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Local Open-Loop Manipulation of Multi-Agent Networks

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Abstract— **The energy required for manipulation of a double-integrator network model via local external actuation is examined. Some previous results on the controllability Gramian, which specifies the required energy, are briefly reviewed. Three new results are then developed. First, the energy required for manipulation along the synchronization manifold over an arbitrary horizon is characterized. Second, several scalar measures that give a global indication of a network's manipulability are analyzed. Based on these measures, we study design of the DIN to prevent or facilitate manipulation.**

I. Introduction

A number of results on the controllability of network synchronization or consensus processes from sparse inputs have been developed, in both the controls and the physics literatures [1], [2], [3], [10]. Many of these results are concerned with relating controllability with the network's underlying graph topology, and/or developing conditions for structural controllability. More recently, researchers have begun to examine the energy required for control from structural and graph-theoretic perspectives, as a means to evaluate the practicality of control and to support actuator placement [3], [10], [9], [4]. A few recent studies have also approached the problem of targeted (output) control [5], have addressed increasingly sophisticated synchronization models [6], or also have extended the controllability analysis to consider input-output properties (e.g., presence of nonminimum-phase zeros) [7]. These various result are proving useful for cyber-physical-systems design as well as infrastructure management applications, because they provide simple graphical rubrics that give rough insight into controllability and actuator placement.

Motivated primarily by cyber-physical-systems applications, our group recently examined the energy required for local control or manipulation of a network synchronization process with planar agents, termed a double-integrator network model [8]. This initial study was focused on developing explicit expressions for the inverse of the controllability Gramian in terms of the spectrum of the network's graph Laplacian. This study also initiated a study of closed- loop manipulation of the network from local observations, which complements the open-loop controllability notions.

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This article continues the study on local manipulation of the double-integrator network model, with a focus on understanding how the model's parameters and the manipulation goal influence the required energy, and hence understanding how to design the network model to facilitate or prevent manipulation. Three main results are developed. First, the energy required for manipulation along the synchronization manifold, wherein the network must be moved to a different synchronized state, is characterized and shown to be small. Second, explicit characterizations of scalar metrics of overall network controllability are undertaken, and used to gain insight into what classes of networks are more easily controlled. Third, a process for designing coupling gains in the double-integrator network to reduce or enhance manipulability is discussed.

The remainder of the article is organized as follows. The manipulation-energy analysis problem for the double-integrator network is reviewed in Section 2. The explicit computations of the inverse Gramian obtained in [8] are reviewed in Section 3. The new results on the manipulation energy are presented in Section 4.

To save space we have omitted the proofs, please see the extended version at http://www.eecs.wsu.edu/ sroy/.

II. Problem Formulation

In this section, local manipulation of a canonical synchronization dynamics, specifically a double-integrator network dynamics, is modeled. Also, the manipulation-energy-analysis problem is formulated. The formulation summarizes, and in places exactly quotes, the formulation given in [8].

Nominally, the *double integrator network* (DIN) model describes the coupled dynamics of n agents, labeled $i = 1, \dots, n$. The model is specified using a weighted digraph $\Psi = (\mathcal{V}, E : \Gamma)$, where $\mathcal{V} = \{1, \dots, n\}$ is the set of vertices, and E is the set of ordered pairs of vertices representing the arcs or directed edges between vertices. Each directed edge (i, j) in the graph has associated with it a positive weight $\gamma_{i,j}$, as specified in the weight set Γ. In this formulation, each vertex i in the graph Ψ is associated with the agent i in the DIN model. Further, each edge $(i, j : \gamma_{i,j})$ specifies the interaction between agents i and j. We associate with each agent i the *position state* $x_i(t)$, which evolves in continuous time ($t \in R^+$) as follows:

$$\ddot{x}_i = \sum_{j \in \mathcal{N}_i} \{\gamma_{ji} k_p (x_j - x_i) + \gamma_{ji} k_v (\dot{x}_j - \dot{x}_i)\} - b\dot{x}_i \qquad \forall\ i = 1, ..., n, \tag{1}$$

where the set $\mathcal{N}_i$ contains the indices of all the vertices j that are *neighbors* of vertex i (i.e., such that (j, i) is an edge), k_p

and k_v are global scaling constants that indicate how strongly each agent weights neighbors' states and state derivatives in their updates, and the nonnegative scalar b captures damping processes. We notice that $v_i = \dot{x}_i(t)$ is also an internal state variable for the agent, which we refer to as the velocity state.

