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DESIGN OF MECHANISMS TO TRACE PLANE CURVES

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ABSTRACT

This paper describes a mechanism design methodology that assembles standard components to trace plane curves that have a Fourier series parameterization. This approach can be used to approximate complex plane curves to interpolate image boundaries constructed from points. We describe three ways to construct a mechanism that generates a curve from a Fourier series parameterization. One uses Scotch yoke linkages for each term of Fourier series which are added using a belt drive. The second approach uses a coupled serial chain for each coordinate Fourier parameterization. The third method uses one constrained coupled serial chain to trace a specified plane curve. This work can be viewed as a version of the Kempe Universality Theorem that states that a linkage exists that can trace any plane algebraic curve. In our case, we include belts and pulleys, and obtain linkages that trace curves that have Fourier parameterizations.

INTRODUCTION

In 1876 Kempe [1] showed that a mechanism exists that can trace any plane curve defined by an algebraic equation. Recent work by Kapovich and Millson [2] confirmed this result. However, the devices generated by this process are hopelessly complex. So here, we consider the problem of constructing simpler mechanisms to trace plane curves. We consider two types of plane curves, f(x,y)=0, (i) those that have an algebraic equation that can be written in polar and therefore have parameterized expressions $\gamma(\theta) = (x(\theta), y(\theta))$, and (ii) those curves that may or may not be algebraic but are parameterized, $\gamma(t) = (x(t), y(t))$. In both cases, we assume that we can find a Fourier series expansion of the functions, $\gamma(\theta)$ and $\gamma(t)$, and then we construct several types of linkage systems that generate these curves by addition of the Fourier coefficients. If there is a large number of Fourier coefficients, then we approximate the curve by truncating the Fourier series. This approach assumes the plane curve is closed, though we are working on methods to expand the class of curves.

LITERATURE REVIEW

Inspired by Watt's application of a four-bar coupler linkage to provide approximate straight line motion on steam engine, Nolle [3, 4] and Koetsier [5, 6] developed the early design methodology of curve-tracing mechanisms. Kempe [1, 7] proved the existence of a mechanism to trace any plane algebraic curve by introducing a standard set of linkages, the Additor, Reversor, Multiplicator and Translator, which he combined to constrain an *RR* chain to trace the specified curve. He remarks that, while the a linkage obtained in this way "would not be practically useful on account of the complexity of the linkwork," this result does show that such linkages exist and he encourages mathematicians and artists to seek simpler versions.

Recently, researchers have revised Kempe's universality theorem, see Kapovich and Millson [2] and Kobel [8]. Saxena [9] provides a step by step demonstration of Kempe's method and

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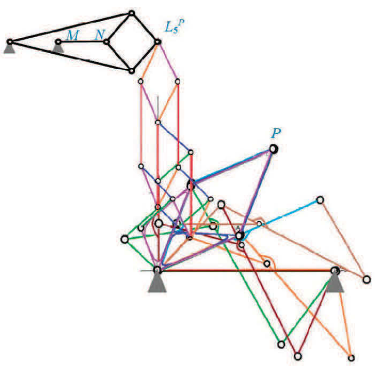
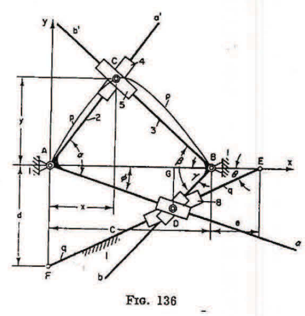
obtains a mechanism with 48 links and 70 joints, see Fig. 1(a) to trace a quadratic curve. Artobolevskii [10] shows that his conograph linkage with eight-bars can be sized to trace any quadratic curve, Fig. 1(b).



(a) Saxena's Kempe Linkage.



(b) Eight-bar Conograph Mechanism.

FIGURE 1. (a) A linkage assembled using Kempe's theorem to trace a quadratic curve requires 48 bars and 70 joints; (b) the same curve can be generated using Artobolevskii's eight-bar conograph mechanism.

