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# Looking Upstream: Optimal Policies for a Class of Capacitated Multi-Stage Inventory Systems

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We consider a multi-stage inventory system with stochastic demand and processing capacity constraints at each stage, for both finite-horizon and infinite-horizon, discounted-cost settings. For a class of such systems characterized by having the smallest capacity at the most downstream stage and system utilization above a certain threshold, we identify the structure of the optimal policy, which represents a novel variation of the order-up-to policy. We find the explicit functional form of the optimal order-up-to levels, and show that they depend (only) on upstream echelon inventories. We establish that, above the threshold utilization, this optimal policy achieves the decomposition of the multidimensional objective cost function for the system into a sum of single-dimensional convex functions. This decomposition eliminates the curse of dimensionality and allows us to numerically solve the problem. We provide a fast algorithm to determine a (tight) upper bound on this threshold utilization for capacity-constrained inventory problems with an arbitrary number of stages. We make use of this algorithm to quantify upper bounds on the threshold utilization for three-, four-, and five-stage capacitated systems over a range of model parameters, and discuss insights that emerge.

*Key words:* multiechelon inventory; optimal policy; capacity constraints; convex decomposition
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## 1. Introduction

Under growing customer demand and escalating costs of manufacturing and information technologies, supply chains are becoming increasingly capacity-constrained. Recently, Chrysler ranked capacity constraints as one of the top risks to its supply chain, because “suppliers ... are operating at maximum capacity, which puts enormous pressure on Chrysler” (Fishstrom 2013). Such developments in the automobile and other industries are raising the awareness of the importance of managing supply chains under capacity constraints, so that “decisions on how to allocate supply chain capacity tend to be hotly debated among the management team, especially when resources are tightly constrained” (Aparajithan et al. 2011). All aspects of the firm’s value chain tend to struggle with capacity constraints, and most existing supply chain planning systems and processes cannot effectively manage capacity constraints (Etheredge and O’Keefe 2009).

While capacity-constrained supply chains present a number of challenges, one of the most fundamental problems concerns optimally managing the flow of inventories in such capacitated systems. That problem is the focus of this study. Formally, we consider a periodic-review, multi-echelon inventory system with stochastic demands and capacity constraints at each stage. We address both the finite horizon and the infinite horizon, discounted-cost setting.

The study of inventory management in capacity-constrained systems under stochastic demand began with research on single-stage models (Wijngaard 1972). Federgruen and Zipkin (1986a,b) show that the optimal policy for a single-stage capacitated problem under finite and infinite horizons is a base stock policy modified by the capacity constraint. Kapuscinski and Tayur (1996) establish the optimal policy for a single-product, capacitated model with stochastic, periodic demand. Özer and Wei (2004) address a capacitated production system under advance demand information and fixed costs, and establish the structure of the optimal policy for the problem. Shi et al. (2014) provide an efficient algorithm for single-stage, capacitated inventory systems with setup costs. Wu and Chao (2014) consider a single-stage capacitated production/inventory system, in which cumulative production and demand follow a two-dimensional Brownian motion process, with a setup cost for switching on the production, and prove the optimality of $(s, S)$ policies.

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With regard to *multi-stage* capacitated systems, Glasserman and Tayur (1994, 1995) show that a base stock-controlled capacitated system is stable when the mean demand is less than the smallest capacity constraint in the system, and provide a derivation of optimal base stock levels. Huh et al. (2010) derive a result on the shortfall processes under echelon base stock policies.

Research on *optimal* inventory policies for multi-stage capacitated systems with stochastic demand have been limited. Speck and van der Wal (1991a,b) provide a numerical example of a capacitated two-stage problem in which the replenishment decision of the downstream stage may depend on the amount of inventory in transfer to the upstream station, so that a base stock policy is not necessarily optimal. In a landmark paper, Parker and Kapuscinski (2004) prove that a modified echelon base stock policy is optimal for a two-stage capacitated system when the smallest capacity is at the downstream stage. They show that this policy, in the spirit of the classic Clark and Scarf (1960) paper, allows the model to be decomposed into two single-dimensional problems. When inventory is discretized, Janakiraman and Muckstadt (2009) rederive this result for a capacitated two-stage system using the customer-unit decomposition approach of Muharremoglu and Tsitsiklis (2008), and show that, when the leadtime at the upstream stage becomes two periods, the optimal policy becomes a "two-tier, base stock policy." They also provide an upper bound on the number of parameters needed to characterize the optimal policy for a capacitated system with more than two stages and show that this bound increases exponentially in the number of stages in the system. Huh and Janakiraman (2010) derive sensitivity results on the general form of the inventory-dependent order quantity. To the best of our knowledge, no other information exists in the literature regarding how to optimally manage capacitated systems with more than two stages. As a result, the structure of the optimal policy for those systems is still unknown, and determining optimal replenishment decisions and associated system costs has remained mired in the curse of dimensionality.

The difficulty in solving capacitated multiechelon inventory problems lies in the fact that the upper bound on the feasible interval for the echelon inventory decision at each stage is determined by two state variables, rather than a single one as is common in multiechelon inventory models solved in the literature. Consequently, the decomposition of the multi-dimensional objective cost function into a sum of single-dimensional convex functions cannot be expected to hold in general. Parker and Kapuscinski (2004) overcome this problem for a two-stage capacitated system by showing that, because the upper stage in a two-stage system has unlimited available supply (so that the increase in the corresponding echelon inventory position is only constrained by the capacity), the upper bound for the feasible interval at the downstream stage can be reduced to a linear function of a single echelon variable. Such an approach does not work for higher-order capacitated systems because feasible increases in echelon inventory positions at intermediate stages in the system are constrained by both capacity and echelon inventory available at the next stage upstream.

In this study, we address a multi-stage system with an arbitrary number of stages, capacity constraints at all stages, and the smallest capacity at the most downstream stage. We refer to such a system as *the original (capacitated) system*. To arrive at the structure of the optimal policy, we identify certain constraints, in the form of lower bounds on the pipeline inventory in the system, that are satisfied by this capacitated system under infinite horizon and sufficiently high utilization. We refer to those bounds as *stocked-up*. Next, we consider a constrained version of the original, finite-horizon, capacitated system with *both* the original capacity constraints *and* the stocked-up constraints. To analyze this stocked-up capacitated problem, we introduce and prove new results in convex optimization pertaining to the preservation of additive convexity under minimization over a series of increasingly complex feasible regions. Using those results, we show that the optimal policy for the stocked-up capacitated problem represents a novel form of the order-up-to policy with state-dependent parameters. Noteworthy characteristics of this policy are that: (i) the structure of the policy at each stage is completely determined by the number of parameters (i.e., critical thresholds) that grows linearly in the number of stages in the system; and (ii) the order up-to level at each stage depends, in a piecewise linear fashion, on the echelon inventories at upstream stages.

We prove that this form of the optimal policy achieves the decomposition of the multi-dimensional objective cost function for the problem into single-variable component functions. We show that this form of the inventory replenishment policy is optimal not only for the stocked-up capacitated problem, but also for the original capacitated system at all utilizations above a certain threshold utilization. By achieving the decomposition of the objective function for the original capacitated problem, the optimal policy identified in this study relieves the curse of dimensionality and makes it possible to quickly numerically solve the underlying dynamic program. This, in turn, allows us to develop a fast algorithm to calculate (tight) upper bounds on the threshold utilization.

The main contribution of our study, therefore, is to identify an explicit form of the optimal policy for a

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class of capacitated multi-stage systems characterized by having the smallest capacity at the most downstream stage, and sufficiently high system utilization. The former is standard in the literature (e.g., Parker and Kapuscinski 2004), while the latter establishes system utilization, defined as the ratio of mean demand to the smallest capacity in the system, as a parameter with key impact on the nature of the optimal inventory policy.

Other related literature includes the classic paper of Clark and Scarf (1960) that showed how a multi-stage inventory model can be reformulated in terms of echelon inventories to allow additive separation of the objective cost function. Federgruen and Zipkin (1984) extend those results to the infinite horizon case. Chao and Zhou (2009) derive the optimal inventory policy for a multiechelon system with batch ordering and fixed replenishment intervals, while Song and Zipkin (2013) consider a setting in which inventory can be held at any point along a continuum, rather than just at discrete stages. Özer (2011) and Luo and Shang (2015) provide comprehensive reviews of multiechelon inventory papers.

## 2. The Capacitated Multiechelon Inventory Model

### 2.1. Model Formulation

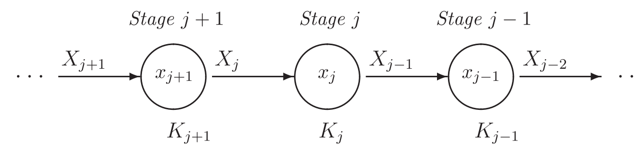
We consider a serial multiechelon system with $N$ stages. Without loss of generality, we make a standard assumption (e.g., Clark and Scarf 1960, Janakiraman and Muckstadt 2009) of a single-period leadtime between successive stages. When an item is first ordered, at the beginning of a period, it is in stage $N + 1$. At the end of the same period this item is in stage $N$. The product flows downstream (though it can be deferred at any stage) until it reaches stage 1, where it is used to satisfy (stochastic) customer demands that are independent across time. Each stage is assumed to have a processing capacity that places a limit on the number of items that can be processed (i.e., received) by that stage in each period. We make use of the following state and decision variables:

$$\begin{aligned} x_{jt} = {} & \text{the on-hand inventory at stage } j, \text{ if } j > 1, \\ & \text{and the net inventory if } j = 1, \text{ at the} \\ & \text{beginning of period } t, \text{ prior to making any} \\ & \text{decisions;} \\ X_{jt} = {} & \text{the amount ordered by stage } j \text{ from stage} \\ & j+1 \text{ in period } t. \end{aligned}$$

Stochastic demand in period $t$ is $\xi_t$, while $K_j$ represents processing capacity at stage $j$. Feasibility requires $X_{jt} \le K_j$ and $X_{jt} \le x_{j+1,t}$ for all $j$, where $x_{N+1,t} := \infty$. The feasible decision set is $\{X_{jt} \mid 0 \le X_{jt} \le \min(x_{j+1,t}, K_j),\ 1 \le j \le N\}$. The state transition equations are

$$x_{j,t+1} = \begin{cases} x_{1t} + X_{1t} - \xi_t, & \text{if } j = 1, \\ x_{jt} + X_{jt} - X_{j-1,t} & \text{if } j = 2, \ldots, N. \end{cases} \tag{1}$$

**Figure 1 Inventory States, Decisions, and Flow of Product in the System**




Let $\mathbf{x}_t := \{x_{1t}, \ldots, x_{Nt}\}$ and $\mathbf{X}_t := \{X_{1t}, \ldots, X_{Nt}\}$ be *the on-hand inventory state*, and *the order schedule*, respectively. The sequence of events in each period $t$ is as follows: (i) on-hand inventory state $\mathbf{x}_t$ is observed; (ii) order schedule $\mathbf{X}_t$ is selected; (iii) ordered amounts are received; (iv) demand is observed and satisfied by the available stock; and, (v) costs are incurred. We assume full backlogging of unsatisfied demand at a (positive) unit backlogging cost $p$ in each period $t$.

Figure 1 illustrates the flow of product, and state and decision variables in the system.

