppose $\mathbb{P} \in j_{0,\omega}(V_\lambda)$ and $\mathbb{P}$ is λ-good in M_ω. Then there exists $G \subseteq \mathbb{P}$ in V such that G is M_ω-generic.*

Here we are only interested in the case that $X = V^\sharp_{\lambda+1}$. A useful sufficient condition for showing λ-goodness is as follows (see [21]): for all

$$\mathscr{D} \subseteq \{D \subseteq \mathbb{P} \mid D \text{ is open dense in } \mathbb{P}\}$$

such that $|\mathscr{D}| < \lambda$, for any $p \in \mathbb{P}$, there are $p^\circ \leq_{\mathbb{P}} p$ and a nondecreasing sequence $\langle \mathscr{D}_{p,i} : i < \omega\rangle$ of subsets of $\mathscr{D}$ such that the following hold

1. $\mathscr{D} = \bigcup\{\mathscr{D}_{p,i} \mid i < \omega\}$,
2. for all $i < \omega$ such that $\mathscr{D}_{p,i} \neq \varnothing$, $\bigcap \mathscr{D}_{p,i}$ is dense below p°, i.e. for any $r \leq_{\mathbb{P}} p^\circ$, there exists $r' \leq_{\mathbb{P}} r$ such that $r' \in D$ for every $D \in \mathscr{D}_{p,i}$.

5.2 Gitik's one extender-based Prikry forcing

Now we describe Gitik's one-extender-based Prikry forcing and show that it is λ-good. The definitions in the next two pages are taken from §3 of Gitik's handbook article [11].[9] However we keep it minimal as far as it is necessary for our later arguments, for further details regarding this forcing, we refer the readers to Gitik's article.

8 The anonymous reviewer points out that λ-goodness of Gitik's forcing was recently also studied by Dimonte–Wu (see [10, Proposition 4.8]). But necessary details are missing in Dimonte–Wu paper, it is worth to go through here in full details.

9 Some small modifications are made for the sake of the proof of λ-goodness.

Let λ, δ be two cardinals such that δ is a strong limit cardinal above λ and λ is $<\delta$-strong. We assume that GCH holds up to δ. Let η be a cardinal $\geq \lambda^{++}$. Then there is a (λ, η)-extender E and a function $f : \lambda \to \lambda$ such that $j(f)(\eta) = \lambda$, where j is the elementary embedding corresponded to E. For every $\alpha \in [\lambda, \eta)$, define a λ-complete ultrafilter U_α as follows: for $X \subseteq \lambda$,

$$X \in U_\alpha \quad \text{iff} \quad \alpha \in j(X).$$

Clearly, each U_α, $\alpha \in [\lambda, \eta)$, is normal. A relevant property is that they are P-point ultrafilters, i.e. for every $f : \lambda \to \lambda$, if f is not constant modulo U_α, then there is a $Y \in U_\alpha$ such that for every $\nu < \lambda$, $|Y \cap f^{-1}\{\nu\}| < \lambda$.

The binary relation $\leq_E$ defined below is a partial order on $[\lambda, \eta)$:

$$\alpha \leq_E \beta \quad \text{iff} \quad \alpha \leq \beta \wedge j_E(f)(\beta) = \alpha \text{ for some } f : \lambda \to \lambda.$$

$([\lambda, \eta), \leq_E)$ is a λ^{++}-directed and $\lambda \leq_E \alpha$ for every $\alpha \in [\lambda, \eta)$. There is a system of mappings $\pi_{\beta,\alpha} : \lambda \to \lambda$, for $\alpha, \beta \in [\lambda, \eta)$ such that $\alpha \leq_E \beta$, with the following properties:[10]

1. $\langle U_\alpha, \pi_{\beta,\alpha} : \lambda \leq \alpha \leq_E \beta < \eta \rangle$ is a $\leq_{\text{RK}}$-commutative system of λ-complete ultrafilters, i.e.

$$\alpha \leq_E \beta \quad \text{iff} \quad \forall X \subseteq \lambda\, (X \in U_\alpha \leftrightarrow \pi^{-1}_{\beta,\alpha}(X) \in U_\beta).$$

2. There is a set $\bar{X}$ such that $\bar{X} \in U_\alpha$ and $\pi_{\alpha,\alpha} \restriction \bar{X} =$ identity, for every $\alpha \in [\lambda, \eta)$.
3. For every $\alpha, \beta, \gamma \in [\lambda, \eta)$ such that $\gamma \leq_E \beta \leq_E \alpha$, $\pi_{\alpha,\gamma}$ agrees with $\pi_{\alpha,\beta} \circ \pi_{\beta,\gamma}$ on a set $Y \in U_\alpha$.
4. For every $\alpha, \beta, \gamma \in [\lambda, \eta)$, if $\alpha, \beta \leq_E \gamma$ and $\alpha < \beta$, then

$$\{\nu \in \lambda \mid \pi_{\gamma,\alpha}(\nu) < \pi_{\gamma,\beta}(\nu)\} \in U_\gamma.$$

5. For $\alpha, \beta \in [\lambda, \eta)$, if $\alpha \leq_E \beta$, then $\pi_{\beta,\lambda}(\nu) = \pi_{\alpha,\lambda}(\pi_{\beta,\alpha}(\nu))$ for all $\nu \in \lambda$.
6. For every $\alpha, \beta \in [\lambda, \eta)$, $\pi_{\alpha,\lambda}(\nu) = \pi_{\beta,\lambda}(\nu)$ for all $\nu \in \lambda$.