Another comparison of the mechanism obtained by Kempe's existence proof and the work by "mathematicians and artist" in the words of Kempe can be seen in the linkage obtained by Kobel [8] to trace the quartic trifolium, Fig. 2. He reports that his algorithm generates too many bars to be able to count. In comparison, Fig. 3 is an eight-bar mechanism designed by Artobolevskii to trace the trifolium. Artobolevskii [10] provides a synthesis theory that yields linkages to trace a large number of plane algebraic curves up to degree four.

Roth and Freudenstein [11] present a different approach of curve tracing. They develop the method of solving loop equations of four bar linkage to obtain the dimensions of a linkage. The coupler point can be driven to go through the nine points on the designed curve. Wampler [12] calculate the complete solution of nine-point synthesis problem. Plecnik [13, 14] shows that

FIGURE 2. Kobel use software Cinderella to generate this linkage to trace trifolium curve.

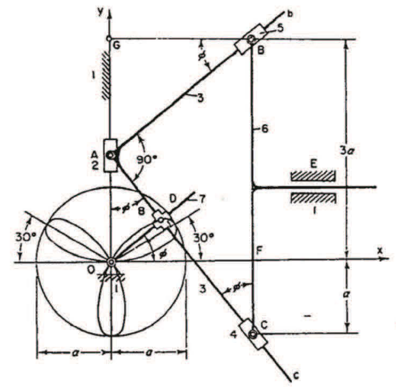


FIGURE 3. Artobolevskii generate this mechanism to trace trifolium curve.

the equations for 15 points six-bar generation has Bezout over 10^{46}.

Nie and Krovi [15] present the method of designing single degree-of-freedom coupled serial chain to trace plane curves. The coupled serial chain is driven by pulleys and belts. They use discrete Fourier transform approach to obtain serial chain that can go through the sample points on a specified curve. The number of links can be reduced using optimization method.

Lord Kelvin invent the first harmonic analyzer in 1872. Miller [16] developed a 32-element harmonic synthesizer in 1916 based on Fourier's Theorem. This device was built according to the calculation of the amplitude and phase of each Fourier component in the curve equation. The continue work was done by Brown [17], a thirty terms harmonic analyzer is constructed to drive a pencil to draw a specified curve on a board.

While it is known that cams can be cut to obtain the desired coordinate functions for a plane curve, or shaped to drive linkages to achieve trace a desired curve [18], our focus is on a simpler realizable way to construct mechanisms to trace plane curves. We increase the set of standard linkages used in synthesis of curve-tracing mechanism to include belts and pulleys,

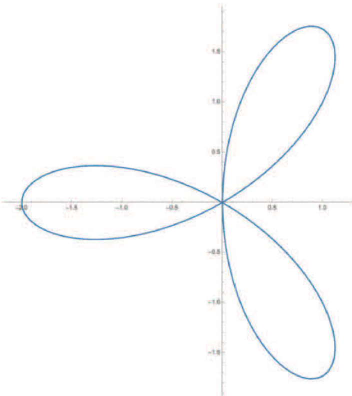


FIGURE 4. Trifolium curve.

but avoid the requirement to cut a cam to achieve a specified function. We use belts and pulleys to add and translate the projection of Scotch yoke mechanisms and to constrain the joints in coupled-serial chain mechanisms to achieve the summation of Fourier coefficients that define a plane curve. The result is a wide range of devices that trace complex plane curves.

FOURIER DECOMPOSITION

In this section, we show how to decompose an algebraic curve to Fourier terms. We assume the curve denoted by Cartesian equations can be converted to polar equation format. This requires the length term that on the left side of the polar equation has to be reduced to power one. The right side has to be the form of summation of cosine and sine terms. We obtain the x and y components by multiplying cosine and sine of the right side of polar equation. We compute the Fourier decomposition of each component in x and y equation. For curves already parameterized in x an y directions, the Fourier decomposition terms can be obtained directly by applying trigonometric identities.

