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Realization of point planar elastic behaviors using revolute joint serial mechanisms having specified link lengths

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ABSTRACT

This paper presents methods for the realization of 2 × 2 translational compliance matrices using serial mechanisms having only revolute joints, each with selectable compliance. The link lengths of the mechanism and the location of the compliant frame relative to the mechanism base are arbitrary but specified. The realizability of a given compliant behavior is investigated, and necessary and sufficient conditions for the realization of a given compliance with a given mechanism are obtained. These realization conditions are interpreted in terms of geometric relationships among the joints. We show that, for an appropriately sized 3R serial mechanism, any single 2 × 2 compliance matrix can be realized by properly choosing the joint compliances and the mechanism configuration. Requirements on mechanism geometry to realize every particle planar elastic behavior at a given location just by changing the mechanism configuration are also identified.

1. Introduction

In order for robots to achieve human-like dexterity in manipulation tasks, a means for regulating interaction force is needed, especially when geometric conflict exists between the robot and its environment. One approach to force regulation is to select or control the robot passive compliance. Passive compliance allows for high speed interaction. Desirable robot passive compliance can be attained by: 1) designing compliant end-of-arm tooling, or 2) designing compliance into the arm itself.

Any desired compliant behavior can be realized through the proper design of an end-effector mounted compliant wrist. It is known that for any specified elastic behavior, an infinite number of parallel [1,2] or serial [3,4] mechanisms having springs at each joint can achieve the desired behavior. Proper compliant wrist design involves selecting the geometry and configuration of the mechanism and selecting the spring rates at each joint. The limitation of this approach is that the mechanism realizes a single specified compliance. If the robot task changes or the desired compliance within the task changes, an entirely different compliant wrist would likely be needed for its realization.

Several strategies exist for designing compliance into the manipulator itself. Series elastic actuators (SEAs) [5], in which passive compliance is provided between the motor and the connected link, can be used to provide a selected amount of compliance in each joint. Variable stiffness actuators (VSAs) [6] are similar, but they allow joint compliance to be changed in real time. This ability is important in robot tasks that require continuously changing compliant behaviors, such as when the nature of the task changes, the task itself changes, or the task environment changes (especially when people are in the robot workspace).

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VSAs enlarge the space of elastic behavior that can be achieved by a robot by changing each joint stiffness. If the manipulator is kinematically redundant, the set of achievable elastic behaviors is increased further. A much larger set of compliances can be attained by adjusting both the joint stiffnesses and the manipulator configuration, without ever changing the endpoint location of the robot.

In some manipulation tasks, such as turning a crank or opening a door, the interaction torque is not important. Since only the relationship between force and translation is important, the interaction can be modeled as point contact, and the compliant behavior can be modeled as an elastically suspended particle, i.e., not an elastically suspended rigid body. This paper presents methods for the evaluation and realization of planar translational elastic behaviors using serial mechanisms having only revolute joints, each with selectable compliance. An example of the type of mechanism considered is illustrated in Fig. 1. The mechanisms considered are arbitrary but have known link lengths.

1.1. Related work

Spatial elastic behavior has been *analyzed* for more than a century. Screw theory [7–11] and Lie groups [12] have been used to analyze and characterize spatial linear elastic behavior (represented by a 6×6 symmetric stiffness matrix $\mathbf{K}$ or compliance matrix $\mathbf{C}$).

More recent work has addressed the *realization* of spatial elastic behavior through the design of passive compliant mechanisms. Previously, the bounds of elastic behaviors achieved with simple parallel mechanisms [1] and with simple serial mechanisms [3] have been identified. A simple parallel mechanism contains only line springs and torsional springs, and a simple serial mechanism contains only prismatic and revolute joints. Synthesis procedures have been developed [1,3,13,14] to realize any elastic behaviors within these bounds. Geometry based screw theory approaches for the realization of an arbitrary realizable stiffness matrix have also been developed [15,16].

The synthesis of an *arbitrary* spatial stiffness matrix with a parallel system has also been addressed [2]. It was shown [2] that in order to realize an arbitrary stiffness, *screw springs* that couple the elastic behavior in translation and rotation in the same direction must be used. The eigen-structure of a stiffness was analyzed using the eigenscrew decomposition of the stiffness matrix [17]. Based on the eigenstiffness analysis, a procedure to calculate the minimum number of screw springs required to realize a specified stiffness matrix in a parallel system was identified [18]. A procedure has been developed [19] to realize an arbitrary spatial stiffness using the minimum numbers of screw springs and simple springs.

This previous work has focused on the realization of a *single* compliance with a mechanism of unspecified geometry at a unspecified configuration. When the desired compliance is changed, a different mechanism with a new configuration must be used. Very little work has addressed: 1) the ability of a single specified mechanism to realize an arbitrary compliance, and 2) the identification of the space of realizable compliances that can be achieved with a mechanism having specified link lengths.

The inability of a 2R compliant joint manipulator to realize general particle planar compliance was noted in [20]. It was also noted that, for a 3R manipulator, the surjectivity of the mapping from 2×2 Cartesian space to joint space compliance is not guaranteed. These limitations were identified as motivation for using optimization (using various norms as objective functions) for approximating the behavior of a targeted Cartesian compliance matrix. More recently, the ability of a manipulator to realize a specified point planar compliance using 3R serial mechanisms with known link lengths was addressed [21]. Optimization was used to identify the combination of mechanism configuration and joint stiffnesses that minimizes the deviation of the attainable Cartesian compliance from a targeted value. The proximity of an optimized compliance to a targeted compliance (measured using the matrix Frobenius norm) was characterized for a set of configurations of the redundant manipulator and for a set of compliant behaviors using a "stiffability map."

The major limitations of using optimization for the realization problem are that: 1) optimization does not ensure that any specified compliance can be achieved even with a large number of redundant joints; 2) the optimal solution may not preserve the desired compliant behavior of the original compliance; and 3) the results obtained from optimization provide little physical insight into the limits of passive compliance realization.

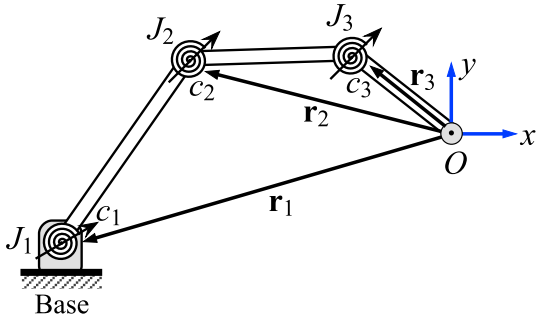

Fig. 1. A 3R serial compliant mechanism with variable stiffness actuators.

In more recent work [22–24], the realization of isotropic compliance in the Euclidian spaces $E(2)$ and $E(3)$ with robotic mechanisms has been addressed. Equations relating the Cartesian compliance entries to the joint compliance and the mechanism geometry were developed. In these papers, the joint compliances are calculated by solving these equations. The approach applies to a general serial mechanism with revolute and prismatic joints. However, the existence of a physical solution ($k_i \geq 0$) to the equations is not guaranteed, and, for many configurations, a solution cannot exist. The set of mechanism configurations for which a solution exists was not identified.

This paper addresses the guaranteed realization of planar translational elastic behaviors with a serial mechanism having specified link lengths. The serial mechanisms considered have only revolute joints each loaded with a joint compliance. The compliance behaviors considered are arbitrary in $E(2)$. This work is motivated by the need for a single manipulator to realize *any* selected planar compliance at any selected endpoint. The paper presents the theory and methods to design a *single* manipulator that is able to achieve *all* compliances by changing its configuration and joint compliances. It also presents the theory and methods for the selection of mechanism configuration and joint stiffnesses needed to achieve any specified elastic behavior. The conditions for the realization of a given compliance are interpreted in terms of geometric relationships among the mechanism joints. The main contributions of the paper are:

1. Necessary and sufficient conditions for a mechanism to realize an arbitrary but given compliance at a given configuration are identified;
2. Synthesis procedures for the realization of a realizable compliance at a given location are developed;
3. Classes of mechanisms that facilitate compliance realization are identified. Necessary and sufficient conditions on the mechanism link lengths that ensure that *all* compliant behaviors can be realized are identified.

1.2. Technical background

Consider a planar serial mechanism having n joints, J_i, with joint compliance $c_i > 0$ and joint twists $\mathbf{t}_i$. Then the compliance matrix $\mathbf{C}$ at the mechanism endpoint is a symmetric positive semidefinite (PSD) matrix that can be expressed [4] as:

$$\mathbf{C} = c_1\mathbf{t}_1\mathbf{t}_1^T + c_2\mathbf{t}_2\mathbf{t}_2^T + \cdots + c_n\mathbf{t}_n\mathbf{t}_n^T \tag{1}$$

where the twists $\mathbf{t}_i$ are described relative to the compliance frame (the reference frame where the compliance matrix is specified). Each compliant joint provides a rank-1 PSD component:

$$\mathbf{C}_i = c_i\mathbf{t}_i\mathbf{t}_i^T. \tag{2}$$

It is known that decomposition (1) is not unique; different decompositions yield different mechanism geometries and configurations that each realizes the given compliance. Thus, decomposition (1) cannot be applied directly to a given mechanism with specified link lengths. A new method that considers mechanism geometry (fixed link lengths) to obtain an appropriate decomposition (1) is needed.

For the mechanism illustrated in Fig. 1, in which the compliance matrix is expressed in the frame at point O, each joint twist is given as:

$$\mathbf{t}_i = \mathbf{r}_i \times \mathbf{k}$$

where $\mathbf{r}_i$ is the position vector from O to joint i and $\mathbf{k}$ is the unit vector perpendicular to the plane. Thus, the joint twist $\mathbf{t}_i$ is determined by the position vector $\mathbf{r}_i$.

To realize a given compliance with a given mechanism, the mechanism configuration and the joint compliance c_i must be adjusted so that Eq. (1) is satisfied.

For a suspended particle, the compliance is a 2×2 PSD matrix. If the position vector for joint-i is:

$$\mathbf{r}_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix},$$

then the corresponding joint twist, perpendicular to $\mathbf{r}_i$, is:

$$\mathbf{t}_i = \begin{bmatrix} y_i \\ -x_i \end{bmatrix}.$$

From Eq. (2), it can be seen that $\mathbf{t}_i$ and $-\mathbf{t}_i$ yield the same joint compliance $\mathbf{C}_i$. Thus, $\mathbf{t}_i$ can be obtained by rotating $\mathbf{r}_i$ either clockwise or counterclockwise about the coordinate frame origin. Mathematically,

$$\mathbf{t}_i = \pm\boldsymbol{\Omega}\mathbf{r}_i \tag{3}$$

where $\mathbf{\Omega}$ is a 2 × 2 anti-symmetric matrix associated with a vector cross-product, given as:

$$\mathbf{\Omega} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{4}$$

It is known that the space of PSD matrices is a cone. All rank-1 PSD matrices lie on the boundary of the cone. Thus, each joint compliance matrix $\mathbf{C}_i$ in Eq. (2) is a point on the boundary of the PSD cone. If c_i is allowed to vary in the range $0 < c_i < \infty$, $\mathbf{C}_i$ represents a ray in the 2 × 2 PSD space, an edge of the PSD cone.

