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Online Combinatorial Optimization for Interconnected Refrigeration Systems: Linear Approximation and Submodularity ⋆

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Abstract: This paper proposes a new control method that can save the energy consumption of multi-case supermarket refrigerators by explicitly taking into account their interconnected and switched system dynamics. Its novelty is a bilevel combinatorial optimization formulation to generate ON/OFF control actions for expansion valves and compressors. The inner optimization module keeps display case temperatures in a desirable range and the outer optimization module minimizes energy consumption. In addition to its energy-saving capability, the proposed controller significantly reduces the frequency of compressor switchings by employing a conservative compressor control strategy. However, solving this bilevel optimization problem associated with interconnected and switched systems is a computationally challenging task. To solve the problem in near real time, we propose two approximation algorithms that can solve both the inner and outer optimization problems at once. The first algorithm uses a linear approximation, and the second is based on the submodular structure of the optimization problem. Both are (polynomial-time) scalable algorithms and generate near-optimal solutions with performance guarantees. Our work complements existing optimization-based control methods (e.g., MPC) for commercial refrigerators, as our algorithms can be adopted as a tool for solving combinatorial optimization problems arising in these methods.

1. INTRODUCTION

Commercial refrigeration systems account for 7% of the total commercial energy consumption in the United States (U.S. Department of Energy [2012]). Therefore, there is a strong need for energy-efficient refrigeration systems, but research and development have focused on improving hardware rather than software, including control systems. Traditionally, hysteresis and set point-based controllers have been used to maintain the display case temperature in a desirable range without considering system dynamics and energy consumption. Over the past decade, however, more advanced control systems have been developed to save energy consumption using real-time sensor measurements and optimization algorithms (see Section 1.1). Advances in new technologies, such as the Internet of Things and cyber-physical systems, enhance the practicality of such an advanced control system with their sensing, communication, and computing capabilities (Graziano and Pritoni [2014]).

Supermarkets are one of the most important commercial sectors in which energy-efficient refrigeration systems are needed. The primary reasons are twofold. First, supermarket refrigerators consume 56% of energy consumed by commercial refrigeration systems (Navigant Consulting, Inc. [2009]). Second, supermarkets operate with very thin profit margins (on the order of 1%), and energy savings thus significantly help their business: the U.S. Environmental Protection Agency estimates that reducing energy costs by $1 is equivalent to increasing sales by $59 (ES2 [2008]). However, improving the energy efficiency of supermarket refrigerators is a challenging task because food products must be stored at proper temperatures. Failure to do so will increase food safety risks. The most popular refrigerators in supermarkets are multi-display case units. An example is illustrated in Fig. 1. Each display case has an evaporator controlled by an expansion valve, and a unit's suction pressure is controlled by a compressor rack, as shown in Fig. 2. In typical supermarket refrigerators, controllers turn ON and OFF expansion valves and compressors to keep display case temperatures in a specific range. Importantly, there are heat transfers between display cases due to the interconnection among them. Note that traditional hysteresis or set point-based controllers do not take into account such heat transfers and therefore perform in a suboptimal way.

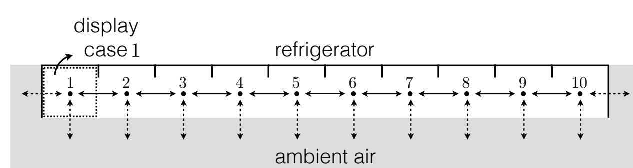

Fig. 1. A supermarket refrigerator, which has 10 display cases. Evaporator i controls the temperature of display case i. Lines with arrows represent heat transfers between neighboring display cases or between a display case and ambient air.

This paper proposes a new control method that can improve the energy efficiency of multi-case supermarket

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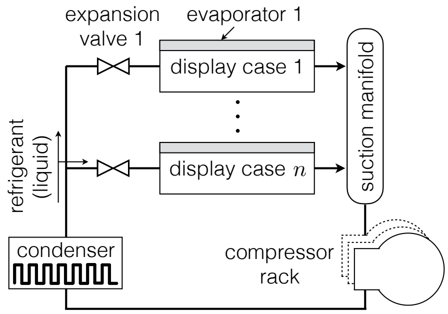

Fig. 2. Schematic diagram of a supermarket refrigerator.

