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Sensor and Actuator Placement for Zero-Shaping in Dynamical Networks

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Abstract— **Placement of sensors and actuators in a linear network model is pursued, with the aim of achieving desirable invariant-zero characteristics for input-output channels (primarily, minimum phase dynamics). Graph-theoretic analyses of the network model's invariant zeros and phase-response properties are undertaken, and used to develop simple insights into and algorithms for sensor and actuator placement.**

I. Introduction

New sensing and actuation technologies are being deployed in engineered and natural networks, which hold promise to revolutionize monitoring and control of dynamical processes ongoing in these networks. At the same time, these same resources can imbue cyber- attackers with increased access with which they can enact destructive impacts. As new sensing and actuation technologies for dynamical networks come to fruition, the question of where to place these devices in built networks is becoming increasingly important. Due to cost, security, and maintenance considerations, typically only a few actuation and sensing devices can be deployed in a network: thus, the design of sparse sensing and actuation schemes for large-scale networks is crucial. Very nice structural techniques for sensor and actuator placement have been developed for linear systems (e.g. [1]). However, these techniques may be difficult to apply in large-scale network applications, due to computational complexity, imperfect or local knowledge of network characteristics by stakeholders, model uncertainties, domain-specific constraints on placement, and other factors. In many of these networks, operators instead currently rely on experience and topological understanding of the network dynamics for placement. Given this, simple topology-based rubrics and algorithms for sensor and actuator placement, which nevertheless achieve specified performance requirements, are very desirable. The research described here contributes to sensor and actuator placement from a topological or graph-theoretic perspective.

There is an incipient research effort on sensor and actuator placement in built dynamical networks from a graph-theory perspective. These initial studies have focused primarily on sensor placement to ensure observability (dually actuator placement to ensure controllability), and subsequently placement to shape observability and controllability metrics [2]–[4]. In many circumstances, however, sensor and/or actuator placement must be undertaken with input-output characteristics in mind. For instance, measurement units in the electric power grid may be used for feedback control to damp oscillations and transients, state monitoring in the presence of unknown inputs, or analysis of disturbance signals. Input-output characteristics, and specifically the zeros of the input-output transfer function, are crucial for addressing these tasks. Hence, sensor and actuator placement in this context needs to be based on input-output properties.

Recently, an exciting research thrust has developed on characterizing the input-output dynamics of a network, including specifically its zeros, from a graph theoretic perspective [5]–[12]. Several recent studies have characterized invariant zeros for canonical models for built networks, including models for synchronization, spread, and multi-agent coordination, with a particular focus on distinguishing minimum-phase and nonminimum-phase characteristics. Zeros of network input-output dynamics have also have been considered in the context of string stability, unknown-input observability/detectability, and security analysis of dynamical systems. Of importance, a key study related to unknown-input observability specifically addresses sensor placement, but focuses on designable algorithms) and on elimination of zeros via sensor placement [11].

The purpose of this study is to apply and enhance the graph-theoretic analyses of network input-output dynamics, to support sensor, actuator, and input-output channel placement in dynamical networks. Specifically, insights into placements that guarantee minimum-phase characteristics are discussed, for several typical network classes and sensor/actuator placement paradigms. Simple placement algorithms are obtained as a result. For collocated single-input single-output channels, some insight into the phase response is also given.

II. Problem Formulation

A dynamical network with n nodes, labeled $1, \dots, n$, is considered. Each node $i = 1, \dots, n$ has associated with it a scalar state $x_i(t)$ which evolves in continuous time. Further, the m nodes in the set $\mathcal{S}$ are subject to actuation, which may represent either control or disturbance inputs. Specifically, each node $i \in \mathcal{S}$ is subject to an additive disturbance input $u_i(t)$. Formally, the state vector $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ is governed by the following linear dynamics:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \tag{1}$$

where $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$, and the m columns of the matrix B are each $0 - -1$ indicator vectors for the elements in $\mathcal{S}$ (i.e., each column is an indicator vector $\mathbf{e}_i$ for a distinct node i listed in $\mathcal{S}$). Further, outputs are taken at the p nodes in the $\mathcal{T}$, which may represent measurements used for feedback or state estimation, response variables of interest, etc. Specifically, the observation or output is modeled as

$$\mathbf{y}(t) = C\mathbf{x}(t), \quad (2)$$

where each row of C is a $0 - -1$ indicator vector $\mathbf{e}_i^T$ for a distinct node $i \in \mathcal{T}$. The state equation (1) and measurement equation (2) are together referred to as the *linear network model*.