In order to demonstrate how to decompose a curve in Cartesian equation, we choose the trifolium curve, see Fig. 4. The Cartesian equation is in the form,

$$(x^2+y^2)[y^2+x(x+a)] = 4axy^2. \tag{1}$$

In order to get the polar equation, we denote the point coordinates been tracing as $\mathbf{P} = (x, y)$ given by,

$$\mathbf{P} = \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} \rho\cos\theta \\ \rho\sin\theta \end{Bmatrix}. \tag{2}$$

Substitute the coordinates $\mathbf{P}$ into Eq. (1) and after simplification we obtain,

$$\rho = -a(4\cos^3\theta - 3\cos\theta). \tag{3}$$

Notice that in Eq. (3), the part in parenthesis can be replaced by power reduction formula,

$$\cos^3\theta = \frac{3\cos\theta + \cos(3\theta)}{4}. \tag{4}$$

The result we obtain is,

$$\rho = -a\cos 3\theta. \tag{5}$$

Now in order to get the equation in x and y direction, we compute the projection of the polar equation onto x and y axis,

$$\mathbf{P} = \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} \rho\cos\theta \\ \rho\sin\theta \end{Bmatrix} = \begin{Bmatrix} -a\cos 3\theta\cos\theta \\ -a\cos 3\theta\sin\theta \end{Bmatrix}. \tag{6}$$

We can reduce the product using the sum and difference identities,

$$\begin{aligned} 2\cos\theta\cos\phi &= \cos(\theta-\phi) + \cos(\theta+\phi), \\ 2\cos\theta\sin\phi &= \sin(\theta+\phi) - \sin(\theta-\phi). \end{aligned} \tag{7}$$

After applying Eq. (7), we have

$$\mathbf{P} = \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} -\frac{a}{2}(\cos 2\theta + \cos 4\theta) \\ -\frac{a}{2}(\sin 4\theta - \sin 2\theta) \end{Bmatrix}. \tag{8}$$

The parameter a only affects the size of the tracing curve, here we set it to be equal to 2. The frequency of the mechanism we design can be free chosen so here we set the unit frequency as $\frac{1}{2\pi}$ rad per second. We can write Eq. (8) in time domain,

$$\mathbf{f}(t) = \begin{Bmatrix} x(t) \\ y(t) \end{Bmatrix} = \begin{Bmatrix} -(\cos 2t + \cos 4t) \\ -(\sin 4t - \sin 2t) \end{Bmatrix}. \tag{9}$$

We can directly using Eq. (9) to design mechanism moving in x and y direction and the projection intersects at the tracing curve. Another approach is using Fourier transform theory, so we can construct one coupled serial chain to trace the curve.

Assuming we have one coupled serial chain, the end point of the last link traces the designed curve. The length of each link can be denoted as $L_1, L_2, L_3 \cdots, L_n$. Each link rotates at a constant speed, the associated angular velocity can be denoted as $\omega_1, \omega_2, \omega_3 \cdots, \omega_n$. Each link can start at any phase, so the corresponding phase is denoted as $\psi_1, \psi_2, \psi_3 \cdots, \psi_n$. Have this set up, we can calculate the x and y coordinates of the end-effector as,

$$\mathbf{f}(t)=\begin{Bmatrix}x(t)\\y(t)\end{Bmatrix}=\begin{Bmatrix}L_1\cos(\omega_1 t+\psi_1)+L_2\cos(\omega_2 t+\psi_2)+\\L_3\cos(\omega_3 t+\psi_3)+\cdots+L_n\cos(\omega_n t+\psi_n)\\L_1\sin(\omega_1 t+\psi_1)+L_2\sin(\omega_2 t+\psi_2)+\\L_3\sin(\omega_3 t+\psi_3)+\cdots+L_n\sin(\omega_n t+\psi_n)\end{Bmatrix}. \tag{10}$$