For $\nu \in \bar{X}$, let $\nu^* = \pi_{\alpha,\lambda}(\nu)$ for some (or equivalently, for all) $\alpha \in [\lambda, \eta)$. Then the following *weak normality* holds for U_α, $\alpha \in [\lambda, \eta)$:

7. If $X_i \in U_\alpha$ for $i < \lambda$, then

$$\Delta^*_{i<\lambda} X_i =_{\text{def}} \{\nu \mid \forall i < \nu^*\, (\nu \in X_i)\} \in U_\alpha.$$

We say that a sequence $\langle \nu_i : i \leq n \rangle$, where $n > 0$ and each $\nu_i < \lambda$, is $*$-*increasing* if $\nu_0^* < \nu_1^* < \cdots < \nu_n^*$, and an ordinal $\nu < \lambda$ is *permitted* for $\langle \nu_i : i < k \rangle$ if $\nu^* > \nu_i^*$ for all $i < k$. A very important fact about members of U_α, $\alpha \in [\lambda, \eta)$, is that if $X \in U_\alpha$, then for every $\nu_0, \nu_1 \in X$ such that $\nu_0^* < \nu_1^*$, $|\{\nu \in X \mid \nu^* < \nu_0^*\}| < \nu_1^*$.

[10] These properties and an example of such a system can be found in Gitik [11,12].

Let $(\Xi, \sqsubseteq)$ denote the tree of all finite $*$-increasing sequences of ordinals in λ, ordered by end-extension. Let f be any one of $\pi_{\beta,\alpha}$, $\alpha \leq_E \beta$. By property 5 and 6 on p. 15, f preserves the $*$-value, namely $(f(\nu))^* = \nu^*$ for $\nu \in \lambda$. Thus such f induces a length-preserving homomorphism of Ξ into itself. Abusing the notation, we use f for the induced homomorphism as well. Below is a frequently used fact about these f's:

Fact 51 *Let $f = \pi_{\beta,\alpha}$ for some $\alpha \leq_E \beta$. Suppose $T_\alpha \subseteq \Xi$ is a U_α-tree and $T_\beta \subseteq \Xi$ is a U_β-tree. Then $T_\alpha \cap f``T_\beta$ is a $U_\alpha$-tree and $T_\beta \cap (f^{-1})``T_\alpha$ is a U_β-tree.*

Now we define the extender-based Prikry-like forcing $\mathbb{P}_{\lambda,\eta}$ that changes the cofinality of λ to ω and at the same time adds η many ω-sequences of ordinals that are cofinal in λ.

Definition 7 A condition $p \in \mathbb{P}_{\lambda,\eta}$ is of the form

$$\{\langle \gamma, p^\gamma \rangle \mid \gamma \in g \backslash \{\max(g)\}\} \cup \{\langle \max(g), p^{\max(g)}, T \rangle\},$$

where

1. $g \subset [\lambda, \eta)$ has cardinality $\leq \lambda$, $\lambda \in g$ and g has a $\leq_E$-maximal element. Denote g by $\mathrm{supp}(p)$, $\max(g)$ by $\mathrm{mc}(p)$, T by T^p, and $p^{\max(g)}$ by p^{mc}.
2. $p^\gamma \in \Xi$, for every $\gamma \in g$.
3. $T \subseteq \Xi$ is a subtree with trunk p^{mc}. All splitting nodes of T are required to be in $U_{\mathrm{mc}(p)}$, i.e. for every $t \in T$ such that $t \geq_T p^{\mathrm{mc}}$,
$$\mathrm{succ}_T(t) =_{\mathrm{def}} \{\nu < \lambda \mid \sigma^\frown \nu \in T\} \in U_{\mathrm{mc}(p)},$$
and further that $t_1 \geq_T t_2 \geq_T p^{\mathrm{mc}} \Rightarrow \mathrm{succ}_T(t_1) \subseteq \mathrm{succ}_T(t_2)$.
4. For every $\gamma \in \mathrm{supp}(p) \cap \mathrm{mc}(p)$, $\max(p^{\mathrm{mc}})$ is not permitted for p^γ.
5. For every $\nu \in \mathrm{succ}_T(p^{\mathrm{mc}})$,
$$|\{\gamma \in g \mid \nu \text{ is permitted for } p^\gamma\}| < \nu^*.$$
6. $\pi_{\mathrm{mc}(p),\lambda}(p^{\mathrm{mc}}) = p^\lambda$.[11]

We will only be concerned with subtrees of Ξ such that all its splitting nodes are in the associated ultrafilter as in item 3 above. So when we say a "*tree at* α", we refer to a subtree of Ξ with the property that all its splitting nodes are in U_α.