At each stage $j$, there is a (positive) unit on-hand inventory holding cost $H_j$ incurred at the end of each period $t$. As in Parker and Kapuscinski (2004), we assume $H_j \ge H_{j+1}$ for each $j$. Total costs in period $t$ can then be expressed as follows (expectation is taken over $\xi_t$)

$$p_t \mathbb{E}\big[(\xi_t - x_{1t} - X_{1t})^+\big] + H_1 \mathbb{E}\big[(x_{1t} + X_{1t} - \xi_t)^+\big] + \sum_{j=2}^{N} H_j (x_{jt} - X_{j-1,t} + X_{jt}). \tag{2}$$

We reformulate this problem using the following echelon variables:

$$\begin{aligned} y_{jt} &:= x_{1t} + x_{2t} + \cdots + x_{jt} \text{ referred to as } \textit{the echelon } j \textit{ inventory}; \\ Y_{jt} &:= y_{jt} + X_{jt} \text{ referred to as } \textit{the echelon } j \textit{ position}. \end{aligned}$$

At the beginning of period $t + 1$, the updated echelon $j$ inventory is $y_{j,t+1} = Y_{jt} - \xi_t$. Let $\mathbf{y}_t := \{y_{1t}, \ldots, y_{Nt}\}$ and $\mathbf{Y}_t := \{Y_{1t}, \ldots, Y_{Nt}\}$ be *the echelon (inventory) state* and *the echelon (inventory) schedule*, respectively. Thus, $\mathbf{Y}_t$ represents the new decision variables of the model. Let $\mathbb{D}(\mathbf{y}_t)$ be the set of feasible echelon schedules for $\mathbf{y}_t$. Let $y_{N+1} := \infty$. Then,

$$\mathbb{D}(\mathbf{y}_t) = \big\{\mathbf{Y}_t \mid y_{jt} \le Y_{jt} \le y_{j+1,t}, Y_{jt} - y_{jt} \le K_j; 1 \le j \le N\big\}.$$

We begin with the finite-horizon formulation of the capacitated multi-stage problem, with time horizon being $T$ periods long. Let $F_t(\mathbf{y}_t)$ denote the minimum expected net present value of the costs over periods $t$

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through $T$, as of the beginning of period $t$, as a function of $\mathbf{y}_t$. Excluding costs unaffected by any decisions made over the time horizon, we get the following dynamic program:

$$F_t(\mathbf{y}_t) = \min_{\mathbf{Y}_t \in \mathbb{D}(\mathbf{y}_t)} \widetilde{F}_t(\mathbf{Y}_t), \tag{3}$$

$$\widetilde{F}_t(\mathbf{Y}_t) := \gamma_t(Y_{1t}) + \sum_{j=1}^{N} h_j Y_{jt} + \alpha \mathbb{E}[F_{t+1}(\mathbf{Y}_t - \xi_t)], \tag{4}$$

where $\mathbf{Y}_t - \xi_t$ denotes the vector $\{Y_{jt} - \xi_t\}$, $\alpha$ is the discount factor, and $h_j := H_j - H_{j+1}$ for each $j$. Also, $\gamma_t(y) := (p + H_1)\mathbb{E}[(\xi_t - y)^+]$, which is convex in $y$. We refer to the dynamic program given in Equations (3) and (4) as *the original capacitated problem*; its solution is the objective of this study.

We make the following assumption about model parameters.

ASSUMPTION 1. (*a*) $K_1 \le K_j$ *for all* $j$; (*b*) $x_{j1} \le K$ *for all* $j > 1$; (*c*) *Demands are IID with mean* $\mu$ ($\mu < \infty$); (*d*) *The discount factor is strictly less than one* ($0 < \alpha < 1$); (*e*) $p + H_1 > (1 - \alpha) \sum_{j=1}^{N} h_j \alpha^{-j}$.

Assumption 1(a) ensures the downstream stage has the smallest capacity in the system. We refer to $K := K_1$ as the *bottleneck capacity*. The inventory state $\mathbf{x}_t$ (order schedule $\mathbf{X}_t$) is called *capacitated* if $x_{jt} \le K$ for all $j > 1$ ($X_{jt} \le K$ for all $j \ge 1$). By part (b), the system starts in a capacitated state in period 1. Part (c) implies a stationary problem setting; part (d) ensures the objective cost function remains finite as the time horizon for the problem goes to infinity. (Note that $\alpha = 0$ implies a single-period, newsvendor-type problem, and is thus omitted from our analysis.) Parts (a)–(d) of Assumption 1 are common to the literature (e.g., Parker and Kapuscinski 2004). Part (e) of Assumption 1 incentivizes the system to meet customer demand. If $p + H_1 \le (1 - \alpha) \sum_{j=1}^{N} h_j \alpha^{j-N}$, it becomes optimal to keep indefinitely postponing new orders for inventory, as the resulting holding costs are not worth the savings from eliminating backlogged demand. (Due to the multiplier $(1 - \alpha)$, part (e) is only mildly restrictive.)

Under Assumption 1, Parker and Kapuscinski (2004) (Lemma 1) show that all capacities the system can be replaced with the bottleneck capacity $K$, without affecting costs, so that our system becomes equivalent to one with the identical capacity at each stage. The feasible set $\mathbb{D}(\mathbf{y}_t)$ becomes

$$\mathbb{D}(\mathbf{y}_t) = \{\mathbf{Y}_t \mid y_{jt} \le Y_{jt} \le y_{j+1,t}, 1 \le j \le N, Y_{jt} - Y_{j-1,t} \le K; 2 \le j \le N\}. \tag{5}$$

The constraint $Y_{jt} - Y_{j-1,t} \le K$ in the above definition for $\mathbb{D}(\mathbf{y}_t)$ implies that $x_{jt} \le K$ in each period $t$. This is because: $y_{j,t+1} = Y_{jt} - \xi_t$, $y_{j,t+1} - y_{j-1,t+1} = x_{j,t+1}$ and $x_{j1} \le K$ by Assumption 1, part (b). Thus, the original capacity constraint on the each stage's order size, $X_{jt} \le \min(x_{j+1,t}, K)$ becomes equivalent to the capacity constraint on on-hand inventory at each stage $j$. This is because any order schedule that ends up leaving more than $K$ units of inventory at any given stage $j > 1$ in any period $t$ simply generates extra unnecessary (holding) costs: since only $K$ units can be ordered from into stage $j - 1$ in period $t + 1$, any inventory at stage $j$ in period $t$ that is in excess of $K$ cannot be moved downstream in period $t + 1$ and is thus cheaper to keep at stage $j + 1$ in period $t$.

By definition of $\mathbb{D}(\mathbf{y}_t)$, the upper bound on the feasible decision at each stage depends on two variables, $y_{j+1,t}$ and $Y_{j-1,t}$. Hence, each optimal echelon position $Y^*_{jt}$ will, in general, depend on all $N$ variables in the echelon state $\mathbf{y}_t$, as this dependence on echelon inventories cascades downstream. Thus, the cost-minimization problem given in Equation (3) suffers from the curse of dimensionality. This dimensionality curse has so far rendered the capacitated multiechelon problem with more than two stages very time consuming to solve numerically, even for systems with only a few stages.

### 2.2. Infinite Horizon Results and Bounds on Optimal Decisions

In what follows, we *first* derive our main results for the infinite-horizon setting, and then identify an equivalent finite-horizon system. We begin by establishing some general properties of the optimal policy for the original capacitated problem. We apply standard definitions for infinite horizon problems, and approximate the infinite horizon problem with a sequence of finite horizon problems of increasing duration. Let $\pi := \{Y_1, Y_2, \ldots\}$ be an infinite-horizon policy for the original capacitated problem, and $\widetilde{\mathcal{F}}_t(\pi)$ be the expected present value of costs of that policy, starting in period $t$ (so that decisions $Y_1, \ldots, Y_{t-1}$ are irrelevant). Let $\mathcal{F}_t$ be the smallest of those over the feasible state space $\mathbb{D}$. Then,

$$\widetilde{\mathcal{F}}_t(\pi) = \lim_{T \to \infty} \sum_{\tau=t}^{T} \alpha^{\tau - t} \mathbb{E}\left[(p + H_1)(\xi_\tau - Y_{1\tau})^+ + \sum_{j=1}^{N} h_j Y_{j\tau}\right] \tag{6}$$

$$\mathcal{F}_t(\mathbf{y}_t) = \inf_{\pi \in \{\pi \mid Y_\tau \in \mathbb{D}(\mathbf{y}_\tau), \tau = t, t+1, \ldots\}} \widetilde{\mathcal{F}}_t(\pi). \tag{7}$$

The above limit is well defined because the RHS of Equation (6) contains only positive terms, so that the summation over $\tau$ is uniformly increasing in $T$. Next, we review the established convergence properties for

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the capacitated, multi-stage, infinite-horizon problem given in Equation (6) and (7).

LEMMA 1. *The following hold for any capacity $K$, $\mu < K < \infty$.*

(*a*) *The finite-horizon objective cost function for the capacitated multi-stage problem converges to a stationary, infinite-horizon counterpart; that is, there exists a convex and bounded function $\mathcal{F}$ such that, for any feasible state $\mathbf{y}_t$, $\mathcal{F}(\mathbf{y}_t) = \mathcal{F}_t(\mathbf{y}_t) = \lim_{T\to\infty} F_t(\mathbf{y}_t)$;*

(*b*) *The optimal policy for the finite-horizon capacitated problem converges to its infinite-horizon counterpart, which is stationary and bounded from above.*

The proof of Lemma 1 can be found in Parker and Kapuscinski (2004): part (a) is proved in their Theorem 7, while part (b) follows directly from their Theorems 8 and 9.

By Lemma 1, the finite-horizon minimum cost function for the original capacitated problem converges to its stationary infinite-horizon counterpart, $\mathcal{F}$. Thus, the infinite-horizon capacitated problem is stationary; as a result, $\mathbf{y}_t$ and $\mathbf{Y}_t$ are stationary, and, going forward in this Section, we use $\mathbf{y} := \{y_1, y_2, \ldots, y_N\}$ and $\mathbf{Y} := \{Y_1, Y_2, \ldots, Y_N\}$ to refer to the echelon state and echelon schedule, respectively, for this problem. From here on, we also explicitly include the dependence of $\mathcal{F}_t$, $\widetilde{\mathcal{F}}_t$ and $\mathbb{D}$ on system capacity $K$; thus, we write $\mathcal{F}_t(y_t)$ as $\mathcal{F}(K, \mathbf{y})$, $\widetilde{\mathcal{F}}_t(Y_t)$ as $\widetilde{\mathcal{F}}(K, \mathbf{Y})$ and $\mathbb{D}(y_t)$ as $\mathbb{D}(K, \mathbf{y})$. The dynamic program for the infinite-horizon capacitated problem therefore becomes:

$$\mathcal{F}(K, \mathbf{y}) = \min_{\mathbf{Y}\in\mathbb{D}(K,\mathbf{y})} \widetilde{\mathcal{F}}(K, \mathbf{Y}) \tag{8}$$

$$\widetilde{\mathcal{F}}(K, \mathbf{Y}) = \gamma(Y_{1t}) + \sum_{j=1}^{N} h_j Y_{jt} + \alpha\mathbb{E}[\mathcal{F}(K, \mathbf{Y} - \xi_t)]. \tag{9}$$

Note that the feasible state space for $\mathcal{F}$ is $(\mu, \infty) \times \Upsilon(K)$, where

$$\Upsilon(K) := \{\mathbf{y} \mid y_j \le y_{j+1} \le \min(y_{j+2}, y_j + K); 1 \le j \le N-1\}. \tag{10}$$

Let $\mathbf{Y}^*(K, \mathbf{y})$ be a solution to the infinite-horizon problem described by the dynamic program in Equations (8)–(9). Thus, given $K$ and $\mathbf{y}$, $\mathbf{Y}^*(K, \mathbf{y})$ is a minimizer of $\widetilde{\mathcal{F}}$ over the feasible set $\mathbb{D}(K, \mathbf{y})$:

$$\mathbf{Y}^*(K, \mathbf{y}) = \arg\min_{\mathbf{Y}\in\mathbb{D}(K,\mathbf{y})} \widetilde{\mathcal{F}}(K, \mathbf{Y}). \tag{11}$$

We now establish the form of the policy for the original, infinite-horizon capacitated system that is asymptotically optimal as capacity approaches mean demand. First, we define a policy that orders $K$, the largest possible capacity-constrained amount, at every stage in the system.

DEFINITION 1. *For the infinite horizon, original capacitated problem, a policy $\mathbf{Y}(K, \boldsymbol{y})$ is called fully-stocked if $Y_j(K, \boldsymbol{y}) - Y_{j-1}(K, \boldsymbol{y}) = K$ for $j = 2, \ldots, N$ and every $\boldsymbol{y} \in \Upsilon(K)$.*

The following proposition establishes the structure of an asymptotically optimal policy when the system capacity approaches mean demand. Proofs are in Appendix S1.

PROPOSITION 1. *For every $\boldsymbol{y} \in \Upsilon(K)$, $\lim_{K\to\mu^+} [Y_j^*(K, \mathbf{y}) - Y_{j-1}^*(K, \mathbf{y})] = \mu$ for every $j = 2, \ldots, N$.*

The fully-stocked policy is therefore asymptotically optimal, as system capacity approaches mean demand (from the right). It then becomes optimal to order the full amount $\mu$ in every period and at each stage in the system; ordering any less than $\mu$ leads to a systemic accumulation of backlogged demand, and echelon inventory levels decrease without bound.

