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SINGULARITY VARIETY OF A 3SPS-1S SPHERICAL PARALLEL MANIPULATOR

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ABSTRACT

In this paper, we study the manifold of configurations of a 3SPS-1S spherical parallel manipulator. This manifold is obtained as the intersection of quadrics in the hypersphere defined by quaternion coordinates and is called its constraint manifold. We then formulate Jacobian for this manipulator and consider its singular. This is a quartic algebraic manifold called the singularity variety of the parallel manipulator. A survey of the architectures that can be defined for the 3SPS-1S spherical parallel manipulators yield a number of special cases, in particular the architectures with coincident base or moving pivots yields singularity varieties that factor into two quadric surfaces.

Keywords: Spherical parallel mechanism; Singularity variety; Jacobian matrix; Dual quaternions; homogeneous polynomial

1 INTRODUCTION

Parallel manipulators can serve better than serially connected manipulators in some applications such as micromanipulation and robotics surgery due to their high positioning accuracy, high speed and high stiffness. And the three-DOF spherical parallel manipulators (SPMs) are a class of parallel manipulators which have a wide range of applications such as orienting devices and wrists [1, 2]. However, one of the disadvantages of parallel manipulators is the singular configurations which may exist within its workspace. The singular configurations where the parallel manipulator loses its stiffness result in the difficulty of trajectory planning [3, 4]. So the singularity analysis is one of the important problems related to SPMs.

Gosselin and Angeles [5] were the first to study the singularities of general parallel manipulators and introduced two Jacobian matrices that define input and output velocities. Then Daniali et al. [6] derived the Jacobian matrices of two classes of planar parallel manipulators and identified the three types of singularities for them. Merlet [7] was the first to apply line geometry to study singular configurations of a six-degree of freedom parallel manipulator. Zlatanov et al. [8] have presented a generalized approach to determine the singular configurations using a velocity equation that includes velocities of active and passive joints. Alici and Shirinzadeh [9] have proposed a systematic method to obtain the singular variety of any kinematics chain whose Jacobian matrices are expressed analytically while without expressing singularity variety mathematically. Bonev and Gosselin [10] have presented the singularity variety of symmetric spherical parallel mechanisms based on the intuitive orientation representation.

Sefrioui and Gosselin [11] have introduced the quadratic nature of the singularity curves of general three-degree-of-freedom planar parallel manipulators and gave a graphical representation of these loci in the manipulator's workspace using the roots of the determinants of the manipulators Jacobian. Collins and McCarthy [12] have found these singularity variety lie on a quartic surface when mapped to the space of planar quaternion. Later, they [13] have generalized the method to spatial parallel manipulators that have triangular base and top platform architectures with 2-2-2 and 3-2-1 actuator configurations.

In this paper we study the Jacobian of 3SPS-1S spherical parallel manipulators composed of three spherical-prismatic-spherical chains and a spherical joint using dual quaternion coordinates. The paper is organized as follows. First, we review the construction and manipulation of dual Quaternion. Next we analyze the structure of the spherical parallel manipulator and define the constraint manifold imposed by an SPS chain. Then we obtain the Jacobian formulation using dual quaternion and present how singularities are distributed within the manipulator workspace as a function of structural parameters. At last, we give a variety of special parallel manipulator architectures, and for the coincident pivots architecture, present the effect of its geometric properties on the function which defines the singularity variety.

2 DUAL QUATERNION COORDINATES

The general spatial displacement between a fixed frame F and a moving frame M is given by

$$\mathbf{Z} = [R]\mathbf{z} + \mathbf{d}. \tag{1}$$

A vector $\mathbf{z}$ measured in the moving frame M can be transformed to a vector $\mathbf{Z}$ in the base frame F. Matrix $[R]$ is a 3×3 rotation matrix and $\mathbf{d} = (d_x, d_y, d_z)$ is the translation vector between the fixed and moving frame.