refrigerators by explicitly taking into account the interconnected and switched dynamics of display case temperatures. The proposed controller receives sensor measurements and optimizes ON/OFF control actions for expansion valves and compressors in near real time. The novelty of this work is a bilevel combinatorial optimization formulation to generate such ON/OFF control signals in which (*i*) the inner combinatorial optimization module is responsible for maintaining display case temperatures in a desirable range, and (*ii*) the outer combinatorial optimization module minimizes energy consumption. The primary advantage of the proposed approach is its energy savings. Because the controller explicitly takes into account the system dynamics and heat transfers, it effectively uses state measurements and optimizes control actions to save energy while guaranteeing desired temperature profiles. In our case studies, the proposed control method saves 7.5–8% of energy compared to a traditional approach. The secondary benefit of the proposed method is to reduce the frequency of compressor switchings. It is known that frequent switchings of compressors accelerate their mechanical wear. We propose a conservative compressor control approach that reduces fluctuations in suction pressure and thus decreases the compressor switching frequency. In our case studies using a benchmark refrigeration system model, the proposed method reduces the switching frequency by 54–71.6%.

The proposed control method, however, presents a theoretical and algorithmic challenge because a bilevel combinatorial optimization associated with a dynamical system must be solved in near real time. To overcome this challenge, we suggest two approximation algorithms that can solve both of the inner and outer optimization problems at once. The first algorithm uses the linear approximation method developed in our previous work (Yang et al. [2016]). The approximate problem is a linear binary program, which can be solved by an efficient and scalable single-pass algorithm. In addition, it simulates the dynamical system model only once to generate control actions at each time point. We also show that the approximate solution obtained by this method has a provable suboptimality bound. The second algorithm is based on the *submodular* structure in the optimization problem. The inner optimization's objective function is submodular because opening an expansion valve when a smaller set of valves are opened gives a greater marginal benefit than opening it when a larger set of valves are already opened. We prove this intuitive submodularity property. Therefore, a greedy algorithm can be adopted to obtain a $(1-\frac{1}{e})$-optimal solution (Nemhauser et al. [1978]). In our case studies, the actual performance of the proposed controller using these two algorithms is 98.9–99.5% of the optimal controller.

1.1 Related Work

Several optimization-based control methods for commercial refrigerators have been developed over the past decade. One of the most popular methods is model predictive control (MPC) although it is computationally challenging to apply standard MPC due to the switched dynamics of refrigeration systems. It is shown that the mixed logical dynamical framework is useful to solve small-size problems with a piecewise affine approximation of a system model (Bemporad and Morari [1999], Larsen et al. [2005]). However, the practicality of this method is questionable due to the high dimensionality of practical problems for supermarket refrigerators, except for limited cases. To overcome this limitation, Sarabia et al. [2009] carefully selects and parametrizes optimization variables to formulate the problem as nonlinear MPC instead of hybrid MPC. Nonetheless this approach is computationally expensive because a nonlinear program with many variables must be solved in each MPC iteration. An alternative approach using hierarchical MPC is proposed in Sonntag et al. [2008]. This method separates time scales into two: in every nonlinear MPC iteration, low-level temperature controllers were employed, and the high-level optimization task is to determine optimal parameters for these controllers. However, this approach still presents the combinatorial growth of the search space. More recently, a sequential convex programming-based method is shown to be computationally efficient in several case studies (Hovgaard et al. [2013]). It iteratively solves an optimization problem using convex programming, replacing the nonconvex cost function with a convex approximation. In several numerical experiments, this heuristic method generates high-quality control signals although it gives no theoretical performance guarantee. We believe that our work is complementary to the aforementioned methods. One of our main contributions is to develop two efficient and scalable algorithms for resolving the computational challenge in discrete optimization problems associated with supermarket refrigeration systems. These algorithms can be adopted as a tool for solving combinatorial optimization problems in the aforementioned methods. We propose one of the most efficient control architectures that use the algorithms.

2. SWITCHED DYNAMICS OF SUPERMARKET REFRIGERATION SYSTEMS

We consider a supermarket refrigerator in which multiple display cases are interconnected with one another. For example, Fig. 1 shows a refrigerator that has 10 display cases. The temperature of each display case is controlled by an evaporator unit, where the refrigerant evaporates absorbing heat from the display case. Let evaporator i be in charge of display case i for $i = 1, \cdots, n$, where n is the number of display cases in all the refrigerators. Several dynamic models of supermarket refrigeration systems have been proposed (Rasmussen and Alleyne [2006], Larsen et al. [2007], Li and Alleyne [2010], Rasmussen [2012], Shafiei et al. [2013] (see also the references therein)). Among those, we use the benchmark model of a typical supermarket refrigeration system proposed in Larsen et al. [2007] and widely used in Sarabia et al. [2009], Sonntag et al. [2008], Yang et al. [2011], Vinther et al. [2015]. This model is useful for simulating display case temperatures and evaluating the performances of several controllers.