For a tree T and $\sigma \in T$, let $T_\sigma =_{\mathrm{def}} \{\tau \mid \sigma^\frown \tau \in T\}$. Next we define the binary relation on $\mathbb{P} = \mathbb{P}_{\lambda,\eta}$.

Definition 8 For $p, q \in \mathbb{P}$, let $p \leq_{\mathbb{P}} q$ iff

1. $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$;
2. For every $\gamma \in \mathrm{supp}(q)$, $p^\gamma \sqsupseteq q^\gamma$;

[11] Here it should be "$\pi_{\mathrm{mc}(p),\lambda}``p^{\mathrm{mc}} = p^\lambda$". But as we said earlier, from here on, we abuse the notation, write $\pi_{\beta,\alpha}$'s as functions on Ξ.

3. $p^{\mathrm{mc}(q)} \in T^q$;
4. For every $\gamma \in \mathrm{supp}(q)$,

$$p^\gamma \backslash q^\gamma = \pi_{\mathrm{mc}(q),\gamma}((p^{\mathrm{mc}(q)} \backslash q^{\mathrm{mc}(q)}) \restriction (|p^{\mathrm{mc}(q)}| \backslash (i_\gamma + 1))),$$

   where i_γ is the largest $i < |p^{\mathrm{mc}(q)}|$ such that $p^{\mathrm{mc}(q)}(i)$ is not permitted for q^γ;
5. $\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}$ projects $T^p_{p^{\mathrm{mc}}}$ into $T^q_{p^{\mathrm{mc}(q)}}$, namely $\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}$"$T^p_{p^{\mathrm{mc}}} \subseteq T^q_{p^{\mathrm{mc}(q)}}$;[12]
6. For every $\gamma \in \mathrm{supp}(q)$ and $\nu \in \mathrm{succ}_{T^p}(p^{\mathrm{mc}})$, if ν is permitted for p^γ, then $\pi_{\mathrm{mc}(p),\gamma}(\nu) = \pi_{\mathrm{mc}(q),\gamma}(\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}(\nu))$.

A remark about item 5. Consider $\pi_{\beta,\alpha}$, $\alpha \leq_E \beta$. Note that $\pi_{\beta,\alpha}$ sends members of U_β to members of U_α. So $\pi_{\beta,\alpha}$ projects a subtree at β to a subtree at α.

Let $p, q \in \mathbb{P}_{\lambda,\eta}$, when $p \leq_{\mathbb{P}} q$ and for every $\gamma \in \mathrm{supp}(q)$, $p^\gamma = q^\gamma$, we say p is a *direct extension of* q and write $p \leq^*_{\mathbb{P}} q$. We will omit the subscript $\mathbb{P}$ in these two partial orders when it causes no confusion. Below we summerize the facts about this forcing in Gitik's article [11].

Fact *Let $\mathbb{P} = \mathbb{P}_{\lambda,\eta}$. Then*

1. *$(\mathbb{P}, \leq)$ is a partial order.*
2. *$(\mathbb{P}, \leq)$ satisfies λ^{++}-c.c.*
3. *$(\mathbb{P}, \leq^*)$ is λ-closed.*
4. *$(\mathbb{P}, \leq, \leq^*)$ satisfies Prikry condition: For every $p \in \mathbb{P}$ and for every sentence φ in the forcing language, there is a $q \leq^* p$ such that q decides φ, i.e. either $q \Vdash \varphi$ or $q \Vdash \neg\varphi$.*

Below is the main theorem in §3 of Gitik's handbook article [11],

Theorem *Suppose δ is a strong limit cardinal, $\lambda < \delta$ is $<\delta$-strong and η is a cardinal in $[\lambda^{++}, \delta)$. Let $\mathbb{P} = \mathbb{P}_{\lambda,\eta}$ as defined above and $G \subseteq \mathbb{P}$ be a V-generic filter. Then the following hold in $V[G]$:*

1. *$\mathrm{cof}(\lambda) = \omega$ and $\lambda^\omega \geq \eta$.*
2. *All the cardinals are preserved.*
3. *No new bounded subsets of λ is added.*

5.3 Gitik's forcing is λ-good

To show that $\mathbb{P}$ is λ-good, we follow the idea in §3.5 of [21], define a notion of rank with respect to this forcing. For the rest of the section, we fix some notations. We use U_p, $\pi_{q,p}$ and $\pi_{p,\gamma}$, for $p, q \in \mathbb{P}$ such that $q \leq p$ and $\gamma \in [\lambda, \eta)$ such that $\gamma \leq_E \mathrm{mc}(p)$, to abbreviate for $U_{\mathrm{mc}(p)}$, $\pi_{\mathrm{mc}(q),\mathrm{mc}(p)}$ and $\pi_{\mathrm{mc}(p),\gamma}$, respectively. For

[12] In Gitik's article, it is "$\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}$ projects $T^p_{p^{\mathrm{mc}}}$ into $T^q_{q^{\mathrm{mc}}}$". This should be an error.