A spatial displacement can also be represented in the form of an 8-dimensional vector known as a dual quaternion [14, 15].The first four components representing the orientation of the moving frame are written as follows,

$$q_1 = S_x \sin(\frac{\theta}{2}), \quad q_2 = S_y \sin(\frac{\theta}{2}), \quad q_3 = S_z \sin(\frac{\theta}{2}), \quad q_4 = \cos(\frac{\theta}{2}), \tag{2}$$

where

$$\cos(\theta) = \frac{Tr[R]-1}{2}. \tag{3}$$

The operator $Tr[R]$ denotes the trace of matrix $[R]$. S_x, S_y, and S_z are extracted from the skew symmetric matrix $[S]$ defined by

$$[S] = \frac{[R]-[R]^T}{2\sin(\theta)}. \tag{4}$$

The translation vector $\mathbf{d}$ combines with the components q_1, q_2, q_3, q_4 to yields four additional components,

$$\begin{aligned} q_5 &= \frac{1}{2}(q_4 d_x + q_3 d_y - q_2 d_z) \\ q_6 &= \frac{1}{2}(-q_3 d_x + q_4 d_y + q_1 d_z) \\ q_7 &= \frac{1}{2}(q_2 d_x - q_1 d_y + q_4 d_z) \\ q_8 &= \frac{1}{2}(-q_1 d_x - q_2 d_y - q_3 d_z). \end{aligned} \tag{5}$$

The eight dual quaternion components satisfy two constraints [15], given by

$$\begin{aligned} q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1, \\ q_1 q_5 + q_2 q_6 + q_3 q_7 + q_4 q_8 = 0. \end{aligned} \tag{6}$$

Points in the 8-dimensional space that satisfy the constraints are dual quaternion coordinates and can be used to represent spatial displacements. Details about the use of these coordinates and the geometry of the corresponding space can be found in [14–16].

The spatial displacement in Eq. 1 can be defined in terms of dual quaternion coordinates as,

$$[R] = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1 q_2 - q_3 q_4) & 2(q_1 q_3 + q_2 q_4) \\ 2(q_1 q_2 + q_3 q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2 q_3 - q_1 q_4) \\ 2(q_1 q_3 - q_2 q_4) & 2(q_2 q_3 + q_1 q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix} \tag{7}$$

and

$$\mathbf{d} = 2 \begin{bmatrix} -q_8 & q_7 & -q_6 & q_5 \\ -q_7 & -q_8 & q_5 & q_6 \\ q_6 & -q_5 & -q_8 & q_7 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}. \tag{8}$$

3 CONSTRAINT MANIFOLD OF THE 3SPS-S

Now consider the 3SPS-1S parallel manipulator shown in Fig.1 [1, 17]. The manipulator is composed of three spherical-prismatic-spherical (SPS) chains and a spherical joint. Each SPS chain consists of a prismatic actuator connected to a fixed platform and a moving platform through two passive spherical joints. The single spherical joint connected to the fixed platform fixes the position of the moving platform of the mechanism while the orientation of the moving platform can be variable. Therefore the last four components of dual quaternion are equal to zero.

For the first chain of the 3SPS-1S mechanism, the motion locus of moving pivot $\mathbf{S}_4$ which lies in the moving platform is a circle of two spheres. One sphere takes point $\mathbf{S}_0$ as the center and $\overline{S_0 S_4}$ as the radius. The other sphere takes point $\mathbf{S}_1$ as the center

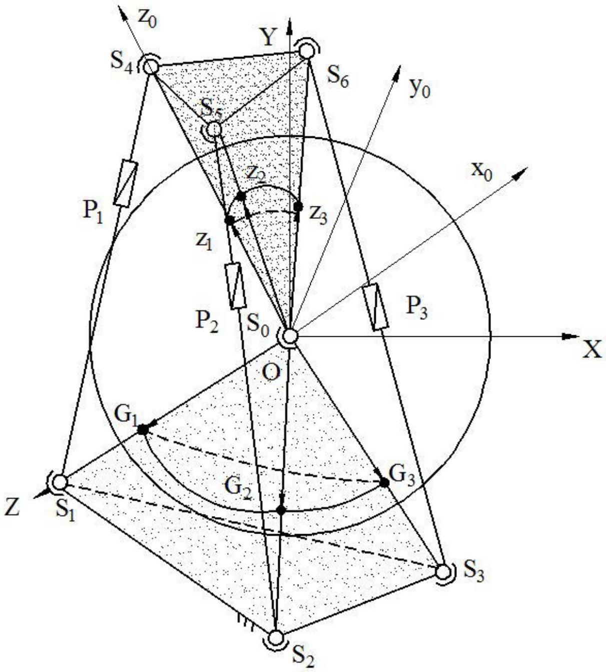

FIGURE 1. 3SPS-1S spherical parallel manipulator. The points $\mathbf{G}_i, i = 1,2,3$ denote the intersection of the lines through $\mathbf{S}_0\mathbf{S}_i$ with the unit sphere. Similarly $\mathbf{z}_i, i = 1,2,3$ denote the intersection of the lines $\mathbf{S}_0\mathbf{S}_4$, $\mathbf{S}_0\mathbf{S}_5$ and $\mathbf{S}_0\mathbf{S}_6$ with the unit sphere.