2.1 Display Cases and Evaporators

Display cases store food products and keep them refrigerated. This refrigeration is due to the heat transfer between the food product and the cold air in the display cases. Let $T_{\text{food},i}$ and $T_{\text{air},i}$ denote the temperatures of the food product and the air in display case i. The heat transfer $Q_{\text{food}\to\text{air},i}$ between the food product and the air in display case i can then be modeled as

$$\begin{aligned} m_{\text{food},i}c_{\text{food},i}\dot{T}_{\text{food},i} &= -Q_{\text{food}\to\text{air},i} \\ &= -k_{\text{food}-\text{air}}(T_{\text{food},i} - T_{\text{air},i}), \end{aligned} \tag{1}$$

where $m_{\text{food},i}$ is the mass of the food product, $c_{\text{food},i}$ is the heat capacity of the food product and $k_{\text{food}-\text{air}}$ is the heat transfer coefficient between the food product and the air.

The display case air temperature is affected by the heat transfers from the food product ($Q_{\text{food}\to\text{air},i}$), the ambient air ($Q_{\text{amb}\to\text{air},i}$), the evaporator ($-Q_{\text{air}\to\text{evap},i}$) and the neighboring display case air ($\sum_{j=1}^n Q_{j\to i}$). The refrigerant flow into an evaporator is controlled by its expansion valve. Let u_i be the valve control variable for evaporator i such that

$$u_i(t) := \begin{cases} 0 & \text{if expansion valve } i \text{ is closed at } t \\ 1 & \text{otherwise.} \end{cases}$$

Expansion valve i controls the refrigerant injection into evaporator i and decreases the pressure of the refrigerant if it is open, as shown in Fig. 2. Then, the dynamics of the display case air temperature can be modeled as the following switched interconnected system:

$$\begin{aligned} &m_{\text{air},i}c_{\text{air},i}\dot{T}_{\text{air},i} \\ &= Q_{\text{food}\to\text{air},i} + Q_{\text{amb}\to\text{air},i} - Q_{\text{air}\to\text{wall},i} + \sum_{j=1}^n Q_{j\to i} \\ &= k_{\text{food}-\text{air}}(T_{\text{food},i} - T_{\text{air},i}) + k_{\text{amb}-\text{air}}(T_{\text{amb}} - T_{\text{air},i}) \\ &\quad - k_{\text{air}-\text{evap}}(T_{\text{air},i} - T_{\text{evap}}u_i) + \sum_{j=1}^n k_{i,j}(T_{\text{air},j} - T_{\text{air},i}), \end{aligned} \tag{2}$$

where T_{amb} is the ambient air temperature, T_{evap} is the refrigerant's evaporation temperature, $k_{\text{amb}-\text{air}}$ is the heat transfer coefficient between the ambient air and the display case air and $k_{i,j}$ is the heat transfer coefficient between display case i's air and display case j's air. Note that $k_{i,j} = 0$ if display cases i and j are not neighbors. For a more detailed model, one can separately consider the dynamics of the evaporator wall temperature Sarabia et al. [2009]. [1] However, the proposed model is a good approximation because the heat transfer coefficient between the evaporator wall and the refrigerant is five to ten times higher than other heat transfer coefficients Larsen et al. [2007].

The mass flow out of the evaporator can be computed as

$$f_i := \frac{1}{\Delta t} m_{\text{ref},i},$$

where the refrigerant mass in the evaporator is controlled by the electronic expansion valve switching

$$m_{\text{ref},i} = \begin{cases} m_{\text{ref}}^{\max} & \text{if } u_i = 1 \\ 0 & \text{if } u_i = 0. \end{cases}$$

Depending on the specification of refrigerators, it takes a nontrivial amount of time to fill up the evaporator by refrigerant. In this case, the dynamics of the refrigerant mass in the evaporator can be explicitly taken into account Sarabia et al. [2009]. Alternatively, one can introduce a delay-time constant, τ, and let $m_{\text{ref},i}(t) = m_{\text{ref}}^{\max} u_i(t-\tau)$ to model the effect of the time to fill up the evaporator.

2.2 Suction Manifold and Compressor Rack

As shown in Fig. 2, the evaporated refrigerant with low pressure from the outlet of the evaporator is compressed by the electric motors in the compressor bank. Each refrigerator could have multiple compressors and each compressor is switched ON or OFF. For example, all the compressors are turned ON when maximal compression is needed. The compressor bank is conventionally controlled by a PI controller to maintain the suction pressure within a bandwidth.