If a desired 2 × 2 PSD compliance matrix $\mathbf{C}$ is to be realized at the endpoint of a redundant 3R mechanism in which each joint has selectable compliance (as illustrated in Fig. 1), then, for a specified configuration, the three joint twists $\mathbf{t}_i$ are known. The desired $\mathbf{C}$ can be realized if and only if there exist non-negative coefficients c_i such that

$$\mathbf{C} = c_1\mathbf{t}_1\mathbf{t}_1^T + c_2\mathbf{t}_2\mathbf{t}_2^T + c_3\mathbf{t}_3\mathbf{t}_3^T. \tag{5}$$

For any 2 × 2 symmetric matrix $\mathbf{A} = [a_{ij}]$, only three entries are independent, therefore, $\mathbf{A}$ can be represented by a 3-vector $\hat{\mathrm{A}} = [a_{11}, a_{12}, a_{22}]^T$. With this operation, Eq. (5) can be written in vector form as:

$$\hat{C} = c_1\hat{T}_1 + c_2\hat{T}_2 + c_3\hat{T}_3 = \left[\hat{T}_1, \hat{T}_2, \hat{T}_3\right]\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},$$

where $\hat{C}$ is the 3-vector representation of $\mathbf{C}$; and $\hat{T}_i$ is the 3-vector representation of $\mathbf{t}_i\mathbf{t}_i^T$. If we denote

$$\hat{\mathbf{T}} = \left[\hat{T}_1, \hat{T}_2, \hat{T}_3\right], \mathbf{c} = [c_1, c_2, c_3]^T,$$

then,

$$\mathbf{c} = \hat{\mathbf{T}}^{-1}\hat{C}. \tag{6}$$

Thus, $\mathbf{C}$ can be realized at the given configuration if and only if $\mathbf{c} \geq 0$.

In the 2 × 2 PSD space, the condition $\mathbf{c} \geq 0$ indicates that $\mathbf{C}$ must be inside the polyhedral convex cone defined by the three edges $\mathbf{t}_1\mathbf{t}_1^T$, $\mathbf{t}_2\mathbf{t}_2^T$, and $\mathbf{t}_3\mathbf{t}_3^T$ as shown in Fig. 2. Because the manipulator is redundant, many configurations yield the same manipulator endpoint. When the mechanism changes configuration, the edge locations on the PSD cone's boundary also change. The solution is tractable because the dimension of the problem is the same as that of the manipulator. Although Eq. (6) can be used to determine the realizability of $\mathbf{C}$ for a *given* mechanism configuration, it does not provide insight into the geometric relations among the joints that are required for compliance matrix realization (i.e, to ensure that $\mathbf{C}$ is contained within the polyhedral convex cone bounded by its edges). For example, if J_1 and J_2 are specified, the space of J_3 locations that allows $\mathbf{C}$ to be realized is unknown. Also, Eq. (6) cannot be used to assess whether a mechanism can realize every compliance at a given location in the workspace.

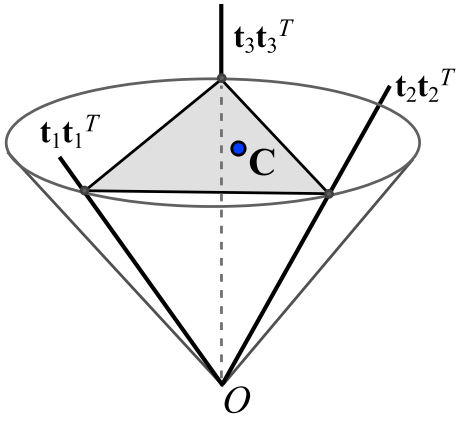

Fig. 2. Realizable condition for a given compliance $\mathbf{C}$ with a 3R mechanism having a specified configuration. $\mathbf{C}$ must be inside of a polyhedral cone with edges $\mathbf{t}_1\mathbf{t}_1^T$, $\mathbf{t}_2\mathbf{t}_2^T$, and $\mathbf{t}_3\mathbf{t}_3^T$.

1.3. Overview

This paper presents means to analyze and realize planar translational elastic behaviors using revolute joint serial mechanisms. The means for analysis and synthesis are based on the geometry of the realizable compliance matrix space and the geometry of the mechanism. This approach provides physical insight into elastic mechanism behavior. The ability of a *single* mechanism with specified link lengths to realize *all* compliance behaviors at a given end-point location is also investigated. In this paper, we assume that the range of joint compliances is $0 < c_i < \infty$. Joint compliances can be obtained via one of the following approaches: 1) providing an appropriate (passive) spring at each joint; 2) providing an appropriate stiffness serial elastic actuator (SEA) at each joint; or 3) providing a variable stiffness actuator (VSA) at each joint.

Section 2 identifies the limits of a 2R mechanism to realize a specified 2×2 compliance matrix. A necessary and sufficient condition for a compliance to be realized by a 2R mechanism is identified. Section 3 presents an approach to realize a given compliance with a 3R mechanism based on mechanism geometry. The method then is extended to n-R mechanisms. Section 4 addresses the ability of a single mechanism to realize all compliances at a given location just by changing the mechanism configuration and/or joint stiffnesses. A necessary and sufficient condition on the mechanism geometry to realize all compliances is identified. Synthesis procedures to achieve a realizable compliance are presented in Section 5. A numerical example illustrating the synthesis procedures for a given mechanism is provided in Section 6. Finally, a brief discussion and summary are presented in Section 7.

2. Limits of a 2R mechanism

To realize a full-rank 2×2 compliance matrix, a mechanism having at least two compliant joints is required. Here, the limitation of a 2R serial mechanism with fixed link lengths and configuration is considered. A necessary and sufficient condition for a specified compliance to be realized with the mechanism is identified. These results allow the characterization of the realizable space of compliances in terms of the geometry of the mechanism.

Consider the 2R serial mechanism shown in Fig. 3. The mechanism endpoint O is the location of interest (where the compliance frame is located) and $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors indicating the locations of joint-1 and joint-2 relative to O. If the position of O relative to J_1 (the location in the workspace) is specified, the configuration of the mechanism cannot change, and the two position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ cannot vary.

The space of realizable elastic behaviors is given by:

Proposition 1. *A compliance matrix* $\mathbf{C}$ *can be realized with a 2R mechanism in a given configuration if and only if:*

$$\mathbf{r}_1^T \mathbf{C} \mathbf{r}_2 = 0. \tag{7}$$

Proof. If $\mathbf{C}$ is realized by the mechanism, then, by Eq. (1), there exist positive constants, the joint compliances, c_1 and c_2, that satisfy

$$\mathbf{C} = c_1 \mathbf{t}_1 \mathbf{t}_1^T + c_2 \mathbf{t}_2 \mathbf{t}_2^T.$$

Since $\mathbf{r}_1 \perp \mathbf{t}_1$ and $\mathbf{r}_2 \perp \mathbf{t}_2$, then $\mathbf{r}_1^T \mathbf{t}_1 = 0$ and $\mathbf{r}_2^T \mathbf{t}_2 = 0$. Thus,

$$\begin{aligned} \mathbf{r}_1^T \mathbf{C} \mathbf{r}_2 &= \mathbf{r}_1^T \left(c_1 \mathbf{t}_1 \mathbf{t}_1^T + c_2 \mathbf{t}_2 \mathbf{t}_2^T \right) \mathbf{r}_2 \\ &= c_1 \left(\mathbf{r}_1^T \mathbf{t}_1 \right) \left(\mathbf{t}_1^T \mathbf{r}_2 \right) + c_2 \left(\mathbf{r}_1^T \mathbf{t}_2 \right) \left(\mathbf{t}_2^T \mathbf{r}_2 \right) = 0, \end{aligned}$$

which proves that Eq. (7) is a necessary condition.

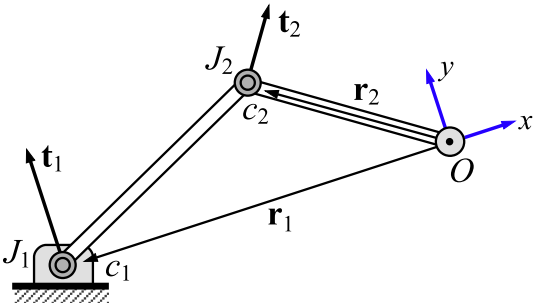

Fig. 3. A 2R mechanism. Each joint twist $\mathbf{t}_i$ is perpendicular to the position vector $\mathbf{r}_i$ relative to the compliance frame Oxy.

To prove that the condition is also sufficient, we show that $\mathbf{C}$ can always be expressed by a known positive combination of $\mathbf{t}_1\mathbf{t}_1^T$ and $\mathbf{t}_2\mathbf{t}_2^T$ if Eq. (7) is satisfied.

Suppose $\mathbf{r}_1^T\mathbf{C}\mathbf{r}_2 = 0$, then $\mathbf{C}\mathbf{r}_2 \perp \mathbf{r}_1$. Due to the symmetry of matrix $\mathbf{C}$, $\mathbf{r}_2^T\mathbf{C}\mathbf{r}_1 = 0$, and $\mathbf{C}\mathbf{r}_1 \perp \mathbf{r}_2$. Using this and Eq. (3), $\mathbf{t}_1 \parallel \mathbf{C}\mathbf{r}_2$ and $\mathbf{t}_2 \parallel \mathbf{C}\mathbf{r}_1$. Thus there exist constants λ and β such that $\lambda\mathbf{t}_1 = \mathbf{C}\mathbf{r}_2$, and $\beta\mathbf{t}_2 = \mathbf{C}\mathbf{r}_1$. Then $\mathbf{t}_1$ and $\mathbf{t}_2$ can be expressed as:

$$\mathbf{t}_1 = \frac{1}{\lambda}\mathbf{C}\mathbf{r}_2, \quad \mathbf{t}_2 = \frac{1}{\beta}\mathbf{C}\mathbf{r}_1. \tag{8}$$

It can be shown that a matrix $\tilde{\mathbf{C}}$ constructed as:

$$\tilde{\mathbf{C}} = \frac{\lambda^2}{\mathbf{r}_2^T\mathbf{C}\mathbf{r}_2}\mathbf{t}_1\mathbf{t}_1^T + \frac{\beta^2}{\mathbf{r}_1^T\mathbf{C}\mathbf{r}_1}\mathbf{t}_2\mathbf{t}_2^T \tag{9}$$

must be equal to $\mathbf{C}$.