The above equation is a generalization version of Eq. (9). Now instead of representing a point in **2D** coordinates as (x,y), we denote it as complex number form $(x+iy)$. So the points consisting the curve are denoted in $\mathbb{C}$ instead of $\mathbb{R}$. We use $z(t)$ to represent the complex format of Eq. (10), it is in the form,

$$\begin{aligned}\mathbf{z}(t)=L_1\cos(\omega_1 t+\psi_1)+L_2\cos(\omega_2 t+\psi_2)+L_3\cos(\omega_3 t+\psi_3)+\\ \cdots+L_n\cos(\omega_n t+\psi_n)+i[L_1\sin(\omega_1 t+\psi_1)+L_2\sin(\omega_2 t+\psi_2)\\ +L_3\sin(\omega_3 t+\psi_3)+\cdots+L_n\sin(\omega_n t+\psi_n)].\end{aligned} \tag{11}$$

Here we apply Euler's formula,

$$e^{ix}=\cos x+i\sin x. \tag{12}$$

We can write Eq. (11) in the form,

$$\begin{aligned}\mathbf{z}(t)=L_1 e^{i(\omega_1 t+\psi_1)}+L_2 e^{i(\omega_2 t+\psi_2)}\\ +L_3 e^{i(\omega_3 t+\psi_3)}+\cdots+L_n e^{i(\omega_n t+\psi_n)}.\end{aligned} \tag{13}$$

We assume infinite terms being added and each angular velocity is being included, so we can write Eq. (13) as

$$\mathbf{z}(t)=\int_{-\infty}^{\infty}\mathbf{L}(\omega)e^{i(\omega t+\psi_\omega)}d\omega. \tag{14}$$

Here we factor out $e^{i\psi_\omega}$ and replace $L(w)e^{i\psi_\omega}$ with $L\psi(\omega)$ and get the result,

$$\mathbf{z}(t)=\int_{-\infty}^{\infty}\mathbf{L}\psi(\omega)e^{i\omega t}d\omega. \tag{15}$$

In Eq. (15), function $L\psi(\omega)$ contains all the information we need to build a single coupled serial chain: link length, angular frequency and phase. As long as $L\psi(\omega)$ has been calculated, we get all the configuration information to construct a coupled serial chain. We can calculate $L\psi(\omega)$ by performing Fourier transform of $z(t)$, we have,

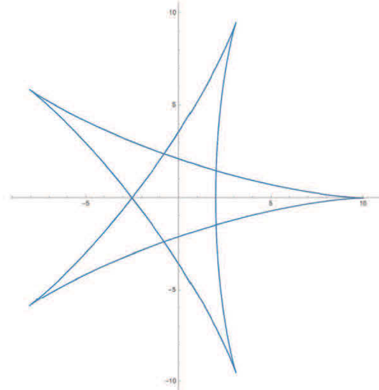


FIGURE 5. Hypocycloid curve.

$$\mathbf{L}\psi(\omega)=\frac{1}{T}\int_{-\infty}^{\infty}z(t)e^{-i\omega t}dt. \tag{16}$$

Now we compute the one coupled serial chain configuration for trifolum curve. We can construct $z(t)$ from Eq. (9), and compute Fourier transform of $z(t)$,

$$\mathbf{L}\psi(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(-(\cos 2t+\cos 4t)-i(\sin 4t-\sin 2t))e^{-i\omega t}dt. \tag{17}$$

After calculation, we get the result that only consists of two terms,

$$\mathbf{L}\psi(\omega)=-\delta(\omega-4)-\delta(\omega+2). \tag{18}$$

For some algebraic curves, their polar equations are not in the standard form we required. But we can easily find their parameterized equations in x and y directions. For these kind of curves, we can still use our design procedure to construct the mechanism to trace the curve. Here we use hypocycloid curve as an example. The polar equation of hypocycloid curve is in the form,

$$\rho^2=(R-mR)^2+(mR)^2+2(R-mR)mR\cos t. \tag{19}$$

The parameterized equation of hypocycloid curve are,

$$\mathbf{f}(t)=\begin{Bmatrix}x(t)\\y(t)\end{Bmatrix}=\begin{Bmatrix}(R-mR)\cos mt+mR\cos(t-mt)\\(R-mR)\sin mt-mR\sin(t-mt)\end{Bmatrix}. \tag{20}$$

