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Synthesis of a linkage to draw a plane algebraic curve

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ABSTRACT

This paper presents a synthesis methodology for a planar linkage that draws a given plane algebraic curve. The existence of such a linkage was demonstrated by A. B. Kempe, who used by specialized linkages, known as additor, mutiplicator and translator linkages, to constrain the two joints of a planar two-link serial chain so that its end point traced the desired curve. Recent research has verified Kempe's results, but yield linkages that are exceedingly complex. In this paper, we replace Kempe's specialized linkages with differentials and cable drives that perform the same functions but simplify the linkage. The result is a construction of Kempe's drawing linkage that illustrates the underlying structure of his approach. Examples are provided that illustrate the theory.

1. Introduction

In 1887 Kempe [1] presented a procedure that begins with the equation of a plane algebraic curve and yields a planar linkage that draws this curve. To do this he introduced four calculating linkages, the reversor, the multiplicator, the additor and the translator, that he used to construct mechanical constraints on the two angles of a planar RR serial chain so that it draws the curve—R denotes a revolute, or hinged, joint.

Kempe recognized that his existence proof yielded complex linkages. He stated, “It is hardly necessary to add, that this method would not be practically useful on account of the complexity of the linkwork…”. However, he posed an interesting question, asking the “mathematical artist to discover the simplest linkworks that will describe particular curves”. This search for the simplest linkage is more easily achieved on a case by case basis, as is found in Artobolevskii [2], who provides a large number of specialized linkages adapted to the geometry of specific classes of algebraic plane curves.

Recent work by mathematicians, Jordan and Steiner [3] and Kapovich and Millson [4] provide a modern proof to Kempe's result, which they call *Kempe's Universality Theorem*. These results expose a direct mathematical connection between algebraic curves and linkages. This connection may obscured by the elementary way the mechanical calculations are implemented in Kempe's method, which results in complex linkage systems.

The goal in this paper is to maintain Kempe's general procedure but simplify the resulting linkages by replacing his four computing linkages with differentials and cable drives. The differential performs addition and the cable drives are configured to reverse, multiply and translate angular values. The result is a set of constraints on the angles of an RR serial chain that cause it draws a given algebraic curve. The resulting linkages are simpler and more clearly illustrate Kempe's approach. However, they remain complex devices that deserve more attention to simplify their construction.

Practical applications for a mechanical system that replaces what is currently done by electrical systems arise in circumstances where size is an issue such as MEMS or nano applications, and where electric power is unavailable or undesirable. Our current focus

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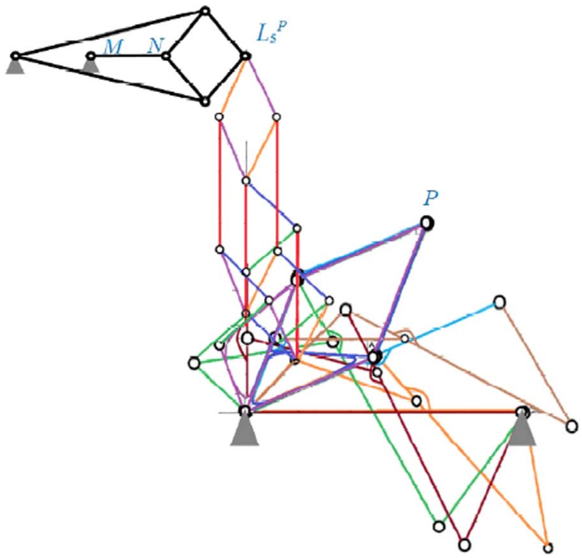


Fig. 1. The linkage obtained by Saxena [9] follows Kempe's procedure to obtain a linkage that moves the point *P* to draw a curve defined by the product of two straight lines. It consists of 48 links and 70 joints.

is on reducing the complexity of this important theoretical result, we look forward to working on applications in the near future.

Examples illustrate the procedure and compare the resulting linkages to previous work.

2. Literature review

The early work on linkages that draw curves is described in Nolle [5,6] and Koetsier [7,8]. In 1876 Kempe showed that a drawing linkage for an arbitrary algebraic curve could be constructed using computing linkage to calculate the angular values of a planar RR serial chain, which he presented as a parallelogram. Saxena [9] presents a step-by-step construction of a drawing linkage, using Kempe's method, for a quadratic curve defined by the product of two lines, and obtains the linkage shown in Fig. 1 that has 48 links and 70 joints.

The difference between Kempe's and Artobolevskii's approach can be seen by comparing Saxena's linkage for a quadratic curve to Artobolevskii's eight-bar conograph linkage which can be adapted to draw any quadratic curve, see Fig. 2.