This is shown by substituting Eq. (8) into Eq. (9) yielding

$$\tilde{\mathbf{C}} = \frac{1}{(\mathbf{r}_2^T\mathbf{C}\mathbf{r}_2)}\mathbf{C}\mathbf{r}_2\mathbf{r}_2^T\mathbf{C} + \frac{1}{(\mathbf{r}_1^T\mathbf{C}\mathbf{r}_1)}\mathbf{C}\mathbf{r}_1\mathbf{r}_1^T\mathbf{C}, \tag{10}$$

then multiplying $\tilde{\mathbf{C}}$ by $\mathbf{r}_1$ from the right yields

$$\begin{aligned}\tilde{\mathbf{C}}\mathbf{r}_1 &= \frac{1}{(\mathbf{r}_2^T\mathbf{C}\mathbf{r}_2)}\mathbf{C}\mathbf{r}_2\mathbf{r}_2^T\mathbf{C}\mathbf{r}_1 + \frac{1}{(\mathbf{r}_1^T\mathbf{C}\mathbf{r}_1)}\mathbf{C}\mathbf{r}_1\mathbf{r}_1^T\mathbf{C}\mathbf{r}_1 \\ &= \frac{1}{(\mathbf{r}_2^T\mathbf{C}\mathbf{r}_2)}\mathbf{C}\mathbf{r}_2\left(\mathbf{r}_2^T\mathbf{C}\mathbf{r}_1\right) + \frac{1}{(\mathbf{r}_1^T\mathbf{C}\mathbf{r}_1)}\mathbf{C}\mathbf{r}_1\left(\mathbf{r}_1^T\mathbf{C}\mathbf{r}_1\right) \\ &= \mathbf{C}\mathbf{r}_1.\end{aligned}$$

Similarly, we have

$$\tilde{\mathbf{C}}\mathbf{r}_2 = \mathbf{C}\mathbf{r}_2.$$

Therefore:

$$\tilde{\mathbf{C}}\left[\mathbf{r}_1, \mathbf{r}_2\right] = \mathbf{C}\left[\mathbf{r}_1, \mathbf{r}_2\right].$$

Since $\mathbf{r}_1$ and $\mathbf{r}_2$ are two linearly independent vectors, then $\tilde{\mathbf{C}} = \mathbf{C}$. Thus, by choosing the joint compliance c_1 and c_2,

$$c_1 = \frac{\lambda^2}{\mathbf{r}_2^T\mathbf{C}\mathbf{r}_2}, \quad c_2 = \frac{\beta^2}{\mathbf{r}_1^T\mathbf{C}\mathbf{r}_1},$$

the compliance matrix $\mathbf{C}$ is realized by the mechanism with the given configuration. Note that $c_i > 0$ since $\mathbf{C}$ is positive definite. This completes the proof.

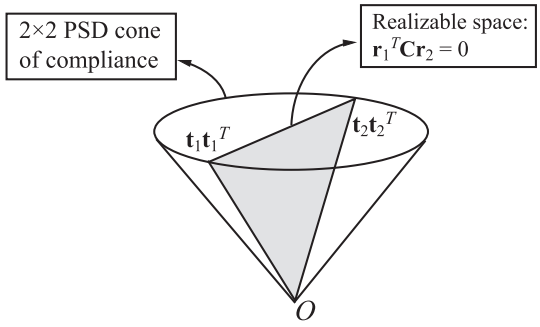

Fig. 4. Realizable space of compliance matrices using a 2R mechanism having variable compliant joints and fixed configuration. The two edges of the subcone are the rank-1 matrices $\mathbf{t}_1\mathbf{t}_1^T$ and $\mathbf{t}_2\mathbf{t}_2^T$ associated with vectors $\mathbf{r}_1$ and $\mathbf{r}_2$.

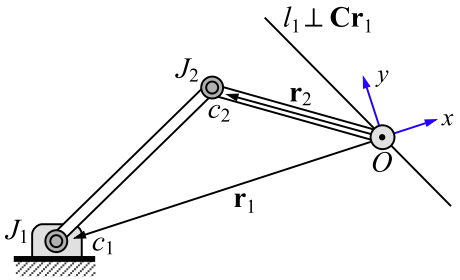

Fig. 5. Limitation of the realization of a given compliance with a 2R mechanism. Joint J_2 must be on the line $l_1 \perp \mathbf{Cr}_1$ to realize the given compliance $\mathbf{C}$.

Condition (7) is very restrictive. It is known that the collection of all point planar compliance matrices is a cone in the space of 2×2 symmetric matrices. Eq. (7) represents a plane in this space. The realizable space of PSD matrices is a subcone determined by the intersection of the PSD cone and the plane defined by Eq. (7) (as shown in Fig. 4). The two edges of the subcone are the rays determined by $\mathbf{t}_1\mathbf{t}_1^T$ and $\mathbf{t}_2\mathbf{t}_2^T$ that are associated with the positions of the two joints, J_1 and J_2. For a given location of the compliance frame, the position of J_1 relative to O, $\mathbf{r}_1$, is specified. The ray $\mathbf{t}_1\mathbf{t}_1^T$ is constant and lies on the boundary of the PSD cone. For a given compliance $\mathbf{C}$, the plane passing through both edge $\mathbf{t}_1\mathbf{t}_1^T$ and compliance $\mathbf{C}$ must intersect the PSD cone at a ray. In order to realize $\mathbf{C}$, the second joint, J_2, must be positioned such that its joint compliance matrix is collinear with that ray.

The restriction on the space of realizable elastic behaviors can be described in terms of the mechanism geometry as illustrated in Fig. 5. Because the position of joint J_1 relative to point O is specified, $\mathbf{r}_1$ is constant. By condition (7), vector $\mathbf{r}_2$ must be perpendicular to vector $\mathbf{Cr}_1$. Let l_1 be the straight line passing through O that is perpendicular to $\mathbf{Cr}_1$. Condition (7) requires that J_2 be on line l_1. Thus, line l_1 represents the locus of all J_2 locations that will allow the given $\mathbf{C}$ to be realized by the mechanism.

Since the realizable space of a 2R mechanism is very limited (zero-measure), in order to realize an arbitrary compliance, serial mechanisms having at least three joints must be considered.

3. Mechanisms with three or more compliant joints

The limits on the space of realizable compliances using a 2-joint mechanism are due to the fact that, once the position of a suspended particle relative to the base location is specified, the configuration of the mechanism is fixed. In this section, mechanisms having three or more compliant joints are considered. Since the degree of freedom is increased, the mobility of the mechanism is no longer zero when the location of O is specified. First, mechanisms having three joints are considered. Mathematical conditions on a compliance $\mathbf{C}$ to be realized with a mechanism having a given configuration are obtained. Next, the set of joint twists needed to realize a specified compliance are obtained. These conditions are then described in terms of the mechanism geometry. Alternate procedures to select the joint compliances based on the mechanism configuration are also provided. Then, the results are extended to mechanisms having more than three joints.

3.1. Conditions for compliance realization

Below, we develop a geometric approach in compliance matrix space that will later be used directly in physical 2-D space to determine the relationships among the three joints required to realize a specified compliance $\mathbf{C}$.

For joint-1 and the corresponding position vector $\mathbf{r}_1$, consider the two vectors defined by

$$\tilde{\mathbf{t}}_1 = \mathbf{Cr}_1, \quad \tilde{\mathbf{r}}_1 = \boldsymbol{\Omega}\mathbf{Cr}_1.$$

Then, by Proposition 1,

$$\mathbf{r}_1^T\mathbf{C}\tilde{\mathbf{r}}_1 = (\mathbf{Cr}_1)^T\boldsymbol{\Omega}(\mathbf{Cr}_1) = 0,$$

and $\tilde{\mathbf{t}}_1$ and $\tilde{\mathbf{r}}_1$ can be viewed as being associated with a nonexistent second joint needed to realize $\mathbf{C}$ with a 2R mechanism. The given compliance $\mathbf{C}$ can be expressed as the positive combination of $\mathbf{t}_1\mathbf{t}_1^T$ and $\tilde{\mathbf{t}}_1\tilde{\mathbf{t}}_1^T$, which means the compliance $\mathbf{C}$ is in the plane P_1 determined by the two rays $\mathbf{t}_1\mathbf{t}_1^T$ and $\tilde{\mathbf{t}}_1\tilde{\mathbf{t}}_1^T$:

$$P_1 = \left\{\mathbf{Q} \in \mathcal{R}^{2\times 2} : \mathbf{Q} = \mathbf{Q}^T, \mathbf{r}_1^T\mathbf{Q}\tilde{\mathbf{r}}_1 = 0\right\}. \tag{11}$$

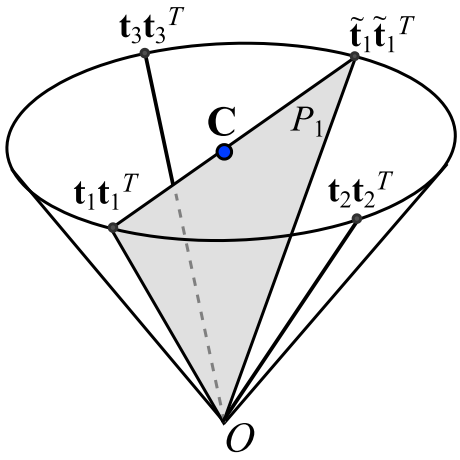

Fig. 6. Condition to realize a given compliance $\mathbf{C}$. Plane P_1 is determined by $\mathbf{C}$ and the joint compliance of J_1, $\mathbf{t}_1\mathbf{t}_1^T$. To realize $\mathbf{C}$, plane P_1 must separate the joint compliances associated with joints J_2 and J_3, $\mathbf{t}_2\mathbf{t}_2^T$ and $\mathbf{t}_3\mathbf{t}_3^T$.

As illustrated in Fig. 6, plane P_1 is uniquely determined by the position of J_1 and the compliance matrix $\mathbf{C}$.