We are free to choose the values of R and m. What the values affect is only the shape of the curve. Here we set R and m to be equal to 10 and 0.4 respectively. We get a star curve that is shown in Fig. 5. After plug in the values we assigned, the parameterized equations become,

$$\mathbf{f}(t) = \begin{Bmatrix} x(t) \\ y(t) \end{Bmatrix} = \begin{Bmatrix} 6\cos 0.4t + 4\cos 0.6t \\ 6\sin 0.4t - 4\sin 0.6t \end{Bmatrix}. \tag{21}$$

We get the complex format $z(t)$ from Eq. (21), and compute the Fourier transform of $z(t)$ to obtain,

$$\mathbf{L}\psi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (6\cos 0.4t + 4\cos 0.6t + i(6\sin 0.4t - 4\sin 0.6t))e^{-i\omega t}dt. \tag{22}$$

The result we get is,

$$\mathbf{L}\psi(\omega) = 4\delta(0.6-\omega) + 6\delta(0.4+\omega). \tag{23}$$

CONSTRUCTING THE MECHANISM

In this section, we present three ways to construct a mechanism to trace a plane curve. The first method uses two serial chains designed to generate the projections of the curve on x and y directions, which are then connected to a single input. The second method uses Scotch yoke mechanisms that compute each term in the Fourier series expansion of the x and y projections, which are then connected to a single input. Finally, we show that the Fourier transformation of the complex coordinates $z(t)$ of the curve can yield a single coupled serial chain that can trace the desired curve. Here we present all three methods for the examples of the trifolium and hypocycloid curves.

Two Coupled Serial Chains

Let $x(t)$ and $y(t)$ be the Fourier expansion of the components of a curve $\mathbf{f}(t)$. Equation (9) defines the trifolium and Eq. (21) defines the hypocycloid.

The procedure we use to define the two coupled serial chains is as follows:

1. Consider the curve $\mathbf{f}(t)$ defined in Eq. (9); the $x(t)$ has two terms, so this serial chain has two links with the lengths, $L_{X1} = 1$ and $L_{X2} = 1$; the angular velocity of the input rotation defines the diameters $D_{X1} = 1/2$ and $D_{X2} = 1/4$ of pulleys attached to each joint relative that are coupled by belts to the drive pulley; finally, the initial configuration is determined by the value $x(0) = -2$;
2. The coupled serial chain for the $y(t)$ component is obtained in the same way; there are two terms, so this serial chain has two links with length, $L_{Y1} = 1$ and $L_{Y2} = 1$; the joint pulley diameters are given by $D_{Y1} = 1/4$ and $D_{Y2} = 1/2$; and the initial configuration is given by $y(0) = 0$, see Table 1;
3. Finally, the end-points of the x coupled serial chain and the y coupled serial chain are connected by horizontal and vertical sliders that intersect at the tracing point, and both chains are driven by the same input rotation. See Fig. 6.

TABLE 1. Two Serial Chain Configuration to Trace Trifolium Curve.

Link Number	Link Length	Phase	Pulley Diameter
L_{X1}	1	$-\pi$	1/2
L_{X2}	1	$-\pi$	1/4
L_{Y1}	1	$-\pi$	1/4
L_{Y2}	1	0	1/2

TABLE 2. Two Serial Chain Configuration to Trace Hypocycloid Curve.

Link Number	Link Length	Phase	Pulley Diameter
L_{X1}	6	0	3
L_{X2}	4	0	2
L_{Y1}	6	0	3
L_{Y2}	4	$-\pi$	2

Applying the same procedure to the hypocycloid curve defined by Eq. (21), we obtain the dimensions listed in Table 2. The result is a convenient procedure for defining a pair of coupled serial chains that trace plane curves, see Fig. 7.