Recent work by Gao et al. [10] and Abbott [11] generalized Kempe's methodology and showed that a curve of degree *n* requires at

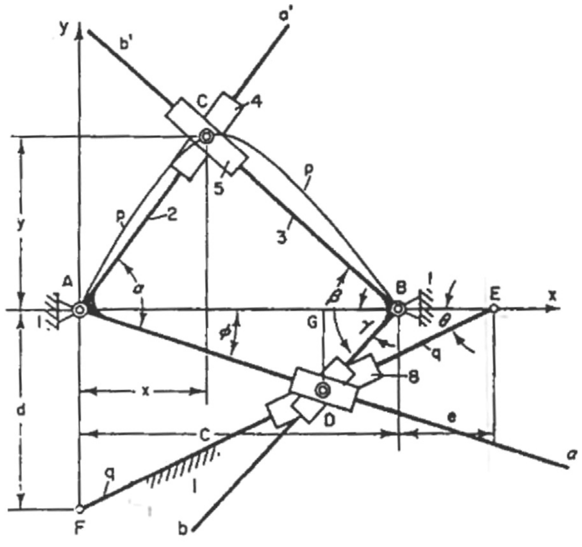


FIG. 136

Fig. 2. Artobolevskii shows that a linkage to draw any planar quadratic curve can be obtained by adjusting the dimensions of this linkage consisting of eight bar and 10 joint, known as a *conograph*. This is simpler than the linkage obtained by Kempe's general method.

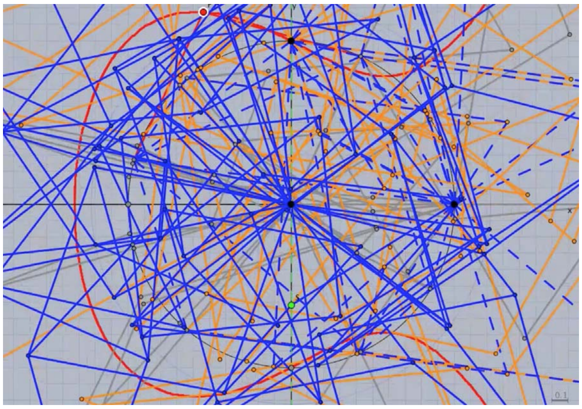


Fig. 3. This linkage that draws an elliptic cubic curve was obtained by Kobel [12] using the dynamic geometry system *Cinderella*. The cubic curve is in the background covered by many linkage elements that guide a point seen near the top of the figure along the curve.

least $O(n^2)$ bars. Kobel [12] used the dynamic geometry system *Cinderella* to construct a number of drawing linkages. See Fig. 3, which is the drawing linkage for a elliptic cubic curve. In what follows, we obtain a drawing linkage for this case and find that the complexity of this figure results primarily from the translator linkages needed to assemble the results.

A different approach to curve-drawing linkages was introduced by Roth and Freudenstein [13], who formulated equations for the dimensions of a four-bar linkage that interpolated a set of nine accuracy points along the designed curve. Also see Ramakrishna and Sen [14], and Bai and Angeles [15].

Wampler et al. [16] obtained the complete solution for four-bar path generation problem and showed that there are as many as 4326 distinct solutions for a prescribed set of nine points on a curve. This point path generation technique was extend by Kim, et al. [17] to six-bar mechanism design to guide the coupler going though 15 accurate points. Recent research by Plecnik [18,19] shows that the equations for 15 precision points six-bar linkage path generation has a Bezout over 10^{46}, which means finding all the six-bar linkages for a given set 15 points is beyond our current computation capabilities.

3. Construction of a drawing linkage

In this section, we introduce Kempe's method to construct a linkage system that draws a given algebraic plane curve. Then, rather than the additor, reversor, multiplicator and translator linkages, that he used to add, negate, multiply and translate angular values, we introduce bevel gear differentials and cable drives, Fig. 4, to accomplish the same operations. In what follows, we provide a detailed construction of the linkage systems that generate each of the two lines of Saxena's example, shown in Fig. 1, so the two versions of the linkage can be compared. We then construct the linkage that draws the elliptic cubic of Kobel's example as shown in Fig. 3.

These examples show that Kempe constructs a constraining linkage and intervening computing linkages in order to coordinate angles of an RR chain so that it draws the desired curve. In our approach, we use differentials and cable drives as the intervening computing linkages to coordinate the angles of the constraining linkage and RR chain.

Consider the plane algebraic curve C be defined as the points (x, y) that are zeros of the polynomial equation,

$$f(x, y) = \sum b_{jk} x^j y^k = 0, \quad (1)$$

where the summation is over non-negative integers j and k with $j + k \le n$, where n is the degree of the polynomial.

Now introduce a planar *RR* chain with link lengths L_1 and L_2, which will draw this curve with its end-point $\mathbf{P} = (x, y)$. The coordinates of $\mathbf{P}$ are related to the joint angles θ and ϕ of the *RR* chain by the equation,

$$\mathbf{P} = \begin{Bmatrix} x(\theta, \phi) \\ y(\theta, \phi) \end{Bmatrix} = \begin{Bmatrix} L_1 \cos\theta + L_2 \cos\phi \\ L_1 \sin\theta + L_2 \sin\phi \end{Bmatrix}. \quad (2)$$

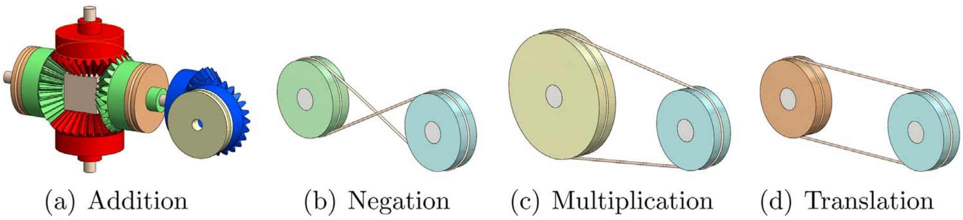


Fig. 4. A bevel gear differential and configurations of cable drives perform the same functions as Kempe's additor, reversor, multiplicator and translator linkages, and simplify the resulting device.