It is easy to derive conditions such that the network nominally achieves *synchronization*, i.e. such that the dynamics is stable in the sense in Lyapunov and further the agents' states x_i and v_i asymptotically converge to the same value. Specifically, synchronization is achieved provided that 1) the network graph Ψ is strongly connected and 2) k_v and k_p are chosen properly (e.g., with k_p chosen sufficiently small compared to k_v) [11], [12]. We assume here that the nominal dynamics achieve synchronization.

This study is focused on manipulation of DIN by a stakeholder who is able to apply an external input u to a single agent, say agent q. The dynamical model for agent q is modified to capture the external input:

$$\ddot{x}_q = \sum_{j \in \mathcal{N}_q} \{\gamma_{jq} k_p (x_j - x_q) + \gamma_{jq} k_v (\dot{x}_j - \dot{x}_q)\} - b\dot{x}_q + u \quad (2)$$

where $y \in \mathbf{R}$ is the output or observation made by the stakeholder.

The full DIN dynamics with the external stakeholder's input and observation included can be written in vector form as:

$$\dot{z} = \left(-L \otimes \begin{bmatrix} 0 & 0 \\ k_p & k_v \end{bmatrix} + I_n \otimes \begin{bmatrix} 0 & 1 \\ 0 & -b \end{bmatrix} \right) z + (e_q \otimes B_2)\, u \quad (3)$$

where $\mathbf{z} = \begin{bmatrix} x_1 & v_1 & \dots & x_n & v_n \end{bmatrix}'$, $B_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}'$, $e_q \in \mathbf{R}^n$ is a standard basis vector with a single unity entry at the q^{th} location, I_n is an $n \times n$ identity matrix, $\otimes$ denotes the Kronecker product, and $(.)'$ denotes the transpose. Further, the matrix L is the weighted *in-degree* Laplacian of the weighted digraph Ψ: for $i \neq j$, $L_{i,j} = -\gamma_{j,i}$ if there is a directed edge from vertex j to vertex i and zero otherwise; meanwhile, $L_{i,i} = -\sum_{j \neq i} L_{i,j}$. For simplicity, the notation $A = -L \otimes \begin{bmatrix} 0 & 0 \\ k_p & k_v \end{bmatrix} + I_n \otimes \begin{bmatrix} 0 & 1 \\ 0 & -b \end{bmatrix}$ is used for the state matrix of the full dynamics, and $B = e_q \otimes B_2$ is used for the input matrix.

The problem of interest is to understand: 1) whether the stakeholder can manipulate the state of the double-integrator network to any desired goal state, which is equivalent to asking whether the state equation (3) is controllable; and 2) how much energy or effort is required to move the state to a particular goal. The controllability of the dynamics can be readily tied to the spectrum of the Laplacian matrix and, in turn, the graph topology, see [8] and the brief review in Section 3 below. The controllability analysis indicates that DIN models are typically controllable from a single input. Thus, our focus here is on characterizing the effort or energy needed for control, to determine whether manipulation from a single input is practical, and to support network design and actuator selection.

A minimum-two-norm metric is used for the required manipulation effort. Specifically, we consider the case that the stakeholder seeks to drive the state away from a nominal synchronization condition (chosen as the origin, without loss of generality). The stakeholder's goal is to move the full state from the origin to a particular final state x_f by time t, under the assumption that the dynamics are in fact controllable. Here, the squared-two-norm of the input, i.e. $\int_{\tau=0}^{t} u^2(\tau) d\tau$, is used to measure the manipulation effort for a particular input signal. The minimum of this squared-two-norm over input signals that achieve the desired final state x_f is chosen as the metric for manipulability. The minimum-energy metric is appealing in that it naturally measures the least input deviation required for manipulation, yet is tractable.