and $\overline{S_1S_4}$ as the radius. The same is true of conditions for two other chains. Therefore, the 3SPS-1S spatial parallel manipulator is a spherical mechanism with the center at point $\mathbf{S}_0$. The three vectors $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$ intersect the unit sphere in three points $\mathbf{G}_1$, $\mathbf{G}_2$ and $\mathbf{G}_3$ on the fixed body, while the other three vectors $\mathbf{S}_4$, $\mathbf{S}_5$ and $\mathbf{S}_6$ intersect the unit sphere in three points $\mathbf{z}_1$, $\mathbf{z}_2$ and $\mathbf{z}_3$ on the moving body.

Let the location of the fixed pivot be specified by $\mathbf{G}_i = (x_i, y_i, z_i)$, $i = 1,2,3$ measured in F and let the moving pivot be $\mathbf{z}_i = (u_i, v_i, w_i)$, $i = 1,2,3$, measured in M. Coordinates in the platform M are related to coordinate in F by the transformation Eq. 7, that is $\mathbf{Z}_i = [R]\mathbf{z}_i$.

Let the angle between each of the vectors $\mathbf{G}_i$ and $\mathbf{Z}_i$ be α_i, $i = 1,2,3$, then we have the three constraint equations for the spherical platform

$$\mathscr{C}_i: \quad \mathbf{G}_i^T\mathbf{Z}_i = \mathbf{G}_i^T[R]\mathbf{z}_i = \cos\alpha_i, \quad i = 1,2,3. \tag{9}$$

These equations define three quadratic constraints on the quaternions $\mathbf{q} = (q_1, q_2, q_3, q_4)$ that define the orientation of the spherical platform [18].

It is convenient to write the equations (9) as the quadratic forms

$$\mathscr{C}_i: \quad \mathbf{q}^T[C_i]\mathbf{q} = \cos\alpha_i, \quad i = 1,2,3, \tag{10}$$

where $[C_i]$ is a symmetric 4×4 matrix with upper triangular coefficients given by

$$\begin{aligned}
C_{11i} &= x_iu_i - y_iv_i - z_iw_i,\\
C_{12i} &= u_iy_i + x_iv_i,\\
C_{13i} &= u_iz_i + x_iw_i,\\
C_{14i} &= v_iz_i - y_iw_i,\\
C_{22i} &= -x_iu_i + y_iv_i - z_iw_i,\\
C_{23i} &= v_iz_i + y_iw_i,\\
C_{24i} &= -u_iz_i + x_iw_i,\\
C_{33i} &= -x_iu_i - y_iv_i + z_iw_i,\\
C_{34i} &= u_iy_i - x_iv_i,\\
C_{44i} &= x_iu_i + y_iv_i + z_iw_i, \quad i = 1,2,3.
\end{aligned} \tag{11}$$

The intersection of the three quadric manifolds in quaternion coordinates defines the constraint manifold of the spherical platform.

4 SINGULARITIES OF THE 3SPS-S

In order to determine the singularities of the spherical platform, we collect the three quadratic equations that define the constraint manifold together with the constraint that the components of a quaternion $\mathbf{q}$ have unit magnitude to obtain,

$$\begin{aligned}
\mathbf{q}^T[C_1]\mathbf{q} &= \cos\alpha_1,\\
\mathbf{q}^T[C_2]\mathbf{q} &= \cos\alpha_2,\\
\mathbf{q}^T[C_3]\mathbf{q} &= \cos\alpha_3,\\
\mathbf{q}^T\mathbf{q} &= 1.
\end{aligned} \tag{12}$$

The time derivative of these equations yields,

$$\begin{bmatrix} \mathbf{q}^T[C_1] \\ \mathbf{q}^T[C_2] \\ \mathbf{q}^T[C_3] \\ q_1,\ q_2,\ q_3,\ q_4 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{Bmatrix} - \begin{Bmatrix} -1/2\sin\alpha_1\dot{\alpha}_1 \\ -1/2\sin\alpha_2\dot{\alpha}_2 \\ -1/2\sin\alpha_3\dot{\alpha}_3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{13}$$

This equation can be written in the form of the Jacobian for a parallel manipulator

$$[A]\dot{\mathbf{q}} - [B]\dot{\mathbf{r}} = 0, \tag{14}$$

TABLE 1. The coordinates of the base and moving pivots used to derive singularity variety for the general 3SPS-S.