Substitute this equation into $f(x, y) = 0$ to obtain,

$$f(x(\theta, \phi), y(\theta, \phi)) = \sum b_{jk}(L_1 \cos\theta + L_2 \cos\phi)^j (L_1 \sin\theta + L_2 \sin\phi)^k = 0. \tag{3}$$

Expand this equation and use trigonometric identities to obtain an equation of the form,

$$f(\theta, \phi) = \sum_{i=1}^{n} A_i \cos(r_i\phi + s_i\theta + \beta_i) - K = 0. \tag{4}$$

where $\beta_i = 0$ or $\pi/2$, and the A_i and K are real constants. This form of the algebraic curve was obtained by Kempe [1].

Kempe recognized that (4) can be viewed as the x'-projection of a serial chain constructed from n links of length A_i, $i = 1, \ldots n$ with n joints at angles $\alpha_i = r_i\phi + s_i\theta + \beta_i$, $i = 1, \ldots, n$ relative to the x'-axis, given by the equation

$$x' = \sum_{i=1}^{n} A_i \cos\alpha_i = K, \tag{5}$$

We call this Kempe's constraining linkage, because the values of the angles α_i are mechanically calculated from θ and ϕ using differentials and cable drives, Fig. 4, so that the RR chain draws the curve. The movement of the end-point of this serial chain along the line $x' = K$ can be viewed as the input to the system. Kempe achieved this constraint by using a Peaucilier linkage, which replace with a prismatic, or sliding, joint.

In what follows we demonstrate this procedure and obtain linkages that reproduce Saxena's and Kobel's examples.

3.1. Drawing linkage for saxena's example

In this section, we use Saxena's [9] example to illustrate our version of Kempe's procedure for constructing curve-drawing linkages. The linkage shown in Fig. 1 draws the quadratic curve generated by the product of two lines given by,

$$f(x, y) = (x - y)(x + y + \sqrt{2}/2) = 0. \tag{6}$$

In what follows, we design a separate linkage for each of these two lines.

3.1.1. Line: $x - y = 0$

Here we show the process of designing one degree of freedom linkage to draw the line defined by

$$g(x, y) = x - y = 0. \tag{7}$$

Select the lengths of the RR chain to be $L_1 = L_2 = 1$. In this case, the trajectory of the end-point $\mathbf{P}$ is given by,

$$\mathbf{P} = \begin{Bmatrix} x(\theta, \phi) \\ y(\theta, \phi) \end{Bmatrix} = \begin{Bmatrix} \cos\theta + \cos\phi \\ \sin\theta + \sin\phi \end{Bmatrix}. \tag{8}$$

Substitute (8) into (7) to obtain,

$$g(\theta, \phi) = \cos\theta + \cos\phi + \cos(\theta + \pi/2) + \cos(\phi + \pi/2) = 0. \tag{9}$$

Kempe interpreted this equation as the x' projection of a simple closed chain consisting of four revolute joints with the end-link constrained by a prismatic joint that slides along the line $x' = K$, that is

$$x' = A_1 \cos\alpha_1 + A_2 \cos\alpha_2 + A_3 \cos\alpha_3 + A_4 \cos\alpha_4 = K, \tag{10}$$

where A_i are the lengths of the individual links, α_i are the angles of these links relative to the x'-axis. Each of the angles α_i is given by

$$\alpha_i = \Delta\alpha_i + \alpha_{i0}, \tag{11}$$

where α_{iO} define the initial configuration of the constraining linkage.

In order to determine the angles α_{iO}, select a point $\mathbf{P}$ on the line $g(x, y)$ that is to be the initial configuration of the system. Compute the initial values θ_0 and ϕ_0 for the joint angles of the RR chain, so that

$$\theta = \Delta\theta + \theta_0, \quad \phi = \Delta\phi + \phi_0. \tag{12}$$

Then the constraint Eq. (10) takes the form,

$$g(\theta, \phi) = \cos(\Delta\theta + \theta_0) + \cos(\Delta\phi + \phi_0) + \cos(\Delta\theta + \theta_0 + \pi/2) + \cos(\Delta\phi + \phi_0 + \pi/2) = 0. \tag{13}$$

For this example, we choose,

$$\mathbf{P} = \begin{Bmatrix} \cos\theta_0 + \cos\phi_0 \\ \sin\theta_0 + \sin\phi_0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \tag{14}$$

and solve to obtain two configurations,

Table 1
Dimensions of Kempe's constraining linkage for the line $x - y = 0$.

