

Handling Uncertainty in Geo-Spatial Data

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Abstract—An inherent challenge arising in any dataset containing information of space and/or time is *uncertainty* due to various sources of imprecision. Integrating the impact of the uncertainty is a paramount when estimating the reliability (confidence) of any query result from the underlying input data. To deal with uncertainty, solutions have been proposed independently in the geo-science and the data-science research community. This interdisciplinary tutorial bridges the gap between the two communities by providing a comprehensive overview of the different challenges involved in dealing with uncertain geo-spatial data, by surveying solutions from both research communities, and by identifying similarities, synergies and open research problems.

I. INTRODUCTION

Current technology trends such as smart phones, general mobile devices, stationary sensors and satellites coupled with a new user mentality of utilizing this technology to voluntarily share information, generate a huge volume of geo-spatial and geo-spatio-temporal data. This data flood, offers a tremendous potential of discovering new and useful knowledge that could advance a plethora of location based services [1]. However, there are certain aspects of reality which render *uncertainty* to be an inevitable component of any geo-spatial application domain: *Location measurements*, regardless whether they are obtained via GPS-enabled device or other tracking devices, are *imprecise*, due to physical limitation of devices. *Contextual information* may be imprecise – e.g., an information stating *in the mall* or *soon*. The quest to *reduce communication bandwidth, energy consumption and storage*, often relies on data reduction which, in many spatio-temporal settings is *lossy* implying “gaps” in both spatial and temporal domains. The attempt to *model a continuous motion with discrete measurements*, ultimately yields an ignorance about what happens in-between consecutive updates.

The main objective of the tutorial is to provide a detailed overview of effective and efficient solutions to various problems related to the management of uncertain geo-spatial data, presented by speakers from both geoinformation-science and data science communities. To provide more concrete motivation and to illustrate the scope of the tutorial, consider the map shown in Figure 1 showing the discrete precipitation measurements on a map.¹ Given only these discrete measurements, answering various queries of interest to the underlying applications becomes challenging due to multiple *types of uncertainty*:

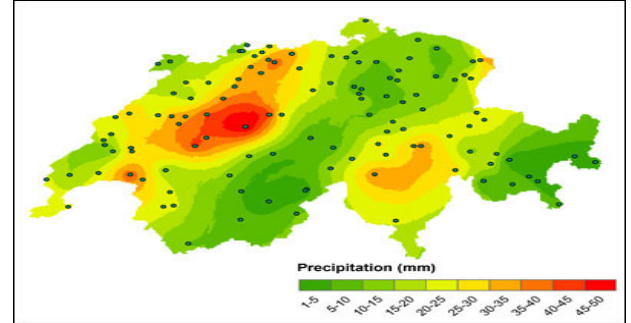


Fig. 1. Interpolated Swiss Rainfall Data.

Spatial Uncertainty is the challenge of inferring knowledge from one location to another. Naive solutions for spatial interpolation do not return any notion of reliability. Given only the map of Figure 1, it is not possible to assess the reliability of the estimated precipitation values. Clearly, in an area having a dense sensor coverage with recent and accurate precipitation measurements, the result is more significant and thus reliable than in an area far from any measurement. To assess this uncertainty information, this tutorial will introduce techniques for Spatial Interpolation including Geostatistical methods such as *Kriging*, in order to provide a measure of certainty and accuracy to the interpolation results.

Temporal Uncertainty is the challenge of deriving meaningful and current information from potentially outdated and obsolete data sources. For instance, some of the discrete precipitation measurements shown in Figure 1 may be several minutes, or even hours old. Clearly, the degree of loss of information of a discrete data source highly depends on the application. Learning how parameters change over time, and how to predict and interpolate parameters through time and space can be done by fitting *Stochastic Processes* on training data of the past to obtain a model describing the current and the future. This tutorial will introduce stochastic processes and show how to apply these to successfully spatio-temporal data sets to reduce the inherent uncertainty.

Attribute Uncertainty is the challenge of dealing with potentially inaccurate and wrong data. For example, some of the data points in both scenarios illustrated in Figure 1 may be reporting, accidentally or deliberately, wrong information. Clearly, a single wrong data record may significantly impact the correctness of the result in a large area. For this purpose, a notion of *Data Reliability* is required to assess the quality of a single data record. This can be done by applying sanity checks to the data source and by learning which of the data sources are trustworthy.

¹Credit for this image goes to Ross Purves, Department of Geography, University of Zurich.