If $\mathbf{C}$ is to be realized by a 3R mechanism, $\mathbf{C}$ must be within a polyhedral convex cone having edges $\mathbf{t}_1\mathbf{t}_1^T$, $\mathbf{t}_2\mathbf{t}_2^T$, and $\mathbf{t}_3\mathbf{t}_3^T$. Hence $\mathbf{C}_2 = \mathbf{t}_2\mathbf{t}_2^T$ and $\mathbf{C}_3 = \mathbf{t}_3\mathbf{t}_3^T$ must be on opposite sides of P_1, which means $\mathbf{r}_1^T\mathbf{C}_2\tilde{\mathbf{r}}_1$ and $\mathbf{r}_1^T\mathbf{C}_3\tilde{\mathbf{r}}_1$ must have opposite signs, which can be expressed as:

$$\left(\mathbf{r}_1^T\mathbf{C}_2\tilde{\mathbf{r}}_1\right)\left(\mathbf{r}_1^T\mathbf{C}_3\tilde{\mathbf{r}}_1\right) < 0. \tag{12}$$

Since $\mathbf{\Omega}^T\mathbf{\Omega} = \mathbf{I}$ (identity),

$$\begin{aligned}\mathbf{r}_1^T\mathbf{C}_2\tilde{\mathbf{r}}_1 &= \mathbf{r}_1^T\left(\mathbf{t}_2\mathbf{t}_2^T\right)\mathbf{\Omega}\mathbf{C}\mathbf{r}_1 = \mathbf{r}_1^T\left(\mathbf{\Omega}\mathbf{r}_2\mathbf{r}_2^T\mathbf{\Omega}^T\right)\mathbf{\Omega}\mathbf{C}\mathbf{r}_1\\ &= \mathbf{r}_1^T\left(\mathbf{\Omega}\mathbf{r}_2\mathbf{r}_2^T\right)\mathbf{C}\mathbf{r}_1.\end{aligned}$$

Similarly,

$$\mathbf{r}_1^T\mathbf{C}_3\tilde{\mathbf{r}}_1 = \mathbf{r}_1^T\left(\mathbf{\Omega}\mathbf{r}_3\mathbf{r}_3^T\right)\mathbf{C}\mathbf{r}_1.$$

Thus Eq. (12) can be expressed as:

$$\left(\mathbf{r}_1^T\mathbf{\Omega}\mathbf{r}_2\mathbf{r}_2^T\mathbf{C}\mathbf{r}_1\right)\left(\mathbf{r}_1^T\mathbf{\Omega}\mathbf{r}_3\mathbf{r}_3^T\mathbf{C}\mathbf{r}_1\right) < 0.$$

The above inequality is a necessary condition. In order to obtain sufficient conditions, the compliance matrix edges associated with the other two joints, $\mathbf{t}_2\mathbf{t}_2^T$ and $\mathbf{t}_3\mathbf{t}_3^T$, should be considered independently using the same procedure. Thus, the conditions for the three joints to realize a given $\mathbf{C}$ are:

$$\left(\mathbf{r}_1^T\mathbf{\Omega}\mathbf{r}_2\mathbf{r}_2^T\mathbf{C}\mathbf{r}_1\right)\left(\mathbf{r}_1^T\mathbf{\Omega}\mathbf{r}_3\mathbf{r}_3^T\mathbf{C}\mathbf{r}_1\right) \le 0, \tag{13}$$

$$\left(\mathbf{r}_2^T\mathbf{\Omega}\mathbf{r}_3\mathbf{r}_3^T\mathbf{C}\mathbf{r}_2\right)\left(\mathbf{r}_2^T\mathbf{\Omega}\mathbf{r}_1\mathbf{r}_1^T\mathbf{C}\mathbf{r}_2\right) \le 0, \tag{14}$$

$$\left(\mathbf{r}_3^T\mathbf{\Omega}\mathbf{r}_1\mathbf{r}_1^T\mathbf{C}\mathbf{r}_3\right)\left(\mathbf{r}_3^T\mathbf{\Omega}\mathbf{r}_2\mathbf{r}_2^T\mathbf{C}\mathbf{r}_3\right) \le 0. \tag{15}$$

Note that these conditions are not independent. If any two of the three inequalities hold, then the remaining inequality must be true.

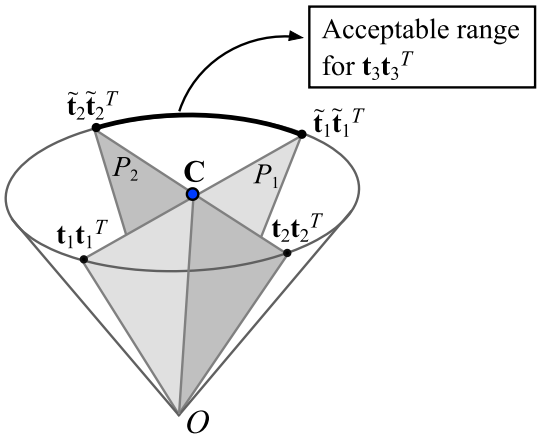

Fig. 7. Locus of acceptable joint compliance of J_3. The compliance associated with joint-3, $\mathbf{t}_3\mathbf{t}_3^T$, must lie on the arc between $\tilde{\mathbf{t}}_1\tilde{\mathbf{t}}_1^T$ and $\tilde{\mathbf{t}}_2\tilde{\mathbf{t}}_2^T$ (which depend on the given positions of J_1 and J_2).

Proposition 2. *An arbitrary compliance* $\mathbf{C}$ *can be realized with three compliant joints with positions* $\mathbf{r}_1$,$\mathbf{r}_2$ and $\mathbf{r}_3$ *if and only if any two inequalities of Eqs.* (13)–(15) *hold.*

3.2. Conditions on configuration for compliance realization

Similar to that for J_1 discussed in Section 3.1, consider the two vectors associated with J_2 given by:

$$\tilde{\mathbf{t}}_2 = \mathbf{C}\mathbf{r}_2, \quad \tilde{\mathbf{r}}_2 = \mathbf{\Omega}\mathbf{C}\mathbf{r}_2,$$

and plane P_2 defined by the two rays $\mathbf{t}_2\mathbf{t}_2$ and $\tilde{\mathbf{t}}_2\tilde{\mathbf{t}}_2$. Since the compliance matrix $\mathbf{C}$ is on both plane P_1 and plane P_2, it must be at the intersection of the two planes as illustrated in Fig. 7.

If $\mathbf{C}$ is realized by a three joint serial mechanism, plane P_1 must separate $\mathbf{t}_2\mathbf{t}_2^T$ and $\mathbf{t}_3\mathbf{t}_3^T$, and plane P_2 must separate $\mathbf{t}_1\mathbf{t}_1^T$ and $\mathbf{t}_3\mathbf{t}_3^T$. Inequalities (13) and (14) require that the joint compliance $\mathbf{t}_3\mathbf{t}_3^T$ be on the arc between $\tilde{\mathbf{t}}_1\tilde{\mathbf{t}}_1^T$ and $\tilde{\mathbf{t}}_2\tilde{\mathbf{t}}_2^T$ as shown in Fig. 7.

Note that joint twist $\mathbf{t}_i$ and joint position $\mathbf{r}_i$ have the relationship:

$$\mathbf{t}_i = \pm\mathbf{\Omega}\mathbf{r}_i, \quad \tilde{\mathbf{t}}_i = \pm\mathbf{\Omega}\mathbf{C}\mathbf{r}_i.$$

Since the two rays associated with $\tilde{\mathbf{t}}_1\tilde{\mathbf{t}}_1^T$ and $\tilde{\mathbf{t}}_2\tilde{\mathbf{t}}_2^T$ set the bounds on $\mathbf{t}_3\mathbf{t}_3^T$ in 2×2 matrix space, the two vectors $\mathbf{\Omega}\mathbf{C}\mathbf{r}_1$ and $\mathbf{\Omega}\mathbf{C}\mathbf{r}_2$ define the bounds on $\mathbf{r}_3$ in 2-vector space and can be represented in terms of the mechanism geometry.

In the plane of the mechanism, consider two lines, l_1 and l_2, passing through the origin O (where the compliance is specified), defined by $l_1 \perp \mathbf{C}\mathbf{r}_1$ and $l_2 \perp \mathbf{C}\mathbf{r}_2$. The two lines separate the space into four subspaces: S_{I}, S_{II}, S_{III}, and S_{IV} as shown in Fig. 8.

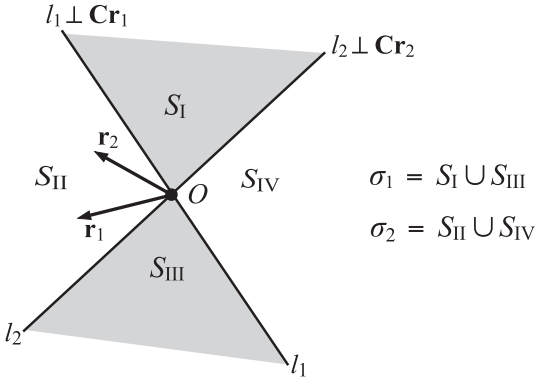

Fig. 8. The acceptable space of $\mathbf{r}_3$ for realizing $\mathbf{C}$. The space of $\mathbf{r}_3$ for the realization of $\mathbf{C}$ is a zone limited by the two lines l_1 and l_2, σ_1 or σ_2. The space is identified by the fact: neither $\mathbf{r}_1$ nor $\mathbf{r}_2$ can be inside of the acceptable space.

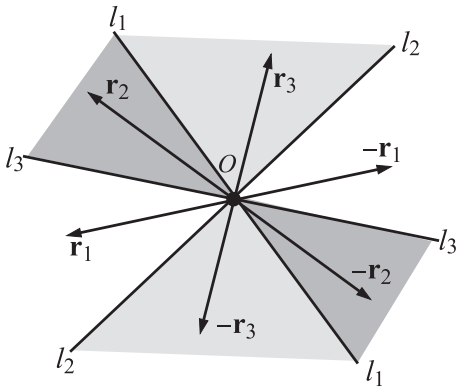

Fig. 9. Necessary and sufficient condition for a compliance to be realized by a 3R mechanism. Position vector $\mathbf{r}_3$ is adjacent to l_1 and l_2; $\mathbf{r}_2$ is adjacent to l_1 and l_3; and $\mathbf{r}_1$ is adjacent to l_2 and l_3.

It can be seen that if vector $\mathbf{r}_3$ satisfies conditions (13)–(14), then $-\mathbf{r}_3$ must also satisfy the same conditions. Thus, if inequalities (13)–(14) are true for S_{I}, they must be also true for S_{III}.

Denote:

$$\sigma_1 = S_{\mathrm{I}} \cup S_{\mathrm{III}}, \tag{16}$$

$$\sigma_2 = S_{\mathrm{II}} \cup S_{\mathrm{IV}}. \tag{17}$$

Then, $\mathbf{r}_3$ must be either in σ_1 or in σ_2 to realize the compliance $\mathbf{C}$. In fact, if any point in σ_1 satisfies inequalities (13)–(14), the following must be true:

(i). Every point in σ_1 satisfies the same conditions;
(ii). No point inside σ_2 satisfies the same conditions.

The above statements (i) and (ii) can be proved by the fact that, for a given $\mathbf{r}_1$ and $\mathbf{r}_2$, the expressions on the left side of inequalities (13)–(14) are continuous functions of $\mathbf{r}_3$, and if one inequality changes sign, $\mathbf{r}_3$ must cross either line l_1 or line l_2.