Link, i	Link Length, A_i	Angle, $\Delta\alpha_i$	Initial config., α_{iO}
1	1	$\Delta\theta$	0
2	1	$\Delta\phi$	$\pi/2$
3	1	$\Delta\theta$	$\pi/2$
4	1	$\Delta\phi$	π

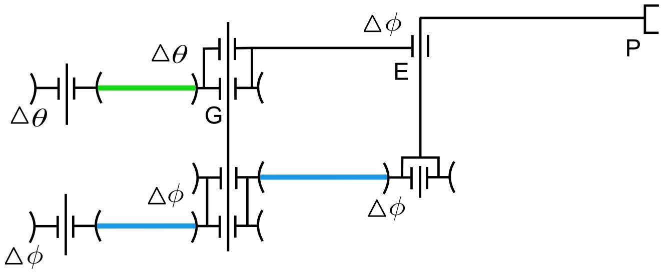


Fig. 5. The green cable translates angle $\Delta\theta$ to link GE and the two blue cables translate angle $\Delta\phi$ to link EP. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

$$\begin{Bmatrix}\theta_0\\ \phi_0\end{Bmatrix} = \begin{Bmatrix}\pi/2\\ 0\end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix}\theta_0\\ \phi_0\end{Bmatrix} = \begin{Bmatrix}0\\ \pi/2\end{Bmatrix}. \tag{15}$$

These solutions define the "elbow-up" and "elbow-down" configurations of the *RR* chain.

Select the elbow down solution and set $\Delta\theta = \Delta\phi = 0$, to obtain the initial angles of the constraining linkage,

$$\alpha_{10} = \theta_0, \quad \alpha_{20} = \phi_0, \quad \alpha_{30} = \theta_0 + \frac{\pi}{2}, \quad \text{and} \quad \alpha_{40} = \phi_0 + \frac{\pi}{2}. \tag{16}$$

Thus, the dimensions of Kempe's constraining linkage are obtained by equating (10) and (13). The results are listed in Table 1.

Cable drives are used to connect the angles $\Delta\theta$ and $\Delta\phi$ to drive the joint angles θ and ϕ of the RR chain, see Fig. 5. The cables connecting $\Delta\alpha_i$ to the joint angles α_i of the constraining linkage are routed in the same way.

The movement of the end-link of the constraining linkage along the line $x' = 0$ generates the curve, see Fig. 6. A construction of a linkage that draws this straight line using Kempe's approach can be found in [12].

3.1.2. Line: $x + y + \sqrt{2}/2 = 0$

In order to define a one degree of freedom drawing linkage for the line,

$$h(x, y) = x + y + \sqrt{2}/2 = 0, \tag{17}$$

substitute (8) into this equation to obtain the constraint equation,

$$h(\theta, \phi) = \cos\theta + \cos\phi + \cos(\theta - \pi/2) + \cos(\phi - \pi/2) = -\sqrt{2}/2. \tag{18}$$

This defines the constraining linkage with dimensions listed in Table 2 and illustrated in Fig. 7.

An initial configuration of the constraining linkage is obtained by setting (8) equal to the point $\mathbf{P} = (-\sqrt{2}/4, -\sqrt{2}/4)$ on the line $h(x, y)$ to determine the angles, θ_0 and ϕ_0. Solve these equations to obtain values for angles α_{iO} for Kempe's constraining linkage listed in Table 2.

The differences between the drawing linkages for the two different lines in Saxena's example consists of the calculation of the angles α_i from θ and ϕ, the value $x' = K$, and the initial configuration.

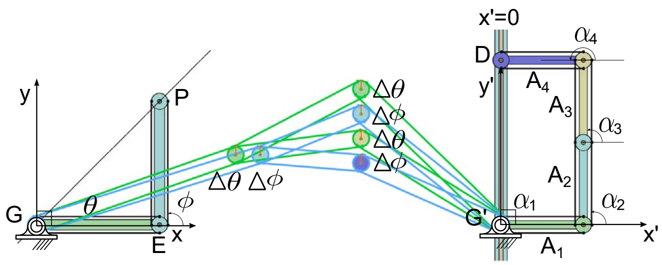


Fig. 6. The end-point of the linkage $A_1A_2A_3A_4$ slides along the vertical guide to impose constraints, through cable drives, on the angles of the RR chain so that the point $\mathbf{P}$ follows the line $x - y = 0$.

Table 2
Dimensions of Kempe's constraining linkage for the line $x + y + \sqrt{2}/2 = 0$.

Link, i	Link length, A_i	Link angle, $\Delta\alpha_i$	Initial config., $\alpha_i{}^0$
1	1	$\Delta\theta$	$-5\pi/12$
2	1	$\Delta\phi$	$11\pi/12$
3	1	$\Delta\theta$	$-11\pi/12$
4	1	$\Delta\phi$	$5\pi/12$

3.2. Drawing linkage for Kobel's example

In this section, we follow the same procedure presented above to obtain the one degree of freedom drawing linkage system for the elliptic cubic curve given by

$$f(x, y) = x^3 - y^2 - x + 1 = 0. \tag{19}$$

(Fig. 8)This is the curve traced by Kempe's linkage as shown in Fig. 3 obtained by Kobel.