To unify all types of uncertainty, recent solutions proposed in spatio-temporal data management as well as state-of-the-art solution of geostatistical simulation are presented. As a case-study application having all these notions of uncertainty at once, is the problem of handling uncertainty in crowd-sourced data. For all of the presented state-of-the-art solutions, both the challenges of effectiveness and efficiency are discussed. The challenge of effectiveness in uncertain data is to correctly determine the set of possible results, each associated with the correct probability of being a result, in order to give a user a confidence about the returned results. The complementary challenge of efficiency is to enable fast computations for these results and corresponding probabilities, allowing for reasonable querying times, even for large uncertain databases. The main objectives of this tutorial are:

- Provide a comprehensive overview of different research issues and solutions addressing various aspects of uncertainty in geo-spatial data. This overview is aimed both at students with no prior experience in the field, as well as at attendants with some background.
- Bridging the gap between data-science and geo-science by surveying and unifying solutions for uncertain geo-spatial data management from both data-science and geo-science.
- Present a comprehensive overview of models, algorithms, solutions and techniques in the field of managing geo-spatial data, catering to a broad audience.
- Teach common paradigms used to manage uncertainty, including techniques for spatial regression, Kriging, sampling, simulation based approaches, and query processing using possible world semantics.
- By bridging data-science and geo-science solutions, this tutorial will identify a number of open research issues on both sides. The tutorial will suggest directions to solve these open issues by exploring techniques from the respectively other research area.

In contrast to existing tutorials on uncertain data management that have been presented in the past [2], [3], [4], this tutorial is the first to unify uncertain data solutions from both geo-science and data-science. The vast majority (approximately 80%) of this tutorial have not been presented at any previous tutorial. This new material includes:

- Approaching uncertain geo-spatial data from a geo-science perspective, thus providing a tutorial on state-of-the-art solutions such as Kriging and simulation, untouched by any previous tutorial that we are aware of.
- The few selected topics, such as uncertainty models and uncertain database management systems, that are inherited from the ICDE 2014 tutorial [2], are enhanced and enriched by the expertise and experience of our geoinformation-science co-authors. Thus, the focus of the tutorial is shifted towards geo-science applications, making it the first of its kind.
- A new case-study of uncertainty in spatial crowdsourcing applications such as Open-Street-Map at the end of the tutorial. This new 30-minute part of the tutorial will unify the previous concepts and techniques and put them into a real application;
- Its unique presentation style – mingling experts from geo-science and data-science to show synergies with the other field.

II. TUTORIAL OUTLINE

Although preparing the materials for the tutorial was a challenging undertaking for which the authors contributed jointly, the presentation will be given by three presenters, who will be aiming at illustrating the geo-science and data-science perspectives, as well as the bridges in-between. These three presenters will be Dr Züfle, Dr. Trajcevski and Dr. Pfoser, who all have a strong scientific background in both geoinformation-science and data-science.

A. Introduction

Motivation - Application Settings – Living in a world of data-driven science, we will begin the tutorial by introducing spatial and spatio-temporal data as well as modern sources of such data, like the Geo-Web 2.0 and Geo-Social data. We motivate the importance of managing and analyzing such data, giving spatial applications such as sensor monitoring [5], location-based services [6]; as well as spatio-temporal applications including RFID tracking [7] and GPS tracking [8]. The importance of analyzing data that arises in such applications is set into the context of the vision of analyzing big data [9]. We briefly introduce spatial and spatio-temporal data and give an overview of existing work that has been done on managing such data, ignoring uncertainty.

B. Geo-Spatial Uncertainty

Uncertainty Models and Possible World Semantics – In the first main part of the tutorial, Dr. Züfle will first introduce the formal categorization of models for uncertain geo-spatial data: discrete uncertainty models [10] and continuous ones [11], [12], along with attribute [13], [14], [15] and existential [16] uncertainty. The concept of Possible World Semantics, widely used by the data-science community, will be discussed as a mathematically sound and intuitive interpretation of uncertain spatial databases. Additionally, a survey of the Equivalent Worlds Paradigm will be given, to tame the exponential number of possible worlds [17], [18], [19] and #P hard query processing [20]. This paradigm allows to answer a large number of spatial query predicates efficiently. We show how these models, which have also been surveyed in a previous tutorial [2], can be extended and applied to uncertain point, line and polygon data.

Uncertainty Within Spatial Interpolation – Approaching uncertainty from the perspective of geoinformation-science, Dr. Züfle will discuss kriging interpolation methods [21] that explicitly measure uncertainty in the interpolation output. Contrary to previous deterministic methods, Kriging's regression-based methodology includes elements of uncertainty in both the prediction of the final surface as well as in the estimated error surface of the predictions [22], [23], [24]. An implementation with Esri's geostatistical analyst ([25], [26]) will be demonstrated to illustrate the important decision steps in the interpolation process. A multi-variogram approach called Empirical Bayesian Kriging ([27]) will be shown that differs from other interpolation and kriging methods by accounting for the error introduced by estimating the underlying semivariogram and selecting the best fit ([28]). He will review Monte Carlo simulation as a well-established and broadly-applied approach for addressing uncertainty in spatial data analysis [29], [30], [31], [32].