COROLLARY 1. *For each $j = 2, \ldots, N$, $\lim_{K\to\mu^+} [Y_j^*(K, \mathbf{y}) - Y_1^*(K, \mathbf{y})] = (j-1)\mu$.*

While Proposition 1 seems intuitive, to the best of our knowledge, it has not been shown before. Next, we define $Q_j(K, \mathbf{y}) := Y_j^*(K, \mathbf{y}) - Y_1^*(K, \mathbf{y})$ for each $(K, \mathbf{y}) \in (\mu, \infty) \times \Upsilon(K)$. Thus, $Q_j(K, \mathbf{y})$ represents the optimal amount of stock to hold from stage 2 to stage $j$ in the next period, given the echelon state $\mathbf{y}$ in this period. Proposition 1 then leads to the following key result.

THEOREM 1. *There exists a threshold capacity $K^*$, $K^* > \mu$, for which the optimal policy $\mathbf{Y}^*(K, \boldsymbol{y})$, for any $K \in (\mu, K^*]$, is such that $Y_j^*(K, \mathbf{y}) - Y_1^*(K, \mathbf{y}) \ge (j-2)K$ for every $j \ge 2$ and all $\boldsymbol{y} \in \Upsilon(K)$.*

By Theorem 1, every capacitated multi-stage, infinite-horizon system subject to Assumption 1 admits a threshold capacity $K^*$, $K^* > \mu$ such that $Q_j(K, \mathbf{y}) \ge (j-2)K$ for every stage $j$, echelon state $\mathbf{y} \in \Upsilon(K)$, and capacity $K \in (\mu, K^*]$. We will make use of this result later in the study.

## 3. The Stocked-up Capacitated Problem

In this section, we consider a system which, in addition to the capacity constraints of the original capacitated system, also satisfies certain lower bounds on pipeline inventory. Those lower bounds are explicitly stated in the new feasible set for the problem $\mathbb{D}_S(\mathbf{y}_t)$ as follows:

$$\mathbb{D}_S(\mathbf{y}_t) = \{\mathbf{Y}_t \mid y_{jt} \le Y_{jt} \le y_{j+1,t}, Y_{jt} - y_{jt} \le K, 1 \le j \le N; \quad Y_{jt} \ge Y_{1t} + (j-2)K, 2 \le j \le N\}.$$

Our motivation for introducing these bounds derives from Theorem 1, in which the optimal policy for the original, infinite-horizon, capacitated problem was found to satisfy those bounds at sufficiently small values of system capacity (i.e., at capacities below the threshold capacity $K^*$). As shown in what follows, those bounds are key to deriving the structure of the optimal policy at higher system utilizations. As already mentioned, we will refer to those bounds as "stocked-up."

We will also refer to any echelon state $\mathbf{y}_t$ (schedule $\mathbf{Y}_t$) that satisfies these lower bounds as *stocked-up*. We assume that the system starts out in period 1 in a stocked-up echelon state.

Assumption 2. *$y_{j1} \ge y_{11} + (j-2)K$ for every $j \ge 2$.*

Since $Y_{jt} \ge Y_{1t} + (j-2)K$ for each $j \ge 2$ in every period $t$, and $y_{j1} \ge y_{11} + (j-2)K$ for every $j \ge 2$, then $y_{jt} \ge y_{1t} + (j-2)K$ for every $j \ge 2$, which we formally state as follows.

Corollary 2. *$y_{jt} \ge y_{1t} + (j-2)K$ for every $j \ge 2$ in every period $t$.*

Let $F_t^S(\mathbf{y}_t)$ be the minimum, over the new feasible set $\mathbb{D}_S(\mathbf{y}_t)$, of the expected present value of costs over periods $t$ through $T$ given $\mathbf{y}_t$. Then,

$$F_t^S(\mathbf{y}_t) = \min_{\mathbf{Y}_t \in \mathbb{D}_S(\mathbf{y}_t)} \left\{ \gamma_t(Y_{1t}) + \sum_{j=1}^{N} h_{jt} Y_{jt} + \alpha \mathbb{E}\left[F_{t+1}^S(\mathbf{Y}_t - \xi_t)\right] \right\}. \tag{12}$$

We will refer to the capacitated problem in Equation (12) as *the stocked-up (capacitated) problem*. We now establish the equivalent of Lemma 1 in Parker and Kapuscinski (2004) for this problem.

Lemma 2. *For each $t$, there exists an optimal order schedule $\mathbf{X}_t$ such that, for all $j > 1$: (a) $x_{jt} \le K$; (b) $y_{jt} - y_{j-1,t} \le K$; and, (c) $Y_{jt} - Y_{j-1,t} \le K$.*

Thus, if $x_{jt} - X_{j-1,t}$, which represents the amount available at stage $j$ after satisfying the order from stage $j-1$, is greater than capacity $K$, no order is placed (at stage $j$) in that period, and the on-hand inventory at any stage $j > 1$ never exceeds $K$. Hence, any policy that is optimal and feasible for the stocked-up capacitated problem is also feasible for the original capacitated problem (3) and (4). The second implication of Lemma 2 is that the feasible decision space $\mathbb{D}_S(\mathbf{y}_t)$ becomes

$$\mathbb{D}_S(\mathbf{y}_t) = \{\mathbf{Y}_t \mid y_{jt} \le Y_{jt} \le y_{j+1,t}, Y_{jt} - Y_{j-1,t} \le K, \quad 1 \le j \le N; Y_{jt} \ge Y_{1t} + (j-2)K, 2 \le j \le N\}$$

Going forward, we make use of the following definition of an additively convex function.

Definition 2. *Let $\mathbf{x} := (x_1, \ldots, x_n) \in \mathbb{X}^n \subset \mathbb{R}^n$, for some $n > 1$, and let $f: \mathbb{X}^n \to \mathbb{R}$ be convex. We say that $f$ is additively convex (on $\mathbb{X}^n$) if there exist convex functions $f_i: \mathbb{R} \underset{n}{\to} \mathbb{R}$ for $i = 1, \ldots, n$, such that, for all $\mathbf{x} \in \mathbb{X}^n$, $f(\mathbf{x}) = \sum_{i=1} f_i(x_i)$.*

The salvage value function for the stocked-up problem is assumed to be additively convex.

Assumption 3. *$F_{T+1}^S(\cdot)$ is additively convex.*

### 3.1. The Three-Stage System

To build intuition for more general results that follow later and provide insight into the analysis that makes those results possible, we first address a stocked-up system with only three stages. In particular, we describe step-by-step how to derive, using basic principles, the structure of the optimal policy for this three-stage system and attain the decomposition of the objective cost function.

Because the salvage value function is additively convex by Assumption 3, we begin by assuming inductively that the objective function in period $t+1$ is additively convex, so that there exist convex functions $F_{j,t+1}^S: \mathbb{R} \to \mathbb{R}$, for $j = 1, 2, 3$, such that $F_{t+1}^S(\mathbf{y}_{t+1}) = \sum_{j=1}^3 F_{j,t+1}^S(y_{j,t+1})$. By Equation (12),

$$F_t^S(\mathbf{y}_t) = \min_{\mathbf{Y}_t \in \mathbb{D}_S(\mathbf{y}_t)} \left\{ \sum_{j=1}^{3} F_{jt}(Y_{jt}) \right\}, \tag{13}$$

where, for the three-stage stocked-up capacitated system, the feasible decision set $\mathbb{D}_S$ becomes

$$\mathbb{D}_S(\mathbf{y}_t) = \{\mathbf{Y}_t \mid y_{jt} \le Y_{jt} \le y_{j+1,t}, Y_{jt} - Y_{j-1,t} \le K, \quad 1 \le j \le 3; Y_{jt} \ge Y_{1t} + (j-2)K, 2 \le j \le 3\},$$

and the convex component functions $F_{jt}$ are given by

$$F_{jt}(Y_{jt}) = \begin{cases} \gamma(Y_{1t}) + h_1 Y_{1t} + \alpha \mathbb{E}\left[F_{1,t+1}^S(Y_{1t} - \xi)\right] & \text{if } j = 1 \\ h_j Y_{jt} + \alpha \mathbb{E}\left[F_{j,t+1}^S(Y_{jt} - \xi)\right] & \text{if } j = 2,3. \end{cases}$$

It follows from Equation (13) by convexity preservation under minimization that $F_t^S(\mathbf{y}_t)$ is (jointly) convex in $\mathbf{y}_t$, since $F_{jt}(Y_{jt})$ is convex in $Y_{jt}$ for each $j$, and $\mathbb{D}_S(\mathbf{y}_t)$ is a convex set. Next, by defining $\mathbb{D}_S^2(\mathbf{y}_t) = \{\mathbf{Y}_t | y_{1t} \le Y_{1t} \le y_{2t} \le Y_{2t} \le y_{3t}; Y_{2t} - Y_{1t} \le K\}$, and $s_{3t} :=$ arg min $F_{3t}(Y_{3t})$, and using $x \vee y$ and $x \wedge y$ to represent max$(x, y)$ and min$(x, y)$, respectively, Equation (13) can be written as

$$F_t^S(\mathbf{y}_t) = \min_{\mathbf{Y}_t \in \mathbb{D}_S^2(\mathbf{y}_t)} \left\{ \sum_{j=1}^{2} F_{jt}(Y_{jt}) + \min_{y_{3t} \vee (Y_{1t}+K) \le Y_{3t} \le Y_{2t}+K} F_{3t}(Y_{3t}) \right\} \tag{14}$$

$$= \min_{\mathbf{Y}_t \in \mathbb{D}_S^2(\mathbf{y}_t)} \left\{ \sum_{j=1}^{2} F_{jt}(Y_{jt}) + F_{3t}(y_{3t} \vee s_{3t} \vee (Y_{1t}+K)) + F_{3t}(s_{3t} \wedge (Y_{2t}+K)) - F_{3t}(s_{3t}) \right\}, \tag{15}$$

due to a result in Karush (1959) (see Lemma 5 in Appendix S1, or Lemma 2 in Parker and Kapuscinski 2004), which we can apply because $y_{3t} \le y_{2t} + K \le Y_{2t} + K$ and $Y_{1t} + K \le Y_{2t} + K$.

It follows from the last term in Equation (14) that the optimal echelon position at stage 3 is given by $Y_{3t}^* = \max[y_{3t}, Y_{1t} + K, \min(s_{3t}, Y_{2t} + K)]$. The cost at stage 3 is given by $F_{3t}(y_{3t} \vee s_{3t} \vee (Y_{1t} + K)) + F_{3t}(s_{3t} \wedge (Y_{2t} + K))$, which depends not only on $y_{3t}$, as is the case in the uncapacitated model of Clark and Scarf (1960), but also on the decisions at stages 1 and 2. As will become apparent, it is this dependence of the cost at each stage on both its own echelon state and multiple echelon positions that complicates the solution of multi-stage capacitated problems.

To proceed, let $\widetilde{F}_{2t}(Y_{2t}) := F_{2t}(Y_{2t}) + F_{3t}(s_{3t} \wedge (Y_{2t} + K))$. Equation (15) then becomes

$$F_t^S(\mathbf{y}_t) = \min_{y_{1t} \le Y_{1t} \le y_{2t}} \left\{ F_{1t}(Y_{1t}) + F_{3t}(y_{3t} \vee s_{3t} \vee (Y_{1t}+K)) + \min_{y_{2t} \le Y_{2t} \le y_{3t} \wedge (Y_{1t}+K)} \widetilde{F}_{2t}(Y_{2t}) \right\} - F_{3t}(s_{3t}). \tag{16}$$

Since $y_{2t} \le y_{1t} + K \le Y_{1t} + K$, we can apply the same result of Karush (1959) to Equation (16) to get

$$F_t^S(\mathbf{y}_t) = \min_{y_{1t} \le Y_{1t} \le y_{2t}} \left\{ F_{1t}(Y_{1t}) + F_{3t}\big(y_{3t} \vee s_{3t} \vee (Y_{1t}+K)\big) + \widetilde{F}_{2t}(s_{2t} \wedge y_{3t} \wedge (Y_{1t}+K)) \right\} + \widetilde{F}_{2t}(y_2 \vee s_{2t}) - \widetilde{F}_{2t}(s_{2t}) - F_{3t}(s_{3t}),$$

where $s_{2t} := \arg\min \widetilde{F}_{2t}(Y_{2t})$. It follows from the minimization over $Y_{2t}$ in Equation (16) that the optimal echelon position at stage 2 is given by $Y_{2t}^* = \max[y_{2t}, \min(s_{2t}, Y_{1t} + K, y_{3t})]$.