The minimum input energy required to move the state to the goal x_f is well known, from standard control-theory methods, to be $x_f' G_r^{-1}(t) x_f$, where the positive definite matrix $G_r(t) = \int_0^t e^{A(t-\tau)} B B' e^{A'(t-\tau)} d\tau$ is the *reachability Gramian* of the system over the interval $[0, t]$ [13]. While this algebraic expression permits computation of the minimum-energy manipulation, it does not directly provide insight into what network characteristics permit or frustrate manipulation. From the expression for the minimum energy, it is clear that the inverse of the reachability Gramian $G_r^{-1}(t)$ plays a central role in deciding ease of manipulation: in general, small inverse Gramians permit easy state manipulation in many directions, while larger inverse Gramians correspond to hard-to-manipulate networks. The asymptotic matrix $G_r^{-1} = \lim_{t \to \infty} G_r^{-1}(t)$ is particularly interesting, since it specifies lower bounds on the manipulation energy that are independent of the time horizon t. In our previous work [8], explicit computations of the inverse Gramian in terms of the spectrum of the Laplacian matrix were developed, see Section 3 for a review. Here, this preliminary analysis of inverse Gramian is invoked to develop three new results on manipulation of the DIN. First, the manipulation energy require for goals states along the consensus manifold is characterized. Second, explicit formulae are obtained scalar metrics for overall controllability of the DIN, specifically the trace and determinant of the inverse Gramian. Third, the dependence of the scalar controllability metrics on the DIN's coupling gain parameters is determined, which facilitates design to enhance or descrease manipulability.

As in [8], our analysis of the manipulation energy is restricted to the case that the Laplacian matrix L is symmetric, or equivalently the network graph is undirected. We stress that the state matrix of the DIN is not symmetric even if L is symmetric, hence the analysis requires considering complex eigenvalues of the state matrix.

III. Summary of Previous Results

The results presented in this article draw on our previous work on the open-loop manipulation of the DIN [8]. For the reader's convenience, the main outcomes of the previous work are summarized in this section. The reader is referred to [8] for proofs and discussion of these results.

The controllability and manipulation-energy analyses given in [8] depend on relationships between the spectrum of the state matrix A and the spectrum of the graph Laplacian

L. The following two lemmas relate the spectra of A and L, for the damped ($b \neq 0$) and undamped ($b = 0$) DIN. For this analysis, we let λ_i be an eigenvalue of L and c_i be its corresponding left eigenvector. We also use μ_i and w_i for each eigenvalue and corresponding left eigenvector for the state matrix A. Lemma 1 expresses two eigenvalues μ_{2i-1} and μ_{2i} (and corresponding left eigenvectors w_{2i-1} and w_{2i}) of the state matrix A in terms of one eigenvalue λ_i (and its corresponding left eigenvector c_i) of the Laplacian matrix L, for the damped DIN.

Lemma 1: Consider the damped DIN ($b \neq 0$). The $2n$ eigenvalues of the state matrix A are given by:

$$\mu_{2i-1} = \frac{-(\lambda_i k_v + b) + \sqrt{(\lambda_i k_v + b)^2 - 4\lambda_i k_p}}{2}$$

$$\mu_{2i} = \frac{-(\lambda_i k_v + b) - \sqrt{(\lambda_i k_v + b)^2 - 4\lambda_i k_p}}{2},$$

for $i = 1, \ldots, n$. Further, for each left eigenvector c_i of L, the matrix A has two corresponding left eigenvectors that are given by $w_{2i-1} = c_i \otimes d_{2i-1}$ and $w_{2i} = c_i \otimes d_{2i}$, where $d_{2i-1} = \begin{bmatrix} \frac{(\lambda_i k_v + b) + \sqrt{(\lambda_i k_v + b)^2 - 4\lambda_i k_p}}{2} \end{bmatrix}$ and $d_{2i} = \begin{bmatrix} \frac{(\lambda_i k_v + b) - \sqrt{(\lambda_i k_v + b)^2 - 4\lambda_i k_p}}{2} \end{bmatrix}$ if $\sqrt{(\lambda_i k_v + b)^2 - 4\lambda_i k_p} \neq 0$. Otherwise, if $\sqrt{(\lambda_i k_v + b)^2 - 4\lambda_i k_p} = 0$, then A has a single corresponding left eigenvector $d_{2i-1} = \begin{bmatrix} \lambda_i k_v + b \\ 2 \end{bmatrix}$ and there is no second eigenvector (i.e., the matrix is defective and so a generalized left eigenvector is needed)[1].