Scotch Yoke Mechanisms

For the second approach, each unit of the Scotch yoke mechanism simulates one term in x or y equation. Assume the positive direction is along x axis in Cartesian coordinates, the output of the Scotch yoke mechanism is the projection of pulley radius on y axis. Equation (9) defines the trifolium and Eq. (21) defines the hypocycloid.

We now demonstrate the procedure of constructing Scotch yoke mechanisms for the trifolium:

1. Consider the curve $\mathbf{f}(t)$ defined in Eq. (9). In order to make the end-effector which moves vertically in y direction to

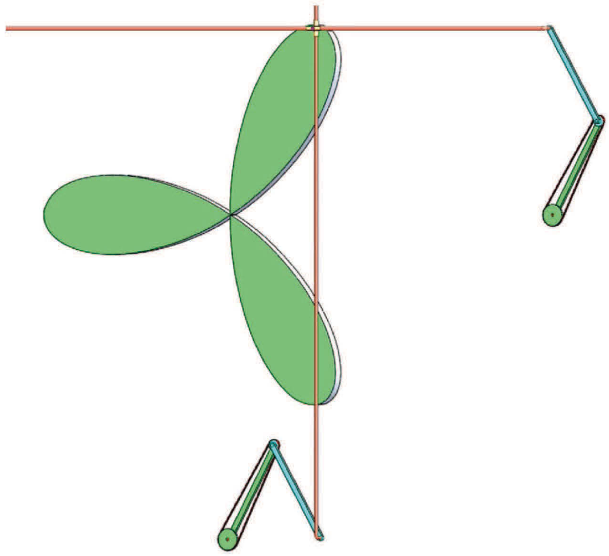


FIGURE 6. Two coupled serial chains combines to trace a trifolium curve.

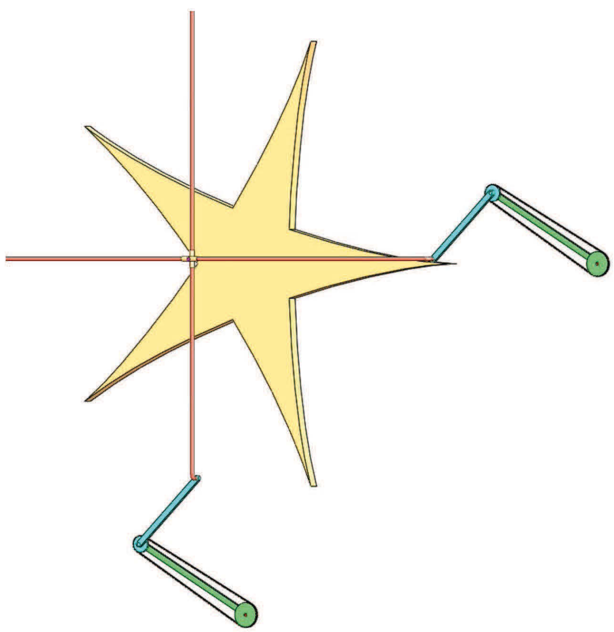


FIGURE 7. Two coupled serial chains combines to trace a Hypocycloid Curve.

trace the curve, we need to drive the board to move horizontally in the opposite direction . So we add a negative sign on $x(t)$ to get,

$$x(t) = \cos 2t + \cos 4t. \tag{24}$$

The output of Scotch yoke mechanism is the radius projection on y axis, we convert $x(t)$ in the format of sine function, we have,