In the above expression for $F_t^S(\mathbf{y}_t)$, the second term $F_{3t}(y_{3t} \vee s_{3t} \vee (Y_{1t} + K))$ is increasing in $y_{3t}$ because higher inventory at echelon 3 results in higher holding costs. Note, however, that the third term $\widetilde{F}_{2t}(s_{2t} \wedge y_{3t} \wedge (Y_1 + K))$ is decreasing in echelon 3 inventory $y_{3t}$. This effect occurs because, when $y_3$ is sufficiently small, stage 3 does not have enough stock to satisfy orders from stage 2, and that results in higher costs for the system due to higher expected shortage costs downstream.

Define $F_{1t}^+(Y_{1t}) := F_{3t}(s_{3t} \vee (Y_{1t} + K)) - F_{3t}(s_{3t})$ and $F_{1t}^-(Y_{1t}) := \widetilde{F}_{2t}(s_{2t} \wedge (Y_{1t} + K)) - \widetilde{F}_{2t}(s_{2t})$. Thus, $F_{1t}^+$ is increasing and $F_{1t}^-$ is decreasing in $Y_{1t}$. The above expression for $F_t^S(\mathbf{y}_t)$ then becomes

$$F_t^S(\mathbf{y}_t) = \min_{y_{1t} \le Y_{1t} \le y_{2t}} \left\{ F_{1t}(Y_{1t}) + F_{1t}^+(Y_{1t} \vee (y_{3t} - K)) + F_{1t}^-(Y_{1t} \wedge (y_{3t} - K)) \right\} + \widetilde{F}_{2t}(y_2 \vee s_{2t}) \tag{17}$$

The optimal echelon 1 position, $Y_{1t}^*$, is thus given by $Y_{1t}^* = \max[y_{1t}, \min(S_{1t}(y_{3t}), y_{2t})]$, where

$$S_{1t}(y_{3t}) := \arg\min_{Y_{1t}} \left\{ F_{1t}(Y_{1t}) + F_{1t}^+(Y_{1t} \vee (y_{3t} - K)) + F_{1t}^-(Y_{1t} \wedge (y_{3t} - K)) \right\}. \tag{18}$$

Hence, $S_{1t}(y_{3t})$ is the unconstrained minimizer of $F_{1t}(Y_{1t}) + F_{1t}^+(Y_{1t} \vee (y_{3t} - K)) + F_{1t}^-(Y_{1t} \wedge (y_{3t} - K)$ over $Y_{1t}$ for each $y_{3t}$. To determine $S_{1t}(y_{3t})$ explicitly, we define base stock levels $s_{21t}$ and $s_{31t}$ as

$$s_{21t} := \arg\min\left[F_{1t}(Y_{1t}) + F_{1t}^+(Y_{1t})\}\right] \text{ and}$$
$$s_{31t} := \arg\min\left[F_{1t}(Y_{1t}) + F_{1t}^-(Y_{1t})\}\right].$$

It follows from this definition of $s_{21t}$ and $s_{31t}$ that $s_{21t} \le s_{31t}$. Note also that

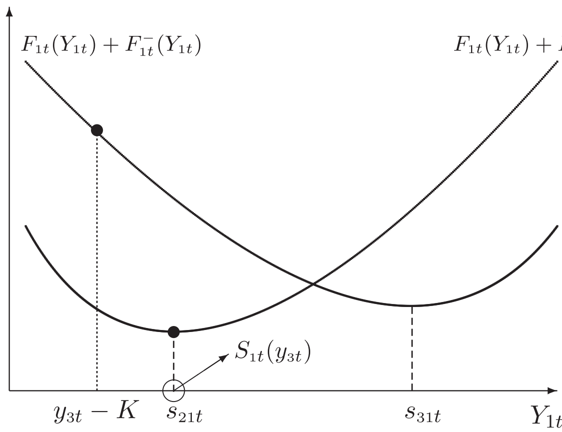
$$\min_{Y_{1t}} \left\{ F_{1t}(Y_{1t}) + F_{1t}^+(Y_{1t} \vee (y_{3t} - K)) + F_{1t}^-(Y_{1t} \wedge (y_{3t} - K)) \right\} = \tag{19}$$

$$\min\left[ \min_{Y_{1t} \ge y_{3t} - K} \left\{ F_{1t}(Y_{1t}) + F_{1t}^+(Y_{1t}) \right\} + F_{1t}^-(y_{3t} - K), \min_{Y_{1t} \le y_{3t} - K} \left\{ F_{1t}(Y_{1t}) + F_{1t}^-(Y_{1t}) \right\} + F_{1t}^+(y_{3t} - K) \right].$$

We now analyze the following three cases that emerge from Equation (19).

CASE 1. $y_{3t} - K \le s_{21t}$. In this case, $\arg\min_{Y_{1t} \ge y_{3t} - K} \{F_{1t}(Y_{1t}) + F_{1t}^+(Y_{1t})\} = s_{21t}$, since $s_{21t}$ is the unconstrained minimizer of $F_{1t}(Y_{1t}) + F_{1t}^+(Y_{1t})$, while being attainable for $Y_{1t} \ge y_{3t} - K$ when $y_{3t} - K \le s_{21t}$. Further, $\arg\min_{Y_{1t} \le y_{3t} - K} \{F_{1t}(Y_{1t}) + F_{1t}^-(Y_{1t})\} = y_{3t} - K$, because $s_{21t} \le s_{31t}$, so that, for $Y_{1t} \le y_{3t} - K \le s_{31t}$, $F_{1t}(Y_{1t}) + F_{1t}^-(Y_{1t})$,

**Figure 2 Derivation of $S_{1t}(y_{3t})$ by Constrained Minimum When $y_{3t} - K \le s_{21t}$ (Case 1)**




being convex, is decreasing in $Y_{1t}$. (Figure 2 shows how these two constrained minima are obtained when $y_{3t} - K \le s_{21t} \le s_{31t}$.) Expression (19) becomes

$$\begin{aligned}\min\big[&F_{1t}(s_{21t}) + F^+_{1t}(s_{21t}) + F^-_{1t}(y_{3t}-K), F_{1t}(y_{3t}-K)\\&+F^-_{1t}(y_{3t}-K) + F^+_{1t}(y_{3t}-K)\big] = F_{1t}(s_{21t}) + F^+_{1t}(s_{21t})\\&+F^-_{1t}(y_{3t}-K),\end{aligned}$$

since $s_{21t}$ is the unconstrained minimum of $F_{1t} + F^+_{1t}$, so that $F_{1t}(s_{21t}) + F^+_{1t}(s_{21t}) \le F_{1t}(y_{3t}-K) + F^+_{1t}(y_{3t}-K)$. Therefore, when $y_{3t} - K \le s_{21t}$, it follows that $S_{1t}(y_{3t}) = s_{21t}$, and the optimal echelon 1 position, by expression (17), is given by $Y^*_{1t} = y_{1t} \vee (s_{21t} \wedge y_{2t})$. Further, expression (17) becomes

$$\begin{aligned}F^S_t(\mathbf{y}_t) = &F_{1t}(y_{1t} \vee (s_{21t} \wedge y_{2t})) + F^+_{1t}(y_{1t} \vee (s_{21t} \wedge y_{2t})\\&\vee (y_{3t}-K)) + F^-_{1t}([y_{1t} \vee (s_{21t} \wedge y_{2t})] \wedge (y_{3t}-K))\\&+\widetilde{F}_{2t}(y_2 \vee s_{2t}) = F_{1t}(y_{1t} \vee (s_{21t} \wedge y_{2t}))\\&+F^+_{1t}(y_{1t} \vee (s_{21t} \wedge y_{2t})) + F^-_{1t}(y_{3t}-K)\\&+\widetilde{F}_{2t}(y_{2t} \vee s_{2t}),\end{aligned} \tag{20}$$

because $y_{2t} \ge y_{3t} - K$ by Lemma 2(b) and $s_{21t} \ge y_{3t} - K$ by assumption, so that $y_{2t} \wedge s_{2t} \ge y_{3t} - K$, which implies $y_{1t} \vee (s_{21t} \wedge y_{2t}) \ge y_{3t} - K$. Next, it is straightforward to verify that

$$\begin{aligned}&F_{1t}(y_{1t} \vee (s_{21t} \wedge y_{2t})) + F^+_{1t}(y_{1t} \vee (s_{21t} \wedge y_{2t}))\\&\quad= F_{1t}(y_{1t} \vee s_{21t}) + F^+_{1t}(y_{1t} \vee s_{21t}) + F_{1t}(s_{21t} \wedge y_{2t})\\&\quad+F^+_{1t}(s_{21t} \wedge y_{2t}) - F_{1t}(s_{21t}) - F^+_{1t}(s_{21t}).\end{aligned}$$

Substituting this expression into Equation (20), we get

$$\begin{aligned}F^S_t(\mathbf{y}_t) = &F_{1t}(y_{1t} \vee s_{21t}) + F^+_{1t}(y_{1t} \vee s_{21t}) + F_{1t}(y_{2t} \wedge s_{21t})\\&+F^+_{1t}(y_{2t} \wedge s_{21t}) + \widetilde{F}_{2t}(y_{2t} \vee s_{2t}) + F^-_{1t}(y_{3t}-K)\\&-F_{1t}(s_{21t}) - F^+_{1t}(s_{21t}),\end{aligned}$$

Hence, when $y_{3t} - K \le s_{21t}$, $S_{1t}(y_{3t}) = s_{21t}$, and the objective function $F^S_t$ is additively convex.

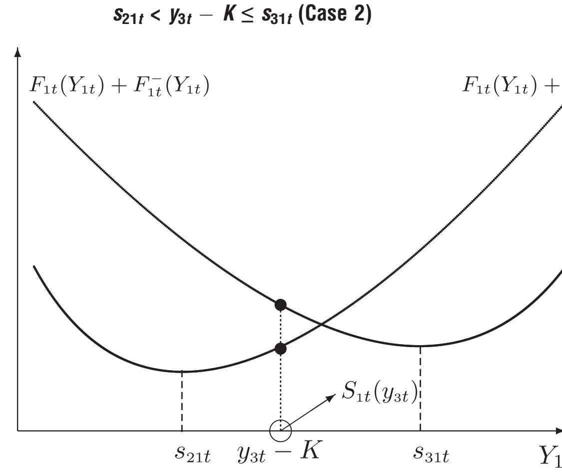
Case 2. $s_{21t} < y_{3t} - K \le s_{31t}$. This time, $\arg\min_{Y_{1t} \ge y_{3t}-K}\{F_{1t}(Y_{1t}) + F^+_{1t}(Y_{1t})\} = y_{3t} - K$, because, when $Y_{1t} \ge y_{3t} - K$, $s_{21t}$ is not attainable, and $F_{1t}(Y_{1t}) + F^+_{1t}(Y_{1t})$ is increasing since $s_{21t} < y_{3t} - K$. Further, $\arg\min_{Y_{1t} \le y_{3t}-K}\{F_{1t}(Y_{1t}) + F^-_{1t}(Y_{1t})\} = y_{3t} - K$, because, for $Y_{1t} < y_{3t} - K$, $s_{31t}$ is not attainable, and $F_{1t}(Y_{1t}) + F^-_{1t}(Y_{1t})$ is decreasing since $y_{3t} - K \le s_{31t}$. (Figure 3 illustrates how the values of those two constrained minima are obtained when $s_{21t} < y_{3t} - K \le s_{31t}$.) Thus, when $s_{21t} < y_{3t} - K \le s_{31t}$, $S_{1t}(y_{3t}) = y_{3t} - K$. Hence, $Y^*_{1t} = y_{1t} \vee ((y_{3t}-K) \wedge y_{2t}) = y_{3t} - K$ because $y_{2t} \ge y_{3t} - K$ by Lemma 2(*b*), and $y_{1t} \le y_{3t} - K$ by Corollary 2. Expression (17) for $F^S_t$ becomes

$$\begin{aligned}F^S_t(\mathbf{y}_t) = &\widetilde{F}_{2t}(y_2 \vee s_{2t}) + F_{1t}(y_{3t}-K) + F^+_{1t}(y_{3t}-K)\\&+F^-_{1t}(y_{3t}-K).\end{aligned}$$

Consequently, when $s_{21t} < y_{3t} - K \le s_{31t}$, $S_{1t}(y_{3t}) = y_{3t} - K$ and $F^S_t(\mathbf{y}_t)$ is additively convex. When $s_{21t} < y_{3t} - K \le s_{31t}$, $F^S_t$ no longer varies with $y_{1t}$, but only with $y_{2t}$ *and* $y_{3t}$. This occurs because it is optimal to bring the echelon 1 position up to $y_{3t} - K$, and thus it is echelon 3 inventory that determines the optimal replenishment level at stage 1, and the resulting cost at that stage.