$p \in \mathbb{P}$ and $\delta \in \operatorname{succ}_{T^p}(p^{\mathrm{mc}})$, let

$$\begin{aligned} p^- &=_{\mathrm{def}} \{\langle \gamma, p^\gamma \rangle \mid \gamma \in \operatorname{supp}(p) \cap \operatorname{mc}(p)\}, \\ t^p &=_{\mathrm{def}} p^- \cup \{\langle \operatorname{mc}(p), p^{\mathrm{mc}} \rangle\}, \\ (p)_\delta &=_{\mathrm{def}} \{\langle \gamma, (p^\gamma)_{\pi_{p,\gamma}(\delta)} \rangle \mid \gamma \in \operatorname{supp}(p) \cap \operatorname{mc}(p)\} \\ &\qquad \cup \{\langle \operatorname{mc}(p), p^{\mathrm{mc}\frown}\langle\delta\rangle, T^{\mathrm{mc}}_{p^{\mathrm{mc}\frown}\langle\delta\rangle} \rangle\}, \end{aligned}$$

where

$$(p^\gamma)_{\pi_{p,\gamma}(\delta)} = \begin{cases} p^{\gamma\frown}\pi_{p,\gamma}(\delta), & \text{if } \delta \text{ is permitted for } p^\gamma; \\ p^\gamma, & \text{otherwise.} \end{cases}$$

So $p = p^- \cup \{\langle \operatorname{mc}(p), p^{\mathrm{mc}}, T^p \rangle\}$, and using the t^p notation, p can be naturally identified as the pair $(t^p, T^p_{p^{\mathrm{mc}}})$. For a $s \in \Xi_{p^{\mathrm{mc}}}$, $(p)_s$ is recursively defined by $p_\varnothing = p$ and $p_{s\restriction i+1} = (p_{s\restriction i})_{s(i)}$ for $i < |s|$. The $(p)_\delta$, $(p)_s$ notations also make sense when p is of the form t^q for some $q \in \mathbb{P}$.

Definition 9 Suppose $D \subseteq \mathbb{P}$ is open. Define R^D_α on $\{t^p \mid p \in \mathbb{P}\}$ as follows:

- Let $H^D_{<0} = H^D_0 = D$ and $R^D_{<0} = R^D_0 = \{t^p \mid p \in D\}$.
- For $\alpha > 0$, let $H^D_{<\alpha} = \bigcup_{\beta<\alpha} H^D_\beta$ and $R^D_{<\alpha} = \bigcup_{\beta<\alpha} R^D_\beta$.
  - Let H^D_α be the set of $p \in \mathbb{P}$ such that $t^{(p)_\delta} \in R^D_{<\alpha}$ for every $\delta \in \operatorname{succ}_{T^p}(p^{\mathrm{mc}})$.
  - Let R^D_α be the set of t^p for $p \in \mathbb{P}$ such that H^D_α is $(\leq, \leq^*)$-dense below p, i.e. for every $q \leq p$, there is a $r \leq^* q$ in H^D_α.

The following properties follow immediately from the definition.

Proposition 1 *The H^D and R^D-hierarchies have the following properties:*

(i) $\alpha \leq \beta$ *implies that* $H^D_\alpha \subseteq H^D_\beta$ *and* $R^D_\alpha \subseteq R^D_\beta$.
(ii) *If* $H_\alpha = H_{\alpha+1}$, *then for any* $\beta \geq \alpha$, $H_\beta = H_\alpha$ *and* $R_\beta = R_\alpha$.
(iii) $R^D_{<\infty} = R^D_{<|\mathbb{P}|^+}$ *and* $H^D_{<\infty} = H^D_{<|\mathbb{P}|^+}$.
(iv) R^D_α *is open with respect to* $(\mathbb{P}, \leq)$, *i.e. if* $q \leq p$ *and* $t^p \in R^D_\alpha$ *then* $t^q \in R^D_\alpha$.
(v) H^D_α *is* $\leq^*$*-open, i.e. if* $p \in H^D_\alpha$ *and* $q \leq^* p$, *then* $q \in H^D_\alpha$.
(vi) H^D_α "$\subseteq$" R^D_α, *i.e.* $\{t^p \mid p \in H^D_\alpha\} \subseteq R^D_\alpha$.
(vii) *If* $t^p \in R^D_\alpha$ *for some* p, *then there exists* $r \leq^* p$ *with* $t^r = t^p$ *such that* H^D_α *is* $(\leq, \leq^*)$*-dense below* r.

Proof (i) First, as D is open, $H^D_0 \subseteq H^D_1$ and $R^D_0 \subseteq R^D_1$. Note that $R^D_{<\alpha} \subseteq R^D_\alpha$ implies that $H^D_\alpha \subseteq H^D_{\alpha+1}$, and $H^D_\alpha \subseteq H^D_{\alpha+1}$ implies that $R^D_\alpha \subseteq R^D_{\alpha+1}$. Therefore (i) follows by induction.
(ii) This is clear from the definitions of H^D_α and R^D_α.
(iii) This follows immediately from (i) and (ii).
(iv) Suppose $p \in R^D_\alpha$ and $q \leq p$. If H^D_α is $(\leq, \leq^*)$-dense below p, it is also $(\leq, \leq^*)$-dense below q. So $q \in R^D_\alpha$.