For this work, the state matrix A of the dynamical model is assumed to be an essentially-nonnegative or Metzler or *M*-matrix, i.e. a matrix whose off-diagonal entries are nonnegative. It is further assumed that the row sums of A are nonpositive, which is sufficient for the system to be neutrally stable. Linear models with essentially-nonnegative state matrices encompass a wide range of network dynamics of interest, including consensus and diffusion processes, conservative and nonconservative flow dynamics, compartmental models, and spread processes.

Our aim here is to develop graph-theoretic insights into the input-output dynamics of the linear network model, and in turn to develop sensor and actuator placement algorithms that achieve desirable input-output characteristics. To develop these results, a formal notion of the network's graph is needed, which encapsulates the interactions among the network's nodes. Formally, we define the *network graph* Γ for the linear network model to be a weighted digraph with n vertices, which correspond to the n network nodes. A directed edge is drawn from vertex j to vertex i in Γ if $A_{ij} > 0$, and is assigned the value A_{ij} as its weight. The edges in the graph thus represent the presence/absence and strengths of direct influences between the nodes in the network dynamics. We note that the diagonal entries in the state matrix are not encapsulated in the network graph.

The remainder of the paper addresses the graph-theoretic analysis of zeros, and the placement of sensors, actuators, and control channels to shape the input-output dynamics (and specifically the zeros). We focus particularly on placement of sensors and actuators to make the input-output dynamics minimum phase, and to move dominant zeros as far left as possible in the complex plane. Results are obtained for several network types and placement paradigms.

III. Results

Graph-theoretic analyses of the finite and infinite zeros of the linear network model are developed, and used to support sensor and actuator placements that have desirable input-output properties. Results are developed in the case that the inputs and outputs are collocated, and then the non-collocated case is considered. The main aim is to develop results for several graph classes and sensing/actuation paradigms. The results developed here use standard definitions for finite- and infinite- zeros, and specifically invariant zeros, which arise from the structural invariants notions originally developed by Morse [13], see also [14]. Our primary approach for characterizing zeros is to transform the dynamics into the *special coordinate basis* for linear systems [14], which allows for an algebraic analysis of the inifinite zeros and invariant zero dynamics. The main effort here is to translate these algebraic analyses to graph-theoretic results, and in turn to consider sensor and actuator placement. The results developed here draw on, but also extend and apply, the graph-theoretic analyses of zeros developed in our previous work [7]–[9].

A. Collocated Inputs and Outputs

In many networks, measurement and actuation capabilities are naturally collocated. In these cases, the placement of collocated sensing and actuation capabilities at multiple network nodes to shape the finite and infinite zero structure is naturally of interest. Here, an algebraic analysis of the invariant zeros for linear networks with collocated inputs and outputs ($\mathcal{S} = \mathcal{T}$) is obtained first, and then several further insights into the zeros are noted as corollaries. These results are used to suggest algorithms for collocated sensor and actuator placement. Finally, the phase characteristics of the input-output transfer function are further characterized, in the case of a single collocated input and output and a diagonally-symmetrizable state matrix.

Before presenting the results, let us introduce some notation and definitions, which are defined for general actuator (input) and sensor (output) locations. Let $d(i, j)$ be the distance (minimum number of directed edges) between the vertices i and j, and $\mathcal{N}_+(i) = \{j : 0 \leq d(j, i) < \infty\}$. In other words $\mathcal{N}_+(i)$ is the set that contains all the vertices from which there is a directed path to the vertex i. Also, let us define $\tilde{\mathcal{N}_b} = \bigcup_{i \in \mathcal{T}, i \notin \mathcal{S}} \mathcal{N}_+(i)$. That is, $\tilde{\mathcal{N}_b}$ contains all the vertices for which there is a directed path to any sensor that does not also have an input. The set $\mathcal{V}$ contains the nodes (vertices) that have both input and output. Note that some vertices in set $\mathcal{V}$ can be included in $\tilde{\mathcal{N}_b}$. Let $\mathcal{N}_b$ be the set of vertices in $\tilde{\mathcal{N}_b}$ but not in $\mathcal{V}$. Finally, the set $\mathcal{N}_a$ are the vertices which do not belong to $\tilde{\mathcal{N}_b}$ or $\mathcal{V}$. We notice that a node i is associated with a state x_i. Consequently, $\tilde{\mathcal{N}_b}$, $\mathcal{N}_b$, $\mathcal{N}_a$, and $\mathcal{V}$ can also represent index sets.