Case 3. $s_{31t} < y_{3t} - K$. This time, $\arg\min_{Y_{1t} \ge y_{3t}-K}\{F_{1t}(Y_{1t}) + F^+_{1t}(Y_{1t})\} = y_{3t} - K$. The base stock level $s_{21t}$ is now not attainable for $Y_{1t} \ge y_{3t} - K$, because $s_{21t} \le s_{31t}$ and $s_{31t} < y_{3t} - K$. Further, when $Y_{1t} \ge y_{3t} - K$, $F_{1t}(Y_{1t}) + F^+_{1t}(Y_{1t})$ is increasing because it is

**Figure 3 Derivation of $S_{1t}(y_{3t})$ by Constrained Minimum When $s_{21t} < y_{3t} - K \le s_{31t}$ (Case 2)**



convex, and $s_{21t} \le s_{31t} < y_{3t} - K$. Also, $\arg\min_{Y_{1t} \le y_{3t} - K} F_{1t}(Y_{1t}) + F^{-}_{1t}(Y_{1t})\} = s_{31t}$, since $s_{31t}$ is the unconstrained minimizer of $F_{1t} + F^{-}_{1t}$ and it is attainable for $Y_{1t} \le y_{3t} - K$ when $s_{31t} < y_{3t} - K$. Thus, expression (19) reduces to

$$\min\Big[F_{1t}(y_{3t}-K)+F^{+}_{1t}(y_{3t}-K)+F^{-}_{1t}(y_{3t}-K), F_{1t}(s_{31t}) + F^{-}_{1t}(s_{31t})+F^{+}_{1t}(y_{3t}-K)\Big] = F_{1t}(s_{31t})+F^{-}_{1t}(s_{31t}) + F^{+}_{1t}(y_{3t}-K),$$

because $s_{31t}$ is the unconstrained minimizer of $F_{1t} + F^{-}_{1t}$. Thus, when $s_{31t} < y_{3t} - K$, $S_{1t}(y_{3t}) = s_{31t}$. It follows that $Y^{*}_{1t} = y_{1t} \vee (s_{31t} \wedge y_{2t})$, and expression (17) becomes

$$\begin{aligned} F^{S}_{t}(\mathbf{y}_t) &= F_{1t}\big(y_{1t} \vee (s_{31t}\wedge y_{2t})\big) + F^{+}_{1t}\big(y_{1t} \vee (s_{31t}\wedge y_{2t}) \\ &\quad \vee (y_{3t}-K)\big) + F^{-}_{1t}([y_{1t} \vee (s_{31t}\wedge y_{2t})] \wedge (y_{3t}-K)) \\ &\quad + \widetilde{F}_{2t}(y_2 \vee s_{2t}) = F_{1t}(y_{1t} \vee (s_{31t}\wedge y_{2t})) \\ &\quad + F^{+}_{1t}(y_{3t}-K) + F^{-}_{1t}(y_{1t} \vee (s_{31t}\wedge y_{2t})) \\ &\quad + \widetilde{F}_{2t}(y_{2t} \vee s_{2t}), \end{aligned} \tag{21}$$

because $s_{31t} < y_{3t} - K$, so that $s_{31t} \wedge y_{2t} \le y_{3t} - K$, and $y_{1t} \le y_{3t} - K$ by Corollary 2. We now use

$$\begin{aligned} &F_{1t}(y_{1t} \vee (s_{31t}\wedge y_{2t})) + F^{-}_{1t}(y_{1t} \vee (s_{31t}\wedge y_{2t})) \\ &\quad = F_{1t}(y_{1t} \vee s_{31t}) + F^{-}_{1t}(y_{1t} \vee s_{31t}) + F_{1t}(s_{31t}\wedge y_{2t}) \\ &\qquad + F^{-}_{1t}(s_{31t}\wedge y_{2t}) - F_{1t}(s_{31t}) - F^{-}_{1t}(s_{31t}) \end{aligned}$$

to reduce expression (21) to the following:

$$\begin{aligned} F^{S}_{t}(\mathbf{y}_t) &= F_{1t}(y_{1t} \vee s_{31t}) + F^{-}_{1t}(y_{1t} \vee s_{31t}) + F_{1t}(y_{2t}\wedge s_{31t}) \\ &\quad + F^{-}_{1t}(y_{2t}\wedge s_{31t}) + \widetilde{F}_{2t}(y_{2t} \vee s_{2t}) + F^{+}_{1t}(y_{3t}-K) \\ &\quad - F_{1t}(s_{31t}) - F^{-}_{1t}(s_{31t}). \end{aligned}$$

Thus, when $s_{31t} < y_{3t} - K$, $S_{1t}(y_{3t}) = s_{31t}$, and $F^{S}_{t}(\mathbf{y}_t)$ is additively convex.

In summary, for the three-stage, stocked-up system, the order-up-to levels of the optimal policy are $S_{1t}(y_{3t}) = \max[s_{21t}, \min(s_{31t}, y_{3t} - K)]$ at stage 1, $s_{2t}$ at stage 2, and $s_{3t}$ at stage 3. Further, the objective cost function in each period $t$ is additively convex so that there exist convex component functions $F^{S}_{jt}$ for $j = 1, 2, 3$, such that $F^{S}_{t}(\mathbf{y}_t) = \sum_{j=1}^{3} F^{S}_{jt}(y_{jt})$. (The convexity of each $F^{S}_{jt}(y_{jt})$ derives from $F^{S}_{t}(\mathbf{y}_t)$ being (jointly) convex in $\mathbf{y}_t$, which was established earlier in this section.)

The dependence of $S_{1t}$, the order-up-to level at stage 1, on $y_{3t}$ comes about due to the particular ordering of $s_{21t}$, $y_{3t} - K$, and $s_{31t}$. This can also be observed in Figures 2 and 3. The existence of multiple base stock levels at a single echelon, established analytically in the above analysis of the three-stage stocked-up system (also derived for multi-stage capacitated systems later in this study) is what distinguishes multi-stage capacitated problems from both the uncapacitated model of Clark and Scarf (1960) as well as the two-stage, capacitated model of Parker and Kapuscinski (2004).

Obtaining the decomposition of the objective cost function in Cases 2 and 3 required stocked-up bounds. In their absence, in Case 2, $Y^{*}_{1t}$ would remain at $y_{1t} \vee (y_{3t} - K)$. The resulting expression for the objective function $F^{S}_{t}$ would contain terms $F_{1t}(y_{1t} \vee (y_{3t} - K))$ and $F^{+}_{1t}(y_{1t} \vee (y_{3t} - K))$, which are not reducible to functions of a single variable. Hence, without stocked-up bounds, $F^{S}_{t}$ would not, in general, be additively convex. Similarly, in Case 3, without stocked-up bounds, $F^{S}_{t}$ could not be expressed as a sum of single-variable functions, as expression (21) would not be obtainable.

### 3.2. The Looking-Upstream Effect

The order-up-to level at stage 1 is state-dependent, since $S_{1t}(y_{3t}) = s_{21t} \vee [(y_{3t} - K) \wedge s_{31t}]$, where $s_{31t} \ge s_{21t}$. Thus, the optimal policy looks upstream for information on echelon 3 inventory. This 'looking upstream' represents a novel aspect of managing inventories in capacitated multi-stage supply systems. We refer to this phenomenon as *the looking-upstream effect.* Note that the lower base stock level $s_{21t}$ is attained for small values of $y_{3t} - K$ (when $y_{3t} - K \le s_{21t}$). This base stock level represents the (usual) optimal tradeoff between the backlogging and holding costs in the system.

As $y_{3t}$ increases, the supply pipeline, being capacity constrained, becomes less able to handle higher future realizations of customer demand. In order to free up the system for the possibility of large customer demands in the future, and thus avoid excessive backlogging costs that could result, it becomes optimal to move additional units from stage 2 to stage 1 beyond those ensured by $s_{21t}$. As a result, the order-up-to level at stage 1 starts to increase with $y_{3t}$. Once $y_{3t}$ reaches the level of $s_{31t} + K$, there is sufficient inventory in the system to accommodate large customer demand realizations potentially faced by stage 1 in the future, and the order-up-to level $S_{1t}(y_{3t})$ settles at the second (higher) base stock level, $s_{31t}$, regardless of any additional increases in $y_{3t}$. Therefore, the looking-upstream effect comes about because stage 1 has to manage not only its own replenishment order in each period, but also the ability of the capacity-constrained supply pipeline to meet future orders. Due to this looking-upstream effect, the optimal echelon 1 position for the stocked-up problem is not given by either the echelon base stock policy of Clark and Scarf (1960) or by the

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modified-echelon base stock policy of Parker and Kapuscinski (2004).

The looking upstream effect also acts to decongest the inventory pipeline to allow larger orders upstream. Suppose that $S_{1t}(y_{3t}) = y_{3t} - K$. Then, if stage 1 orders any positive amount (which occurs when $x_{2t} > 0$ and $x_{2t} + x_{3t} > K$), stage 3 is compelled to also order more inventory in order to maintain the stocked-up condition in the system. Further, stage 3 needs to order *at least* the amount ordered into stage 1, so that $X_{3t} \geq X_{1t}$. Thus, by ordering up to $y_{3t} - K$, stage 1 compels stage 3 to order at least as much. In that manner, the optimal order at stage 1 brings additional inventory into the system and acts to protect the supply chain against excessive backlogging costs in the future. (The same phenomenon occurs when $S_{1t}(y_{3t}) = s_{31t} \geq y_{3t} - K$.).

### 3.3. Structure of the Optimal Policy

While we were able to derive the optimal policy and established the decomposition of the objective cost function for the three-stage stocked-up system using basic principles, such an approach does not suffice when solving a stocked-up system with an arbitrary number of stages. Instead, it becomes necessary to derive some intermediate results that facilitate the analysis. In particular, instead of enumerating step-by-step all the possible cases that arise in such a system like we did with the three-stage stocked-up system, we need a mechanism to: (a) to capture all the different cases that emerge from the ordering of $s_{ijt}$'s and $y_{jt} - (j-2)K$ at each stage $j$; and, (b) to generalize the original result of Karush (1959) to multi-dimensional feasible regions and multi-variable convex functions. Such a mechanism is provided in a set of intermediate results stated and proved in Appendix S2. These results are used in the remainder of this section. They are, to the best of our knowledge, new to the literature, and as such represent a contribution to convex optimization.