Introduce the *RR* chain with link lengths, $L_1 = L_2 = 1$ and substitute the equation for the trajectory of its end-point $\mathbf{P}(\theta, \phi)$ into (19) to obtain the constraint equation,

$$f(\theta, \phi) = \cos^3\theta + \cos^3\phi + 3\cos^2\theta\cos\phi + 3\cos^2\phi\cos\theta + \cos^2\theta + \cos^2\phi - 2\sin\theta\sin\phi - \cos\theta - \cos\phi - 1 = 0. \tag{20}$$

Eliminate the powers of cosine using the identities,

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}, \quad \cos^3\theta = \frac{3\cos\theta + \cos(3\theta)}{4}. \tag{21}$$

And use the trigonometric sum and difference identities to obtain the constraint equation,

$$\begin{aligned} f(\theta, \phi) = {} & 1.25\cos\theta + 1.25\cos\phi + 0.50\cos 2\theta + 0.50\cos 2\phi + 0.25\cos 3\theta + 0.25\cos 3\phi + \cos(\theta - \phi + \pi) + \cos(\theta + \phi) \\ & + 0.75\cos(\theta - 2\phi) + 0.75\cos(\theta + 2\phi) + 0.75\cos(2\theta - \phi) + 0.75\cos(2\theta + \phi) = 0. \end{aligned} \tag{22}$$

This equation is used to define Kempe's constraining linkage which has the x component defined by,

$$x' = \sum_{i=1}^{n} A_i\cos\alpha_i = K, \tag{23}$$

where A_i are the link lengths, the angles $\alpha_i = \Delta\alpha_i + \alpha_{i0}$ are measured relative to the x'-axis, and $x' = K$ locates the slider that guides the end-point.

In order to determine the initial configuration of the system, set $\mathbf{P} = (1, 1)$ and solve the equations,

$$\mathbf{P} = \begin{Bmatrix} \cos\theta_0 + \cos\phi_0 \\ \sin\theta_0 + \sin\phi_0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}. \tag{24}$$

Select the elbow down solution to obtain

$$\theta_0 = 0, \quad \phi_0 = \pi/2. \tag{25}$$

Substitute these initial angles into the constraint equation (22) to obtain,

$$\begin{aligned} f(\theta, \phi) = {} & 1.25\cos\Delta\theta + 1.25\cos(\Delta\phi + \pi/2) + 0.50\cos 2\Delta\theta + 0.50\cos(2\Delta\phi + \pi) + 0.25\cos 3\Delta\theta + 0.25\cos(3\Delta\phi + 3\pi/2) \\ & + \cos(\Delta\theta - \Delta\phi + \pi/2) + \cos(\Delta\theta + \Delta\phi + \pi/2) + 0.75\cos(\Delta\theta - 2\Delta\phi - \pi) + 0.75\cos(\Delta\theta + 2\Delta\phi + \pi) \\ & + 0.75\cos(2\Delta\theta - \Delta\phi - \pi/2) + 0.75\cos(2\Delta\theta + \Delta\phi + \pi/2) = 0. \end{aligned} \tag{26}$$

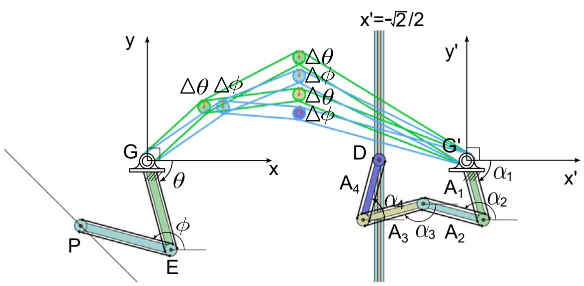


Fig. 7. This mechanical system uses Kempe's theory to design a mechanical system to trace a straight line. Cable drives is used to provide the transmission to constrain the RR chain to trace this curve.

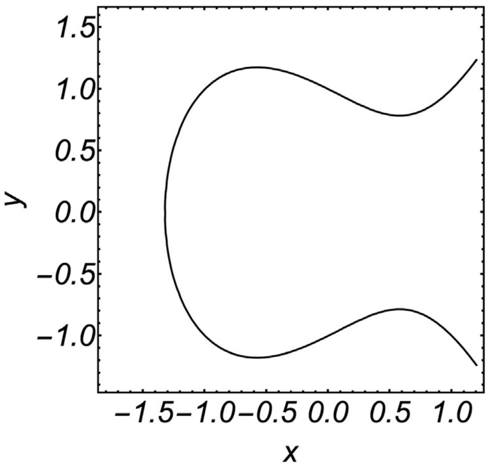


Fig. 8. A plot of the elliptic cubic curve defined by $f(x, y) = x^3 - y^2 - x + 1 = 0$.

Table 3
Dimensions for Kempe's constraining linkage that defines the elliptic cubic curve $x^3 - y^2 - x + 1 = 0$.