$$x(t) = \sin(2t + \pi/2) + \sin(4t + \pi/2). \tag{25}$$

2. The x Scotch yoke mechanisms is obtained from Eq. (25) which has two terms, so we need two Scotch yoke mechanisms, which have the radius $R_{X1} = 1$ and $R_{X2} = 1$; the frequencies of each term define the rotating velocities $\omega_{X1} = 2$ and $\omega_{X2} = 4$ for each Scotch yoke mechanism; the initial configuration is determined by the phase in Eq. (25) that $\phi_{X1} = \pi/2$ and $\phi_{X2} = \pi/2$;
3. The y Scotch yoke mechanisms is obtained in the same way; $R_{Y1} = 1$ and $R_{Y2} = 1$; the rotating velocities of the Scotch yoke mechanism are given by $\omega_{Y1} = 2$ and $\omega_{Y2} = 4$; the starting position is determined by $\phi_{Y1} = 0$ and $\phi_{Y2} = \pi$. See Table 3;
4. Finally, one belt add x Scotch yoke mechanisms outputs together and drive the board horizontally; one belt add y Scotch yoke mechanisms outputs together and drive the end-effector to trace a trifolium curve. See Fig. 8.

TABLE 3. Scotch Yoke Mechanisms Configuration to Trace Trifolium Curve.

Pulley Number	Pulley Radius	Phase	Angular Velocity
R_{X1}	1	$\frac{\pi}{2}$	2
R_{X2}	1	$\frac{\pi}{2}$	4
R_{Y1}	1	0	2
R_{Y2}	1	π	4

TABLE 4. Scotch Yoke Mechanisms Configuration to Trace Hypocycloid Curve.

Pulley Number	Pulley Radius	Phase	Angular Velocity
R_{X1}	6	$-\frac{\pi}{2}$	2
R_{X2}	4	$-\frac{\pi}{2}$	3
R_{Y1}	6	0	2
R_{Y2}	4	π	3

Applying the same procedure to the hypocycloid curve defined by Eq. (21), we obtain the dimensions listed in Table 4.

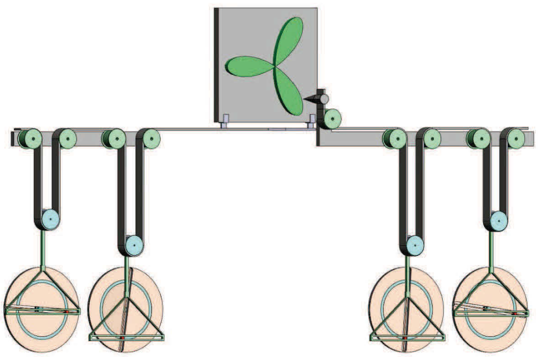


FIGURE 8. A system of Scotch yoke mechanisms to trace a trifolium curve.

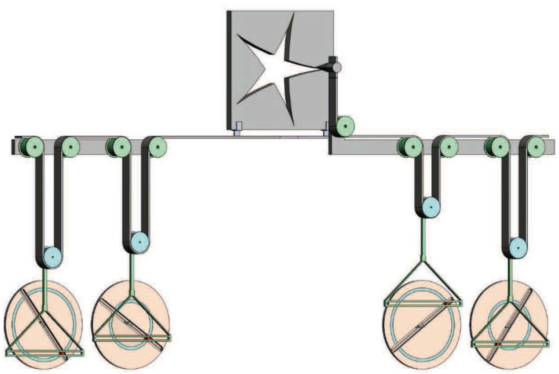


FIGURE 9. A system of Scotch yoke mechanisms to trace a Hypocycloid Curve.

The result is a convenient procedure for defining Scotch yoke mechanisms to trace hypocycloid curve, see Fig. 9.

Single Coupled Serial Chain

The third approach is constructing one constrained coupled serial chain to trace a designed curve. Equation (18) defines the trifolium and Eq. (23) defines the hypocycloid. Function $L\psi(\omega)$ is in the format of summation of delta functions. The link's angular frequency is the ω that makes the integral of delta function not equal to zero. The sign of ω determines the rotating direction of each link.

