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LIMIT CYCLE ANALYSIS AND CONTROL OF THE DISSIPATIVE CHAPLYGIN SLEIGH

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ABSTRACT

There are many types of systems in both nature and technology that exhibit limit cycles under periodic forcing. Sometimes, especially in swimming robots, such forcing is used to propel a body forward in a plane. Due to the complexity in studying a fluid system it is often useful to investigate the dynamics of an analogous land model. Such analysis can then be useful in gaining insight about and controlling the original fluid system. In this paper we investigate the behavior of the Chaplygin sleigh under the effect of viscous dissipation and sinusoidal forcing. This is shown to behave in a similar manner as certain robotic fish models. We then apply limit cycle analysis techniques to predict the behavior and control the net translational velocity of the sleigh in a horizontal plane.

1 INTRODUCTION

The Chaplygin sleigh is a canonical system in the study of nonholonomic mechanics [1], [2]. There are multiple works which discuss control of the sleigh by various means such as a sliding mass or an internal rotor [3], [4], [5]. However a disadvantage of the classical Chaplygin sleigh is that the energy can only increase so full control over velocity is not possible. In [6], [7], Coulomb friction was considered as a means of energy dissipation. However this was accomplished through stick-slip motion. The sleigh was found to exhibit piecewise-smooth dynamics which is not ideal for finding a useful means of control. In this work we consider a variant of the Chaplygin sleigh in which we take into account viscous friction. Velocity based dissipation can be incorporated into a model with the use of Rayleigh dissipation functions and allows one to look for an efficient means of velocity and heading control.

The introduction of viscous friction is motivated by similarity between the Chaplygin sleigh and the hydrodynamic foil. In recent work [8], [9], interaction between the foil and the fluid through vortex shedding has been shown to be an affine nonholonomic constraint. Moreover, this constraint has a formal similarity to that of the constraint on the Chaplygin sleigh. In this work we show with simulations that the sleigh with viscous dissipation and periodic input from an internal rotor emulates the behavior of the foil under similar forcing [10]. Sinusoidal forcing causes the sleigh to propel itself forward with average velocity approaching a constant value. The sleigh being a comparatively simple mechanical system allows us to apply known analytical techniques to study the dynamics and find an efficient means of control. From this analysis we then get intuition similar on nonholonomic systems like the foil.

The proposed Chaplygin sleigh system is nonlinear, time-varying, underactuated, and contains drift terms. However due to the consistent limit cycle behavior of the sleigh, we are able to apply the harmonic balance method [11], [12] to predict its asymptotic behavior. With a slight modification to this method it becomes possible to control the average translational velocity of the sleigh by choosing the amplitude of the input signal. Our work can be applied to similar systems that exhibit periodic be-

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havior to produce some net motion.

2 Equations of Motion

A diagram of the sleigh system with all relevant kinematic variables may be found in Fig. 1 (a). The sleigh has mass m and moment of inertial I. Point P represents the sharp knife edge at which the sleigh is not allowed to slip in the transverse direction. The axes X_b and Y_b are body fixed where X_b is aligned with the line between P and the center of gravity. The position of the center of the sleigh is denoted by (x,y) and the orientation of the sleigh is θ. The distance between P and the center of gravity is b. The sleigh carries a balanced rotor (I_r), whose center coincides with the center of mass of the sleigh. The relative angle that the rotor makes with the body axes is denoted by ϕ. The configuration space for the system may be written as $Q = SE2 \times S^1$. The equations of motion are derived herein using the Lagrange multiplier method. The configuration space of the system is parameterized by the variables $q = (x,y,\theta,\phi)$ and $\dot{q} = (\dot{x},\dot{y},\dot{\theta},\dot{\phi})$. Since there are no potential forces, the Lagrangian turns out to be the kinetic energy (1).

$$\mathscr{L} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}I_r(\dot{\theta}+\dot{\phi})^2 \tag{1}$$

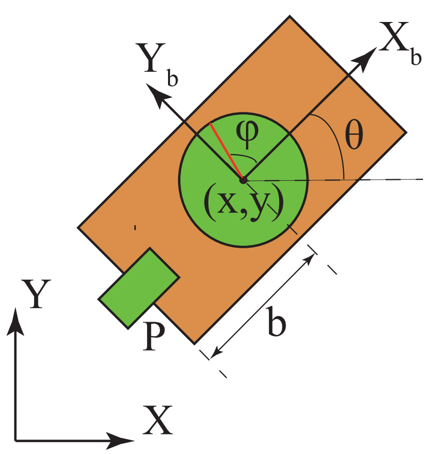

FIGURE 1. CHAPLYGIN SLEIGH WITH A BALANCED ROTOR. THE ROTOR IS PLACED AT DISTANCE OF b FROM THE REAR CONTACT.