The suction manifold pressure P_{suc} evolves with the following dynamics:

$$\dot{P}_{\text{suc}} = \frac{1}{V_{\text{suc}} r_{\text{suc}}} \left(\sum_{i=1}^n f_i - \rho_{\text{suc}} \sum_{i=1}^{n_c} F_{c,i} \right), \tag{3}$$

where V_{suc} is the volume of the suction manifold, ρ_{suc} is the density of the refrigerant in the suction manifold, and $r_{\text{suc}} := d\rho_{\text{suc}}/dP_{\text{suc}}$. The variable $F_{c,i}$ denotes the volume flow out of the suction manifold controlled by compressor i. Let $u_{c,i}$ be the control variable for compressor i, where $u_{c,i} = 0$ represents that compressor i is OFF and $u_{c,i} = 1$ represents that compressor i is ON. The volume flow $F_{c,i}$ is then given by

$$F_{c,i} = k_c u_{c,i} := \frac{\eta V_{\text{comp}}}{n} u_{c,i},$$

where η is the volumetric efficiency of each compressor, and V_{comp} denotes the compressor volume.

The total power consumption by the compressor rack is given by

$$p = \rho_{\text{suc}}(h_{oc} - h_{ic}) \sum_{i=1}^{n_c} F_{c,i},$$

where h_{ic} and h_{oc} are the enthalpies of the refrigerant flowing into and out of the compressor, respectively. The compressed refrigerant flows to the condenser and is liquefied by generating heat, as shown in Fig. 2. The liquefied refrigerant flows to the expansion valve, and as a result, the refrigeration circuit is closed.

2.3 Traditional Set-Point/PI-Based Control

A widely used control method consists of (i) a set-point based control of expansion valves, and (ii) a PI control of compressors Larsen et al. [2007], Sarabia et al. [2009]. The specific simulation setting used in this paper is contained in the extended version of this paper (Yang [2016]). We perturbed the mass of food products in each display case by $\pm 20\%$ from the nominal value $\bar{m}_{\text{food},i}$. Despite

[1] Alternatively, one can introduce a delay parameter, τ, and replace $T_{\text{evap}}u_i(t)$ with $T_{\text{evap}}u_i(t-\tau)$ to explicitly take into account the effect of the evaporator wall temperature.

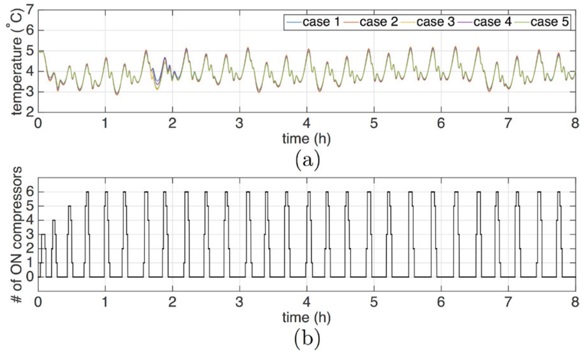

Fig. 3. (a) The food temperatures in display cases 1–5, operated by the PI controller over 8 hours. The temperatures in all display cases are almost identical. (b) The number of ON compressors operated by the PI controller.

this heterogeneity, the set point-based controller almost identically turns ON and OFF all the expansion valves and therefore all the display case temperatures have almost the same trajectory as shown in Fig. 3 (a). This synchronization is due to the decentralized nature of the set point-based controller: the control decision for expansion valve i depends only on its local temperature. Intuitively, this decentralized controller is suboptimal because it does not actively take into account the heat transfer between neighboring display cases. This inefficiency of the traditional control approach motivates us to develop a new optimization-based control method that explicitly considers the interdependency of display case temperature dynamics.

Another disadvantage resulting from the synchronization of expansion valves is the significant fluctuation of suction pressure. Since the PI controller integrates the deviation of suction pressure from its reference, the output $u_{PI}(t)$ presents large and frequent variations. As a result, the number of ON compressors frequently varies as shown in Fig. 3 (b). A frequent switching of compressors is a serious problem because it accelerates the mechanical degradation of the compressors. Our strategy to discourage frequent compressor switchings is twofold: (i) our conservative compressor control method tries to maintain $P_{\mathrm{suc}}(t) = \bar{P}_{\mathrm{suc}}$, not fully utilizing the pressure bandwidth $\pm DB$, and (ii) our online optimization-based controller indirectly desynchronizes the ON/OFF operation of expansion valves. The details about the two control methods are presented in the following sections.