Also, let $A_{[\mathcal{N}_a]}$ be a principal submatrix of A obtained by deleting all the rows and columns $i \notin \mathcal{N}_a$. In the special case where $\mathcal{S} = \mathcal{T}$, we note that $\tilde{\mathcal{N}_b}$ and $\mathcal{N}_b$ are empty, and hence $\mathcal{N}_a$ contains all vertices that are not in $\mathcal{V}$. It follows that $A_{[\mathcal{N}_a]}$ is simply the principal submatrix of A for which all the rows and columns corresponding to the collocated input/output vertices are removed (or equivalently the rows and columns corresponding to the remaining vertices are maintained).

The following lemma characterizes the invariant zeros of the linear network model, in the case of collocated inputs and outputs.

Lemma 1: If $\mathcal{S} = \mathcal{T}$, then the invariant zeros of the linear network model (1) and (2) are the eigenvalues of $A_{[\mathcal{N}_a]}$.

Proof: The analysis draws on the special coordinate basis (SCB) transformation, a state transformation that decomposes the system in four interconnected subsystems: 1) infinite zeros chains (state $\mathbf{x}_d$), which are driven by the inputs and directly impact the output; 2) chains (state $\mathbf{x}_b$) that are not driven by the inputs and directly influence the outputs; 3) a state $\mathbf{x}_c$ that is directly influenced by the inputs but does not directly influence the output; and 4) a state $\mathbf{x}_a$, which is neither directly controlled by any input nor does it directly affects any output [14]. It is important to highlight that the state $\mathbf{x}_b$ is not driven by any state in $\mathbf{x}_a$.

Here, we do not fully transform the network system into the SCB, however, the definitions of all the subspaces are used to identify the state dynamics of $\mathbf{x}_a$ (zero dynamics) and consequently the invariant zeros. Without loss of generality let the state vector have the following form $\mathbf{x} = \begin{bmatrix}\mathbf{x}_d' & \mathbf{x}_a'\end{bmatrix}'$ such that $\mathbf{x}_a = \{x_i : i \in \mathcal{N}_a\}$, and $\mathbf{x}_d = \{x_i : i \in \mathcal{V}\}$. Then the network dynamics can be written as follows:

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\mathbf{x} + \begin{bmatrix}I_m\\ 0\end{bmatrix}\mathbf{u}\\ \mathbf{y} &= \begin{bmatrix}I_m & 0\end{bmatrix}\mathbf{x}\end{aligned}$$

where I_q is the $q \times q$ identity matrix, 0 indicates a zero matrix of appropriate dimension, and A_{ij}, for $i, j = 1, 2$, is a submatrix of A related with the states $\mathbf{x}_d$ and $\mathbf{x}_a$ respectively. We note that because $\mathcal{S} = \mathcal{T}$ there is no state $\mathbf{x}_b$ and states in $\mathbf{x}_d$ correspond to the outputs of the system. Therefore, the zero dynamics is given by

$$\dot{\mathbf{x}}_a = A_{22}\mathbf{x}_a + A_{21}\mathbf{x}_d$$

Note that according to the definition of $A_{[\mathcal{N}_a]}$ and A_{22}, we have $A_{[\mathcal{N}_a]} = PA_{22}P^{-1}$ for some permutation matrix P, and hence the eigenvalues of both matrices are the same. Further, since the invariant zeros are the eigenvalues of the zero dynamics [14], the result follows. ■

Remark: The coordinate transform further shows that the model is uniform-rank-1, in the case that $\mathcal{S} = \mathcal{T}$.

The result above can be derived straightforwardly from Lemma 4 in [11], and also has been approached from a different perspective in [15]. The above presentation is helpful because it makes explicit the connection to the special coordinate basis, and also shows that the zeros are the eigenvalues of a particular principal submatrix of the state matrix A when the inputs and outputs are collocated. Because the state matrix A of the linear network model has a special structure (essentially nonnegative matrix with nonpositive row sums), the zeros of the input-output dynamics can easily be characterized. Specifically, two further results on the finite invariant zeros follow immediately from the lemma above, together with basic properties of principal submatrices of essentially-nonnegative matrices (see [16]):

Corollary 1: If $\mathcal{S} = \mathcal{T}$ and the network graph Γ is strongly connected, the dominant invariant zero (the one with largest real part) is real and strictly negative.