The system is subject to the following nonholonomic constraint (2), which ensures that the transverse velocity (along the Y_b direction) of the point of contact P be equal to zero,

$$-\dot{x}\sin\theta + \dot{y}\cos\theta - b\dot{\theta} = 0 \tag{2}$$

with Pfaffian one form being

$$-\sin\theta dx + \cos\theta dy - bd\theta = 0.$$

We further assume the presence of viscous dissipation at the rear wheel. This can be accounted for using the following Rayleigh dissipation function

$$R_w = \frac{1}{2}c(\dot{x}\cos(\theta) + \dot{y}\sin(\theta)).$$

The Euler-Lagrange equations are of the following form

$$\frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot{q}_k}\right) - \frac{\partial\mathscr{L}}{\partial q_k} = C_k\lambda + Q_{q_k} \tag{3}$$

where λ is the Lagrange multiplier, C_k is the coefficient corresponding to one forms dq_k and $Q_{q_k} = -\frac{\partial R_w}{\partial q_k}$ is the dissipation force due to dissipation at the wheel. The dissipation forces are calculated as

$$\begin{aligned} Q_x &= -c(\dot{x}\cos^2(\theta) + \dot{y}\sin(\theta)\cos(\theta)) \\ Q_y &= -c(\dot{y}\sin^2(\theta) + \dot{x}\sin(\theta)\cos(\theta)) \\ Q_\omega &= 0. \end{aligned}$$

With this formulation the following Euler-Lagrange equations are readily obtained to be

$$\begin{aligned} m\ddot{x} &= -\lambda\sin(\theta) + Q_x \\ m\ddot{y} &= \lambda\cos(\theta) + Q_y \\ m\ddot{\theta} &= -b\lambda - I_r\ddot{\phi} + Q_\theta. \end{aligned}$$

In order to eliminate the constraint force and obtain the reduced equations of motion, the velocities of the center of the cart can be written in terms of the velocity of the point P and the angular velocity of the sleigh, $\omega = \dot{\theta}$.

$$\dot{x} = u\cos\theta - \omega b\sin\theta \tag{4}$$
$$\dot{y} = u\sin\theta + \omega b\cos\theta \tag{5}$$

$$\ddot{x} = \dot{u}_x\cos\theta - u_x\omega\sin\theta - \omega^2 b\cos\theta - \dot{\omega}b\sin\theta$$
$$\ddot{y} = \dot{u}_x\sin\theta + u_x\omega\cos\theta - \omega^2 b\sin\theta + \dot{\omega}b\cos\theta$$

Using the above expressions, the equations of motion can be reduced to

$$\dot{u} = b\omega^2 - \frac{k}{m}u \tag{6}$$
$$\dot{\omega} = \frac{-mbu\omega - I_r\ddot{\phi}}{I + I_r + mb^2} \tag{7}$$
$$\dot{\theta} = \omega. \tag{8}$$

In the absence of any actuation from the rotor, i.e. $\ddot{\phi}$, the system has one globally asymptotically stable fixed point at $(u,\omega) = (0,0)$. It was found through simulation that sinusoidal inputs produce net forward motion in the sleigh. Furthermore, this motion reaches a 'steady state' with the numerical simulation indicating that a limit cycle is born in the (u,ω,θ) space. We describe this behavior in detail in section 3.

3 Limit Cycle Behavior of the Chaplygin Sleigh

The input to the dynamical system (6)-(8) is $\ddot{\phi}$ or equivalently the torque exerted by the rotor on the sleigh, $-I_r\ddot{\phi}$. Figure 2 shows the results of a simulation of (6)-(8) under input of the form

$$-I_r\ddot{\phi} = A\sin(\Omega t) \tag{9}$$

In Fig. 2 (a) we see that sinusoidal forcing causes the sleigh to perform a snake-like motion in some fixed direction θ_c. We also see from Fig. 2 (b), (c), and (d) that there appears to be a stable limit cycle in (u,ω) as well as in (u,ω,θ). This becomes clear when we plot the trajectory in the (u,ω) plane.