Link, i	Link length, A_i	Link angle, $\Delta\alpha_i$	Initial config., α_{iO}
1	1.25	$\Delta\theta$	0
2	1.25	$\Delta\phi$	$\pi/2$
3	0.50	$2\Delta\theta$	0
4	0.50	$2\Delta\phi$	π
5	0.25	$3\Delta\theta$	0
6	0.25	$3\Delta\phi$	$3\pi/2$
7	1	$\Delta\theta - \Delta\phi$	$\pi/2$
8	1	$\Delta\theta + \Delta\phi$	$\pi/2$
9	0.75	$\Delta\theta - 2\Delta\phi$	$-\pi$
10	0.75	$\Delta\theta + 2\Delta\phi$	π
11	0.75	$2\Delta\theta - \Delta\phi$	$-\pi/2$
12	0.75	$2\Delta\theta + \Delta\phi$	$\pi/2$

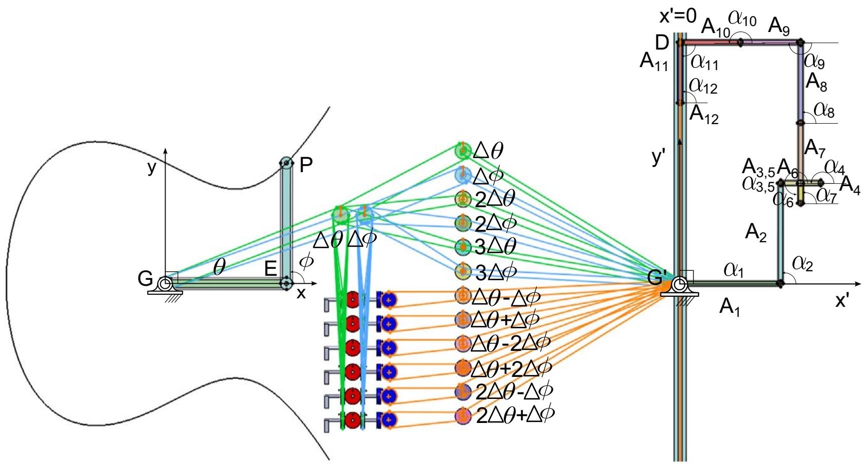


Fig. 9. The drawing linkage for the elliptic cubic curve presented by Kobel consists of the RR serial chain constrained by the linkage consisting 12 hinged links that end in a prismatic joint that moves along line $x' = 0$.

From this equation we determine the angles $\alpha_i = \Delta\alpha_i + \alpha_{i0}$ that define the constraining linkage. See Table 3. Fig. 9 shows the linkage system that draws this elliptic cubic curve.

4. Counting the number of parts

In this section, we compute the number of parts in our drawing and compare it to the number of links using Kempe's approach.

4.1. The number of links, differentials and cable drives

There are two links in the RR linkage that draws the curve, and n links in Kempe's serial chain, corresponding to the n terms in (4), so $c_l = n + 2$.

Let a_i, $i = 1,\ldots,n$ be the number of addition operations in the i^{th} term, which is equal to 0, 1 or 2, therefore,

$$c_a = \sum_i^n a_i. \tag{27}$$

Similarly, let v_i denote the number of multiplications in the i^{th} term, which again will be 0, 1 or 2, that is,

$$c_m = \sum_i^n v_i. \tag{28}$$

Recall that we perform the actual multiplications $r_i\phi(r_i \geq 2)$ and $s_i\theta(s_i \geq 2)$ using the relative sizes of pulleys in cable drives.

Three cable drives are needed to position the *RR* chain that draws the curve. The i^{th} joint of Kempe's serial chain requires i cable drives to constrain it. Thus, the number of cables drives for Kempe's serial chain is,

$$c_t = n(n + 1)/2 + 3. \tag{29}$$

The total number of parts for our drawing linkage for n terms is given by,

$$p = c_l + c_t + c_a + c_m = (n^2 + 3n + 10)/2 + \sum_i^n a_i + \sum_i^n v_i. \tag{30}$$

4.2. The number of links needed in Kempe's method

In order to estimate the number of links for the drawing linkage obtained using Kempe's method, notice that as before Kempe's serial has n links, one for each terms in (4) together with two links for the RR chain that draws the curve, thus $c_l = n + 2$.

Kempe's multiplicator is a series of scaled contra-parallelograms, which performs a multiplication by q using $2q + 2$ links. This linkage is needed only if there is a multiplication by 2 or more, that is $q \geq 2$. Recall that the i^{th} term has the multiplication $r_i\phi$ and $s_i\theta$. Thus, the number of links for multiplication is given by,

$$c_m = \sum_i^n (2r_i + 2) + \sum_i^n (2s_i + 2), \tag{31}$$

where each term is counted only if the multiplication is by factor of 2 or more.

Kempe's additor has 11 links. Let a_i, $i = 1, \ldots n$ denote the number of additions in the i^{th} term. Notice that in Kempe's construction, additors are required for additions between θ or ϕ with a constant angle. This means the number of links used for additions is,

$$c_a = \sum_i^n 11a_i. \tag{32}$$

Kempe's translator is constructed from a series of parallelograms linkage that are positioned along Kempe's serial chain to constrain its angular values. The i^{th} link requires $i - 1$ parallelograms to connect it to the calculating linkages. This can be assembled by adding two additional links for parallelogram, which yields,

$$c_t = n(n - 1). \tag{33}$$

Thus, the number of links required for Kempe's drawing linkage can be estimated to be,

$$p = c_l + c_t + c_a + c_m = n^2 + 2 + \sum_i^n 11a_i + \sum_i^n (2r_i + 2) + \sum_i^n (2s_i + 2), \tag{34}$$

where n is the number of links in Kempe's serial chain.