	1	2	3
	$\mathbf{G}_1$	$\mathbf{G}_2$	$\mathbf{G}_3$
X	0	0	$\sin\lambda$
Y	0	$-\sin\mu$	$-\sin\nu\cos\lambda$
Z	1	$\cos\mu$	$\cos\nu\cos\lambda$
	$\mathbf{z}_1$	$\mathbf{z}_2$	$\mathbf{z}_3$
x_0	0	0	$\sin\sigma$
y_0	0	$-\sin\tau$	$-\sin\rho\cos\sigma$
z_0	1	$\cos\tau$	$\cos\rho\cos\sigma$

where $\dot{\mathbf{r}} = (\dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}_3, 0)$. The configurations of the manipulator for which the determinant of the coefficient matrix $[A]$ is zero are known as "type 2 singularities" of the manipulator [5].

The elements of $[A]$ are linear in the quaternion coordinates $\mathbf{q} = (q_1, q_2, q_3, q_4)$ therefore,

$$\mathscr{S}: \quad \det[A] = 0, \tag{15}$$

defines is a quartic algebraic manifold that we call the singularity variety for the platform.

5 SINGULARITY VARIETY OF THE GENERAL 3SPS-S

The coordinates of the base S-joints of the platform, $\mathbf{S}_i$, $i = 1,2,3$, define lines through the fixed S joint that intersect the unit sphere in the base points $\mathbf{G}_1$, $\mathbf{G}_2$ and $\mathbf{G}_3$, see Figure 1. Let the coordinates of the points $\mathbf{G}_i$ be defined as shown in Table 1.

Similarly, the moving S-joints of the platform, $\mathbf{S}_j$, $j = 4,5,6$, define lines though the fixed S joint that intersect the unit sphere in the moving points $\mathbf{z}_i$, $i = 1,2,3$, Figure 1. Let the coordinates of $\mathbf{z}_1$, $\mathbf{z}_2$ and $\mathbf{z}_3$ be specified as shown in Table 1.

Substitute these values of the coordinates into the matrix $[A]$ of the Jacobian, to obtain,

$$[A] = \begin{bmatrix} -q_1 & -q_2 & q_3 & q_4 \\ D_1 & D_2 & D_3 & D_4 \\ D_5 & D_6 & D_7 & D_8 \\ q_1 & q_2 & q_3 & q_4 \end{bmatrix}, \tag{16}$$

where

$$\begin{aligned}
D_1 &= q_4\sin(\mu-\tau) - q_1\cos(\mu-\tau),\\
D_2 &= -q_3\sin(\mu+\tau) - q_2\cos(\mu+\tau),\\
D_3 &= -q_2\sin(\mu+\tau) + q_3\cos(\mu+\tau),\\
D_4 &= q_1\sin(\mu-\tau) + q_4\cos(\mu-\tau),\\
D_5 &= q_4\cos\lambda\cos\sigma\sin(\nu-\rho) + q_3(\cos\rho\cos\sigma\sin\lambda\\
&\quad +\cos\lambda\cos\nu\sin\sigma) - q_1[\cos\lambda\cos\sigma\cos(\nu-\rho)\\
&\quad -\sin\lambda\sin\sigma] - q_2(\cos\sigma\sin\lambda\sin\rho + \cos\lambda\sin\nu\sin\sigma),\\
D_6 &= -q_3\cos\lambda\cos\sigma\sin(\nu+\rho) + q_4(\cos\rho\cos\sigma\sin\lambda\\
&\quad -\cos\lambda\cos\nu\sin\sigma) - q_2[\cos\lambda\cos\sigma\cos(\nu+\rho)\\
&\quad +\sin\lambda\sin\sigma] - q_1(\cos\sigma\sin\lambda\sin\rho + \cos\lambda\sin\nu\sin\sigma),\\
D_7 &= -q_2\cos\lambda\cos\sigma\sin(\nu+\rho) + q_1(\cos\rho\cos\sigma\sin\lambda\\
&\quad +\cos\lambda\cos\nu\sin\sigma) + q_3[\cos\lambda\cos\sigma\cos(\nu+\rho)\\
&\quad -\sin\lambda\sin\sigma] + q_4(\cos\sigma\sin\lambda\sin\rho - \cos\lambda\sin\nu\sin\sigma),\\
D_8 &= q_1\cos\lambda\cos\sigma\sin(\nu-\rho) + q_2(\cos\rho\cos\sigma\sin\lambda\\
&\quad -\cos\lambda\cos\nu\sin\sigma) + q_4[\cos\lambda\cos\sigma\cos(\nu-\rho)\\
&\quad +\sin\lambda\sin\sigma] + q_3(\cos\sigma\sin\lambda\sin\rho - \cos\lambda\sin\nu\sin\sigma)
\end{aligned} \tag{17}$$