In Fig. 3 we see that, indeed u and ω appear to be converging to a stable limit cycle that looks like a figure 8 in (u,ω) space. The figure 8 shape can be explained by the fact that the frequency of oscillations in u is double the frequency of oscillations in ω.

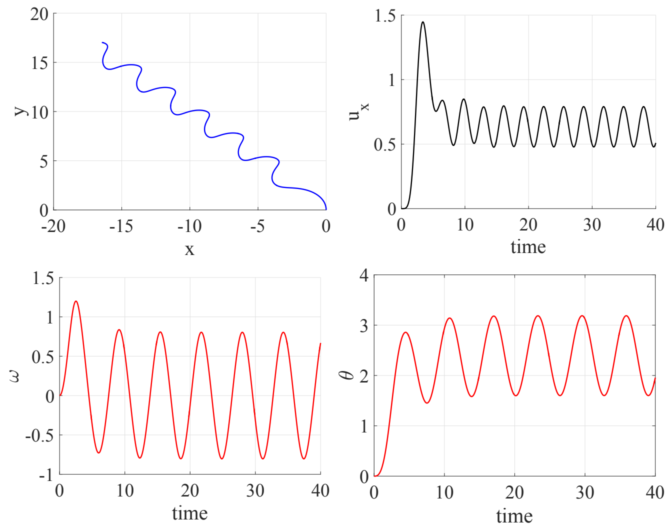

FIGURE 2. SIMULATION OF CHAPLYGIN SLEIGH WITH $(u(0),\omega(0)) = (0,0)$. THE INPUT IS IN FORM (9) WITH $A = 2$ AND $\Omega = 1$.

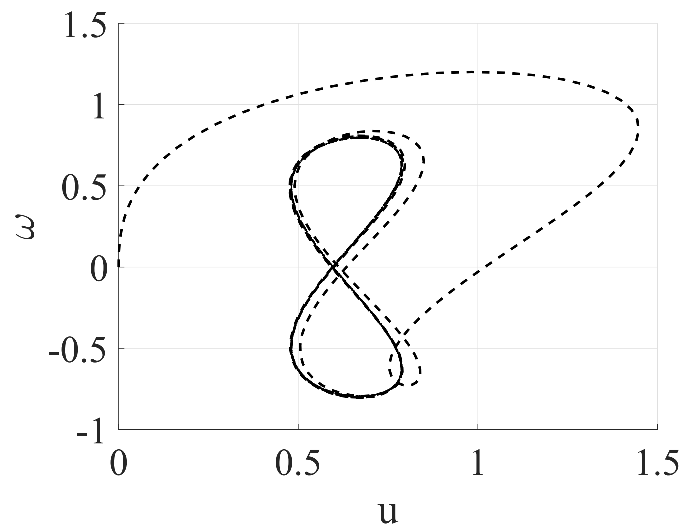

FIGURE 3. TRAJECTORY OF CHAPLYGIN SLEIGH WITH $(u(0),\omega(0)) = (0,0)$ (DASHED LINE) AND PREDICTED LIMIT CYCLE FROM HARMONIC BALANCE METHOD (SOLID LINE). THE INPUT IS IN FORM (9) WITH $A = 2$ AND $\Omega = 1$.

From Fig. 2 (a) it is clear that sinusoidal forcing can cause the sleigh to exhibit useful motion. Furthermore, it appears that on average the sleigh is translating along θ_c with some fixed average velocity v_{net}. Let us denote the period of the limit cycle by T. We define v_{net} as follows. Let $(x(t_1), y(t_1))$ be a point on the limit cycle, then v_{net} can be expressed as

$$v_{net} = \frac{1}{T}\sqrt{(x(t_1+T)-x(t_1))^2+(y(t_1+T)-y(t_1))^2}. \quad (10)$$

The variable v_{net} is the average translational velocity of the center of the sleigh. It can be seen equivalently as the translation of the sleigh over one limit cycle divided by the time period. The average velocity v_{net} is the velocity which would actually be useful in developing motion planning for the sleigh. The instantaneous velocity of the center only tells us how quickly the sleigh is moving along its path whereas v_{net} gives the net velocity in the (x,y) plane. Since v_{net} is defined in terms of the time period of the limit cycle, it must be constant on the limit cycle.