Corollary 2: Consider the dominant invariant zero z for a particular set of collocated inputs and outputs, say $\mathcal{S} = \mathcal{T} = \mathcal{Q}$, where $\mathcal{Q}$ contains a subset of the network's nodes. Now say that further collocated input-output channels are added, i.e. $\mathcal{S} = \mathcal{T} = \tilde{\mathcal{Q}}$, where $\mathcal{Q} \in \tilde{\mathcal{Q}}$. The dominant invariant zero $\tilde{z}$ for the system with augmented inputs and outputs satisfies $\tilde{z} \leq z$, i.e. the dominant zero moves left.

The above theorem and corollaries immediately suggest a metric for placing control channels (in this case, collocated input-output pairs), given a budget of r channels allowed. Noting that that the dominant zero governs the settling rate of the closed-loop system upon application of a high-gain static control, a natural design metric is to select the r channels that move the dominant zero as far as possible to the left. This amounts to selecting the $(n-r)\ times(n-r)$ submatrix of A whose dominant eigenvalue is most negative. This search problem is combinatorial, requiring analysis of $n_C r$ matrices. However, a greedy algorithm, wherein single rows and columns are reduced so as to move the dominant eigenvalue as far left as possible, works well in practice. Based on the convexity of the dominant eigenvalue of A with respect to its diagonal entries, we conjecture that a submodularity argument can be used to prove that the greedy algorithm works nearly optimally, but leave it to future work to develop this in detail. In cases where actuators are in place and sensors need to be placed (or vice versa), the above results show that placing the sensors to cover the actuators guarantees minimum-phase dynamics.

In special cases, the model can not only be shown to be minimum phase, but the phase characteristics of the system's transfer function can be further characterized. The following result characterizes the phase response of a single-input single-output linear network model with collocated input and output, and diagonally-symmetrizable state matrix A:

Lemma 2: Consider a linear network model with a single input and single output, which are collocated. Assume that the state matrix A of the linear network model is diagonally symmetrizable. Then the phase of the frequency response satisfies $-90 \leq \angle H(j\omega) \leq 0$ for all frequencies ω.

The proof is omitted to save space, see [17]. The lemma shows that choosing collocated inputs and outputs not only guarantees that the zeros are in the left half plane, but guarantees a phase margin of at least 90^o (provided that the state matrix is diagonally symmetrizable). Thus, any chosen input-output channels has desirable robustness properties, and will not oscillate when subject to feedback.

Illustrative Example: A 5-node linear network model with $A = \begin{bmatrix}-2 & 1 & 1 & 0 & 0\\ 1 & -3 & 1 & 1 & 0\\ 1 & 1 & -3 & 1 & 0\\ 0 & 1 & 1 & -3 & 1\\ 0 & 0 & 0 & 1 & -1\end{bmatrix}$ is considered. The network is assumed to have a single input and single output, both at node 1. The invariant zeros are the eigenvalues of the submatrix of A formed by removing the first row and column. The invariant zeros are found to be -0.32, -1.46, -4, and -4.21. Also, the frequency response for this SISO system is shown in Figure 1. The phase response is entirely within -90^o and 0^o.

B. Non-Collocated Inputs and Outputs

In many networks, channels with non-collocated inputs and outputs must be leveraged for control or other purposes. Thus, a study of zeros for the general case where inputs and

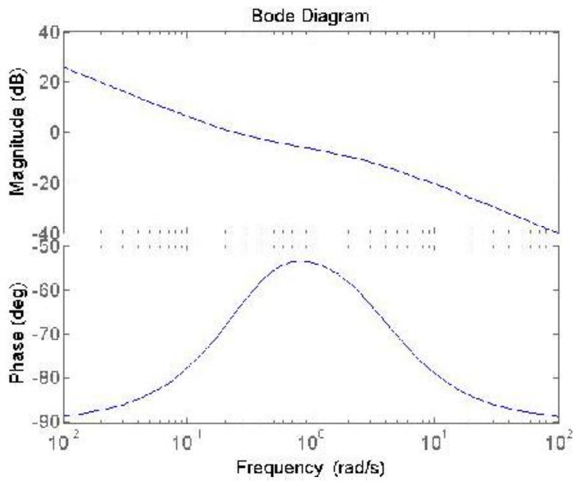

Fig. 1: The frequency response for the example system is shown. The phase remains between -90^o and 0^o for all frequencies.

outputs may be placed at different locations is of interest. As a starting point, it is natural to consider a SISO system where the input and output are non-collocated ($|\mathcal{S}| = |\mathcal{T}| = 1$); some results for more general MIMO systems can be developed thereof. (For SISO systems, the various different notions for finite zeros coincide, so we simply refer to them as zeros rather than invariant zeros).