(v) The case H_0^D is trivial. Suppose $p \in H_\alpha^D$ and $q \leq^* p$. For every $\zeta \in \text{succ}_{T^q}(q^{\text{mc}})$, $(q)_\zeta \leq^* (p)_{\pi_{q,p}(\zeta)}$. Since $R_{<\alpha}^D$ is open with respect to $(\mathbb{P}, \leq^*)$, $t^{(q)_\zeta} \in R_{<\alpha}^D$. Therefore $q \in H_\alpha^D$.
(vi) Suppose $p \in H_\alpha^D$ and $q \leq p$. Let $r = q$ and $\zeta \in \text{succ}_{T^r}(r^{\text{mc}})$. Then $(r)_\zeta \leq^* (p)_s$ for some $s \in T_{p^{\text{mc}}}^p \backslash \{\varnothing\}$. As $p \in H_\alpha^D$, $t^{(p)_{\min(s)}} \in R_{<\alpha}^D$. By (iv), $t^{(p)_s} \in R_{<\alpha}^D$ and $t^{(r)_\zeta} \in R_{<\alpha}^D$. Therefore, $r \in H_\alpha^D$. So H_α^D is $(\leq, \leq^*)$-dense below p, hence $t^p \in R_\alpha^D$.
(vii) By the definition of R_p^D, there is a $r \leq^* p$ in H_α^D. $r \leq^* p$ implies that $t^r = t^p$. To see that H_α^D is $(\leq, \leq^*)$-dense below r, take any $q \leq r$. Then $q \leq p$. Since H_p^D is $(\leq, \leq^*)$-dense below p, there is a $r' \leq^* q$ in H_α^D. So H_α^D is $(\leq, \leq^*)$-dense below r. □

Definition 10 For $p \in \mathbb{P}$, $\mathring{\rho}_D(t^p)$, the *D-semi-rank of* t^p, is the least ordinal α such that $t^p \in R_\alpha^D$, if it exists; otherwise $\mathring{\rho}_D(t^p) = \infty$.[13] We often write the relativized notation $\mathring{\rho}_{p,D}(s)$, in which case is called (p, D)-semi-rank of s, to abbreviate for $\mathring{\rho}_D(t^{(p)_s})$, for $s \in T_{p^{\text{mc}}}^p$, although its value only depends on t^p.

Here are some quick facts about semi-ranks.

Proposition 2 *Suppose $D \subseteq \mathbb{P}$ is open and $p, q \in \mathbb{P}$.*

(i) If $\mathring{\rho}_D(t^p) < \infty$, then $\mathring{\rho}_D(t^p) < |\mathbb{P}|^+$.
(ii) If $\mathring{\rho}_D(t^p) < \infty$ and $q \leq p$, then $\mathring{\rho}_D(t^q) \leq \mathring{\rho}_D(t^p)$.

Proof (i) This is immediate from Proposition 1-(iii).
(ii) If $q \leq p$ and $\mathring{\rho}_D(t^p) < \infty$, then by Proposition 1-(iv),

$$\varnothing \neq \{\alpha \in \text{Ord} \mid t^p \in R_\alpha^D\} \subseteq \{\alpha \in \text{Ord} \mid t^q \in R_\alpha^D\}.$$

Thus $\mathring{\rho}_D(t^q) \leq \mathring{\rho}_D(t^p)$. □

Definition 11 Suppose $D \subseteq \mathbb{P}$ is open and $p \in \mathbb{P}$. We say that p is *D-good* if $p \in H_\alpha^D$ and for every $s \in T_{p^{\text{mc}}}^p$ and for $\beta \leq \alpha$,

$$(p)_s \in H_\beta^D \implies (p)_{s^\frown\langle\delta\rangle} \in H_{<\beta}^D, \text{ for all } \delta \in \text{succ}_{T_{p^{\text{mc}}}^p}(s).$$

Clearly if p is D-good, then so is $(p)_s$ for every $s \in T_{p^{\text{mc}}}^p$.

Proposition 3 *Suppose $D \subseteq \mathbb{P}$ is open. Let $E_D =_{\text{def}} \{p \in \mathbb{P} \mid p \text{ is } D\text{-good}\}$. Then E_D is $\leq^*$-dense below any p with $\mathring{\rho}_D(t^p) < \infty$; or equivalently, for every p such that $\mathring{\rho}_D(t^p) < \infty$, there is a $q \leq^* p$ in E_D.*

Proof Take an $N \prec V_\mu$ for a sufficiently large μ and such that $|N| = \lambda^+$, $N^\lambda \subseteq N$. Let $\kappa < \eta$ be an ordinal such that $\kappa \geq_E \zeta$ for all $\zeta \in N \cap [\lambda, \eta)$. We write $R_\alpha^{D,N}$ and $H_\alpha^{D,N}$ for the corresponding notions defined in N, and write $\mathring{\rho}_D^N(t^p)$ and $\mathring{\rho}_{p,D}^N(s)$,[14]

[13] We demand that $\infty > \alpha$ for all $\alpha \in \text{Ord}$.