C. Spatio-Temporal Uncertainty

This part of the tutorial adds the time dimension to uncertain geo-spatial data. As we will have discussed in the previous part of the tutorial, the complexity of querying uncertain geo-spatial data is exponential. Considering time, this problem becomes even more complex. Consequently, the main scope of this part of the tutorial is to explore approximate solutions for handling uncertain spatio-temporal data such as trajectory data. Dr. Trajcevski will be presenting this part of the tutorial.

Traditional Approaches for Uncertain Spatio-Temporal Data Dr. Trajcevski will review traditional models to cope with these new challenges by bounding the possible locations of objects over time by spatio-temporal cylinders [33], [34], diamonds [35] or beads [36], [37]. Based on these models/types, corresponding algorithms for processing certain query categories have been proposed, e.g. range queries [38], [34], kNN queries [33], [39], etc. which will be overviewed in this part of the tutorial. Additionally, some foundational works on location dependency will be reviewed [39], [33], [35], [40] in order to motivate the use of more advanced models. State-of-the-art approaches for querying traffic network data will also be discussed, along with the issues related to spatio-temporal data compression [41], [42], [43] and the fusion of uncertain location data from different sources [44].

Data Science Approaches for Spatio-Temporal Uncertainty – Dr. Trajcevski will present models for uncertain spatio-temporal data that describe objects by stochastic processes [7] to model the motion of objects in space and time. The notion of stochastic processes will be brought in line with possible world semantics, describing a possible spatio-temporal database as an instantiation of a global stochastic process consisting of all spatio-temporal objects in the database. We will show how a Bayesian learning approach can add additional information, such as discrete object observations, to improve the motion model of individual objects [8].

D. Case Study: Uncertainty in Crowd-Sourced Data

Crowdcouring and Spatial Data Quality – Spatial uncertainty has been described as “the Achilles’ Heel of GIS, the dark secret that once exposed will bring down the entire house of cards” [45]. Dr. Pfoerster will show that with the advent of Volunteered Geographic Information (VGI) [46], [47] as an alternative mechanism for the acquisition and compilation of geographic information the problems of uncertainty and data quality have multiplied. We will analyze geocrowdsourcing quality issues using Openstreetmap as a case study [48], [49]. While such studies give useful insights into the accuracy of VGI, but only indirectly help to identify mechanisms for assuring and improving data quality. As authoritative data gets increasingly out of date, we will discuss various crowd-based, social, and geographic approaches to assure and improve quality in VGI [50], [51]. Crowdsourcing approaches converge on a solution by relying on frequently edits of crowdsourced facts [52], [53]. When discussing social approaches, we rely on reputation as a means to assess the reliability of contributions [54] and, in a broader context, on trust and credibility of VGI [55]. Finally, any volunteered geographic information needs to adhere to certain rules that govern geographic knowledge. Aspects we can exploit to assure data quality include Tobler’s first law of geography [56], the fractal dimension of spatial features [57],

or the central place theory [58]. We will conclude this part of the tutorial by discussing working systems that have been put in place to improve crowdsourced geospatial data quality.

III. PRESENTER BIOS

Andreas Züfle received his PhD in Computer Science from LMU, Munich in 2013 under the direction of Dr. Hans-Peter Kriegel. Since January 2016, he is an Assistant Professor at the Department of Geography and Geoinformation Science, George Mason University, USA. His research interests include searching and mining of uncertain geo-spatial data. In September 2016, Dr. Züfle was awarded \$507,852.00 by the National Science Foundation as a co-Pi together with Dr. Pfoerster, for their research project NSF/AitF: Collaborative Research: Modeling movement on transportation networks using uncertain data. Dr. Züfle was PC co-chair of the ACM SIGMOD Workshop on Managing and Mining Enriched Geo-Spatial Data in 2014, 2015 and 2016.

Dieter Pfoerster received his PhD in Computer Science from Aalborg University, Denmark in 2000. He is currently an Associate Professor at the Department of Geography and Geoinformation Science, George Mason University, USA. His research focusses on spatiotemporal databases, shortest-path algorithms and data mining methods for user-generated geospatial content such as map matching and map construction methods. He has organized and chaired conferences and workshops. He has co-authored more than 100 papers and his research has been supported by various funding agencies in Europe and the US. The most recent award, which is highly related to the topic of this tutorial, is an award of \$507,852.00 by the National Science Foundation in the role of PI.

Gocce Trajcevski received his PhD from the University of Illinois at Chicago in 2002 and his main research interests are in the areas of Mobile Data Management, Uncertainty Management and Sensor Networks. His research has been funded by the NSF, DoD as well as industry (Northrop Grumman and BEA). He has co-authored over 95 publications in refereed conferences and journals and was part of the organizing committees of ACM SIGMOD 2006, IEEE MDM2011, 2012, 2014, 2015, ACM GIS 2011, 2012, 2015, a General Co-Chair of ICDE 2014, PC CoChair ADBIS 2014 and PC CoChair of ACM GIS 2016.

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