Setting the determinant of $[A]$ to zero, we obtain the algebraic equation of the singularity variety of the general 3SPS-S manipulator as,

$$\begin{aligned}
\mathscr{S}: \quad & c_1q_1q_2q_3q_4 + c_2q_1^3q_3 + c_3q_2^3q_4 + c_4q_3^3q_1 + c_5q_4^3q_2\\
&+ c_6q_1^2q_3^2 + c_7q_2^2q_4^2 + c_8q_1^2q_3q_4 + c_9q_1^2q_2q_3 + c_{10}q_1^2q_2q_4\\
&+ c_{11}q_2^2q_3q_4 + c_{12}q_2^2q_1q_3 + c_{13}q_2^2q_1q_4 + c_{14}q_3^2q_2q_4 + c_{15}q_3^2q_1q_2\\
&+ c_{16}q_3^2q_1q_4 + c_{17}q_4^2q_1q_2 + c_{18}q_4^2q_2q_3 + c_{19}q_4^2q_1q_3 = 0.
\end{aligned} \tag{18}$$

The coefficients c_1 to c_{19} are constants defined by the coordinates $\mathbf{G}_i$ and $\mathbf{z}_i$, $i = 1,2,3$ that define the platform. The singularity variety is a quartic surface in the homogeneous coordinates q_1, q_2, q_3 and q_4. Its geometric properties are a function of the parameters defining the kinematic architecture of the spherical parallel manipulator.

6 SINGULARITY VARIETIES FOR SPECIAL CASES

The geometric properties of the singularity variety are characterized by the locations of the platform fixed and moving pivots. Besides the above general architecture, we have studied the singularity variety for the following special architectures.

Type 1 Pivots lie on a great circle. There are three cases: (a) the points $\mathbf{G}_1, \mathbf{G}_2$ and $\mathbf{G}_3$ on the fixed body lie on a great circle; (b) the moving points $\mathbf{z}_i$, $\mathbf{z}_2$ and $\mathbf{z}_3$ lie on a great circle; and (c) both the bases points and the moving points lie on great circles. See Figure 2.

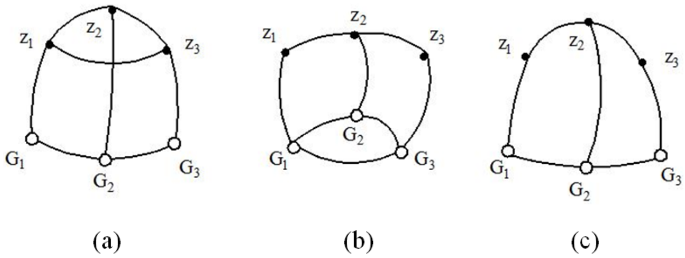

FIGURE 2. Type 1: (a) the points $\mathbf{G}_i$ lie on a great circle, (b) the points $\mathbf{z}_i$ lie on a great circle, and (c) both sets of points $\mathbf{G}_i$ and $\mathbf{z}_i$ lie on great circles.

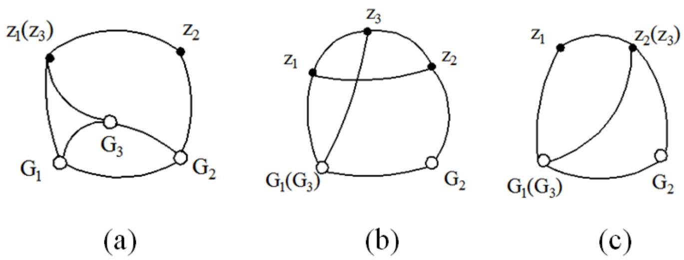

FIGURE 3. Type 2: (a) two of the points $\mathbf{z}_i$ on the moving body are coincident, (b) two of the points $\mathbf{G}_i$ on the fixed body are coincident, and (c) there are two sets of coincident points on both the moving and fixed bodies.

Type 2 Pivots are coincident. There are three cases, (a) Two of moving points $\mathbf{z}_i$, are coincident in the moving body, (b) two of the base points $\mathbf{G}_i$ are coincident in the fixed body, or (c) both base points and moving points are coincident. See Figure 3.

Type 3 Pivots are both coincident and on great circles. There are two cases: (a) the points $\mathbf{z}_i$ are on a great circle and two fixed points are coincident; and (b) the points $\mathbf{G}_i$ are on a great circle and two of the moving points are coincident. See Figure 4.