[14] More precisely, should be $\mathring{\rho}_{D\cap N}^N(t^p)$ and $\mathring{\rho}_{p,D\cap N}^N(s)$.

$s \in T^p_{p^{\mathrm{mc}}}$, for the corresponding notions computed in N. By the elementarity of N, these notions are absolute between N and V, more precisely, $R^{D,N}_\alpha = R^D_\alpha \cap N$, $H^{D,N}_\alpha = H^D_\alpha \cap N$ for $\alpha \in \mathrm{Ord} \cap N$, and $\mathring{\rho}^N_D(t^p) = \mathring{\rho}_D(t^p)$ for $p \in \mathbb{P} \cap N$. Proposition 3 follows from the following lemma.

Lemma 1 *Suppose $p \in \mathbb{P} \cap N$ and T is a U_κ-tree with trunk s_κ and such that $t^p \cup \{\langle \kappa, s_\kappa, T\rangle\} \leq^* p$. Suppose $\mathring{\rho}^N_D(t^p) < \infty$. Then there are a $q \in N$ and a U_κ-subtree $T^r \subseteq T$ such that $r = q \cup \{\langle \kappa, s_\kappa, T^r\rangle\}$ is a D-good direct extension of p.*

Grant Lemma 1. Suppose $p \in N$ and $\mathring{\rho}^N_D(t^p) < \infty$. By Lemma 1, there is a $q \in V$ that is D-good and directly extends p. Since $\mathring{\rho}_D(\cdot)$ is absolute between N and V, for every $p \in \mathbb{P} \cap N$ with $\mathring{\rho}_D(t^p) < \infty$, there is a D-good direct extension of p in V. By elementarity, for every $p \in \mathbb{P} \cap N$ with $\mathring{\rho}^N_D(t^p) < \infty$, there is a D-good direct extension of p in N. Using elementarity again, every $p \in \mathbb{P}$ in V with $\mathring{\rho}_D(t^p) < \infty$ has a D-good direct extension. Thus the set E_D is $\leq^*$-dense below p. □

Now we prove Lemma 1.

Proof of Lemma 1 The proof proceeds by induction on $\alpha = \mathring{\rho}^N_D(t^p)$ in N. For $\alpha = 0$, it is trivial. We follow the idea in Gitik's proof of his Lemma 3.12 in [11, p. 1387]. Assume that for all $\beta \in \alpha \cap N$, the claim holds.

Assume $p \in \mathbb{P}$ and $t^p \in R^D_\alpha \cap N$. By Proposition 1-(viii), we may replace p with a $p' \leq^* p$ in N with least $\alpha \leq \mathring{\rho}^N_D(t^p)$ in N such that $p' \in H^D_\alpha \cap N$. As H^D_α is $\leq^*$-open, we may in addition assume that $\mathring{\rho}^N_D(t^q) = \mathring{\rho}^N_D(t^p) = \alpha$ for any $q \leq^* p$ in N. Thus, by elementarity, for any $q \leq^* p$, $\mathring{\rho}_D(t^q) = \mathring{\rho}_D(t^p) = \alpha$. Let $A = \mathrm{succ}_T(s_\kappa)$. We shall construct inductively $\langle (p_\xi, T^\xi) : \xi \in A\rangle$. To simplify the presentation, we may assume that $p^- = \varnothing$ and $s_\kappa = \varnothing$.

Suppose we already have $\langle (p_\delta, T^\delta) : \delta \in A \cap \zeta\rangle$. Now we construct p_ζ and T^ζ. Let $p'_\zeta = p \cup (\bigcup\{p_\delta \mid \delta \in A \cap \zeta\})$ and

$$r'_\zeta = t^{p'_\zeta} \cup \{\langle \kappa, \varnothing, \bigcup\{T_{\langle \xi\rangle} \mid \xi \in A \setminus \zeta\}\rangle\}.$$

A little calculation (see the proof of Claim 4.9 in [10]) shows that $(r'_\zeta)_\zeta \leq^* (p)_{\pi_{p,\kappa}(\zeta)}$. As $t^{(p)_{\pi_{p,\kappa}(\zeta)}} \in R^{D,N}_\beta$ for some $\beta \in \alpha \cap N$, $\mathring{\rho}^N_D(t^{(r'_\zeta)_\zeta}) \leq \beta$, by the inductive hypothesis, there are a $q \in N$ and a U_κ-subtree $T_\zeta \subseteq T_{\langle \zeta\rangle}$ such that $q \cup \{\langle \kappa, \langle \zeta\rangle, T_\zeta\rangle\}$ is a D-good direct extension of $(r'_\zeta)_\zeta$. Let

$$p_\zeta = p'_\zeta \cup \{\langle \iota, q^\iota\rangle \mid \iota \in \mathrm{supp}(q) \setminus \mathrm{supp}(r'_\zeta)\}.$$

This completes the inductive construction.