The main outcome of this section is give several sufficient conditions for the SISO linear network model to be minimum phase, which give insight into sensor and actuator placement. The results address particular graph classes or sensing/actuation paradigms (e.g., adjacent input and output). Some results on placement of multiple sensors or actuators, which derive from the SISO case, are also presented. These results build on several key lemmas, which 1) give an explicit algebraic expression for the state matrix of the finite zero dynamics using the *special coordinate basis for linear systems* [14]; 2) relate the structure of the zero-dynamics state matrix to the network graph; and 3) hence give conditions on the graph which guarantee that the input-output dynamics are minimum phase. Because these results have been developed in detail in our previous work, we do not present them here even though they are necessary for proving our new results. Instead, we give a self-contained development of the new results (without proof) as a series of lemmas, and then give a sketch of how these results can be developed based on the earlier work.

First, conditions on the network graph are presented that guarantee minimum phase dynamics, even if inputs and output are remote. The main intuition is that the invariant zeros are guaranteed to be in the OLHP if there is only a single directed path in the network graph between the input and output, i.e. a single distinct sequence of vertices between the input and output (no matter the distance between the two). Thus, minimum-phase dynamics are guaranteed no matter the input and output location if the the network graph Γ is a strongly connected and a "tree", in the sense that there is only a single directed path between the input and output. The notion is formalized in the following lemma:

Lemma 3: Consider a SISO linear network model whose network graph Γ 1) is strongly connected and 2) for each pair of vertices has a single directed path between them (equivalently, the graph with edge directions ignored is a tree). The linear network model is minimum phase for any input and output location.

Noting that the inclusion of further input or output channels does not introduce new invariant zeros compared to the SISO case, the above lemma can be immediately generalized:

Lemma 4: Consider a single-input multiple-output (or multiple-input single-output) linear network model whose network graph Γ 1) is strongly connected and 2) for each pair of vertices has a single directed path between them (equivalently, the graph with edge directions ignored is a tree). The linear network model is minimum phase no matter where in the network inputs and outputs are placed.

The above two lemmas show the sensors or actuators can be placed at will in the linear network model to achieve minimum-phase dynamics, if the corresponding graph with directions ignored is a tree.

In fact, guaranteeing minimum-phase dynamics only depends on there being a single path between the input and output (regardless of the rest of the network topology), as formalized in the following lemma.

Lemma 5: Consider a linear network model with a single input at node i and a single measurement at node j. Then the linear network model is strongly detectable if there is a single path between the input and output in the network graph Γ (i.e., a single sequence of distinct vertices with directed edges between them between the input and output).

Notice that the above result is applicable to both directed and undirected (symmetric) networks, and allows any part of the network except the input-to-output path to have cycles of length greater than 2. Unlike the result for the tree-like graph, the above lemma does not guarantee minimum-phase dynamics for any input and output. However, according to the lemma, if the edges between the input and output are each a cutset of the graph. Such input and output locations could be chosen by finding minimum cutsets using a standard algorithm, and placing the input and output a single-edge cutset.

In many settings, it is unrealistic to expect that an input and output can be placed so that there is a single path between them. Next, we consider placing the input and output in a more tightly connected graph, but for the special case where the input and output are adjacent. In the case where the input and output are adjacent, minimum phase dynamics can also be guaranteed provided that the edge between the input and output is sufficiently strong. This notion is formalized in the following lemma.

Lemma 6: Consider a SISO linear network model with input at location i and output at location j, where j is adjacent to i in the network graph Γ. The linear network model is guaranteed to be minimum phase if: $A_{ki}A_{ji} \geq$

$A_{ik}(\sum_r A_{rj})$ for all k.