3. ONLINE COMBINATORIAL OPTIMIZATION

3.1 Conservative Compressor Control

We control the compressor rack to (approximately) maintain the suction pressure as the reference $\bar{P}_{\mathrm{suc}}$, i.e.,

$$P_{\mathrm{suc}}(t) \approx \bar{P}_{\mathrm{suc}} \quad \forall t.$$

In other words, to make $\dot{P}_{\mathrm{suc}} \equiv 0$ in (3), we set $u_c := (u_{c,1}, \cdots, u_{c,n_c})$ such that the refrigerant outflow from the suction manifold is equal to the inflow:

$$\rho_{\mathrm{suc}} \sum_{i=1}^{n_c} k_c u_{c,i} \approx \sum_{i=1}^{n} f_i. \tag{4}$$

In practice, we may not be able to exactly satisfy this equality because each $u_{c,i}$ is either 0 or 1. However, we assume that the compressor control action u_c can be chosen to make the difference between the outflow and the inflow negligible. This compressor control rule is suboptimal: it induces a conservative operation of the compressor rack that does not fully utilize the pressure bandwidth. However, this conservative control approach has a practical advantage: it does not create significant compressor switchings. Therefore, it can potentially decelerate the mechanical wear of compressors. Under this compressor control rule, the total power consumption can be computed as

$$p = (h_{oc} - h_{ic}) \sum_{i=1}^{n} f_i = \frac{(h_{oc} - h_{ic}) m_{\mathrm{ref}}^{\max}}{\Delta t} \sum_{i=1}^{n} u_i. \tag{5}$$

3.2 Bilevel Optimization Formulation

We consider a receding-horizon online optimization approach to generate control signals for expansion valves and compressors. Let $\{t_0, t_1, \cdots, t_k, t_{k+1}, \cdots\}$ be the time steps at which the control action is optimized. For the sake of simplicity, we describe a one-step look-ahead optimization method; however, this approach can be easily extended to multiple-step look-ahead optimization (see Remark 2).

Inner problem for temperature management At time t_k, we control the expansion valves to minimize the following quadratic deviation from the upper-bound $T_i^{\max}$, $i = 1, \cdots, n$:

$$J(\alpha) = \sum_{i=1}^{n} \int_{t_k}^{t_{k+1}} (T_{\mathrm{air},i} - T_i^{\max})_+^2 dt,$$

where $(a)_+^2 = a^2 \cdot \mathbf{1}_{\{a \geq 0\}}$, assuming T_{air} is evolving with (1) and (2). Specifically, the expansion valve action at t_k is generated as a solution to the following combinatorial optimization problem:

$$\min_{\alpha \in \{0,1\}^n} \quad J(\alpha) \tag{6a}$$

$$\text{s.t.} \quad \dot{x} = Ax + Bu + C, \quad x(t_k) = \mathbf{x}_{\mathrm{meas}} \tag{6b}$$

$$u(t) = \alpha, \quad t \in (t_k, t_{k+1}] \tag{6c}$$

$$\|\alpha\|_0 = \sum_{i=1}^{n} \alpha_i \leq K. \tag{6d}$$

Here, $x := (T_{\mathrm{food}}, T_{\mathrm{air}})$ and (6b) gives a linear system representation of the dynamics (1) and (2). Note that $\mathbf{x}_{\mathrm{meas}}$ represents $(T_{\mathrm{food}}, T_{\mathrm{air}})$ measured at $t = t_k$.[2] As specified in (6c), the control action over $(t_k, t_{k+1}]$ is fixed as the solution α. The last constraint (6d) essentially limits the power consumed by the refrigeration system as $K(h_{oc} - h_{ic}) m_{\mathrm{ref}}^{\max}/\Delta t$ due to (5). Therefore, the choice of K is important to save energy: as K decreases, the power consumption lessens.

Outer problem for energy efficiency To generate an energy-saving control action, we minimize the number K of open expansion valves while guaranteeing that the quadratic deviation $J(\alpha)$ from the upper-bound $T_i^{\max}$, $i =$

[2] If T_{food} is not directly measured, an observer needs to be employed to estimate the state. Then, the control system uses the estimate T_{food}^{est} instead of its actual measurement.