We now demonstrate the procedure for the trifolium:

1. Start with the frequency domain function $\mathbf{L}\psi(\omega)$ defined in Eq. (18); the single coupled serial chain is obtained from $\mathbf{L}\psi(\omega)$ which has two terms, so the serial chain has two links, which have the lengths $L_1 = 1$ and $L_2 = 1$; the frequencies calculated from each delta function define the diameters $D_1 = 1/2$ and $D_2 = 1/4$ of pulleys attached each joint relative that are coupled by cables to unit diameter drive pulley;
2. The sign of the frequencies determine the rotating direction of each link, L_1 is clockwise and L_2 is counterclockwise; the initial configuration is determined by the arctangent of imaginary part and real part of the coefficient in each term that $\phi_1 = -\pi$ and $\phi_2 = -\pi$, see Table 5;
3. Finally, the end-point of the single coupled serial chain can trace the trifolium curve. See Fig. 10.

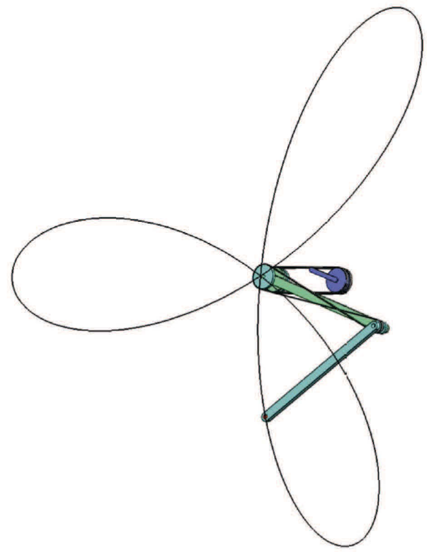


FIGURE 10. A constrained coupled serial chain to trace a trifolium curve.

TABLE 5. Configuration of One Constrained Coupled Serial chain Mechanisms to Trace Trifolium Curve .

Link Number	Link Length	Phase	Pulley Diameter
L_1	1	$-\pi$	1/2
L_2	1	$-\pi$	1/4

TABLE 6. Configuration of One Constrained Coupled Serial chain Mechanisms to Trace Hypocycloid Curve.

Link Number	Link Length	Phase	Pulley Diameter
L_1	6	0	3
L_2	4	0	2

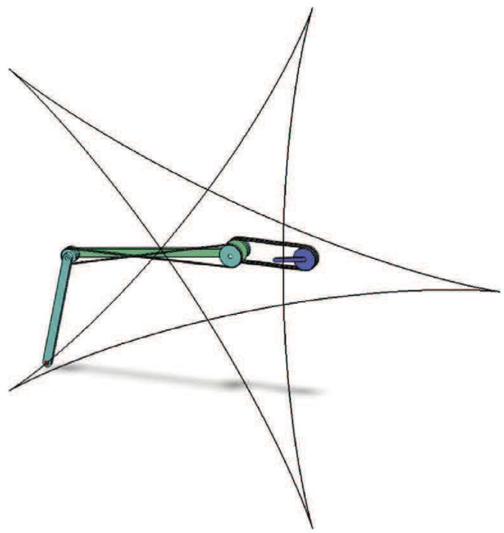


FIGURE 11. A constrained coupled serial chain to trace a hypocycloid curve.

Applying the same procedure to the hypocycloid curve defined by Eq. (23), we obtain the dimensions listed in Table 6. The result is a convenient procedure for defining a single coupled serial chain that trace hypocycloid curve, see Fig. 11.

CONCLUSION

In this paper, we present a methodology to design mechanical devices that trace plane curves. Our approach applies to plane algebraic curves that have polar form, and to parameterized plane curves. In both cases, expand the functions defining these curves with a Fourier series and construct devices that add the Fourier components. We present three ways to design these devices, (i) the combination of two sets of coupled serial chains,(ii) the summation of the outputs of Scotch yoke mechanisms, and (iii) the trace of the end-point of a single coupled serial chain. Two examples are used throughout to demonstrate the procedure. These are preliminary results, our goal is to obtain similar devices for general algebraic curves, and perhaps curves obtained as the interpolation of sets of points.

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