The following lemma characterizes the eigenstructure in the undamped case ($b = 0$):

Corollary 1: When the DIN is undamped ($b = 0$), A has an eigenvalue at 0 with algebraic multiplicity of 2, which corresponds to the zero eigenvalue of L, ($\lambda_1 = 0$). Further, the left eigenvector and left generalized eigenvector associated with the zero eigenvalues are:

$$w_1 = \vec{1} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix}'$$
$$w_2 = \vec{1} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix}'$$

where $\vec{1} \in \mathbf{R}^n$ is the all ones vector.

Next, following the development in [8], the controllability of the DIN is characterized in terms of the Laplacian spectrum:

Theorem 1: The DIN with the external input applied to agent q is controllable if and only if all left eigenvectors of L contain nonzero entries at the q^{th} position.

Theorem 1 shows that the controllability of (A, B) is equivalent to the controllability of (L, e_q). From the equivalence, it is clear that controllability is entirely dependent on the graph topology of the DIN, and the input location relative to the graph. The condition for controllability is met under broad conditions, see [8] for details.

The following theorems give explicit expressions for the finite-horizon reachability Gramian $G_r(t)$ of a reachable DIN (Theorems 2, and 4), and then the inverse of the infinite-horizon Gramian $G_r^{-1} = \lim_{t\to\infty} G_r^{-1}(t)$ of a reachable DIN (Theorems 3 and 5), in terms of the spectrum of A. The expressions obtained for the undamped ($b = 0$) and damped ($b \neq 0$) cases are distinct, because the state matrix has a repeated eigenvalue at the origin in the undamped case.

Theorem 2: The finite-horizon reachability Gramian $G_r(t)^2$ for the undamped case ($b = 0$) is:

$$G_r(t) = W^{-1} W_{2q} \widehat{G}_r W_{2q} (W^{-1})', \tag{4}$$

where

$$\widehat{G}_r = \begin{bmatrix} R(t) & Q'(t) \\ Q(t) & P(t) \end{bmatrix}, R(t) = \begin{bmatrix} t^3/3 & t^2/2 \\ t^2/2 & t \end{bmatrix},$$

$$Q'(t) = \begin{bmatrix} \left(\frac{1}{[\mu_3^*]^2} - \frac{t}{\mu_3^*}\right) e^{\mu_3^* t} - \frac{1}{\mu_3{}^2} & \cdots & \left(\frac{1}{[\mu_{2n}^*]^2} - \frac{t}{\mu_{2n}^*}\right) e^{\mu_{2n}^* t} - \frac{1}{[\mu_{2n}^*]^2} \\ \frac{e^{\mu_3^* t} - 1}{\mu_3^*} & \cdots & \frac{e^{\mu_{2n}^* t} - 1}{\mu_{2n}^*} \end{bmatrix},$$

$$P_{ij}(t) = \frac{e^{(\mu_{i+2}^* + \mu_{j+2})t} - 1}{\mu_{i+2}^* + \mu_{j+2}} \ ; i, j = 1, \ldots, 2n-2$$

$W_{2q} = \text{diag}(1, 1, w_{3,(2q)}, \ldots, w_{2n,(2q)})$ is a diagonal matrix whose $(i)^{th}$ diagonal entry $w_{i,(2q)}$, is the $2q^{th}$ entry of the eigenvector associated with μ_i where $i = 3, ..., 2n$. W is the matrix whose i^{th} row is the eigenvector w_i'. Each eigenvector is assumed to be normalized to unit length, i.e. its two-norm is 1.

Remark: The matrix $P(t)$ is related to the family of Cauchy matrices. This is the essential structure that is exploited to characterize the inverse of the Gramian, see [8] for details.