In this paper we address the problem of controlling v_{net} by choosing A. First we employ the harmonic balance technique to show that given A the limit cycle can be accurately predicted. Then we take A to be an unknown and apply averaging to reduce the problem of controlling v_{net} to solving a nonlinear system of equations. We then employ the Newton-Raphson algorithm to solve the system and obtain A. Finally, we check the accuracy of our prediction with simulations.

4 Limit Cycle Prediction Using Harmonic Balance Method

The harmonic balance technique assumes the outputs of a system to be sinusoidal and attempts to use the equations of motion to predict the limiting trajectory. From simulations we see that the assumption of a sinusoidal solution is not unreasonable. We can also use the fact that u appears to be on frequency 2Ω and assume solutions of the form

$$u = u_c + A_u\sin(2\Omega t) + B_u\cos(2\Omega t) \quad (11)$$

$$\omega = A_w\sin(\Omega t) + B_w\cos(\Omega t). \quad (12)$$

In order for this solution to exist it must satisfy (6)-(7). Substituting (11) and (12) into (6)-(7) and simplifying the expressions we obtain

$$\begin{aligned}\dot{u} &= A_w^2bm + B_w^2bm - 2cu_c + (2A_wB_wb - 2A_uc)\sin(2\Omega t)\\ &\quad + (-A_w^2bm + B_w^2bm - 2B_uc)\cos(2\Omega t)...\\ \dot{\omega} &= \frac{(-A_uB_wbm + A_wB_ubm - 2A_wbmu_c + 2A)}{mb^2+I+I_r}\sin(\Omega t)\\ &\quad + \frac{(-A_uA_wbm - B_uB_wbm - 2B_wbmu_c)}{mb^2+I+I_r}\cos(\Omega t)...\end{aligned}$$

The higher harmonics are neglected. From simply differentiating (11) and (12), we get

$$\begin{aligned}\dot{u} &= -2\Omega B_u\sin(2\Omega t) + 2\Omega A_u\cos(2\Omega t)\\ \dot{\omega} &= -\Omega B_w\sin(\Omega t) + \Omega A_w\cos(\Omega t).\end{aligned}$$

To determine u_c and the coefficients A_u, B_u, A_w and B_w we simply equate the coefficients of the above two systems. At this point we define $\alpha = mb^2 + I + I_r$ to avoid long expressions. This yields the following system of nonlinear equations

$$\begin{aligned}0 &= A_w^2bm + B_w^2bm - 2cu_c\\ -4m\Omega B_u &= 2A_wB_wb - 2A_uc\\ 4m\Omega A_u &= -A_w^2bm + B_w^2bm - 2B_uc\\ -2\alpha\Omega B_w &= -A_uB_wbm + A_wB_ubm - 2A_wb^2c - 2A_wbmu_c + 2A\\ 2\alpha\Omega A_w &= -A_uA_wbm - B_uB_wbm - 2B_wb^2c - 2B_wbmu_c. \quad (13)\end{aligned}$$

The equations are highly coupled, so an algebraic solution is difficult to find. We employ the Newton-Raphson to solve the equations numerically. In general it is not possible to guarantee a solution using this method, however it was found to be consistent for this problem. By chosing the parameters and the input to be the same as for Fig. 3 we can solve (13) and verify whether the expected limit cycle matches closely with the simulated trajectory. Performing the calculation yields $(u_c, A_u, B_u, A_w, B_w) = (0.6341, 0.1103, 0.1071, 0.2069, -0.7689)$ after 4 iterations. Since we do not have any information about the phase difference in solutions, we may define $C_u = \sqrt{A_u^2+B_u^2}$ and $C_w = \sqrt{A_w^2+B_w^2}$, the net amplitudes of the signals. This allows us to define the error between the solution and the actual trajectory as

$$e = \sqrt{(u_c-u_c^*)^2+(C_u-C_u^*)^2+(C_w-C_w^*)^2} \quad (14)$$

where superscript $*$ denotes the amplitude and center values obtained from the simulations. The error was found to be $e = 8.6e-3$. We see that there is close agreement between the simulations and the values obtained from harmonic balance method. In Fig. 3 we can see further agreement between the analytical solution of the limit cycle and one obtained through direct numerics. The dotted graph shows a trajectory with generic initial values of (u,ω) converging to the analytically predicted limit cycle (solid line).