Thus, once the positions of J_1 and J_2 are specified, the acceptable space for J_3 to realize $\mathbf{C}$ is a zone with vertical (opposite) angles and boundaries of l_1 and l_2, either the space σ_1 or the space σ_2 shown in Fig. 8. Although the space can be identified by testing conditions (13)–(14) using an arbitrary point, the space can also be identified directly by testing the position of J_1 or J_2. Neither $\mathbf{r}_1$ nor $\mathbf{r}_2$ can be in the interior of the space acceptable for $\mathbf{r}_3$. This can be proved by setting $\mathbf{r}_3 = \mathbf{r}_1$ in condition (13), which violates the inequality. Therefore, as shown in Fig. 9, $\mathbf{r}_3$ or $-\mathbf{r}_3$ must be separated from $\mathbf{r}_1$ by l_1. By the same reasoning, $\mathbf{r}_3$ or $-\mathbf{r}_3$ must be separated from $\mathbf{r}_2$ by l_2, and $\mathbf{r}_1$ or $-\mathbf{r}_1$ must be separated from $\mathbf{r}_2$ by line $l_3 \perp \mathbf{C}\mathbf{r}_3$. Fig. 9 shows the geometric relations among position vectors $\mathbf{r}_i$s and the lines l_is for a mechanism configuration that realizes a given compliance $\mathbf{C}$.

In summary, we have:

Proposition 3. *Consider a compliance matrix* $\mathbf{C}$ *and a 3R mechanism having joint positions* $\mathbf{r}_i$. *Let* l_i *be a line passing through O and perpendicular to vector* $\mathbf{C}\mathbf{r}_i$ $(i = 1, 2, 3)$. *Then,*

a) *If* $\mathbf{r}_1$ *and* $\mathbf{r}_2$ *are specified, the space of* $\mathbf{r}_3$ *that can realize* $\mathbf{C}$ *is the zone* σ_1 *that does not contain* $\mathbf{r}_1$ *(or* $\mathbf{r}_2$*) as shown in Fig. 8;*
b) *If* $\mathbf{r}_1$, $\mathbf{r}_2$ *and* $\mathbf{r}_3$ *are specified,* $\mathbf{C}$ *can be realized if and only if*: $\pm\,\mathbf{r}_1$ *is between and adjacent to lines* l_2 *and* l_3; $\pm\,\mathbf{r}_2$ *is between and adjacent to lines* l_1 *and* l_3; *and* $\pm\,\mathbf{r}_3$ *is between and adjacent to lines* l_1 *and* l_2 *as shown in Fig. 9.*

Similar to conditions (13)–(15), the three conditions on mechanism configuration in Proposition 3b are not independent. If any two conditions in Proposition 3b are true, the remaining condition must also be true.

3.3. Example

An application of these results is illustrated in Fig. 10. This 3R mechanism example shows how to determine the range of the locations of J_3 when the locations of the other two joints are specified.

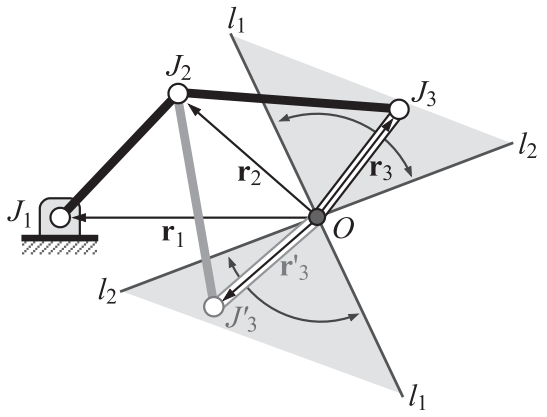

Fig. 10. Configurations of a 3R mechanism that realizes the example compliance. Joint position vector J_3 could be anywhere in the two identified regions bounded by l_1 and l_2.

If a desired compliance matrix is given as:

$$\mathbf{C} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \times 10^{-2}\text{m/N},$$

and the locations of J_1 and J_2 are specified as:

$$\mathbf{r}_1 = \begin{bmatrix} -0.2 \\ 0 \end{bmatrix}\text{m}, \quad \mathbf{r}_2 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}\text{m}.$$

The two vectors $\mathbf{Cr}_1$ and $\mathbf{Cr}_2$ are:

$$\mathbf{Cr}_1 = \begin{bmatrix} -4 \\ -2 \end{bmatrix} \times 10^{-3}\text{m}^2/\text{N}, \quad \mathbf{Cr}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \times 10^{-3}\text{m}^2/\text{N}.$$

The two lines $l_1 \perp \mathbf{Cr}_1$ and $l_2 \perp \mathbf{Cr}_2$ each passing through point O are illustrated in Fig. 10. For this case, $\mathbf{r}_1$ is along the x-axis. The space of acceptable $\mathbf{r}_3$ to realize the given $\mathbf{C}$ is identified to be the two shaded areas that meet $\mathbf{r}_1$ only at the origin O.

Fig. 10 illustrates the locations of the three joints in a mechanism. When J_1 and J_2 are specified, J_3 must be in the shaded area bounded by l_1 and l_2 in order to realize the given $\mathbf{C}$. Two possible acceptable configurations $J_1J_2J_3$ and $J_1J_2J'_3$ for the given mechanism are shown.

3.4. Determination of the joint compliances

Below, means for selecting the joint compliance values for a given mechanism configuration and realizable compliance $\mathbf{C}$ are derived. Unlike the calculation of Eq. (6), the matrices are not converted to vectors and a matrix inversion is not required. Only the compliance matrix and joint locations are needed.

Suppose $\mathbf{C}$ is to be realized by a 3R mechanism with joint twists $\mathbf{t}_i$, then by Eq. (5),

$$\mathbf{C} = c_1\mathbf{t}_1\mathbf{t}_1^T + c_2\mathbf{t}_2\mathbf{t}_2^T + c_3\mathbf{t}_3\mathbf{t}_3^T.$$

Multiplying $\mathbf{C}$ by $\mathbf{r}_2^T$ from the left and by $\mathbf{r}_3$ from the right yields

$$\mathbf{r}_2^T\mathbf{Cr}_3 = c_1\mathbf{r}_2^T\mathbf{t}_1\mathbf{t}_1^T\mathbf{r}_3 + c_2\left(\mathbf{r}_2^T\mathbf{t}_2\right)\mathbf{t}_2^T\mathbf{r}_3 + c_3\mathbf{r}_2^T\mathbf{t}_3\left(\mathbf{t}_3^T\mathbf{r}_3\right).$$

Since $\mathbf{r}_2^T\mathbf{t}_2 = 0$ and $\mathbf{t}_3^T\mathbf{r}_3 = 0$,

$$\mathbf{r}_2^T\mathbf{Cr}_3 = c_1\mathbf{r}_2^T\mathbf{t}_1\mathbf{t}_1^T\mathbf{r}_3.$$

The value for the compliance of joint-1 is:

$$c_1 = \frac{\mathbf{r}_2^T\mathbf{Cr}_3}{\mathbf{r}_2^T\mathbf{t}_1\mathbf{t}_1^T\mathbf{r}_3} = \frac{\mathbf{r}_2^T\mathbf{Cr}_3}{\mathbf{r}_2^T\boldsymbol{\Omega}\mathbf{r}_1\mathbf{r}_1^T\boldsymbol{\Omega}^T\mathbf{r}_3}.$$

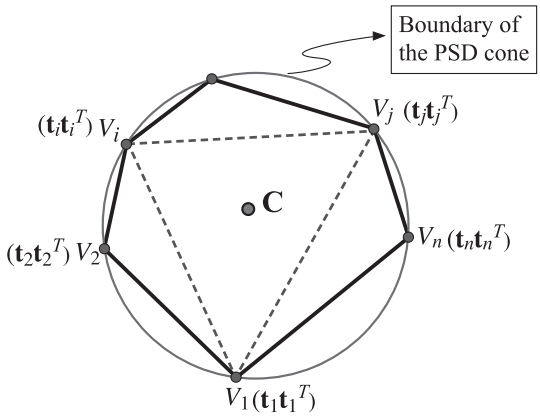

Fig. 11. Cross section of the PSD cone containing the compliance **C**. The vertexes V_i of the polygon are associated with joint compliance $\mathbf{t}_i\mathbf{t}_i^T$. **C** must be inside the polygon to be realized with the mechanism.

Repeating the process for the pairs $(\mathbf{r}_1, \mathbf{r}_3)$ and $(\mathbf{r}_1, \mathbf{r}_2)$ respectively, the equations for coefficients of c_2 and c_3 are obtained. Thus, for a given configuration $(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ and a realizable **C**, the joint compliances are calculated using:

$$c_1 = \frac{\mathbf{r}_2^T\mathbf{C}\mathbf{r}_3}{\mathbf{r}_2^T\mathbf{\Omega}\mathbf{r}_1\mathbf{r}_1^T\mathbf{\Omega}^T\mathbf{r}_3}, \tag{18}$$

$$c_2 = \frac{\mathbf{r}_3^T\mathbf{C}\mathbf{r}_1}{\mathbf{r}_3^T\mathbf{\Omega}\mathbf{r}_2\mathbf{r}_2^T\mathbf{\Omega}^T\mathbf{r}_1}, \tag{19}$$

$$c_3 = \frac{\mathbf{r}_1^T\mathbf{C}\mathbf{r}_2}{\mathbf{r}_1^T\mathbf{\Omega}\mathbf{r}_3\mathbf{r}_3^T\mathbf{\Omega}^T\mathbf{r}_2}. \tag{20}$$

Note that once the configuration is specified, the compliance values are each unique. As stated in Proposition 2, if **C** is realizable, then inequalities (13)–(15) must be satisfied. It can be proved that inequalities (13)–(15) ensure that each c_i in Eqs. (18)–(20) is positive. Thus, if **C** is realizable, the coefficients calculated using Eqs. (18)–(20) are all positive. In fact, $c_i > 0$ $(i = 1, 2, 3)$ is also a necessary and sufficient condition for **C** to be realized with the mechanism at the given configuration.

3.5. Mechanisms with more than three compliant joints

Consider a mechanism having n $(n > 3)$ revolute joints. At a configuration, joint i has position $\mathbf{r}_i$ and joint twist $\mathbf{t}_i$, $i = 1, 2, \ldots, n$. If a compliance **C** is realized by the mechanism at the configuration, then **C** must be in the polyhedral convex cone bounded by the edges (rays) associated with the rank-1 joint compliance $\mathbf{t}_i\mathbf{t}_i^T$, where each edge is on the boundary of the PSD cone.