$1, \cdots, n$ is bounded by the threshold Δ. More precisely, we consider the following outer optimization problem:

$$\min\{K \in \{0, \cdots, n\} \mid J(\alpha^{opt}(K)) \leq \Delta\}, \tag{7}$$

where $\alpha^{opt}(K)$ is a solution to the expansion valve optimization problem (6). Let K^{opt} be a solution to this problem. Then, $\alpha^{opt}(K^{opt})$ is the expansion valve control action that saves energy the most while limiting the violation of the food temperature upper-bound $T_i^{\max}$, $i = 1, \cdots, n$. This outer optimization problem can be easily solved by searching K from 0 in an increasing order. Once we find $\hat{K}$ such that $J(\alpha^{opt}(\hat{K})) \leq \Delta$, we terminate the search and obtain the solution as $K^{opt} := \hat{K}$. In the following section, we will show that this procedure can be integrated into approximation algorithms for the inner optimization problem.

Then, as specified in (6c), the controller chooses $u^{opt}(t) := \alpha^{opt}(K^{opt})$ for $t \in (t_k, t_{k+1}]$. Furthermore, it determines the compressor control signal u_c^{opt} such that $\sum_{i=1}^{n_c} u_{c,i}^{opt} \approx m_{\text{ref}}^{\max}/(\rho_{\text{suc}} k_c \Delta t)$ using (4). If $P_{\text{suc}}(t) < \bar{P}_{\text{suc}}$, the controller rounds $m_{\text{ref}}^{\max}/(\rho_{\text{suc}} k_c \Delta t)$ to the next smaller integer and then determines the number of ON compressors as the integer. If $P_{\text{suc}}(t) \geq \bar{P}_{\text{suc}}$, the controller rounds $m_{\text{ref}}^{\max}/(\rho_{\text{suc}} k_c \Delta t)$ to the nearest integer greater than or equal to it.

Remark 1. Our objective function $J(\alpha)$ only takes into account the violation of temperature upper-bounds. This choice is motivated by the fact that the food temperature in each display case increases as we close more expansion valves, which is summarized in Proposition 1. In other words, as we reduce the number K of open valves in the outer optimization problem, the possibility of violating temperature upper-bounds increases, while it is less likely to violate temperature lower-bounds. This monotonicity property of food temperatures justifies our focus on temperature upper-bounds.

Proposition 1. Let $T_{\text{food},j}^{\alpha}$ and $T_{\text{air},j}^{\alpha}$ denote the food and air temperatures in display case j when the control action α is applied. Then, for any $\alpha, \beta \in \mathbb{R}^n$ such that

$$\alpha_i \leq \beta_i, \quad i = 1, \cdots, n,$$

we have

$$T_{\text{food},j}^{\alpha} \geq T_{\text{food},j}^{\beta} \text{ and } T_{\text{air},j}^{\alpha} \geq T_{\text{air},j}^{\beta}, \quad j = 1, \cdots, n.$$

Its proof can be found in Appendix A.

4. APPROXIMATION ALGORITHMS

We present two approximation methods for the inner optimization problem. One is based on linear approximation, and another utilizes submodularity. These will give approximate solutions with guaranteed suboptimality bounds. We further show that, by simply modifying these approximation algorithms, we can obtain a near-optimal solution to the outer optimization problem.

4.1 Linear Approximation

We first consider a linear approximation-based approach to the inner combinatorial optimization problem (6). It is convenient to work with the following value function:

$$V(\alpha) = J(0) - J(\alpha). \tag{8}$$

Initialization:
$\alpha \leftarrow 0$;
Construction of d:
Compute $DV(0)$;
Sort the entries of $DV(0)$ in descending order;
Construct $\mathbf{d} : \{1, \cdots, n\} \to \{1, \cdots, n\}$ satisfying (10);
Solution of (9)**:**
while $[DV(0)]_{\mathbf{d}(i)} > 0$ and $i \leq K$
$\alpha_{\mathbf{d}(i)} \leftarrow 1$;
$i \leftarrow i + 1$;
end

Algorithm 1. Algorithm for the approximate problem (9)

The value $V(\alpha)$ represents the reduction in the quadratic deviation from from the upper-bound $T_i^{\max}$, $i = 1, \cdots, n$, when expansion valve j is chosen to be open only for j such that $\alpha_j = 1$. Note that this value function is normalized such that $V(0) = 0$. The Taylor expansion of V at 0 gives $V(\alpha) = DV(0)^\top \alpha + O(\alpha^2)$ assuming the derivative DV is well-defined. This motivates us to consider the following first-order approximation of the expansion valve optimization problem (6):

$$\max_{\alpha \in \{0,1\}^n} \{DV(0)^\top \alpha : \|\alpha\|_0 \leq K\}. \tag{9}$$

The ith entry $[DV(0)]_i$ of the derivative represents the marginal benefit of opening expansion valve i. Therefore, the approximate problem (9) can be interpreted as maximizing the marginal benefit of valve operation while guaranteeing that the number of open valves is less than or equal to K. A detailed method to define and compute the derivative can be found in Yang et al. [2016]. Computing the derivative should also take into account the dependency of the state x on the binary decision variable α. For example, an adjoint-based approach can be used to handle this dependency (Kokotović and Heller [1967]).