In what follows, we provide the singularity variety for each of these cases. The coordinates of the points $\mathbf{G}_i$ and $\mathbf{z}_i$ are defined as shown in Table 2, where s and c denote the sine and cosine functions, respectively.

The points $\mathbf{G}_i$ lie on a great circle. In this case, the singularity variety of the platform is given by,

$$\begin{aligned}\mathscr{S}:\quad & k_1q_1^3q_3-k_1q_4^3q_2+k_2q_3^3q_1-k_2q_2^3q_4\\ &+k_3q_1^2q_3^2-k_3q_2^2q_4^2+k_4q_1^2q_2q_3+k_5q_2^2q_1q_4+k_6q_1^2q_3q_4\\ &+k_7q_2^2q_3q_4+k_8q_1^2q_2q_4+k_9q_2^2q_1q_3+k_5q_3^2q_1q_4+k_7q_3^2q_1q_2\\ &+k_{10}q_3^2q_2q_4+k_4q_4^2q_2q_3+k_6q_4^2q_1q_2+k_{11}q_4^2q_1q_3=0, \qquad (19)\end{aligned}$$

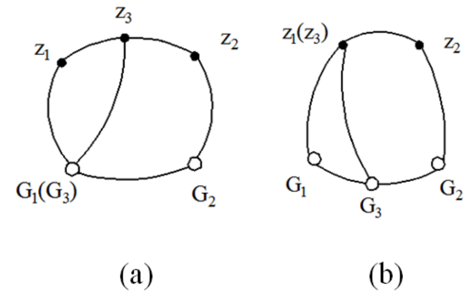

FIGURE 4. Type 3: (a) the points $\mathbf{z}_i$ are on a great circle and two fixed points are coincident, (b) the points $\mathbf{G}_i$ are on a great circle and two of the moving points are coincident.

where the coefficients k_i, $i=1,\dots 11$ depend on the coordinates of $\mathbf{G}_i$ and $\mathbf{z}_i$.

The points $\mathbf{z}_i$ lie on a great circle. In this case, the singularity variety of the platform is given by,

$$\begin{aligned}\mathscr{S}:\quad & k_1q_1^3q_3+k_1q_2^3q_4+k_2q_3^3q_1+k_2q_4^3q_2\\ &+k_3q_1^2q_3^2-k_3q_2^2q_4^2+k_4q_1^2q_2q_4+k_5q_2^2q_1q_3+k_6q_1^2q_2q_3\\ &+k_7q_2^2q_1q_4-k_8q_1^2q_3q_4+k_9q_2^2q_3q_4+k_5q_3^2q_2q_4+k_7q_3^2q_1q_4\\ &-k_9q_3^2q_1q_2+k_4q_4^2q_1q_3+k_6q_4^2q_2q_3+k_8q_4^2q_1q_2=0. \qquad (20)\end{aligned}$$

where the coefficients k_i, $k=1,\dots 9$ depend on the coordinates of $\mathbf{G}_i$ and $\mathbf{z}_i$.

The points $\mathbf{G}_i$ and $\mathbf{z}_i$ lie on great circles. In this case, the singularity variety of the platform is given by,

$$\begin{aligned}\mathscr{S}:\quad & k_1(q_1^2q_3^2-q_2^2q_4^2)\\ &+k_2(q_1^2q_2q_3+q_4^2q_2q_3)+k_3(q_2^2q_1q_4+q_3^2q_1q_4)=0. \qquad (21)\end{aligned}$$

where the coefficients k_i, $i=1,2,3$ depend on the coordinates of $\mathbf{G}_i$ and $\mathbf{z}_i$.

Two of the points $\mathbf{z}_i$ are coincident. In this case, the singularity variety of the platform is given by,

$$\mathscr{S}:\quad P_1P_2=4\sin\tau(-q_1q_3\cos\mu+q_2q_4\cos\mu+q_1q_2\sin\mu+q_3q_4\sin\mu)$$
$$(q_2q_3\sin\lambda-q_1q_4\sin\lambda+q_1q_3\cos\lambda\sin\nu+q_2q_4\cos\lambda\sin\nu)=0. \qquad (22)$$

TABLE 2. The coordinates of the base and moving pivots used to derive singularity variety for each of the special cases. s and c denote the sine and cosine functions.