5 Feedforward velocity control of the sleigh

The harmonic balance method has proven successful in predicting the limit cycle trajectory of the dissipative Chaplygin sleigh. However, in order for this technique to be useful for control, it would be beneficial to have the ability to prescribe a desired motion for the sleigh and obtain the input required to produce that behavior. The asymptotic angle θ_c is due to the transient phase of the motion and it is not predicted by the Harmonic balance approach. However special cases can treated and in this paper we develop a technique for controlling v_{net}.

5.1 Derivation of v_{net} and control approach

Equations (6) and (7) are independent of θ, so the eventual value of the heading does not influence the average speed of the sleigh. Therefore in calculating v_{net} we may set $\theta_c = 0$. This simplifies v_{net} to

$$v_{net} = \frac{1}{T}(x(t_1+T) - x(t_1)). \tag{15}$$

This is equivalent to the averaging problem

$$v_{net} = \frac{1}{T}\int_{t_1}^{t_1+T} \dot{x}dt \tag{16}$$

which can then be put in terms of u and ω using (4)

$$v_{net} = \frac{1}{T}\int_{t_1}^{t_1+T} u\cos\theta - \omega b \sin\theta dt.$$

Since v_{net} is defined for any t_1 we may chose $t_1 = 0$. We also split up the above integral using the distributive property

$$v_{net} = \frac{1}{T}\int_0^T u\cos\theta dt - \frac{1}{T}\int_0^T \omega b\sin\theta dt. \tag{17}$$

Recall that $\omega = \dot{\theta}$. Using this and $\frac{d\theta}{dt} = \dot{\theta}$ the second integral of (17) simplifies to

$$-\frac{b}{T}\int_{\theta(0)}^{\theta(T)} \sin\theta d\theta = \frac{b}{T}(\cos(\theta(T)) - \cos(\theta(0)).$$

Now we may use the fact that v_{net} is defined on the limit cycle again and the result we obtained using harmonic balance method to assume the solutions for u and ω defined by (11)-(12) must hold. Furthermore we can obtain $\theta(t)$ by integrating ω

$$\begin{aligned}\theta(t, A_w, B_w) &= \int \omega dt \\ &= \int A_w \sin(\Omega t) + B_w\cos(\Omega t)dt \\ &= \frac{B_w}{\Omega}\sin(\Omega t) - \frac{A_w}{\Omega}\cos(\Omega t)\end{aligned}$$

Notice that from (17) we now get an expression for v_{net} in terms of u_c, A_u, B_u, A_w, and B_w.

$$\begin{aligned}v_{net} = &\frac{1}{T}\int_0^T u(t, u_c, A_u, B_u)\cos(\theta(t, A_w, B_w))dt \\ &+ \frac{b}{T}\big(\cos(\theta(T, A_w, B_w)) \\ &- \cos(\theta(0, A_w, B_w))\big).\end{aligned} \tag{18}$$

The integral leftover in (18) cannot be evaluated analytically, however in our formulation it will not matter since an iterative algorithm can solve the integral numerically during each iteration. By now taking A to be an unknown (18) together with (13) form a system of 6 equations and 6 unknowns $(u_c, A_u, B_u, A_w, B_w, A)$. Solving this system for some desired v_{net} and finding the A required will allow us to control the velocity of the sleigh. In order to attempt to solve the system we once again employ the Newton-Raphson algorithm.

5.2 Feedforward control of the Chaplygin sleigh

Suppose we want the sleigh to be on a limit cycle such that $v_{net} = 0.2$. Solving system (18), (13) yields the solution $(u_c, A_u, B_u, A_w, B_w, A) = (0.2106, 0.0466, 0.0208, -0.0400, 0.4571, -1.1403)$ after 9 iterations. This means that we must chose $A = -1.1403$ to get a translational speed of $v_{net} = 0.2$. A natural definition of the error in the analytical approximation due to the Harmonic balance approach is

$$e_v = ||v_{net} - v_{net}^*|| \tag{19}$$

where v_{net}^* is the v_{net} found from the simulation using (15). A simulation of the sleigh under the chosen input is shown in Fig. 4.