At the end, let $q = \bigcup_{\xi < \lambda} p_\xi$. For $i < \lambda$, let

$$C_i = \begin{cases} \bigcap\{\mathrm{succ}_{T^\xi}(\langle \xi\rangle) \mid \xi \in A \wedge \xi^* = i\}, & \text{if } \exists \xi \in A\, (\xi^* = i); \\ A, & \text{otherwise.} \end{cases}$$

Since the set of $\xi \in A$ such that $\xi^* = i$ is bounded, $C_i \in U_\kappa$ for every $i < \lambda$. Set $A^* = A \cap (\Delta^*_{i<\lambda} C_i)$. By the weak normality for U_κ, $A^* \in U_\kappa$. Let T^r be the tree obtained from $\bigcup\{T_\xi \mid \xi \in A^*\}$ by intersecting all its levels with A^*. Then by Claim 3.12.1 in Gitik's [11, p. 1388], $r = q \cup \{\langle \kappa, \varnothing, T^r\rangle\}$ is in $\mathbb{P}$ and directly extends p.

By our construction, for each $\zeta \in A^*$, $T^{(r)_\zeta}$ is a U_κ-subtree of T_ζ, so $(r)_\zeta$ is D-good and directly extends $(p)_{\pi_{p,\kappa}(\zeta)}$. It suffices to check that α is least such that $(r)_\varnothing = r \in H^D_\alpha$. As R^D_α is open with respect to $(\mathbb{P}, \leq)$, $t^{(p)_{\pi_{p,\kappa}(\zeta)}} \in R^D_{<\alpha}$ implies that $t^{(r)_\zeta} \in R^D_{<\alpha}$. So $r \in H^D_\alpha$. By our additional assumption on p, $\mathring{\rho}_D(t^r) = \mathring{\rho}_D(t^p) = \alpha$. This means that $r \notin H^D_{<\alpha}$. So r is D-good.

The Prikry condition for $\mathbb{P}$ (see Lemma 3.12, [11]) can be stated in terms of our semi-rank notion as follows.

Proposition 4 (Gitik) *Suppose $D \subseteq \mathbb{P}$ is dense and open. Let $\mathbb{1}_\mathbb{P}$ denote the largest element of $\mathbb{P}$. Then $\mathring{\rho}_D(t^{\mathbb{1}_\mathbb{P}}) < \infty$; or equivalently, for every $p \in \mathbb{P}$, there is a $q \leq^* p$ in $H^D_{<\infty}$.*

Proof Rerun Gitik's proof but with "p decides σ" replaced by "$p \in H^D_{<\infty}$". □

Next we define a notion of rank on members of E_D to isolate a set of "D-better" conditions. For every $p \in E_D$, we define a rank function $\rho_{p,D}(\cdot)$ on $T^p_{p^{\mathrm{mc}}}$ inductively as follows:

- if $(p)_s \in D$, then $\rho_{p,D}(s) = 0$;
- if $(p)_s \notin D$, then $\rho_{p,D}(s)$ is the least α such that there is a U_p-measure one $A \subseteq \mathrm{succ}_{T^p_{p^{\mathrm{mc}}}}(s)$ such that $\alpha \geq \rho_{p,D}(s^\frown\langle\delta\rangle) + 1$ for all $\delta \in A$.

By the definition of D-goodness, if $p \in E_D$, then the set $\{s \in T^p_{p^{\mathrm{mc}}} \mid \mathring{\rho}_{p,D}(s) > 0\}$ is a wellfounded subtree of $T^p_{p^{\mathrm{mc}}}$. Thus $\rho_{p,D}(s)$ is defined for all $s \in T^p_{p^{\mathrm{mc}}}$ if $p \in E_D$.

Below is a simple observation to be used in our proof of λ-goodness for $\mathbb{P}$.

Proposition 5 *Suppose $D \subseteq \mathbb{P}$ is open and $p \in E_D$. If $q \leq p$, then for every $s \in T^q_{q^{\mathrm{mc}}}$, $\mathring{\rho}_D(t^{(q)_s}) \leq \mathring{\rho}_D(t^{(p)_{\pi_{q,p}(q^{\mathrm{mc}\frown} s)}})$ and $\rho_{q,D}(s) \leq \rho_{p,D}(\pi_{q,p}(q^{\mathrm{mc}\frown} s))$.*

Proof It suffices to consider only the case $q \leq^* p$. The proof proceeds by induction on $\rho_{q,D}(s)$. We leave the details to the readers. □

Lemma 2 *Suppose $D \subseteq \mathbb{P}$ is open and $p \in E_D$. Then $\rho_{p,D}(\varnothing) < \omega$. More precisely, there is a U_p-subtree $S_p \subseteq T^p_{p^{\mathrm{mc}}}$ such that for every $s \in S_p$, $\rho_{p,D}(s) =$* $\max(\rho_{p,D}(\varnothing) - |s|, 0)$.