The first advantage of the proposed approximation approach is that it gives an approximate solution with a provable suboptimality bound. The bound is *a posteriori*, which does not require the globally optimal solution α^{opt} but the solution $\alpha^\star$ of (9).

Theorem 1. (Yang et al. [2016]). Let $\alpha^\star$ be a solution to the approximate problem (9). If $DV(0)^\top \alpha^\star \neq 0$, then the following suboptimality bound holds:

$$\rho V(\alpha^{opt}) \leq V(\alpha^\star),$$

where

$$\rho = \frac{V(\alpha^\star)}{DV(0)^\top \alpha^\star} \leq 1.$$

If $DV(0)^\top \alpha^\star = 0$, then $V(\alpha^{opt}) = V(0) = 0$, i.e., 0 is an optimal solution.

Its proof is contained in Appendix B. This theorem suggests that the approximate solution's performance is greater than $(\rho \times 100)\%$ of the globally optimal solution's performance.

The second advantage of the proposed method is that it yields an efficient algorithm to solve the approximate problem (9). Specifically, we design a very simple algorithm based on the ordering of the entries of $DV(0)$. Let $\mathbf{d}(\cdot)$ denote the map from $\{1, \cdots, n\}$ to $\{1, \cdots, n\}$ such that

$$[DV(0)]_{\mathbf{d}(i)} \geq [DV(0)]_{\mathbf{d}(j)} \tag{10}$$

for any $i, j \in \{1, \cdots, n\}$ such that $i \leq j$. Such a map can be constructed using a sorting algorithm with $O(n \log n)$ complexity. Such a map may not be unique. We let $\alpha_{\mathbf{d}(i)} = 1$ for $i = 1, \cdots, K$ if $[DV(0)]_{\mathbf{d}(i)} > 0$. A more detailed algorithm to solve this problem is presented in Algorithm 1. Note that it is a single-pass algorithm, i.e., does not require multiple iterations. For large problems, one may be able to use an $O(n)$ algorithm without sorting (Balas and Zemel [1980]).

Remark 2. The proposed linear approximation method is applicable to multi-period optimization problems, in which the objective is given by $J(\alpha) := \sum_{k=1}^{N_{period}} J_k(\alpha^k)$ and the control variable is time-varying, i.e., $\alpha = (\alpha^1, \cdots, \alpha^{N_{period}}) \in \mathbb{R}^{n \times N_{period}}$. In such a case, we compute the derivative DV_k of $V_k(\alpha^k) := J_k(0) - J_k(\alpha^k)$ for each k. The objective function can be approximated as $\sum_{k=1}^{N_{period}} DV_k(0)^\top \alpha^k$, which is still linear in α.

4.2 Submodularity

The second approach gives another approximate solution of the expansion valve optimization problem (6) with a suboptimality bound. This solution is generally different from the solution obtained by the first approach. Let $\Omega := \{1, \cdots, n\}$ be the set of expansion valves to be controlled. We define a set function, $\mathcal{V} : 2^\Omega \to \mathbb{R}$, as

$$\mathcal{V}(X) = V(\mathbb{I}(X)),$$

where the value function V is defined as (11) and $\mathbb{I}(X) := (\mathbb{I}_1(X), \cdots, \mathbb{I}_n(X)) \in \{0,1\}^n$ is the indicator vector of the set X such that $\mathbb{I}_i(X) := 0$ if $i \notin X$ and $\mathbb{I}_i(X) := 1$ if $i \in X$. In other words, $\mathcal{V}$ is a set function representation of V. The expansion valve optimization problem (6) is equivalent to selecting the set $X \subseteq \Omega$ such that $|X| \leq K$ to maximize the value function $\mathcal{V}(X)$, i.e.,

$$\max_{X \in 2^\Omega} \{\mathcal{V}(X) : |X| \leq K\}. \quad (11)$$

We observe that the value function $\mathcal{V}$ has a useful structure, which is called the *submodularity*. It represents a *diminishing return* property such that opening an expansion valve when a smaller set of valves is opened gives a greater marginal benefit than opening it when a larger set of valves is already opened.