Fig. 11 shows a cross section of the PSD cone containing the compliance **C**. In this plane, the edges (rays) associated with the joint compliances $\mathbf{t}_i\mathbf{t}_i^T$s are represented by points V_i on the boundary of the PSD cone, and the polygon with vertexes V_i is the cross section of the polyhedral convex cone of all realizable compliances. It can be seen that the compliance **C** can be realized with the mechanism at the configuration *if and only if* **C** is within the polygon with vertexes V_i.

Consider the vertex V_1 in the plane shown in Fig. 11. If **C** is in the polygon $V_1V_2\cdots V_n$, there must exist two vertexes V_i and V_j such that **C** is the triangle $V_1V_iV_j$. This means that the compliance can be realized by three compliant joints J_1, J_i and J_j. This statement is valid for any joint. As such we have:

Proposition 4. *For an n-R mechanism having joint positions* $\mathbf{r}_i$ $(i = 1, 2, \cdots, n, n > 3)$*, a compliance* **C** *can be realized with mechanism at the configuration if and only if the following equivalent conditions are true:*

a) *There exists three joints in the mechanism that realize* **C***;*
b) *For every joint* J_i*, there exist two joints* J_j *and* J_k *in the mechanism such that the three joints* (J_i, J_j *and* J_k) *realize* **C***.*

Thus, for an n-R mechanism, to determine whether a given compliance can be realized at a given configuration, condition 4a or 4b can be used. Since only three joints are involved for each test set, conditions (13)–(15) can be used. If the compliance is realizable, one can always choose just three joints in the mechanism to realize the behavior. The corresponding values of the three joint compliances can be calculated using Eqs. (18)–(20). It can be seen that for a realizable compliance at a configuration, the set of joints that realizes the compliance is not unique.

Note that if compliance is provided at more than three joints, there are an infinite number of solutions for the set of joint compliances that achieve a realizable compliance. A closed form solution cannot be obtained if no additional constraints are considered.

As shown above, the kinematics of a mechanism plays a significant role in compliance realization. In the following section, mechanism geometries that facilitate realization of an arbitrary compliance are identified.

4. Realization of any compliance with a given mechanism

Section 3 addressed the realizability of a single given compliance for a given mechanism configuration. Since an n-R mechanism has n degrees-of-freedom, when $n \geq 3$, the configuration can change when the position of endpoint O relative to the mechanism base J_1 is specified. A compliance unrealizable with one configuration could be realizable by changing the configuration. Here, the ability of a given n-R mechanism to realize every particle planar compliance at its endpoint without changing the endpoint location is investigated.

First, mechanisms having three joints are considered. A requirement on the twist spaces associated with joints J_2 and J_3 is identified. Next, the requirement on the twist space is converted to geometric conditions on the mechanism. We show that if a 3R mechanism has appropriate link length ratios (including the distance from J_1 to O), all compliances can be realized at the given endpoint location. Then, the results for 3R mechanisms are extended to n-R mechanisms. A discussion on the workspace and the space of endpoint locations that ensure the mechanism ability to realize all compliances is presented.

4.1. Realization conditions for every compliance

Consider a 3R serial mechanism in which the endpoint location O relative to the base J_1 is specified. The mechanism can be viewed as a four-bar linkage with J_1O grounded. The position of J_1 relative to the compliance frame, $\mathbf{r}_1$, is constant.

If a compliance matrix $\mathbf{C}$ is given, line $l_1 \perp \mathbf{C}\mathbf{r}_1$ is specified. When the configuration of the mechanism changes, two joints, J_2 and J_3, change their positions. The position vectors of J_2 and J_3, $\mathbf{r}_2$ and $\mathbf{r}_3$, span different ranges depending on the mechanism geometry. We show that the capability of a mechanism to realize an arbitrary compliance depends on the ranges spanned by $\mathbf{r}_2$ and $\mathbf{r}_3$. In fact, we have:

Proposition 5. *Consider a mechanism having three compliant revolute joints with positions $\mathbf{r}_i$, $i = 1, 2, 3$.*

a) *A given compliance $\mathbf{C}$ can be realized with the mechanism if $\mathbf{r}_2$ or $\mathbf{r}_3$ can cross line $l_1 \perp \mathbf{C}\mathbf{r}_1$;*
b) *Every compliance matrix can be realized with the mechanism if and only if the union of spaces spanned by $\mathbf{r}_2$ and $\mathbf{r}_3$ is no less than a half plane.*

The proof of Proposition 5a is based on the fact that one can always choose a configuration such that $\mathbf{r}_2$ or $\mathbf{r}_3$ is on line l_1. The compliance can be realized with two compliant joints: either using J_1 and J_2 with $c_3 = 0$, or using J_1 and J_3 with $c_2 = 0$. Note that the realization condition in Proposition 5a is also necessary if link-1 cannot make a full rotation. For such case, the ranges of $\mathbf{r}_2$ and $\mathbf{r}_3$ are connected. Thus, the ranges of joint compliances $\mathbf{t}_2\mathbf{t}_2^T$ and $\mathbf{t}_3\mathbf{t}_3^T$ are connected. If neither $\mathbf{r}_2$ nor $\mathbf{r}_3$ can cross l_1, $\mathbf{t}_2\mathbf{t}_2^T$ and $\mathbf{t}_3\mathbf{t}_3^T$ must always be on the same side of plane P_1 defined in Eq. (11). Thus, the given $\mathbf{C}$ cannot be realized at that endpoint location.

Proposition 5b is sufficient as a direct result of Proposition 3a. If the union of ranges of $\mathbf{r}_2$ and $\mathbf{r}_3$ continuously spans more than a half plane, then any straight line is within the range. Thus, for *any* compliance, one of the two vectors $\mathbf{r}_2$ or $\mathbf{r}_3$ must be able to cross line l_1. To prove that Proposition 5b also provides a necessary condition to realize all compliances, it can be shown that all possible joints compliances $\mathbf{C}_i = c_i\mathbf{t}_i\mathbf{t}_i^T$ will not fill the boundary of the PSD cone if the union of ranges of $\mathbf{r}_2$ and $\mathbf{r}_3$ does not span a half space or more. Thus there are some compliances that are not comprised of a positive combination of $\mathbf{t}_1\mathbf{t}_1^T$, $\mathbf{t}_2\mathbf{t}_2^T$, and $\mathbf{t}_3\mathbf{t}_3^T$ for all possible configurations of the mechanism having the given endpoint, which proves Proposition 5b.

The ability of a 3R mechanism to realize an arbitrary compliance depends on the mobility of the mechanism, especially the ranges spanned by $\mathbf{r}_2$ and $\mathbf{r}_3$. It should be noted that while the range of $\mathbf{r}_3$ is determined by a single rotation angle (angle of link-3), the range spanned by $\mathbf{r}_2$ is not directly determined by the rotation angle of a single link and is, therefore, more complicated to calculate. The ability to realize any compliance can be readily assessed by evaluating the rotation range of link-3. If link-3 can rotate more than 180°, then any compliance matrix can be realized with the mechanism.

In the following, geometric conditions on a 3R mechanism that ensure a sufficiently large twist space are developed.

4.2. Conditions on the mechanism link lengths

Consider the 3R serial mechanism shown in Fig. 12. If the position of the base J_1 relative to the end-point O is specified, the system is kinematically equivalent to a four-bar mechanism. The geometry of the four-bar mechanism is characterized by the three link lengths L_1, L_2, L_3 and the distance between the base joint J_1 and the endpoint O, L_0.

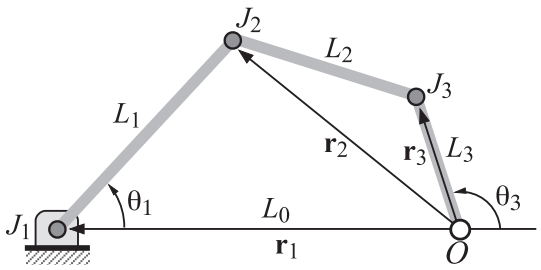

Fig. 12. A 3R serial mechanism with specified endpoint. If the location of endpoint O relative to base J_1 is specified, the system is kinematically equivalent to a four-bar mechanism.

It is known that if a mechanism satisfies the Grashof criterion, then at least one link is capable of turning a full rotation. For the mechanism shown in Fig. 12, if L_3 or L_0 is the shortest link, link-3 can make a full rotation [25]. For this case, any compliance can be realized with the mechanism.

For a non-Grashof mechanism, none of the links can make a full rotation. Both link-1 and link-3 rock between limits but can always cross line OJ_1 either clockwise or counterclockwise [25]. Due to the restrictions on $\mathbf{r}_1$ and $\mathbf{r}_3$, the twist spaces associated with J_2 and J_3 are limited. However, some mechanism geometries will still allow the space spanned by $\mathbf{r}_2$ and $\mathbf{r}_3$ to be more than a half plane. It can be shown that if L_2 is the longest link, this condition is satisfied. Thus any compliance can be realized.

As stated previously, if link-3 in a mechanism can rotate more than 180°, then any compliance can be realized. It is known that the range of a link in a four-bar mechanism is determined by the dead points (extreme positions) of the link [25]. For the mechanism shown in Fig. 12, extreme positions of link-3 can be evaluated with minimum and maximum values of θ_3, $\theta_{3\min}$ and $\theta_{3\max}$.

It can be shown that if the following two conditions are satisfied

$$(L_1 + L_2)^2 \geq L_0^2 + L_3^2, \tag{21}$$

$$|L_2 - L_1| \leq |L_0 - L_3|, \tag{22}$$

then, $\theta_{3\min} \leq 90^\circ$ and link-3 can rotate more than 180° through $\theta_3 = 180^\circ$. If the following two conditions are satisfied:

$$(L_1 - L_2)^2 \leq L_0^2 + L_3^2, \tag{23}$$

$$L_1 + L_2 \geq L_0 + L_3, \tag{24}$$

then, $\theta_{3\max} \geq 90^\circ$ and link-3 can rotate more than 180° through $\theta_3 = 0^\circ$.