Type 1(a)	$\mathbf{G}_1 = (0,0,1)$	$\mathbf{z}_1 = (0,0,1)$
	$\mathbf{G}_2 = (0,-s\mu,c\mu)$	$\mathbf{z}_2 = (0,-s\tau,c\tau)$
	$\mathbf{G}_3 = (0,-s\nu,c\nu)$	$\mathbf{z}_3 = (s\sigma,-s\rho c\sigma,c\rho c\sigma)$
Type 1(b)	$\mathbf{G}_1 = (0,0,1)$	$\mathbf{z}_1 = (0,0,1)$
	$\mathbf{G}_2 = (0,-s\mu,c\mu)$	$\mathbf{z}_2 = (0,-s\tau,c\tau)$
	$\mathbf{G}_3 = (s\lambda,-s\nu c\lambda,c\nu c\lambda)$	$\mathbf{z}_3 = (0,-s\rho,c\rho)$
Type 1(c)	$\mathbf{G}_1 = (0,0,1)$	$\mathbf{z}_1 = (0,0,1)$
	$\mathbf{G}_2 = (0,-s\mu,c\mu)$	$\mathbf{z}_2 = (0,-s\tau,c\tau)$
	$\mathbf{G}_3 = (0,-s\nu,c\nu)$	$\mathbf{z}_3 = (0,-s\rho,c\rho)$
Type 2(a)	$\mathbf{G}_1 = (0,0,1)$	$\mathbf{z}_1 = (0,0,1)$
	$\mathbf{G}_2 = (0,-s\mu,c\mu)$	$\mathbf{z}_2 = (0,-s\tau,c\tau)$
	$\mathbf{G}_3 = (s\lambda,-s\nu c\lambda,c\nu c\lambda)$	$\mathbf{z}_3 = (0,0,1)$
Type 2(b)	$\mathbf{G}_1 = (0,0,1)$	$\mathbf{z}_1 = (0,0,1)$
	$\mathbf{G}_2 = (0,-s\mu,c\mu)$	$\mathbf{z}_2 = (0,-s\tau,c\tau)$
	$\mathbf{G}_3 = (0,0,1)$	$\mathbf{z}_3 = (s\sigma,-s\rho c\sigma,c\rho c\sigma)$
Type 2(c)	$\mathbf{G}_1 = (0,0,1)$	$\mathbf{z}_1 = (0,0,1)$
	$\mathbf{G}_2 = (0,-s\mu,c\mu)$	$\mathbf{z}_2 = (0,-s\tau,c\tau)$
	$\mathbf{G}_3 = (0,0,1)$	$\mathbf{z}_3 = (0,-s\tau,c\tau)$
Type 3(a)	$\mathbf{G}_1 = (0,0,1)$	$\mathbf{z}_1 = (0,0,1)$
	$\mathbf{G}_2 = (0,-s\mu,c\mu)$	$\mathbf{z}_2 = (0,-s\tau,c\tau)$
	$\mathbf{G}_3 = (0,0,1)$	$\mathbf{z}_3 = (0,-s\rho,c\rho)$
Type 3(b)	$\mathbf{G}_1 = (0,0,1)$	$\mathbf{z}_1 = (0,0,1)$
	$\mathbf{G}_2 = (0,-s\mu,c\mu)$	$\mathbf{z}_2 = (0,-s\tau,c\tau)$
	$\mathbf{G}_3 = (0,-s\nu,c\nu)$	$\mathbf{z}_3 = (0,0,1)$

Two of the points $\mathbf{G}_i$ are coincident. In this case, the singularity variety of the platform is given by,

$$\mathscr{S}: \quad P_1P_2 = 4\sin\mu(q_1q_3\cos\tau + q_2q_4\cos\tau - q_1q_2\sin\tau + q_3q_4\sin\tau)$$
$$(q_1q_3\cos\sigma\sin\rho - q_2q_4\cos\sigma\sin\rho + q_2q_3\sin\sigma + q_1q_4\sin\sigma) = 0. \quad (23)$$

There are two sets of coincident points on both the moving and fixed bodies. The singularity variety of the platform is given by,

$$\mathscr{S}: \quad P_1P_2 = 4\sin\mu\sin\tau(q_1q_3 - q_2q_4)$$
$$(q_1q_3\cos\tau + q_2q_4\cos\tau - q_1q_2\sin\tau + q_3q_4\sin\tau) = 0. \quad (24)$$

The points $\mathbf{z}_i$ are on a great circle and two fixed points are coincident. In this case, the singularity variety of the platform is given by,

$$\mathscr{S}: \quad P_1P_2 = 4\sin\mu\sin\rho(q_1q_3 - q_2q_4)$$
$$(q_1q_3\cos\tau + q_2q_4\cos\tau - q_1q_2\sin\tau + q_3q_4\sin\tau) = 0. \quad (25)$$

The points $\mathbf{G}_i$ are on a great circle and two of the moving points are coincident. In this case, the singularity variety of the platform is given by,

$$\mathscr{S}: \quad P_1P_2 = 4\sin\nu\sin\tau(q_1q_3 + q_2q_4)$$
$$(-q_1q_3\cos\mu + q_2q_4\cos\mu + q_1q_2\sin\mu + q_3q_4\sin\mu) = 0. \quad (26)$$

Notice that for each of the cases that has a coincident pair of base pivots or moving pivots factors into the product of quadric surfaces $P_1P_2 = 0$.