The actual net translational velocity was found to be $v_{net}^* = 0.20049$ giving us an error of $4.9e-4$. A plot of the error against

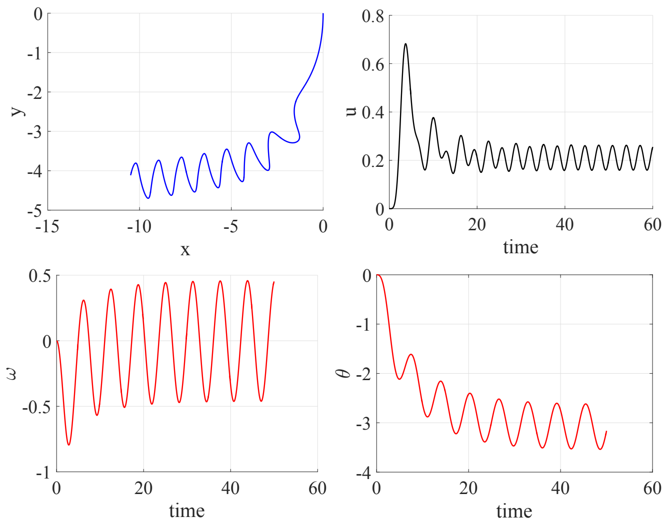

FIGURE 4. SIMULATION OF CHAPLYGIN SLEIGH WITH $(u(0), \omega(0)) = (0,0)$. THE INPUT IS IN FORM (9) WITH $A = -1.1403$ AND $\Omega = 1$.

t_2 can be seen in Fig. 5. It was shown that the proposed approach can quickly and accurately calculate required input amplitudes to produce desired motion in the (x, y) plane.

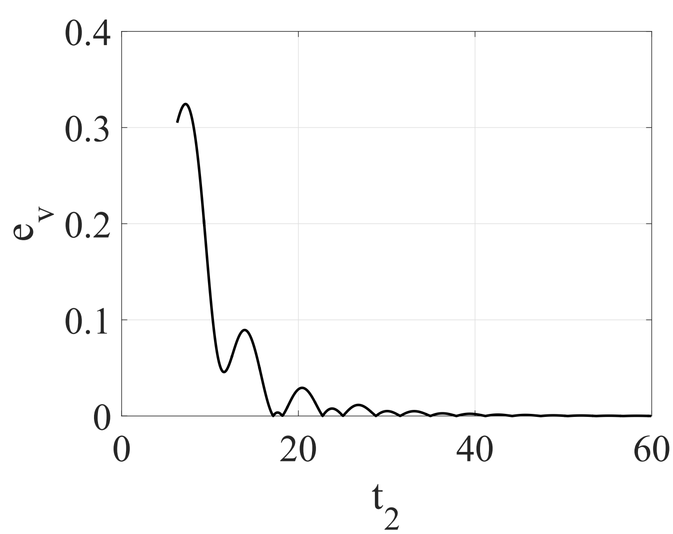

FIGURE 5. ERROR IN v_{net} FOR THE CHAPLYGIN SLEIGH WITH $(u(0), \omega(0)) = (0,0)$. THE INPUT IS IN FORM (9) WITH $A = -1.1403$ AND $\Omega = 1$.

6 Conclusion and Discussion

The Chaplygin sleigh was found to exhibit a stable limit cycle under the effect of viscous dissipation. We were able to calculate a limit cycle solution with the use of the harmonic balance method. Furthermore, with a small adjustment to the harmonic balance method we were able to predict its net translational velocity v_{net} and control it by adjusting the amplitude of the input signal. The utility of the proposed approach was demonstrated through simulations and error estimates. Control of heading (θ_c) will be discussed in future work, however preliminary work suggests this can be accomplished with some type of state feedback.

The harmonic balance method is robust to both time variant and time invariant systems. Our work can be applied other nonholonomic dynamic systems that are subjected to periodic inputs and exhibit limit cycle behavior.

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