Next, define a sequence of feasible sets $\mathbb{D}_S^j$, $j = 1, \ldots, N$, (with $\mathbb{D}_S^N \equiv \mathbb{D}_S$) as:

$$\mathbb{D}_S^1(\mathbf{y}_t) := \{Y_{1t} \mid y_{1t} \leq Y_{1t} \leq y_{2t}\}; \tag{22}$$

$$\begin{aligned}\mathbb{D}_S^j(\mathbf{y}_t) := \{ & Y_{1t}, \ldots, Y_{jt} \mid y_{it} \leq Y_{it} \leq y_{i+1,t},\\ & Y_{it} - Y_{i-1,t} \leq K, Y_{jt} \geq Y_{1t} + (j-2)K; i \leq j\}.\end{aligned} \tag{23}$$

Proposition 2. *Assume that, in period $t+1$, $F_{t+1}^S$ is additively convex so that there exist convex functions $F_{j,t+1} : \mathbb{R} \to \mathbb{R}$, for $j = 1, \ldots, N$, such that*

$$F_t^S(\mathbf{y}_t) = \min_{\mathbf{Y}_t \in \mathbb{D}_S(\mathbf{y}_t)} \left\{ \sum_{j=1}^N F_{i,t+1}(Y_{it}) \right\}.$$

*Then, for each $j \leq N$, there exist convex increasing functions $F_{ij,t+1}^+ : \mathbb{R} \to \mathbb{R}$ and convex decreasing functions $F_{ij,t+1}^- : \mathbb{R} \to \mathbb{R}$, defined for every $i \in \{j+1, \ldots, N\}$,* such that

(*a*) $$\begin{aligned} F_t^S(\mathbf{y}_t) = \min_{\mathbf{Y}_t \in \mathbb{D}_S^j(\mathbf{y}_t)} \Bigg\{ & \sum_{i=1}^j F_{i,t+1}(Y_{it}) \\ & + \sum_{i=j+1}^N \Big[ F_{ij,t+1}^+\big(Y_{1t} \vee (y_{it} - (i-2)K)\big) \\ & + F_{ij,t+1}^-\big(Y_{jt} \wedge (y_{i+1,t} - (i-j)K)\big)\Big]\Bigg\}; \end{aligned}$$

(*b*) *The optimal echelon-1 position $Y_{1t}^*(\mathbf{y}_t)$ is given by*

$$Y_{1t}^*(\mathbf{y}_t) = \begin{cases} y_{1t} & \text{if } y_{1t} \geq S_{1t}(\mathbf{y}_t) \\ S_{1t}(\mathbf{y}_t) & \text{if } y_{1t} < S_{1t}(\mathbf{y}_t) \leq y_{2t} \\ y_{2t}, & \text{if } S_{1t}(\mathbf{y}_t) > y_{2t} \end{cases} \tag{24}$$

*where*

$$\begin{aligned} S_{1t}(\mathbf{y}_t) = \arg\min_{Y_{1t}} \Bigg\{ & F_{1,t+1}(Y_{1t}) + F_{N1,t+1}^-(Y_{1t}) \\ & + \sum_{i=3}^N \Big[ F_{i1,t+1}^+\big(Y_{1t} \vee (y_{it} - (i-2)K)\big) \\ & + F_{i-1,1,t+1}^-(Y_{1t} \wedge (y_{it} - (i-2)K))\Big]\Bigg\}; \end{aligned}$$

(*c*) *Given $Y_{1t}^*(\mathbf{y}_t)$, the optimal echelon-j position $Y_{jt}^*(\mathbf{y}_t)$, for $j = 2, \ldots, N$, is given recursively by*

$$Y_{jt}^*(\mathbf{y}_t) = \begin{cases} y_{jt} \vee \big(Y_{1t}^*(\mathbf{y}_t) + (j-2)K\big) & \textit{if } y_{jt} \vee \big(Y_{1t}^*(\mathbf{y}_t) + (j-2)K\big) \geq S_{jt}(\mathbf{y}_t) \\ S_{jt}(\mathbf{y}_t) & \textit{if } y_{jt} \vee \big(Y_{1t}^*(\mathbf{y}_t) + (j-2)K\big) < S_{jt}(\mathbf{y}_t) \leq y_{j+1,t} \wedge \Big(Y_{j-1,t}^*(\mathbf{y}_t) + K\Big) \\ y_{j+1,t} \wedge \Big(Y_{j-1,t}^*(\mathbf{y}_t) + K\Big), & \textit{if } S_{jt}(\mathbf{y}_t) > y_{j+1,t} \wedge \Big(Y_{j-1,t}^*(\mathbf{y}_t) + K\Big) \end{cases} \tag{25}$$

*where* $S_{jt}(\mathbf{y}_t) := \arg\min_{Y_{jt}}\{F_{j,t+1}(Y_{jt}) + F_{Nj,t+1}^-(Y_{jt}) + \sum_{i=j+2}^N F_{i-1,j,t+1}^-(Y_{jt} \wedge (y_{it} - (i-j-1)\,K))\}$.

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When the objective function for the stocked-up problem is additively convex in period $t+1$, Proposition 2 establishes the optimality of an order-up-to policy with state-dependent control parameters. Further, the state-dependent order-up-to level $S_{jt}(\cdot)$ depends on echelon inventories at all stages upstream of stage $j+1$. Going forward, $S_{jt}(\cdot)$ will be referred to as *the echelon j threshold function*. Proposition 2 also implies that this optimal policy is constructed from *the bottom-up*: once $Y^*_{1t}(\mathbf{y}_t)$ has been determined, the remaining optimal echelon positions are constructed recursively, starting at stage 2, and moving upstream, according to the expression $Y^*_{jt}$ given in part (c).

Part (a) of Proposition 2 follows from the repeated use of Lemma 15, where $u$ is replaced with $Y_{j-1,t}+K$, $x_{n+1}$ is replaced with $y_{jt}$ and $x_j$ is replaced with $y_{j+1,t}$. Part (b) derives from the convexity in $Y_{1t}$ of the function being minimized for $j=1$ in part (a). Similarly, part (c) follows from the convexity in $Y_{jt}$ of the function being minimized for $j>1$ in part (a) (details are in Appendix S1). We now further characterize the optimal policy for the stocked-up problem by formally defining a new class of order-up-to policies with state-dependent parameters.

Definition 3. *For a multi-stage optimization problem, a policy is called a branching echelon base stock (BEBS) policy if there exist, in each period t and at each stage $j \in \{1, \ldots, N\}$, two sequences of numbers, $\{a_{ijt}\}$ and $\{s_{ijt}\}$, with $a_{i+1,jt} \geq a_{ijt}$ and $s_{i+1,jt} \geq s_{ijt}$ for all $i \in \{j+1, \ldots, N\}$, such that the order-up-to level of the policy at stage j is a threshold function $S_{jt}(\cdot)$ given by*

$$S_{jt}(\mathbf{y}_t) = s_{Njt} \wedge \bigwedge_{i=j+2}^{N} \left[ \left(y_{it} - a_{ijt}\right) \vee s_{i-1,jt} \right]. \tag{26}$$

Consequently, a BEBS policy represents an order-up-to policy uniquely determined by a set of ordered constants $a_{ijt}$, and a set of ordered base stock levels $s_{ijt}$, $i \in \{j+1, \ldots, N\}$, for each stage $j$. A BEBS policy is also a special case of the multitier (echelon) base stock policy first introduced by Janakiraman and Muckstadt (2009). The base stock levels of the BEBS policy establish each threshold function $S_{jt}$ as a piecewise linear function of upstream state variables from stage $j+2$ to stage $N$. The term "branching" in BEBS refers to the fact that the set of base stock parameters needed to characterize the replenishment decision at each stage is sprouting a new branch from one stage to another, going downstream; that is, the number of such parameters needed to describe each threshold function increases (by one), looking downstream from stage $j+1$ to stage $j$. The total number of those parameters needed at stage $j$ is $N-j$.

Proposition 3. *The following hold in each period t.*

(*a*) *$F^S_t$ is additively convex and there exist convex increasing functions $G^+_{it} : \mathbb{R} \to \mathbb{R}$ for $i \in \{1, \ldots, N\}$, and convex decreasing functions $G^-_{it} : \mathbb{R} \to \mathbb{R}$ for $i \in \{2, \ldots, N\}$, such that*

$$F^S_t(\mathbf{y}_t) = \sum_{i=1}^{N} G^+_{it}(y_{it}) + \sum_{i=2}^{N} G^-_{it}(y_{it}). \tag{27}$$

(*b*) *There exists a BEBS policy, with $a_{ijt} := (i-j-1)K$, optimal for $F^S_t$.*

The additive convexity of the objective function for the stocked-up capacitated problem is thus preserved even though the order-up-to levels of the policy are not constants, but rather functions of (upstream) echelon inventories. The objective cost function represents a sum of $N$ single-variable, convex increasing functions, and $N-1$ single-variable, convex decreasing functions. The latter represent induced-penalty functions similar to those of Clark and Scarf (1960). Those convex decreasing functions are associated with inventory levels at upstream stages, as they induce a penalty on each upstream stage for not carrying sufficient inventory to fulfil orders from the downstream stage. As in Clark and Scarf (1960), those convex decreasing functions represent penalties charged to upstream stages for not having sufficient inventory to meet downstream orders.

It is worthwhile to understand how that preservation comes about when optimal decisions, in the form of echelon positions such as $Y^*_{1t}(\mathbf{y}_t)$ for example, can depend on the full echelon state $\mathbf{y}_t$. The key to this relationship is the fact that $Y^*_{1t}(\mathbf{y}_t)$ is piecewise linear in individual components of $\mathbf{y}_t$. In particular, since $Y^*_{1t}(\mathbf{y}_t) = \max[y_{1t}, \min(S_{1t}(\mathbf{y}_t), y_{2t})]$, and $S_{1t}(\mathbf{y}_t)$ is given by expression (26) with $a_{i1t} := (i-2)K$, then, on any such given interval $[s_{i1t}, s_{i-1,1t}]$, the optimal echelon 1 position $Y^*_{1t}(\mathbf{y}_t)$ is equal to only a single echelon inventory level $y_{it}$, minus the constant term $(i-2)K$. It is this dependence of optimal inventory decisions on actually only *a single* echelon inventory variable (on any such interval) that allows the additive convexity of the objective cost function to be preserved.

By Definition 3, the threshold functions for the top two stages in the system are constants. Hence, the optimal BEBS policy for a two-stage capacitated problem reduces to exactly the modified echelon base stock (MEBS) policy of Parker and Kapuscinski (2004). For stocked-up capacitated systems with more than two stages, order-up-to levels at each lower stage start to vary with echelon inventories

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at upstream stages. This looking upstream for information on echelon inventory at upstream stages (also encountered in our analysis of the three-stage stocked-up system) represents a novel aspect of managing inventories in multi-stage supply chains.

### 3.4. Infinite Horizon Convergence

We now formulate the stocked-up infinite-horizon, capacitated problem. We follow the same steps used in the formulation of the original infinite-horizon, capacitated multi-stage problem.

Let $\pi := \{\mathbf{Y}_1, \mathbf{Y}_2, \ldots\}$ be an arbitrary infinite horizon policy for the stocked-up capacitated problem, and $\widetilde{\mathcal{F}}_t^S(\pi)$ be the expected present value of costs of implementing that policy, starting in period $t$. Let $\mathcal{F}_t^S$ be the smallest of those over the feasible space $\mathbb{D}_S$. Then,

$$\widetilde{\mathcal{F}}_t^S(\pi) = \lim_{T\to\infty} \sum_{\tau=t}^{T} \alpha^{\tau-t} \mathbb{E}\Big[(p + H_1)(\xi_\tau - Y_{1\tau})^+ + \sum_{j=1}^{N} h_j Y_{j\tau}\Big] \tag{28}$$

$$\mathcal{F}_t^S(\mathbf{y}_t) = \inf_{\pi\in\{\pi|\mathbf{Y}_\tau\in\mathbb{D}_S(\mathbf{y}_\tau),\tau=t,t+1,\ldots\}} \widetilde{\mathcal{F}}_t^S(\pi). \tag{29}$$

PROPOSITION 4. *There exists a continuous, bounded and additively convex function $\mathcal{F}^S : \mathbb{R}^N \to \mathbb{R}$ such that $\mathcal{F}_t^S = \mathcal{F}^S$ for all $t$. Further, there exists a stationary BEBS policy optimal for $\mathcal{F}^S$.*

The dynamic program for the infinite-horizon restricted problem therefore becomes

$$\mathcal{F}^S(K, \mathbf{y}) = \min_{\mathbf{Y}\in\mathbb{D}_S(K,\mathbf{y})} \widetilde{\mathcal{F}}^S(K, \mathbf{Y}). \tag{30}$$

The infinite-horizon minimum cost function for the capacitated problem over the stocked-up feasible set $\mathbb{D}_S$ is stationary, and there exists a stationary BEBS policy that is optimal for it. Thus, the BEBS policy optimal for the finite-horizon, stocked-up, capacitated problem converges to a stationary BEBS policy that is optimal for the infinite-horizon, stocked-up, capacitated problem.

## 4. Problem/Policy Equivalence and Threshold Utilization

We now establish the equivalence of the stocked-up, infinite-horizon, capacitated problem and the original, infinite-horizon, capacitated problem for any capacity below $K^*$, and in that way complete the solution of the latter for any such capacity level.