We dehomogenize the quaternion coordinates with respect to q_4 to visualize the surfaces. This is done by making the substitution $x = q_1/q_4, y = q_2/q_4$ and $z = q_3/q_4$. For Type 1(c), set $\mu = 30^\circ$, $\nu = 60^\circ$, $\tau = 30^\circ$, $\rho = 60^\circ$ to obtain the quartic surface shown in Fig. 5. For Type 2(c), set $\tau = 60^\circ$ to obtain the two quadric surfaces shown in Figure 6. For Type 3(b), set $\mu = 60^\circ$ to obtain the two quadric surfaces shown in Figure 7.

7 CONCLUSIONS

In this paper we derive the constraint manifold of the 3SPS-S spherical parallel manipulator in quaternion coordinates. This formulation allows the derivation of the Jacobian and the calculation of a quartic singularity variety for this manipulator.

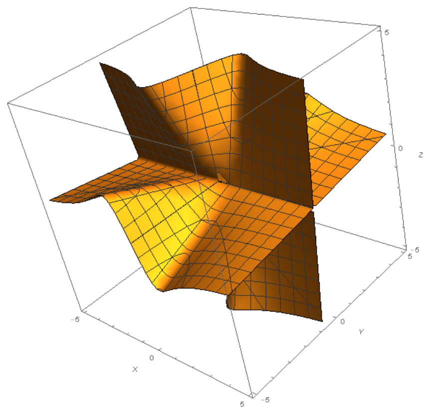

FIGURE 5. The singularity variety of the 3SPS-S spherical parallel manipulator that has both sets of points $\mathbf{G}_i$ and $\mathbf{z}_i$ lie on great circles.

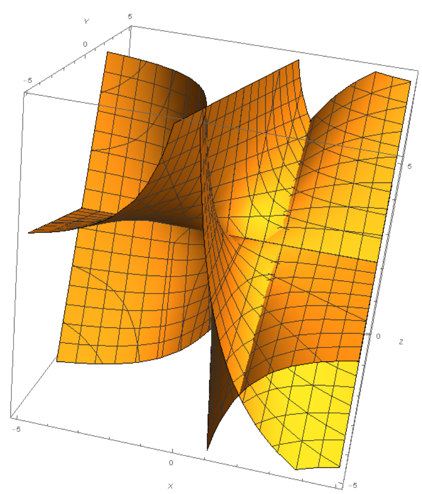

FIGURE 6. The singularity variety of the 3SPS-S spherical parallel manipulator that has two coincident points on the base and two coincident points on the platform, consists of two quadric surfaces P_1=0 and P_2=0.

We present the general case of the singularity variety and then consider eight special cases of these singularity varieties determined by the locations of the base and moving pivots of the 3SPS-S manipulator. In the cases where either a pair of base pivots or moving pivots are coincident, we find the singularity variety factors into two quadric surfaces.

Understanding how singularities are distributed within the manipulator workspace as a function of architectural parameters provides insight to singularity free design of parallel manipulators.

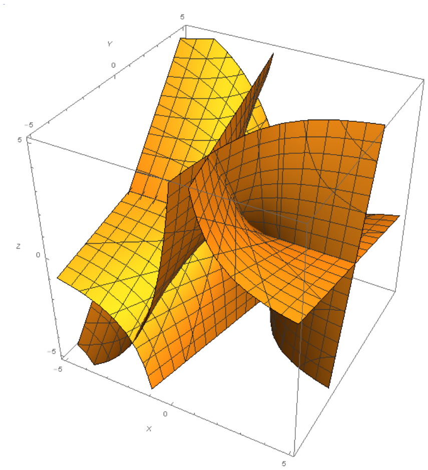

FIGURE 7. The singularity variety of the 3SPS-S spherical parallel manipulator that the points $\mathbf{G}_i$ are on a great circle and two of the moving points are coincident, consists of two quadric surfaces P_1=0 and P_2=0.

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