Theorem 3: The inverse of the reachability Gramian $G_r(t)$ over the infinite horizon for the undamped case ($b = 0$) is:

$$G_r^{-1} = W' W_{2q}^{-1} \widehat{G}_r^{-1} W_{2q}^{-1} W, \tag{5}$$

where $\widehat{G}_r^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & P^{-1} \end{bmatrix}$. The 0 notations within the matrices above represent zero matrices of appropriate dimensions, and the entries of P^{-1} are:

$$\{P^{-1}\}_{i,j} = -\left(\mu_{i+2}^* + \mu_{j+2}\right) \prod_{\substack{m=3 \\ m \neq j+2}}^{2n} \frac{\mu_m + \mu_{i+2}^*}{\mu_{j+2} - \mu_m} \prod_{\substack{p=3, \\ p \neq i+2}}^{2n} \frac{\mu_p^* + \mu_{j+2}}{\mu_{i+2}^* - \mu_p^*}, \tag{6}$$

for $i = 1, \ldots, 2n-2$ and $j = 1, \ldots, 2n-2$.

Theorem 4: The finite-horizon reachability Gramian $(G_r(t))$ for the damped case ($b \neq 0$) is:

[1]In the case that L is defective (has generalized eigenvectors), the analysis of the eigenvectors of A becomes more subtle. Details are omitted here, see [**?**]. Relevant to our development here, it can be shown that the above theorem accounts for all eigenvectors of A.

[2]For convenience, we omit the atypical case that non-zero eigenvalues of A are in Jordan blocks. This may happen in a controllable system if a single non-zero eigenvalue λ_i of the graph Laplacian corresponds to repeated eigenvalues of A, i.e. $(\lambda_i k_v + b)^2 - 4\lambda_i k_p = 0$. The analysis can be generalized to this case with some additional work, or the analysis here can be used to obtain an arbitrarily-close approximation via a small perturbation of k_p.

$$G_r(t) = W^{-1} W_{2q} \tilde{G}_r W_{2q} (W^{-1})^{'}, \tag{7}$$

where

$$\tilde{G}_r = \begin{bmatrix} t & M'(t) \\ M(t) & N(t) \end{bmatrix}, M'(t) = \begin{bmatrix} \frac{e^{\mu_2^* t}-1}{\mu_2^*} & \dots & \frac{e^{\mu_{2n}^* t}-1}{\mu_{2n}^*} \end{bmatrix},$$

$$N_{ij}(t) = \frac{e^{(\mu_{i+1}^* + \mu_{j+1})t} - 1}{\mu_{i+1}^* + \mu_{j+1}} \ ; \ i,j = 1, \dots, 2n-1$$

$W_{2q} = \text{diag}(w_{1,(2q)}, \dots, w_{2n,(2q)})$ is a diagonal matrix whose i^{th} diagonal $w_{i,(2q)}$, is the $2q^{th}$ entry of the eigenvector associated with μ_i, W is the matrix whose i^{th} row is the eigenvector w_i' and each eigenvector has 2-norm equals 1.

Theorem 5: The inverse of the reachability Gramian $G_r(t)$ over the infinite horizon for the damped case ($b \neq 0$) is:

$$G_r^{-1} = W^{'} W_{2q}^{-1} \tilde{G}_r^{-1} W_{2q}^{-1} W, \tag{8}$$

where, $\tilde{G}_r^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & N^{-1} \end{bmatrix}$. The 0 notations within the matrices above represent zero matrices of appropriate dimensions, and the entries of the inverse of the Cauchy matrix $\tilde{P}^{-1}$ are:

$$\{N^{-1}\}_{i,j} = -\left(\mu_{i+1}^* + \mu_{j+1}\right) \prod_{\substack{m=2 \\ m \neq j+1}}^{2n} \frac{\mu_m + \mu_{i+1}^*}{\mu_{j+1} - \mu_m} \prod_{\substack{p=2, \\ p \neq i+1}}^{2n} \frac{\mu_p^* + \mu_{j+1}}{\mu_{i+1}^* - \mu_p^*} \tag{9}$$

IV. New Results

Several new results are presented, which are focused on measuring how easy or hard it is to manipulate a DIN, and designing the DIN to prevent or facilitate manipulation.