THEOREM 2. *Let $\mathcal{F}$ be the objective cost function for the original, infinite-horizon capacitated problem given in Equation (7), and $\boldsymbol{Y}^*$ be the optimal policy for $\mathcal{F}$. Let $\mathcal{F}^S$ be the objective cost function for the stocked-up, infinite-horizon capacitated problem with identical model parameters, and $\boldsymbol{Y}^{S*}$ be the optimal policy for $\mathcal{F}^S$. Then, for every $K$ such that $K \le K^*$, where $K^*$ is as defined in Theorem 1, and every $\boldsymbol{y} \in \Upsilon(K)$: (a) $\mathbb{D} \equiv \mathbb{D}_S$; (b) $\mathcal{F}(K, \mathbf{y}) = \mathcal{F}^S(K, \mathbf{y})$; and, (c) $\boldsymbol{Y}^*(K, \boldsymbol{y}) = \boldsymbol{Y}^{S*}(K, \boldsymbol{y})$.*

Because the objective cost function for an infinite-horizon problem does not depend on either a starting state nor any finite-horizon terminal function, Theorem 2 relies only on Assumption 1. This theorem, together with Proposition 4, also leads to the following corollary.

COROLLARY 3. *For every $K \le K^*$:*

*(a) There exists a stationary BEBS policy that is optimal for $\mathcal{F}$;*
*(b) $\mathcal{F}$ is continuous, bounded, and additively convex.*

Therefore, we can apply all the results concerning the structure of the optimal policy and properties of the objective function for the stocked-up, infinite-horizon, capacitated problem to the original infinite-horizon capacitated problem. Thus, for any member of the class of multi-stage problems defined by having the smallest capacity and system capacity below $K^*$, the optimal policy is a BEBS policy, and the objective cost function is additively convex.

### 4.1. Reformulation through Threshold Utilization

Because different practical settings have inherently different average demands and different capacity scales, then, in order to refer to a range of such settings without specifying the capacity level for each, it is common in industry to refer to "the utilization" rather than the capacity of a system. Therefore, in order to render our results more accessible and intuitive, we now reparametrize our problem to make use of this system performance measure universal to all capacitated systems regardless of the size of average demand. For that purpose, we now formally define *system utilization*.

DEFINITION 4. *System utilization $\rho$ for a capacitated system is defined as the ratio of mean demand to the bottleneck (system) capacity. Thus, $\rho = \mu/K$.*

COROLLARY 4. *Given an infinite-horizon, capacitated, multi-stage system, there exists a threshold utilization $\rho^*$, $\rho^* < 100\%$, such that, for any utilization $\rho$, $\rho \in [\rho^*, 100\%)$, there exists an optimal BEBS policy, and the objective cost function $\mathcal{F}$ is additively convex.*

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Corollary 4 follows directly from Theorem 1 and Corollary 3 by defining the threshold utilization as $\rho^* := \mu/K^*$. This corollary completes the characterization of the optimal policy for the class of capacitated multi-stage systems considered in this study. There exists a utilization above which the optimal policy for these systems (under the infinite horizon) is a BEBS policy, and optimal cost function is additively convex. This makes it possible to numerically solve the capacitated multi-stage problem above the threshold utilization, for any number of stages encountered in practice, since its optimization reduces to minimizing only single-variable functions.

Recall that it is possible to interpret the behavior of the stocked-up system under the optimal policy as attempting to free up (i.e., "decongest") stages upstream, so that as many as possible of the needed units can flow downstream rather than getting stuck upstream. This is why, in making its replenishment decision, every stage $j$ has associated with it a base stock level $s_{ij}$ for each stage $i$ upstream of it. By expression (26), the BEBS policy optimal for the capacitated problem above the threshold utilization $\rho^*$, is characterized, at each stage $j$, by the threshold function

$$S_j(\mathbf{y}) = s_{Nj} \wedge \bigwedge_{i=j+2}^{N} \left[ (y_i - (i-j-1)K) \vee s_{i-1,j} \right].$$

Consider, for instance, stage 1, and suppose first that the above expression reduces to $S_1(\mathbf{y}) = s_{N,1} \wedge [(y_k - (k-2)K) \vee s_{k-1,1}]$ for some particular $k \geq 3$, with $s_{N,1} \geq s_{k-1,1}$. It follows that

$$k = \arg \min_{i=3,\ldots,N} \bigwedge_i \left[ (y_i - (i-2)K) \vee s_{i-1,1} \right].$$

Then, the base stock level $s_{k-1,1}$ represents the (usual) optimal tradeoff between the backlogging and holding costs in the system, attained for small values of $y_k - (k-2)K$. However, as $y_k$ increases, the supply pipeline, being capacity constrained, becomes less able to handle higher future realizations of customer demand. Once $y_k$ reaches the level of $s_{k-1,1} + K$, then, in order to free up the system for the possibility of large customer demands in the future, it becomes optimal to move additional units from stage 2 to stage 1 beyond those ensured by $s_{k-1,1}$. The order-up-to level at stage 1 then starts to increase with $y_k$. As soon as that happens, and if stage 1 orders any positive amount (that is, if $X_1 > 0$ for which it suffices that $x_2 > 0$ and $x_2 + x_3 + \cdots + x_k > K$), stage $k$ becomes compelled to also order additional inventory in order to maintain the stocked-up condition of the system. Further, stage $k$ will order *at least* the amount ordered into stage 1, in order to preserve the stocked-up condition in the next period, so that $X_k \geq X_1$. Thus, by ordering up to $y_i - (i-2)K$, stage 1 effectively compels stage $k$ to order at least as much, thus decongesting the stages upstream of it by drawing additional inventory down from them. (This effect also occurs if $S_1(\mathbf{y}) = s_{N,1}$.) The similar process is repeated when it comes to the replenishment order at stage $k$, where we have $S_k(\mathbf{y}) = s_{Nk} \wedge [(y_m - (m - k - 1)K) \vee s_{m-1,k}]$ for some $m \geq k+2$. Therefore, unless $S_k(\mathbf{y}) = s_{m-1,k}$, stage $k$ acts to draw additional inventory into stage $m$ and thus decongest stages upstream of it.

What drives the selection of a particular $s_{ij}$ for the optimal replenishment of stage $j$ is the size of $s_{ij}$ relative to $y_{it} - (i-j-1)K$. This is because $(i-j-1)K$ represents the maximum amount by which echelon inventory at that stage $i$ can be decongested before any of the stock sent downstream reaches stage $j+1$ and becomes available to stage $j$. As long as the maximum feasible drawing down of inventory reduces echelon inventory below the corresponding base stock level $s_{ij}$, it is not necessary for stage $j$ to draw down extra inventory into stage $i$. (In the above example, this corresponds to $S_1(\mathbf{y}) = s_{N,1} \wedge s_{k-1,1} = s_{k-1,1}$.) In that manner, the BEBS policy has the effect of drawing down extra inventory from upstream stages (unless that inventory is already low enough). This is also why the optimal order-up-to level at each stage $j$ is characterized by $N - j$ basestock levels $s_{ij}$ that are decreasing in $i$, and why the choice of a particular $s_{ij}$ to use for replenishment at stage $j$ is determined by the comparison of each $s_{ij}$ to $y_{it} - (i-j-1)K$.

In summary, the structure of the BEBS policy reflects the need to decongest the system in order to preserve the stocked-up condition of the pipeline. This stocked-up nature of pipeline inventory, in turn, allows the optimal BEBS policy to achieve the decomposition of the objective cost function, much like turning on a faucet connected to a long hose that is sufficiently, though not necessarily completely, full of water allows the water to immediately starts coming out the other end. Because the pipeline is sufficiently stocked only when the system is sufficiently utilized, this provides a perspective on why BEBS policies are optimal only at sufficiently high utilizations.

By comparison, with sufficiently low system utilizations, each stage may have sufficient space to order the amount needed to satisfy customer demand. What complicates the solution of the original capacitated problem when utilization is below $\rho^*$ is not, however, the possibility that *all* stages in the system may have sufficient space to meet optimal orders, which is something that would occur at sufficiently low utilizations, but rather the possibility that *some* stages may and *other* stages may not have that space, which is something that can occur at utilizations below the threshold utilization $\rho^*$. Ordering up to the base stock level $s_{ij}$ at stage $j$ would then not act to initiate a decongestion at some stage $i > j$, if there existed a stage between $j$ and $i$ that had abundant space to accommodate future orders. This is because a stage with ample space for new orders could not pass on the decongesting information from the stage below it, much like a sufficiently empty section of the hose would prevent the water from immediately coming out the other end of the hose, once the faucet is turned on. Hence, when system utilization is below the threshold utilization, the optimal replenishment order at each stage cannot, in general, be expressed through a series of increasing base stock levels, and the decomposition of the objective function cannot be achieved.

The BEBS policy identified in this study is a special case of the multitier base stock policy introduced in Janakiraman and Muckstadt (2009), who show that a capacitated multi-stage system with discretized inventory and identical capacities can be decomposed into single-unit, single-customer pairs. They also provide an upper bound on the number of parameters needed to characterize the optimal replenishment decision at each stage, and that upper bound is shown to increase exponentially in the number of stages in the system. In particular, for a capacitated system with $N$ stages that are one period apart, the optimal decision at each stage is determined, in their model, by at most $2^N$ parameters. While Janakiraman and Muckstadt (2009) were the first to provide insight into the structure of the optimal policy for a capacitated multi-stage system, they do not provide a method for finding those parameters and the resulting cost functions. In contrast, we consider continuous inventory states, and provide explicit expressions for the structure of the optimal policy, its parameters, and associated cost functions. Further, the number of parameters needed to characterize the BEBS policy at each stage $j$ is exactly $N - j$, which grows linearly in the number of stages in the system. The decomposition achieved in our study is of the objective function into single-variable cost functions, rather than into single-unit single-customer pairs. Our results, on the other hand, apply only at system utilizations above the threshold utilization, as already discussed.

### 4.2. Implications for the Finite-Horizon Capacitated Problem

We conclude with a standard result concerning the relationship between an infinite-horizon dynamic programming problem and a corresponding finite-horizon one. The next theorem completes the solution of the original finite-horizon, capacitated problem formulated in expressions (3) and (4).

THEOREM 3. *Given an infinite-horizon, capacitated, multi-stage system that satisfies Assumptions* 1 *and* 2, *let* $\rho^*$ *be its threshold utilization. Let* $\rho$ *be such that* $\rho \in [\rho^*, 100\%)$, *and let* $K$ *be the corresponding system capacity. Let* $\mathbf{Y}^*(K)$ *be the BEBS policy optimal for that infinite-horizon, capacitated multi-stage problem. Consider a corresponding finite-horizon problem, with the same (stationary) model parameters, identical capacity* $K$, *and objective cost function* $F_t(K, \cdot)$. *Then, for any time horizon* $T$,

(*a*) *There exists an additively convex, terminal value function* $F_{T+1}(K, \cdot)$ *such that* $\mathbf{Y}_t^*(K) = \mathbf{Y}^*(K)$ *is optimal in each period for that finite-horizon capacitated problem;*
(*b*) $F_t(K, y_t)$ *is additively convex in* $y_t$ *in each period* $t$.

Thus, under Assumptions 1–2, there exists a terminal value function such that a BEBS policy is optimal for any finite-horizon, stationary, capacitated multi-stage problem with utilization in excess of the threshold utilization for the corresponding infinite-horizon capacitated problem. One such terminal value function is the infinite-horizon objective cost function for which the same BEBS policy is optimal. A linear salvage value function will not, in general, lead to the optimality of a BEBS policy for the finite-horizon, capacitated problem, because any such terminal value function will result in an optimal policy in period $T$ that is either not capacitated or not stocked-up.

## 5. Quantifying the Threshold Utilization

We now make use of our results to quantify the value of the threshold utilization for a set of infinite-horizon, capacitated three-stage systems, and numerically explore the sensitivity of that threshold utilization to model parameters. Customer demand follows Erlang distribution, and capacity at each stage is 10. Our procedure, results, and related discussion are in Appendix S3. In Tables 2 and 3 of Appendix S3, we determine the threshold utilization as a function of the unit backlogging cost, unit inventory holding costs, and the coefficient of variation. The corresponding threshold utilization is found to be below 50% in most cases, reaching as low as 28%, when the coefficient of variation (CV) is high, and there is a large difference ($\delta$) between unit holding costs at adjacent stages. Thus, it is for any system utilization above those values shown in Tables 2 and 3 of Appendix S3 that our results apply.