Proof Clearly, the range of $\rho_{p,D}(\cdot)$ is an ordinal. The lemma follows from the fact that U_p is countably complete. Assume towards a contradiction that $\rho_{p,D}(\varnothing) \geq \omega$, then there is an $s \in T^p_{p^{\mathrm{mc}}}$ such that $\rho_{p,D}(s) = \omega$. But due to the countably completeness of U_p, there is a finite number k such that $\rho_{p,D}(s^\frown\langle\delta\rangle) < k$ for a U_p-measure one set of $\delta \in \mathrm{succ}_{T^p_{p^{\mathrm{mc}}}}(s)$. Therefore $\rho_{p,D}(s) \leq k < \omega$. Contradiction!

Using the idea in §3.4 of [21], by trimming off nodes s in $T^p_{p^{\mathrm{mc}}} \setminus \{\varnothing\}$ such that $\rho_{p,D}(s) \geq \rho_{p,D}(s \upharpoonright (|s| - 1)) > 0$, one obtain a U_p-subtree $S_p \subseteq T^p_{p^{\mathrm{mc}}}$ such that for every $s \in S_p$, either $\rho_{p,D}(s) = 0$ or $\rho_{p,D}(s) = \sup\{\rho_{p,D}(s^\frown\langle\delta\rangle) + 1 \mid \delta \in \mathrm{succ}_{S_p}(s)\}$. It is easy to see that this S_p works as desired. □

Definition 12 Suppose $D \subseteq \mathbb{P}$ is open. For a $p \in E_D$, we say p is *D-better* if $T^p_{p^{\mathrm{mc}}}$ satisfies the condition that for every $s \in T^p_{p^{\mathrm{mc}}}$, $\rho_{p,D}(s) = \max\{\rho_{p,D}(\emptyset) - |s|, 0\}$.

Let $B_D =_{\mathrm{def}} \{p \in E_D \mid p \text{ is } D\text{-better}\}$. From Lemma 2, we have

Corollary 7 *B_D is $\leq^*$-dense in E_D, therefore $\leq^*$-dense in $\mathbb{P}$.*

Now we are ready to prove the main result of this section.

Lemma 3 *$\mathbb{P}$ is λ-good.*

Proof Fix a $p \in \mathbb{P}$ and $\mathscr{D}$, a collection of dense open subsets of $\mathbb{P}$ with $|\mathscr{D}| < \lambda$. Enumerate $\mathscr{D}$ as $\{D_\iota \mid \iota < |\mathscr{D}|\}$. Start with p, we inductively construct a $\leq^*$-decreasing sequence $\langle p_\iota : \iota < |\mathscr{D}|\rangle$ and a sequence of integers $\langle k_\iota : \iota < |\mathscr{D}|\rangle$ as follows:

First, let p_0 be a D_0-better direct extension of p and $k_0 = \rho_{p_0,D}(\emptyset)$. Suppose we have constructed the two sequences up to some $\iota > 0$, i.e. $\langle p_\zeta : \zeta < \iota\rangle$ and $\langle k_\zeta : \zeta < \iota\rangle$. Since $(\mathbb{P}, \leq^*)$ is λ-closed, there is a $q_\iota \in \mathbb{P}$ such that $q_\iota \leq^* p_\zeta$ for all $\zeta < \iota$. Let p_ι be a D_ι-better direct extension of q_ι and $k_\iota = \rho_{p_\iota,D_\iota}(\emptyset)$.

At the end, pick a $p^\circ \in \mathbb{P}$ such that $p^\circ \leq^* p_\iota$ for all $\iota < |\mathscr{D}|$. For each $k < \omega$, let $\mathscr{D}_{p,k} = \{D_\iota \mid k_\iota \leq k\}$. We may assume that $\mathscr{D}_{p,k} \neq \emptyset$ for all $i < \omega$. We claim that $\bigcap \mathscr{D}_{p,k}$ is dense below p° for all $k < \omega$.

Fix a $k < \omega$. Suppose $r \leq p^\circ$. We are going to prove that for any sufficiently long s, $(r)_s \in D$ for every $D \in \mathscr{D}_{p,k}$. Note that by Proposition 5 for any $s \in T^r_{r^{\mathrm{mc}}}$ and any $\xi < |\mathscr{D}|$ such that $D_\xi \in \mathscr{D}_{p,k}$, $\mathring{\rho}_D(t^{(r)_s}) \leq \mathring{\rho}_D(t^{(p_\xi)_{\pi_{r,p_\xi}(r^{\mathrm{mc}} \frown s)}})$ and $\rho_{r,D_\xi}(s) \leq \rho_{p_\xi,D_\xi}(\pi_{r,p_\xi}(s)) = k - |s|$. Pick an $s \in T^r_{r^{\mathrm{mc}}}$ such that $|s| \geq k$, then $\rho_{r,D}(s) = 0$ for every $D \in \mathscr{D}_{p,k}$. Hence, $(r)_s \in D$ for every $D \in \mathscr{D}_{p,k}$. $(r)_s \leq r$, so this shows that $\bigcap \mathscr{D}_{p,k}$ is dense below p°. $\square$

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