The condition on the edge-weights for minimum phase dynamics depends only on the edges adjacent to vertex i and vertex j in the graph. The above lemma suggests a simple, myopic rubric for placing sensors, actuators, or control channels that are minimum phase. In particular, the lemma shows that any two adjacent nodes can be chosen as input and output, provided that the edge between them is sufficiently strong compared to the other edges adjacent to these vertices (as specified in the lemma). We note that this check only requires local information about the network topology: the rest of the network besides the two vertices and their edges in and out need not be considered. Also, checking the condition requires very little computation, hence the check can be easily done for every edge in the network.

Even when the unknown input and the measurement location are not adjacent, minimum phase dynamics can be guaranteed if the shortest path between the input and measurement is sufficiently strong compared to alternative longer paths. Conversely, if the shortest path between the input and measurement is weak compared to other longer paths, then the input-to-measurement transfer function is necessarily nonminimum phase (see [7]). Unfortunately, it is difficult to obtain a crisp numerical bound on the relative strengths of the paths that guarantees either minimum-phase or nonminimum-phase dynamics.

Finally, we study whether sensors can be placed at multiple network nodes to guarantee minimum phase dynamics, when there is only a single input. The following lemma indicates that sensors can be placed in a way that guarantees minimum phase dynamics, specifically by separating the input from parts of the graph containing loops using the measurements.

Lemma 7: Consider a linear network model with a single unknown input location i. Say that the monitor measures a set of locations $\mathcal{T}$. Now consider forming a reduced graph $\widehat{\Gamma}$ from the network graph Γ, by removing from Γ all nodes and edges which are separated from i by $\mathcal{T}$ (i.e., there is no path from i to the node or edge which does not contain a vertex in $\mathcal{T}$). If the graph $\widehat{\Gamma}$ is strongly connected and there is a single directed edge between each pair of vertices, then the linear network model is minimum phase.

As a special case, minimum phase dynamics are guaranteed if all of the neighbors of the input location are outputs. Thus, another general strategy for achieving minimum phase dynamics is to surround the unknown-input location with sensors. We note that this approach is akin to the separating-set sensor placement for strong observability introduced by Sundaram et al [11], but does not require the presence of independent paths to all vertices.

a) Sketch of Proofs for Single-Input Lemmas: The lemmas developed in this section build on algebraic and graph-theoretic characterizations of the zero dynamics of the SISO linear network model, which were developed in a sequence of previous studies [7]–[9]. The algebraic characterization of the state matrix of the zero dynamics, which uses the special coordinate basis for linear systems, is rather intricate. To avoid redundancy and give a concise presentation, we do not re-develop these results in detail here but rather only list the main outcomes. We then sketch how these results are used to prove the lemmas above.

The lemmas presented here draw on the following four results, which are quoted directly from [8]:

1) The relative degree (number of infinite zeros) is given by $n_d = d+1$, where d is the distance from the input to the output vertex in G (see also [5]). In the SCB formulation, the states associated with the vertices in the shortest directed input-output path form the chain of n_d integrators. Let us call this path the *special input-output path*.

2) The dimension of the zero dynamics (number of finite zeros) is $n_a = n - d - 1$. The states of the zero dynamics can be defined via a transform of the states corresponding to vertices that are not on the special input-output path. We define a set V_1 containing these n_a vertices. We also use the notation G_1 for the induced subgraph of G on V_1. WLOG, the vertices on the special input-output path are labeled $n-d, n-d+1, \dots, n$, where vertex $n-i$ is at a distance i to the output, while the vertices in V_1 are labeled $1, 2 \dots, n_a$.

3) The network's finite-zero dynamics is given by:

$$\dot{x}_0 = A_{aa}x_0 + (\sum_{i=0}^{n_d-1} A_{aa}^{n_d-i-1} A_{n_{ad}} Z_{nd}^{-1} e_{n_d-i})\tilde{y} \qquad (3)$$

where $A_{aa} = A_{n_a} - \Delta$ and $\Delta = A_{n_{ad}} Z_{nd}^{-1} Z_{n_{ad}}$. The matrix A_{n_a} is a principal submatrix of the A formed by the rows and columns corresponding to the vertices in V_1. The matrix $A_{n_{ad}}$ is an off-diagonal submatrix of A, while Z_{nd}^{-1} and $Z_{n_{ad}}$ can be computed explicitly in terms of powers of the graph matrix (see [8] Appendix A2); due to space constraints, we omit the full expressions here. Since A_{n_a} specifies the graph G_1, we refer to A_{n_a} as the *reduced graph matrix*. The eigenvalues of the zero-dynamics state matrix A_{aa} are the finite zeros of the network model [14].