In summary, we have:

Proposition 6. *Consider an arbitrary 3R serial mechanism with specified location of endpoint O relative to the base J_1.*

a) *If the mechanism is Grashof with L_3 or L_0 being the shortest link, then any compliance matrix can be realized.*
b) *If the mechanism is non-Grashof with L_2 being the longest link, then any compliance matrix can be realized.*
c) *If the mechanism satisfies conditions (21)–(22) or conditions (23)–(24), then any compliance can be realized.*

4.3. Mechanisms with more than three joints

The results obtained for 3R mechanisms can be extended to n-R mechanisms ($n > 3$). Similar to Proposition 5, we have:

Proposition 7. *Consider a mechanism having n ($n > 3$) compliant revolute joints. Suppose $\mathbf{r}_i$ is the position vector of joint J_i, $i = 1, 2, \cdots, n$.*

a) *A given compliance $\mathbf{C}$ can be realized with the mechanism if there is a joint J_i that can cross line $l_1 \perp \mathbf{Cr}_1$;*
b) *Every compliance matrix can be realized with the mechanism if and only if the union of spaces spanned by $\mathbf{r}_i$s ($2 \leq i \leq n$) is no less than a half plane.*

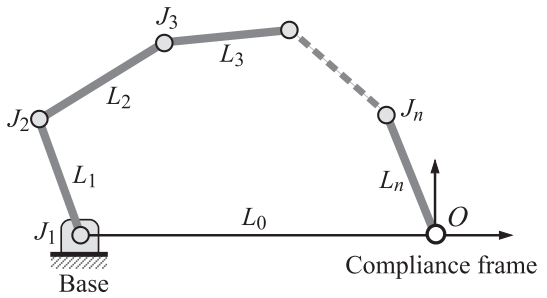

Fig. 13. An n-R mechanism. When the mechanism base J_1 and the end-point O are specified, the mechanism is kinematically equivalent to an $(n+1)$-bar linkage with L_0 grounded.

Similar to the 3R case discussed in Section 4.2, when the mechanism base and the end-point (where the compliance frame is located) are specified, an n-R mechanism is kinematically equivalent to an $(n+1)$-bar linkage. Due to the increase in degrees of freedom, the description of the space spanned by each $\mathbf{r}_i$ is much more complicated. However, for some cases, the mechanism ability to realize all compliances can be assessed by the rotation range of the nth link. If link-n can rotate more than 180°, then every compliance can be achieved with the mechanism. Below, a necessary and sufficient condition for link-n to have this ability is obtained using the theory presented in [26].

Consider an n-R mechanism with link length L_i as shown in Fig. 13. Suppose that the distance between the base joint J_1 and the end-point O is specified as L_0.

Denote:

$$L_{ij} = \min(L_i, L_j), \qquad L_{\max} = \max(L_0, L_1, \cdots, L_n).$$

Then, the two adjacent links L_i and L_j can revolve relative to each other if and only if the sum of L_{ij} and $L_{\max}$ is no greater than the sum of all remaining link lengths [26]. For the two adjacent links L_0 and L_n, the condition can be expressed as:

$$L_{\max} + \min(L_n, L_0) \le \frac{1}{2}\sum_{i=0}^{n} L_i. \tag{25}$$

Since L_0 is grounded, condition (25) is a necessary and sufficient condition for link-n to make a full rotation. Thus, condition (25) is a sufficient condition for a mechanism to achieve all compliances.

4.4. Discussion

A 3R serial mechanism is an open chain having 3 DOF. Although it is kinematically equivalent to a four-bar linkage when the endpoint O (the location of the compliance frame) is specified, the endpoint is clearly not fixed in space. The implications of this analogy on the manipulator workspace are described below.

Fig. 14 shows, for two different cases of manipulator geometry, both the manipulator workspace and the space of endpoint locations for which all compliances can be realized.

In Fig. 14a, the link lengths of the manipulator satisfy $L_1 > L_2 > L_3$ and $L_2 + L_3 > L_1$. The manipulator workspace is a circle of radius $R_w = L_1 + L_2 + L_3$. It can be proved that if the location of the mechanism endpoint is inside of the circle of radius R_c, where

$$R_c = \sqrt{(L_1 + L_2)^2 - L_3^2},$$

then any compliance can be realized with the manipulator at that location.

In Fig. 14b, the link lengths of the manipulator satisfy $L_1 > L_2 > L_3$, but $L_2 + L_3 < L_1$. The manipulator workspace is an annulus between the circle of radius $R_w = L_1 + L_2 + L_3$ and the circle of radius $r_w = L_1 - L_2 - L_3$. It can be proved that if the location of the endpoint is inside of the annulus between the circle of radius R_c and the circle of radius r_c, where

$$R_c = \sqrt{(L_1 + L_2)^2 - L_3^2},$$
$$r_c = \sqrt{(L_1 - L_2)^2 - L_3^2},$$

then any compliance can be realized with the manipulator at that location.

It can be seen that, for both cases, if link-3 is short, the space of endpoint locations for which all compliances can be realized is close to the mechanism workspace.

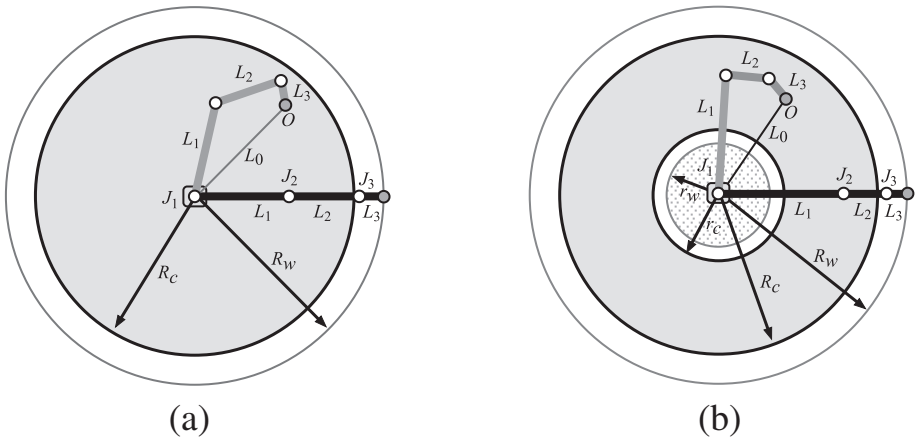

Fig. 14. The workspace and the space of endpoint locations that ensure the mechanism ability to realize any compliance matrix. (a) Case 1: $L_1 > L_2 > L_3$ and $L_2 + L_3 > L_1$. (b) Case 2: $L_1 > L_2 > L_3$ and $L_2 + L_3 < L_1$.

5. Synthesis of a realizable compliance

If it is possible to realize a compliance at a given location with a given mechanism, a procedure for finding an appropriate configuration and associated joint compliance coefficients is needed. In this section, two types of synthesis procedures for any given realizable compliance are provided. First, a synthesis procedure for the realization of a compliance using a 3R mechanism having only two compliant joints is developed. Then, a synthesis procedure for the general case in which all three joints of a 3R mechanism are compliant is presented.

The synthesis procedures presented in this section for 3R mechanisms can be applied to n-R mechanisms using the results presented in Section 3.5.

5.1. Synthesis with a mechanism having two compliant joints

Since a non-singular 2×2 compliance $\mathbf{C}$ is rank 2, to realize a given $\mathbf{C}$ at least two compliant joints are required. Proposition 4 shows that if a compliance $\mathbf{C}$ can be realized with a 3R mechanism, $\mathbf{C}$ can always be realized with only two compliant joints, either J_1 and J_2 or J_1 and J_3, with the remaining joint having $c_i = 0$. If a mechanism satisfies the conditions in Propositions 6 and 7, then any compliance can be realized with joints J_1 and J_3 alone. For these mechanisms, adding compliance to the intermediate joint (joint-2) does not enlarge the realizable compliance space of the mechanisms.

Although loading all three joints with compliance is not necessary to realize a given compliance, mechanisms with three compliant joints have the advantage that a significant amount of compliances (a large "neighborhood" in the PSD space) can be reached with the mechanism by just changing the joint compliances while keeping the configuration unchanged. For a mechanism with only two compliant joints, when a compliance varies in the realizable space, both the mechanism configuration and the joint compliances need to be changed.

Below a synthesis procedure is developed to realize an arbitrary compliance with a 3R mechanism having compliance at joints J_1 and J_3. For a given compliance $\mathbf{C}$, the procedure determines the mechanism configuration and the joint compliances.

Consider a given mechanism having link lengths L_1, L_2, and L_3 as shown in Fig. 12. The distance between the base joint J_1 and compliance frame origin is specified as L_0.

A 2×2 compliance matrix $\mathbf{C}$ to be realized is expressed in the coordinate frame Oxy having the form:

$$\mathbf{C} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}.$$

In the coordinate frame Oxy, the position of J_1 and the corresponding twist are:

$$\mathbf{r}_1 = \begin{bmatrix} -L_0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{t}_1 = \begin{bmatrix} 0 \\ L_0 \end{bmatrix}.$$

The 3-step synthesis procedure is:

1. Decompose **C** into two rank-1 matrices:

$$\mathbf{C} = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & d - \frac{b^2}{a} \end{bmatrix} + \begin{bmatrix} a & b \\ b & \frac{b^2}{a} \end{bmatrix} = \mathbf{C}_1 + \mathbf{C}_3.$$

Since **C** is a PSD matrix, $a > 0$, $d > 0$, and

$$ad - b^2 \geq 0 \Rightarrow d - \frac{b^2}{a} \geq 0,$$

and $\mathbf{C}_1$ and $\mathbf{C}_3$ are PSD matrices.

2. Synthesize $\mathbf{C}_1$ with compliant joint J_1.
Choose

$$c_1 = \frac{d - \frac{b^2}{a}}{L_0^2} > 0, \tag{26}$$

then, $\mathbf{C}_1 = c_1 \mathbf{t}_1 \mathbf{t}_1^T$.

3. Synthesize $\mathbf{C}_3$ with compliant joint J_3.
Express $\mathbf{C}_3$ in dyad form:

$$\mathbf{C}_3 = \begin{bmatrix} a & b \\ b & \frac{b^2}{a} \end{bmatrix} = \begin{bmatrix} \sqrt{a} \\ \frac{b}{\sqrt{a}} \end{bmatrix} \left[\sqrt{a}, \frac{b}{\sqrt{a}} \right].$$

- Calculate angle θ_3.
The slope of the vector $\mathbf{Cr}_1$ is $\frac{b}{a}$. In order for link-3 to be on line $l_1 \perp \mathbf{Cr}_1$,

$$\theta_3 = \tan^{-1} \frac{b}{a} \pm 90^\circ. \tag{27}$$

- Determine the joint compliance c_3.
The joint compliance can be calculated by:

$$c_3 = \frac{a + \frac{b^2}{a}}{L_3^2}. \tag{28}$$

Then, $\mathbf{C}_3 = c_3 \mathbf{t}_3 \mathbf{t}_3^T$.

With the final step, the mechanism configuration and joint compliances are determined. It is noted that Eq. (27) yields two configurations. Each realizes the same compliance **C**.

5.2. Synthesis with a mechanism having three compliant joints

A synthesis procedure is developed to realize a given compliance **C** in which all three joints of a 3R mechanism are compliant. This development is based on the procedure used for the 2-compliant joint case (described in Section 5.1).