4.3. Part count for the straight line

Links, differentials and cable drives: The number of parts for our mechanism to draw a straight line can be estimated as follows. Here we use the straight line defined in (7) as an example. Examine (13) to see that there are n=4 terms, the number of additions c_a and multiplications c_m are both zero. Thus, the part count for the straight line drawing linkage is computed to be,

$$p_{1a} = (n^2 + 3n + 10)/2 + \sum_i^n a_i + \sum_i^n v_i = 19, \tag{35}$$

where the subscript $1a$ denotes our drawing linkage for the first example.

Links in Kempe's method: Kempe's drawing linkage for the straight line requires two additors that $a_i = 1,\ i = 3, 4$ for the addition of θ or ϕ with a constant angle. There is no mutiplicators needed in this case. Thus, the number of parts used in Kempe's construction is computed to be,

$$p_{1b} = n^2 + 2 + \sum_i^n 11a_i + \sum_i^n (2r_i + 2) + \sum_i^n (2s_i + 2) = 18 + 22 = 40. \tag{36}$$

Thus, we can estimate Kempe's construction to trace the straight line requires at least 40 links.

4.4. Part count for the elliptic cubic curve

Links, differentials and cable drives: The number of parts for the drawing linkage for the elliptic cubic curve is obtained from (22). We see that there are n=12 terms, the number of additions is c_a=6 and the number of multiplications is c_m=8. Thus, the part count for the elliptic cubic curve drawing linkage is

$$p_{2a} = (n^2 + 3n + 10)/2 + \sum_i^n a_i + \sum_i^n v_i = 95 + 6 + 8 = 109, \tag{37}$$

where the subscript $2a$ denotes our drawing linkage for our second example.

Links in Kempe's method: Finally, the number of links in Kempe's linkage for the elliptic cubic curve is obtained from Eq. (26). It has n=12 terms, which require six additors needed are $a_i = 1,\ i = 7, \ldots 12$, and eight mutiplicators, which are $r_4 = 2,\ r_6 = 3,\ r_9 = 2,\ r_{10} = 2$ and $s_3 = 2,\ s_5 = 3,\ s_{11} = 2,\ s_{12} = 2$. Thus, the number of links for the elliptic cubic curve drawing linkage is estimated to be

$$p_{2b} = n^2 + 2 + \sum_i^n 11a_i + \sum_i^n (2r_i + 2) + \sum_i^n (2s_i + 2) = 146 + 66 + 26 + 26 = 264. \tag{38}$$

Thus, we can estimate Kempe's construction to trace the elliptic cubic curve requires at least 264 links.

This part count comparison shows that our use of differentials and cable drives in the drawing linkage reduces the part count by more than one-half.

5. Workspace analysis for the drawing linkage

The plots of a number of algebraic curves are in the region $-\infty < x < \infty$ or $-\infty < y < \infty$. But the mechanism drawing the algebraic curve is designed to work in a specific range. In this section, the moving range of the drawing linkage and the relationship between θ and ϕ are analyzed.

The extreme points are hit when the *RR* chain is fully stretched out. This constrains the moving range of the curve drawing mechanism. Thus, angles θ and ϕ for the extreme position can be obtained by solving,

$$\theta = \phi, \quad \text{and} \quad f(\theta, \phi) = \sum_i^n A_i \cos(r_i\phi + s_i\theta + \alpha_i) - K = 0. \tag{39}$$

Here we use the straight line defined by (7) as an example. In order to obtain the moving range of the mechanical system constructing from our approach, we solve for Eq. (9) by setting $\theta = \phi$ which yields,

$$\theta = \phi = 0.785 + 2k\pi \quad \text{and} \quad 3.927 + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{40}$$

The first solution gives the configuration of the *RR* chain when it reaches the maximum y value while the second solution is for the situation it reaches the minimum y value. Substituting Eq. (40) into Eq. (8) to compute the two extreme points that the *RR* chain can reach are

$$\mathbf{P}_1 = (1.414, 1.414) \quad \text{and} \quad \mathbf{P}_2 = (-1.414, -1.414). \tag{41}$$

Eq. (41) provides us the *RR* chain's workspace. The part of the straight line been drawn is within the boundary that $-1.414 < x < 1.414, -1.414 < y < 1.414$.