4) We can view the zero-dynamics state matrix A_{aa} as a perturbation of the matrix A_{n_a} by the matrix Δ. The key idea is that the zero structure is specified by the reduced graph G_1 but with a modification arising via its connection to the special input-output path. The perturbation Δ is a sparse matrix whose nonzero entries are identifiable by the graph structure [8]. Specifically, $\Delta_{ij} = 0$ ($\{A_{aa}\}_{i,j} = \{A_{n_a}\}_{i,j}$) unless the following two conditions are satisfied:

a) There is an arc (directed edge) from a vertex in the special input-output path (excluding the output) to the vertex $i \in V_1$.
b) There is a directed path from the vertex $j \in V_1$ to the output vertex whose length is less than or equal to $d - d_i + 1$, where d_i is the distance from the input to the vertex $i \in V_1$ satisfying condition 1.

These four results are a starting point for proving the lemmas listed above. Specifically, Lemmas 3 and 5 can be proved by first considering reduced graph $\widehat{\Gamma}$. The conditions for both lemmas specify that there is only a single path between the input location i and the measurement location j. It thus follows that the the reduced graph $\widehat{\Gamma}$ comprises (at most) d disconnected subgraph or components, which correspond to the subgraphs of Γ attached to each vertex on the special input-output path. It thus follows that the reduced graph matrix A_{n_a} is block diagonal, with each diagonal block corresponding to the subgraph of Γ attached to each vertex on the special input-output path. Without loss of generality, assume that these diagonal blocks are order based on the distance of their connecting node on the special input-output path to the input (so the first block corresponds to the subgraph connected to the input, the next block corresponds to the next node on the special input-output path, and so forth). Next, let us consider the perturbation matrix Δ. The entry Δ_{qr} is non-zero only if the distance from vertex $q \in V_1$ to the input vertex plus the distance from the output vertex to vertex $r \in V_1$ is as most $d+1$ in the graph Γ. However, this means that Δ_{qr} can be greater than zero only if the vertex i is connected to the special input-output path at a closer point to the input than vertex j. It thus follows that the state matrix of the zero dynamics $A_{aa} = A_{n_a} + \Delta$ is block upper triangular, with the diagonal blocks equal to those of A_{n_a}. Thus, the zeros are simply the eigenvalues of the diagonal blocks of the reduced graph matrix A_{n_a}. However, these blocks are essentially-nonnegative matrices which correspond to connected graphs, have row sums less than or equal to zero, and have one row sum that is strictly less than 0. It follows from standard properties of essentially nonnegative matrices that the eigenvalues of these diagonal blocks, and hence the zeros, are in the OLHP.

To prove Lemma 6, first note that $d = 1$ since the input i and output j are adjacent. The special input-output path comprises only the input vertex and output vertex. Let us again consider the state matrix of the zero dynamics, $A_{aa} = A_{n_a} + \Delta$, which has dimension $n-2$ in this case. The matrix A_{n_a} is an essentially nonnegative matrix, whose row sums are less than or equal to zero. Further, for any vertex k that is a neighbor of i (specifically, $A_{ki} > 0$), the corresponding row of A_{n_a} has sum less than or equal to $-A_{ki}$. Meanwhile, since the input and output are adjacent, Δ_{qr} can be nonzero only if q is a neighbor of the input i ($A_{iq} > 0$) and r is a neighbor of the output j ($A_{rj} > 0$). Let us consider the perturbations on a particular row, say k, where the corresponding vertex is adjacent to i. From the detailed expressions for the entries in the perturbation matrix (omitted), it can be shown that the absolute sum of the entries in Δ in this row equals $\frac{A_{ik} \sum_r A_{rj}}{A_{ji}}$. From Gersgorin's disk theorem, it this follows that the eigenvalues of A_{aa} are in the OLHP if $A_{ki} < A_{ik} \sum_r A_{rj} A_{ji}$ for all k. Thus, the theorem statement is recovered.

The proof of Lemma 7 requires generalizing the graph-theoretic analysis of A_{aa} to the single-input multiple-output case. Lemma 7 is an analog to the case that there is a single input-output path when there is a single input and output, and proof follows in a similar fashion albeit with some technical issues to address the non-right-invertibility of the system. This analysis is lengthy between it requires reworking the entire algebraic computation of the zero dynamics from the special coordinate basis, as done for the SISO case in [7]. Details are omitted in the interest of space. □

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