When the position of the endpoint relative to the joint base J_1 is specified, the position vector $\mathbf{r}_1$ is constant and line $l_1 \perp \mathbf{Cr}_1$ is determined. Since the mechanism has one DOF, the locations of J_2 and J_3 are related by the geometry of a four-bar linkage. In order to ensure J_3 is in an acceptable position, the process presented in Section 4.1 can be used to estimate the acceptable space for $\mathbf{r}_3$. First, consider the configuration that realizes the given compliance **C** using two compliant joints J_1 and J_3. As shown in 4.1, the position of J_2, $\mathbf{r}_2'$, and line $l_2' \perp \mathbf{Cr}_2'$ are obtained. The two lines l_1 and l_2' can be used to estimate the acceptable space of $\mathbf{r}_3$ based on Proposition 3a. A location for J_3 can be chosen in the acceptable space.

The 4-step synthesis procedure is:

1. Calculate the vector $\mathbf{Cr}_1$ and determine line $l_1 \perp \mathbf{Cr}_1$.
2. Choose an angle of $\mathbf{r}_3$ (or a location of J_3) such that $\mathbf{r}_3$ is in an acceptable zone described in Section 3.2.

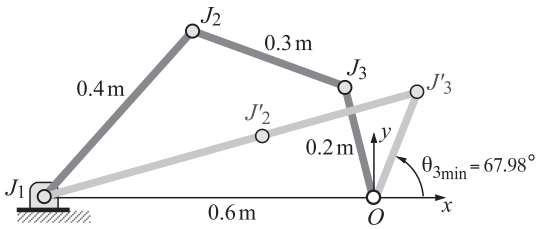

Fig. 15. A specified 3R mechanism having given link lengths L_1, L_2 and L_3. The position of joint base J_1 relative to the compliance frame origin O is specified. At an extreme configuration $J_1J'_2J'_3O$, $\theta_{3\min} = 67.98°$.

The angle θ_3 can be found through the following:

(a) Evaluate the dead points of link-3 and determine the rotation range of θ_3.
(b) Calculate the angle θ_3' corresponding to line l_1 using Eq. (27).
(c) Calculate the position of J_2, $\mathbf{r}_2'$, assuming $\theta_3 = \theta_3'$.
(d) Determine the line $l_2' \perp \mathbf{Cr}_2'$.
(e) Choose $\mathbf{r}_3$ in the zone bounded by l_1 and l'_2. The zone is identified using Proposition 3a.

The angle θ_3 associated with $\mathbf{r}_3$ is now determined.

3. Calculate $\mathbf{r}_2$ associated with the configuration yielding θ_3.
4. Determine the joint compliance constants using Eqs. (18)–(20) presented in Section 3.4.

With the final step the configuration and joint compliances of the mechanism are determined. It is noted that when $\mathbf{r}_2$ is obtained in Step 3, line $l_2 \perp \mathbf{Cr}_2$ needs to be constructed to confirm that $\mathbf{r}_3$ is between l_1 and l_2 and satisfies the condition in Proposition 3a. If the conditions are not satisfied, a new position of $\mathbf{r}_3$ closer to l_1 should be selected. It is also noted that the synthesis solution is not unique.

A numerical example is used to to illustrate the synthesis procedures in Section 6.

6. Synthesis procedure examples

An example is provided to illustrate the synthesis procedures presented in Section 5. The compliance matrix to be realized is the same $\mathbf{C}$ used in Section 3.3:

$$\mathbf{C} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \times 10^{-2}\text{m/N}.$$

A manipulator to be used for the realization of $\mathbf{C}$ is shown in Fig. 15. The link lengths of the mechanism are given as: $L_1 = 0.4$ m, $L_2 = 0.3$ m, $L_3 = 0.2$ m. The distance between the base joint J_1 and the compliance frame origin O is specified as $L_0 = 0.6$ m. Since the mechanism geometry satisfies conditions (21)–(22), any compliance can be realized by the manipulator at this endpoint location.

With the mechanism geometry, the minimum angle of link-3 is determined to be $\theta_{3\min} = 67.98°$. The extreme position of the mechanism is also shown in Fig. 15.

Since the synthesis of $\mathbf{C}$ using all three compliant joints is based on line l_1 and line l'_2 obtained from the configuration at which the compliance is realized with two compliant joints J_1 and J_3, the synthesis of $\mathbf{C}$ with these two compliant joints is first performed. Then, the synthesis of $\mathbf{C}$ using all three compliant joints of the mechanism is presented.

6.1. Synthesis of $\mathbf{C}$ with two compliant joints

The procedure presented in Section 5.1 is first used to synthesize compliance matrix $\mathbf{C}$.

Using Eq. (26), the joint compliance c_1 for J_1 is determined:

$$c_1 = \frac{d - \frac{b^2}{a}}{L_0^2} = \frac{(4 - 1/2) \times 10^{-2}}{0.6^2} = 0.0972\text{m/N}.$$

Using Eqs. (27)–(28), the angle θ_3 and the joint compliance c_3 for J_3 are determined:

$$\theta_3 = \tan^{-1}\frac{b}{a} \pm 90° = 116.6° \text{or} -63.4°,$$
$$c_3 = \frac{a + \frac{b^2}{a}}{L_3^2} = \frac{(2 + 1/2) \times 10^{-2}}{0.2^2} = 0.625\text{m/N}.$$

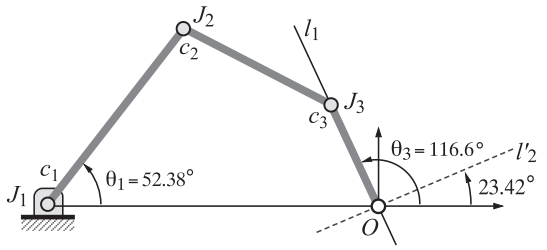

Fig. 16. Configuration of the mechanism: Synthesis of **C** with two compliant joints J_1 and J_3. The joint compliances are $c_1 = 0.0972$ m/N, $c_3 = 0.125$ m/N, and $c_2 = 0$ m/N.

If $\theta_3 = 116.6^\circ$ is chosen, the configuration of the mechanism is determined. Using the four-bar kinematic relations, the angle θ_1 is calculated to be 52.38°. The configuration of the mechanism is illustrated in Fig. 16. Line $l'_2 \perp \mathbf{Cr}_2$ associated with the configuration is also illustrated in the figure. This line is needed in the synthesis with three compliant joints described below.

6.2. Synthesis of **C** with three compliant joints

Following the procedure presented in Section 5.2, the configuration of the mechanism and joint compliances needed to realize the compliance are identified.

1. Calculate $\mathbf{Cr}_1$ and determine line $l_1 \perp \mathbf{Cr}_1$, as described in Section 5.1.
2. Choose a position of joint-3 such that $\mathbf{r}_3$ is in the realizable zone between l_1 and l'_2. An angle of link-3, θ_3, can be selected based on the following conditions:
   (a) The extreme position is $\theta_{3\min} = 67.98^\circ$.
   (b) The angle θ_3' associated with l_1 is calculated in Section 6.1 as $\theta_3' = 116.6^\circ$.
   (c) The position of J_2 associated with $\theta_3 = 116.6^\circ$ is calculated as $\mathbf{r}'_2 = [-0.3559\text{ m}, 0.3169\text{ m}]^T$.
   (d) Line $l'_2 \perp \mathbf{Cr}'_2$ is determined. The angle between the line and the x-axis is 23.42° (shown in Fig. 16).
   (e) Choose a position between l_1 and l'_2 using Proposition 3a that also satisfies kinematics constraints (greater than $\theta_{3\min}$). Here, line l'_2 is used to estimate the acceptable zone of $\mathbf{r}_3$. It can be seen that the configuration $\theta_3 = 90^\circ$ satisfies the above conditions and is therefore chosen to realize the compliance.
3. Determine the positions of the joints.
   For $\theta_3 = 90^\circ$, the positions of three joints are determined by the kinematics of a four-bar mechanism. The joint positions are:

$$\mathbf{r}_1 = \begin{bmatrix} -0.6\text{m} \\ 0 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} -0.2943\text{m} \\ 0.2580\text{m} \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} 0 \\ 0.2\text{m} \end{bmatrix}.$$

4. Determine the joint compliance coefficients using Eqs. (18)–(20).

$$c_1 = 0.0794\text{m/N},$$
$$c_2 = 0.1317\text{m/N},$$
$$c_3 = 0.2809\text{m/N}.$$

With the final step, the mechanism configuration with $\theta_3 = 90^\circ$ and the joint compliances are obtained. The mechanism configuration is illustrated in Fig. 17. To confirm that the mechanism configuration satisfies the realization conditions, line $l_2 \perp \mathbf{Cr}_2$ corresponding to the final configuration is also illustrated in the figure.

Note that, when synthesizing a given compliance with two compliant joints J_1 and J_3, the configuration and joint compliance of the mechanism are uniquely determined; when synthesizing a given compliance with three compliant joints, the configuration and joint compliance of the mechanism are non-unique. There are an infinite number of configurations and sets of joint compliances for the given mechanism that realize the same compliance.

7. Discussion and conclusions

The decomposition process presented in the paper is theoretical. For a given Cartesian compliance, the process yields the mechanism configuration and the joint compliances. In this paper, we assume that any set of joint compliances can be attained via a conventional (passive) spring at each joint, or SEAs/VSAs. In practical application, the prescribed global compliance matrix will be influenced by the following factors: 1) gravity or loading of joint compliances, which changes the mechanism equilibrium

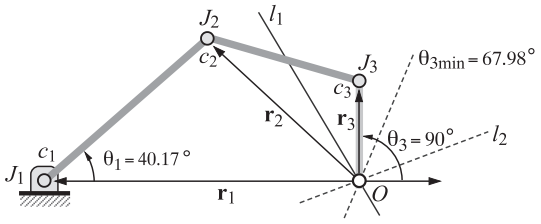

Fig. 17. Configuration of the mechanism: Synthesis of **C** with three compliant joints J_1, J_2, and J_3.

and causes non-symmetry of the global compliance; 2) a limited range of the joint spring stiffness or SEA/VSA stiffness, which would limit the space of global compliances achieved by the mechanism. The influence of these factors is negligible if the overall mechanism stiffness is sufficiently high and the range of joint compliances is sufficiently large.

In this paper, methods to realize an arbitrary 2×2 elastic behavior using serial mechanisms having only revolute joints are presented. The ability of any specified 3R mechanism to realize an arbitrary compliance behavior is characterized. It is shown that if a mechanism has appropriately sized relative link lengths, every compliance matrix can be realized by the mechanism at the specified endpoint location. This ability allows one to realize *all* particle compliant behaviors with a *single* mechanism by properly selecting the joint compliances and the mechanism configuration. Geometric appreciation of compliances associated with a mechanism provides insight into the design of manipulators that realize desired elastic behaviors. In robotic applications, a time-varying compliance can be achieved with a mechanism with VSAs by properly changing the mechanism configuration and adjusting the joint compliances.

Acknowledgments

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