We follow the same procedure to get the mechanical system's workspace for the elliptic cubic curve. Solving Eq. (22) by setting $\theta = \phi$ to obtain,

$$\theta = \phi = 0.824 + 2k\pi, \quad \text{and} \quad 5.459 + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{42}$$

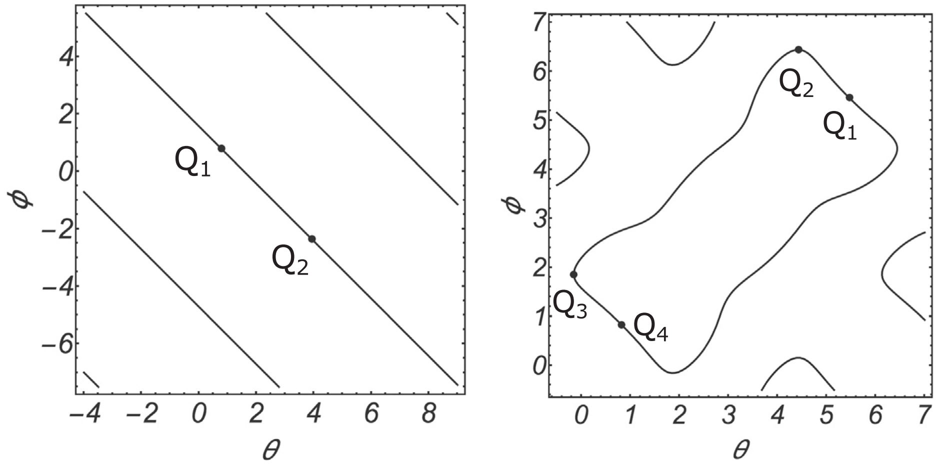
Substituting Eq. (42) into Eq. (8) to compute the two extreme points the *RR* chain can reach on the elliptic cubic curve are,

$$\mathbf{P}_1 = (1.359, 1.467), \quad \text{and} \quad \mathbf{P}_2 = (1.359, -1.467). \tag{43}$$

We set y=0 in Eq. (19) to obtain the minimum x value to be -1.325. The elliptic cubic curve is symmetrical with respect to y=0. Thus the part of elliptic cubic curve been drawn is within the boundary that $-1.325 < x < 1.359, -1.467 < y < 1.467$.

Note that the region of the curve been drawn is dependent on the lengths of the *RR* chain. The above workspace calculations for the straight line and elliptic cubic curve are both based on our construction that $L_1 = L_2 = 1$.

Since we have constrained angle θ and ϕ through Kempe's serial chain, the degree of freedom of the resulting mechanical system



(a) Straight line: The relationship between θ and ϕ for $k = 0$ in Eq (40).

(b) Elliptic: The relationship between θ and ϕ for $k = 0$ in Eq (42).

Fig. 10. In Eqs. (40) and (42), setting k=0 to obtain two specific plots. In (a), angle θ and ϕ change linearly from $\mathbf{Q}_1$ to $\mathbf{Q}_2$. In (b), points $\mathbf{Q}_1$ and $\mathbf{Q}_4$ are two boundary points where the *RR* chain changes elbow directions; points $\mathbf{Q}_2$ and $\mathbf{Q}_3$ are places θ and ϕ reach its maximum or minimum values.

is one. We now analyze the relationship between θ and ϕ.

In order to see how θ or ϕ change as the *RR* chain draws the algebraic curve, we plot the relationship between θ and ϕ from Eqs. (9) and (22). We focus on the plot for θ and ϕ when k=0 in Eqs. (40) and (42), see Fig. 10.

In Fig. 10(a), all the lines defined by equation $\phi = -\theta + \pi/2 + 2k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$ represent the relationship of θ and ϕ to constrain the RR chain to draw the straight line. From $\mathbf{Q}_1$ to $\mathbf{Q}_2$, the RR chain moves from one fully stretched-out position with maximum y value to the other fully stretched-out position with minimum y value. During this movement, angle θ increases linearly from $\pi/4$ to $5\pi/4$ while ϕ decreases linearly from $\pi/4$ to $-3\pi/4$.

In Fig. 10(b), the upper left part from $\mathbf{Q}_1$ counter-clockwise to $\mathbf{Q}_4$ shows the constraint between θ and ϕ for the *RR* chain starting from "elbow-down" configuration while the lower right part from $\mathbf{Q}_1$ clockwise to $\mathbf{Q}_4$ shows the constraint with "elbow-up" starting configuration. Now we focus on the upper left part resulting from our construction for elliptic cubic curve in Fig. 9. It shows θ keeps decreasing from point $\mathbf{Q}_1$ to $\mathbf{Q}_3$ and starts increasing from $\mathbf{Q}_3$ to $\mathbf{Q}_4$. Thus we can obtain that θ decreases from 5.459 to −0.156 and then increases to 0.824. While ϕ starts increasing from point $\mathbf{Q}_1$ to $\mathbf{Q}_2$ and keeps decreasing from $\mathbf{Q}_2$ to $\mathbf{Q}_4$. We can compute that ϕ increases from 5.459 to 6.440 and then decreases to 0.824.

6. Conclusion

In this paper, we present a simplified version of Kempe's method for the design of a drawing linkage for a general plane algebraic curve. We replace his additor, reversor, multiplicator and translator computing linkages, with differential and cable drives to perform the equivalent mechanical calculations. In order to compare these two approaches, we estimate the part count and find that our approach uses less than one-half the number of parts. Kempe's universality theorem guarantees the existence of a drawing linkage for any algebraic curve, therefore our simplified linkage exists as well. We use Saxena's and Kobel's examples to illustrate our method, which has the added benefit of illustrating the central role played by Kempe's constraining linkage in controlling the movement of a RR chain that draws the curve.

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