A. Manipulation along the consensus manifold

In many application domains, the stakeholder's ability to manipulate the double-integrator network along the consensus manifold, i.e. to a goal state where all agents have the same value, is of particular interest. For instance, in distributed decision-making applications, a stakeholder may wish to ensure that consensus is reached, but manipulate the agreed-upon value based on his/her selfish motivation. Likewise, in multi-vehicle-team control problems, an adversary may seek to manipulate a vehicle formation away from a nominal location toward an alternate target. Intuition suggests that manipulation along the consensus manifold should require less effort than manipulation along other coordinate directions, since the double-integrator network intrinsically approaches and maintains synchronization without requiring an external drive. In fact, the asymptotic expression for the inverse Gramian immediately shows that the energy required to manipulate the state along the consensus manifold approaches zero, for a sufficiently long manipulation horizon. Here, we characterize the minimum energy required for manipulation along the consensus manifold over a finite time horizon, in terms of the spectrum of L, finite time horizon t, and number of agents n. The analysis is done separately for the damped and undamped model. We also separately develop results for the case where only the position states are to be manipulated away from the origin, before addressing the general case where positions and velocities are to be manipulated. The following theorem addresses position manipulation along the consensus manifold, for the undamped case:

Theorem 6: The minimum energy required for manipulation of the undamped ($b = 0$) DIN to the goal state $z_{new} = \bar{K} \begin{bmatrix} 1 & 0 & 1 & 0 & \dots & 1 & 0 \end{bmatrix}'$ over the interval $[0, t]$ is upper-bounded by $12n\bar{K}^2 t^{-3}$. Further, the expression is exact in the asymptote of large t.

The result shows that manipulation along the consensus manifold requires little effort given a sufficiently long horizon, even when the network has a significant number of agents. This makes sense conceptually because a small input can be applied to the manipulated agent to move it slowly to the desired new state $\bar{K}$, whereupon the remaining agents will naturally follow and remain in consensus.

In some circumstances, manipulation of both the agents' position and their velocity states to a new point along the consensus manifold, i.e. to a goal state of the form $z_{new} = \begin{bmatrix} \bar{K} & \hat{K} & \dots & \bar{K} & \hat{K} \end{bmatrix}$, may be of interest. Using a similar argument, the manipulation energy for this more general case can be upper bounded by
$n\left(4\hat{K}^2 t^{-1} - 12\bar{K}\hat{K}t^{-2} + 12\bar{K}^2 t^{-3}\right)$

The energy required for manipulation along the consensus manifold can also be characterized for the damped DIN:

Theorem 7: The minimum energy required for manipulation of the damped ($b \neq 0$) DIN to the goal state $z_{new} = \bar{K} \begin{bmatrix} 1 & 0 & 1 & 0 & \dots & 1 & 0 \end{bmatrix}'$ over the interval $[0, t]$ is upper-bounded by $\bar{K}^2 \frac{b^2 n}{b^2+1} t^{-1}$. Further, the expression is exact in the asymptote of large t.

The proof for the damped DIN is similar to that for the undamped DIN, so we omit it to save space.

B. Scalar measures: trace and determinant

Scalar measures defined from the inverse Gramian, including its trace and determinant, are important global measures of the energy required for manipulation. These global measures are important as integrative metrics of the manipulability of dynamical networks (the DIN in our case), and are also a stepping stone toward designing networks which are easy or hard to manipulate. Here, we pursue analysis of these scalar measures.

The trace of the inverse Gramian indicates average energy required for manipulation over all goal states on the unit ball (with two-norm equal to 1). The following theorem expresses the trace of the inverse Grammian directly in terms of the spectrum of L, k_p, and k_v.

Theorem 8: The trace of the inverse of reachability Gramian G_r when the system has no damping ($b = 0$) is:

$$
\begin{aligned}
\text{trace}(G_r^{-1}) = \sum_{i=2}^{n} & \left(\frac{\left[(\lambda_i k_v + b) + \sqrt{(\lambda_i k_v + b)^2 - 4\lambda_i k_p}\right]^2 + 4}{[w_{2q}^{2i-1}]^2} \right) \left[P^{-1}\right]_{2i-3,2i-3} \\
& + \left(\frac{4[1 + \lambda_i k_p]}{w_{2q}^{2i-1} w_{2q}^{2i}} \right) \left[P^{-1}\right]_{2i-3,2i-2} + \left(\frac{4[1 + \lambda_i k_p]}{w_{2q}^{2i-1} w_{2q}^{2i}} \right) \left[P^{-1}\right]_{2i-2,2i-3} \\
& + \left(\frac{\left[(\lambda_i k_v + b) - \sqrt{(\lambda_i k_v + b)^2 - 4\lambda_i k_p}\right]^2 + 4}{[w_{2q}^{2i}]^2} \right) \left[P^{-1}\right]_{2i-2,2i-2},
\end{aligned}
\tag{10}
$$

where $\left[P^{-1}\right]_{i,j}$ is given by the equation 6.