### 5.1. Base Stock Levels of the Optimal BEBS Policy

Having found threshold utilizations above which BEBS policies are optimal for which a three-stage capacitated

**Table 1 Basestock Levels of the Optimal Policy**

| | Unit backlogging cost ($p$) | | | | |
|---|---|---|---|---|---|
| Coefficient of variation | 2 | 4 | 6 | 8 | 10 |
| 0.10 | (11, 19, 20, 29) | (11, 20, 21, 30) | (11, 21, 21, 31) | (11, 21, 21, 31) | (11, 21, 21, 31) |
| 0.20 | (12, 20, 21, 30) | (13, 23, 23, 33) | (14, 25, 24, 35) | (15, 25, 25, 35) | (15, 26, 26, 36) |
| 0.33 | (13, 23, 23, 33) | (17, 28, 27, 38) | (19, 31, 30, 41) | (21, 33, 32, 43) | (22, 34, 33, 44) |
| 0.50 | (15, 26, 26, 36) | (23, 35, 33, 45) | (27, 39, 38, 49) | (30, 42, 41, 52) | (32, 45, 43, 55) |
| 0.71 | (19, 30, 29, 40) | (30, 43, 41, 53) | (37, 50, 48, 60) | (41, 55, 53, 65) | (45, 59, 57, 69) |
| 1.00 | (23, 35, 33, 45) | (40, 55, 52, 65) | (51, 67, 63, 77) | (58, 74, 71, 84) | (64, 80, 77, 90) |

system studied in Tables 2 and 3 of Appendix S3, we choose one such utilization level to determine base stock levels $s_{ij}$ of the optimal BEBS policy. We solve the same three-stage capacitated system studied in Table 1, and vary the same model parameters (i.e. the unit backlogging cost and the coefficient of variation). Each cell in Table 1 displays the four base stock levels for each such three-stage systems: ($s_{21}$, $s_{31}$, $s_2$, $s_3$), where $s_{21}$ and $s_{31}$ are the two base stock levels at stage 1, while $s_2$ and $s_3$ are the base stock levels at stages 2 and 3, respectively. The utilization level is 90% throughout. Base stock levels of the optimal policy are thus found to be increasing in both the unit backlogging cost and the coefficient of variation, as costlier stockouts and higher demand volatility necessitate more inventory optimally held in the system to offset the increased (expected) cost of stocking out.

### 5.2. Upper Bound on the Threshold Utilization

We quantified the threshold utilization for a three-stage system by comparing the objective cost functions for the original problem and the stocked-up problem. For capacitated systems with more than three stages this type of comparison is not feasible due to the curse of dimensionality. (Solving a four-stage problem without the decomposition of the objective function achieved by a BEBS policy, would take a full month of CPU time. With even more stages, numerical solution becomes completely unattainable.) It is therefore necessary to develop a method to quickly calculate the bounds on the threshold utilization. In particular, it is the upper bound that is of interest, because for any utilization above the threshold utilization a BEBS policy is optimal and the decomposition of the objective function is achieved. The following theorem provides such a method.

THEOREM 4. *Let $\mathcal{F}(K)$ be the objective cost function for the original, infinite-horizon capacitated problem given in (7), and $\boldsymbol{Y}^*(K)$ be the optimal policy for $\mathcal{F}(K)$, as a function of the bottleneck capacity K. Let $\mathcal{F}^S(K)$ be the objective cost function for the stocked-up, infinite-horizon capacitated problem with identical model parameters, and $\boldsymbol{Y}^{S^*}(K)$ be the optimal policy for $\mathcal{F}^S(K)$. Let $\mathbf{S}(K) := \{s_{ij}\}$ be the set of base stock levels associated with the optimal (stationary) BEBS policy $\boldsymbol{Y}^{S^*}(K)$ (where each $s_{ij}$ is a function of the bottleneck capacity K). If $\mathbf{S}(K)$ is such that $s_{Nj} - s_{N1} \geq (j - 2)K$ for every $j > 2$, then $K \leq K^*$, and $\boldsymbol{Y}^*(K) = \boldsymbol{Y}^{S^*}(K)$.*

Therefore, if, given capacity $K$, the largest of the base stock levels at each stage are stocked-up, then such capacity $K$ is below the threshold capacity $K^*$. As a result, the BEBS policy optimal for the stocked-up multi-stage capacitated problem with capacity $K$ is also optimal for the original multi-stage capacity problem with the identical capacity $K$. Thus, Theorem 4 provides a sufficient condition for the optimality of BEBS policies for the original capacitated multi-stage problem. This condition can be verified quickly and reliably. To do so, it suffices to numerically determine $\mathbf{S}(K)$ for the corresponding stocked-up capacitated problem with the given capacity $K$, and then check that the largest of the base stock levels at each stage $j$ is stocked-up (i.e., $s_{Nj} - s_{N1} \geq (j - 2)K$).

We make use of Theorem 4 as follows. Given a capacitated multi-stage problem, we start at a utilization close to 1 (e.g., 99%), calculate $\mathbf{S}$ for the corresponding stocked-up capacitated problem, then verify that the stocked-up condition for the largest base stock level at each stage is satisfied. If it is, we decrease the utilization by a small amount, such as 1%, and repeat the process. When, during this process, at least one of the largest base stock levels is no longer stocked-up, we stop. The upper bound on the threshold utilization thus obtained is the last utilization at which all the largest base stock levels were stocked-up. In this way, an upper bound on the threshold utilization for a capacitated multi-stage system is quickly determined. Tables 4–10 in Appendix S3 display those upper bounds for a range of 3-stage, 4-stage and 5-stage capacitated systems. using model parameters established in the literature (e.g., Parker and Kapuscinski 2004, DeCroix 2013).

Note that Theorem 4 provides only an upper bound on threshold utilization $\rho^*$, and not the actual threshold utilization. This is because it is possible for $\mathbf{Y}^*(K)$ to have stocked-up optimal decisions, and therefore be identical to $\mathbf{Y}^{S^*}(K)$ without having base stock levels $s_{Nj}$ be stocked-up as well. Nevertheless, the upper bound specified by Theorem 4 is actually quite tight (as evaluated for three-stage capacitated systems for which we can compute the exact threshold utilization).

In summary, the algorithm proposed in this section provides fast, practical, and tight upper bounds on the threshold utilization for BEBS policies identified in this study. This upper bound makes it possible to quickly determine the range of optimality of these policies. As most of those bounds do not exceed 75%, they are still below utilizations encountered in industry (Federal Reserve 2014), so that our results can be said to apply at utilizations levels commonly seen in practice.

## 6. Discussion

### 6.1. Limitations and Extensions

In this study, we assumed a single-period leadtime between successive stages in the system. While this assumption may appear restrictive, in reality it does not result in any loss of generality. This is because our results apply to capacitated systems with an arbitrary number of stages, which allows a multi-period leadtime between any two stages to be accommodated within our model as follows. Suppose there is a leadtime of $L$ periods between stages $j$ and $j + 1$ associated with every order placed by stage $j$. Suppose also that stage $j$ incurs the holding cost on those units in transit, so that the holding cost for every such unit is $H_j$. Then, to capture that leadtime effect, it suffices to insert $L - 1$ additional stages between the original stages $j$ and $j + 1$, with a single-period leadtime between each, and associate unit inventory holding cost $H_j$ with each of those additional stages.

The original system with $N$ stages is thus transformed into another system with $N + L - 1$ stages, so that, for example, the original stage $j + 1$ becomes stage $j + L$ in the new system. The transformed system is equivalent in every way to the original system since all the costs and optimal decisions are the same. In particular, it will never be optimal to hold any inventory in stages $j + 1$ through $j + L - 1$ in the transformed system, as the holding cost at each of those stages is the same as that at the next stage downstream, and inventory at a lower stage is closer to the customer. Each of those additional stages will therefore simply pass each arriving unit downstream until it reaches stage $j$, so that the new system becomes identical to the original one with the leadtime of $L$ periods between stages $j$ and $j + 1$. Since the transformed system satisfies all the conditions assumed in this study, there will exist, by Theorem 2, a BEBS policy optimal above a certain threshold utilization, and all other results of this study go through as well accordingly.

The implication for control parameters of the optimal BEBS policy from transforming the system by adding such stages (when, for example, $L$ such stages are added after stage $j$) is that for each such stage $i$, $i = j + 1, \ldots, j + L - 1$, the base stock level $s_{Ni}$, the largest base stock level at stage $i$, is smaller than both $s_{N,i-1}$, the largest of the base stock levels at stage $i - 1$, and $s_N - (N - i)K$, where $s_N$ is the base stock level at stage $N$. Stage $i - 1$ will then aim to bring its echelon inventory up to $s_{N,i-1}$, thus ensuring that all inventory from stage $i$ moves downstream to stage $i - 1$.

Our results apply to capacitated systems with the series structure. While the series structure prevails in the multiechelon literature, most supply chains encountered in practice tend to have more complex topologies. Without solving the capacitated series problem, Angelus and Zhu (2013) show that an assembly system with capacity constraints at each node can be reduced to an equivalent series system, when the bottleneck capacity is at the most downstream, final-assembly stage. Consequently, our results are directly applicable to assembly systems. Capacitated distribution systems, however, remain an open research area. Because very little is known about such systems beyond the lower and upper bounds and heuristic policies established in Atali and Özer (2012), extending our results in that direction represents a worthwhile generalization of our work.

Our model also assumes the smallest capacity in the system is at the most downstream stage. While this assumption is common in the literature (e.g., Janakiraman and Muckstadt 2009, Parker and Kapuscinski 2004), it is nevertheless restrictive. To the best of our knowledge, there is no information in the literature about the structure of the optimal policy for capacitated systems in which the bottleneck capacity is located at an upstream stage. Parker and Kapuscinski (2004) present a numerical example of one such system, and illustrate how the form of the optimal policy for even a two-stage system, when the bottleneck capacity is upstream, is very difficult to ascertain. Extending our results in the direction of multi-stage systems with general capacity configurations would represents an important, though potentially challenging, next step in this research area.

Finally, as already described, our results hold for capacitated systems whose utilization is above a certain threshold level, which restricts the range of optimality of BEBS policies to system utilizations above that threshold utilization and represents a limitation of our model. At the same time, in our numerical studies, this range of optimality lies in the domain of utilizations typically encountered in practice. Nevertheless, determining the structure of the optimal policy for capacitated system below

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our threshold utilization remains an important open research question.

### 6.2. Concluding Remarks

In this study, we identify a class of capacitated multi-stage systems for which we establish the structure of the optimal policy. This optimal policy, a branching echelon base stock policy, is a special case of the order-up-to policy, in which the order-up-to level at each stage is a piecewise-linear function of upstream echelon inventories. The class of capacitated systems for which a BEBS policy is found to be optimal is characterized by having the bottleneck capacity at the most downstream stage, and system utilization above a particular threshold utilization. To the best of our knowledge, system utilization is therefore, for the first time, identified as a parameter with key impact on the structure of the optimal inventory policy for capacitated multi-stage problems. Because the optimal BEBS policy achieves the decomposition of the multi-variable objective function into single-variable component functions, and thus removes the curse of dimensionality for the problem, we are also able to develop a fast algorithm to determine tight upper bounds on that threshold utilization.

In the process of arriving at the main theorems of our study, we developed new results in convex optimization pertaining to the preservation of additive convexity over feasible regions whose boundaries are determined by sets of ordered variables. Those results are applicable to other problems in multiechelon inventory theory. For example, in solving a multiechelon inventory problem with short-term take-or-pay contracts, Goh and Porteus (2016) rely on their lemma 1, derived independently of us, which represents a special case of our Lemma 14. We believe there are other open multiechelon problems whose analysis can be facilitated by our new results on convex optimization. These results are also potentially applicable to other multi-stage optimization problems whose research context may be different from ours, but whose general structure bears similarity, such as multi-period, multi-location models with separable cost functions.

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## Supporting Information

Additional supporting information may be found online in the supporting information tab for this article:

**Appendix S1:** Proofs.
**Appendix S2:** Intermediate Results.
**Appendix S3:** Threshold Utilization Quantified.

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