The trace can also be computed in the damped case:

Theorem 9: The trace of the inverse of reachability Gramian G_r when the system has damping ($b \neq 0$) is:

$$
\begin{aligned}
\text{trace}(\tilde{G}_r^{-1}) = & \left(\frac{\left[(\lambda_1 k_v + b) - \sqrt{(\lambda_1 k_v + b)^2 - 4\lambda_1 k_p}\right]^2 + 4}{[w_{2,2q}]^2} \right) \left[N^{-1}\right]_{1,1} + \\
& \sum_{i=2}^{n} \left(\frac{\left[(\lambda_i k_v + b) + \sqrt{(\lambda_i k_v + b)^2 - 4\lambda_i k_p}\right]^2 + 4}{[w_{2i-1,2q}]^2} \right) \left[N^{-1}\right]_{2i-2,2i-2} + \\
& \left(\frac{4[1 + \lambda_i k_p]}{w_{2i-1,2q} w_{2i,2q}} \right) \left[N^{-1}\right]_{2i-2,2i-1} + \left(\frac{4[1 + \lambda_i k_p]}{w_{2i-1,2q} w_{2i,2q}} \right) \left[N^{-1}\right]_{2i-1,2i-2} \\
& + \left(\frac{\left[(\lambda_i k_v + b) - \sqrt{(\lambda_i k_v + b)^2 - 4\lambda_i k_p}\right]^2 + 4}{[w_{2i,2q}]^2} \right) \left[N^{-1}\right]_{2i-1,2i-1}
\end{aligned}
\tag{11}
$$

Where, $\left[N^{-1}\right]_{i,j}$ is given by the equation 9.

The determinant of the reachability Gramian indicates the volume of states reachable with unit energy (to within a fixed scale factor), and hence also serves as an important global metric for manipulability. For the DIN, theorems 6 and 7 indicate that any goal state on the consensus manifold can be reached with arbitrarily small amount of manipulation energy over an infinite horizon. Hence, the volume of the reachable states becomes infinite for large t. Since the energy required to reach states along the consensus manifold is independent of the model's parameters, a more convenient and insightful global metric is the volume of reachable states perpendicular to the consensus manifold, which can be found as the determinant of a projection of the reachability Gramian. The following theorems give explicit formulae for this volume metric, in the damped and undamped cases.

Theorem 10: For the undamped DIN ($b = 0$), the volume of states perpendicular to the consensus manifold that are reachable with unit energy is given by:

$$
\left[\prod_{i=3}^{2n} \left(w_{i,2q}\right)^2 \right] \left[\frac{\prod_{i=2}^{2n-2} \prod_{i=1}^{i-1} \left(\mu_{i+2}^* - \mu_{j+2}^*\right) \left(\mu_{i+2} - \mu_{j+2}\right)}{\prod_{i=1}^{2n-2} \prod_{i=1}^{2n-2} \left(\mu_{i+2}^* + \mu_{j+2}\right)} \right]
$$

Theorem 11: For the damped DIN ($b = 0$), the volume of states perpendicular to the consensus manifold that are reachable with unit energy is given by:

$$
\left[\prod_{i=2}^{2n} \left(w_{i,2q}\right)^2 \right] \left[\frac{\prod_{i=2}^{2n-1} \prod_{i=1}^{i-1} \left(\mu_{i+1}^* - \mu_{j+1}^*\right) \left(\mu_{i+1} - \mu_{j+1}\right)}{\prod_{i=1}^{2n-1} \prod_{i=1}^{2n-1} \left(\mu_{i+1}^* + \mu_{j+1}\right)} \right]
$$

The proofs of these results follow readily from the Cauchy-matrix structure of the Gramian. Details are excluded to save space.

The expressions for the scalar metrics give structural insight into how easy or hard it is to manipulate a DIN using local actuation. A basic outcome, which is particularly clear from the determinant expressions, is that manipulating a large networkat will from a single input is difficult. In particular, a number of studies have shown that the eigenvalue terms in the determinant expressions necessarily become small for large networks [15], [16], under broad conditions.These results indicate that few goal states can be reached with limited energy from a single input, for a large double-integrator network. Also, the trace and determinant expressions indicate that the networkis difficult to manipulate fully when two eigenvalues of the state matrix A are close to each other, regardless of the location of the actuation. We notice that A necessarily has nearby eigenvalues if L has nearby eigenvalues, which thus causes difficulty in manipulation. Conditions on the graph Ψ such that the eigenvalues of L are close or small have been developed, see e.g. [14], [17], [18]. Also, the expressions indicate that manipulation is difficult, if the entries in the eigenvectors of A (and hence of L) corresponding to the measurement location are small. Recently, graph-theoretic results on the eigenvector components of the Laplacian have also been established, see e.g. [19], [20]. These results indicate the slow modal dynamics can be most easily manipulated from extreme points in the graph, but overall controllability is often strongest near the center; details are omitted.

C. A Design Result

The above characterizations of the manipulation energy allow us to study whether and how the DIN can be made easier or harder to manipulate, by designing or tuning the network's parameters. Interestingly, the above results show that manipulation along the consensus manifold is easy regardless of the DIN parameters. However, the average effort required to manipulate the network to an arbitrary goal state, as measured by the trace of the inverse Gramian, is amenable to design via tuning of the DIN parameters. As an initial study, the dependence of the manipulation effort on the gain parameters k_v and k_p is analyzed. In a number of applications (e.g. autonomous-vehicle-network control), the gain parameters are naturally amenable to tuning. More abstractly, the gain parameters indicate the strength of the interfaces among the network's agents. Thus, it is natural to study how the ease of manipulation of the double-integrator network can be modulated using the gain parameters. In the following theorem, it is shown that a certain scaling of the gains can be used to systematically make the DIN harder to manipulate.

Theorem 12: Consider the DIN model given in Equation (3). Now consider scaling the global gain parameter k_v by a factor $\xi \geq 1$, and the global gain parameter k_p by ξ^2. The trace of the inverse Gramian for the scaled DIN model monotonically increases with the scaling parameter ξ.

V. EXAMPLE

A DIN is considered with five agents connected in a line (see the graph in Figure1). The example to used to illustrate the manipulation effort depends on the input location.

Fig. 1. Example network

Table I shows the Laplacian matrix L of the network, the left eigenvector matrix C of the matrix L and the other DIN parameters selected for this specific example.

Parameter name	Value
L	$\begin{bmatrix} 10 & -10 & 0 & 0 & 0 \\ -10 & 20 & -10 & 0 & 0 \\ 0 & -10 & 20 & -10 & 0 \\ 0 & 0 & -10 & 20 & -10 \\ 0 & 0 & 0 & -10 & 10 \end{bmatrix}$
C	$\begin{bmatrix} -0.1954 & 0.5117 & -0.6325 & 0.5117 & -0.1954 \\ -0.3717 & 0.6015 & 0 & -0.6015 & 0.3717 \\ -0.5117 & 0.1954 & 0.6325 & 0.1954 & -0.5117 \\ -0.6015 & -0.3717 & 0 & 0.3717 & 0.6015 \\ 0.4472 & 0.4472 & 0.4472 & 0.4472 & 0.4472 \end{bmatrix}$
t	10
k_p	100
k_v	50
b	0.5
$z(0)$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}'$
$z(10)$	$\begin{bmatrix} 100 & 0 & 100 & 0 & 100 & 0 & 100 & 0 & 100 & 0 \end{bmatrix}'$

TABLE I
EXAMPLE: PARAMETER VALUES USED

From the left-eigenvector matrix, it is immediate that manipulation is possible from agents 1,2,4 and 5 but not from agent 3. The following table compares the trace of the inverse Gramian for each agent.

Manipulated agent	Open-loop energy
1	2.5745×10^3
2	2.6391×10^3
3	-
4	2.6408×10^3
5	2.5859×10^3

TABLE II
EXAMPLE: RESULTS

The results shown in the table II indicate that manipulation is much easier from the agents located at the edges of the network.

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