Theorem 2. The set function $\mathcal{V} : 2^\Omega \to \mathbb{R}$ is submodular, i.e., for any $X \subseteq Y \subseteq \Omega$ and any $\mathbf{a} \in \Omega \setminus Y$,

$$\mathcal{V}(X \cup \{\mathbf{a}\}) - \mathcal{V}(X) \geq \mathcal{V}(Y \cup \{\mathbf{a}\}) - \mathcal{V}(Y).$$

Furthermore, it is monotone, i.e., for any $X \subseteq Y \subseteq \Omega$

$$\mathcal{V}(X) \leq \mathcal{V}(Y).$$

See Appendix C for a proof.

Initialization:
$X \leftarrow \emptyset$;
Greedy algorithm:
while $i \leq K$
$\mathbf{a}^* \in \arg\max_{\mathbf{a} \in \Omega \setminus X} \mathcal{V}(X \cup \{\mathbf{a}\})$;
$X \leftarrow X \cup \{\mathbf{a}^*\}$;
$i \leftarrow i + 1$;
end

Algorithm 2. Greedy algorithm for (11)

The submodularity of $\mathcal{V}$ guarantees that Algorithm 2, which is a greedy algorithm, provides an $(1 - \frac{1}{e})$-optimal solution. In other words, the approximate solution's performance is greater than $(1 - \frac{1}{e}) \approx 63\%$ of the oracle's performance. In our case study, the actual submodularity is 98.9%, which is significantly greater than this theoretical bound.

Theorem 3. (Nemhauser et al. [1978]). Algorithm 2 is a $\left(1 - \frac{1}{e}\right)$-approximation algorithm. In other words, if we let $X^\star$ be the solution obtained by this greedy algorithm, then the following suboptimality bound holds:

$$\left(1 - \frac{1}{e}\right) \mathcal{V}(X^{opt}) \leq \mathcal{V}(X^\star),$$

where X^{opt} is an optimal solution to (11).

Algorithm 2 makes a locally optimal choice at each iteration. Therefore, it significantly reduces the search space, i.e., it does not search over all possible combinations of open expansion valves.

while $[DV(0)]_{\mathbf{d}(i)} > 0$ and $J(\alpha) > \Delta$
$\alpha_{\mathbf{d}(i)} \leftarrow 1$;
$i \leftarrow i + 1$;
end

Algorithm 3. Modified version of Algorithm 1 for the outer optimization problem (7)

4.3 Modified Algorithms for the Outer Problem

We now modify the two approximation algorithms for the inner problem (6) to solve the full bilevel optimization problem. In both Algorithms 1 and 2, the expansion valve chosen to be open at iteration i is *independent of the selections at later iterations*. This independency plays an essential role in incorporating the outer optimization problem into the algorithms. To be more precise, we compare the cases of $K = l$ and $K = l+1$. Let α^l and α^{l+1} be the solutions in the two cases obtained by Algorithm 1. Since the expansion valve selected to be open at iteration $l+1$ does not affect the choices at earlier iterations, we have $\alpha^l_{\mathbf{d}(i)} = \alpha^{l+1}_{\mathbf{d}(i)}$ for $i = 1, \cdots, l$. Therefore, we do not have to re-solve the entire inner optimization problem for $K = l+1$ if we already have the solution for $K = l$; it suffices to run one more iteration for $i = l + 1$ to obtain $\alpha^{l+1}_{\mathbf{d}(l+1)}$. This observation allows us to simply modify the last part of Algorithm 1 as Algorithm 3. We select expansion valves to be open until the temperature upper-bound violation $J(\alpha)$ is less than or equal to the threshold Δ. Similarly, we modify the last part of Algorithm 2 as Algorithm 4 to solve the outer problem.

while $J(\mathbb{I}(X)) > \Delta$
$\mathbf{a}^* \in \arg\max_{\mathbf{a} \in \Omega \setminus X} \mathcal{V}(X \cup \{\mathbf{a}\})$;
$X \leftarrow X \cup \{\mathbf{a}^*\}$;
$i \leftarrow i + 1$; **end**

Algorithm 4. Modified version of Algorithm 2 for the outer optimization problem (7)

5. CASE STUDIES

In this section, we examine the performance of the proposed online optimization-based controllers. For fair comparisons with the traditional controller, we use the parameter data reported in the extended version (Yang [2016]).

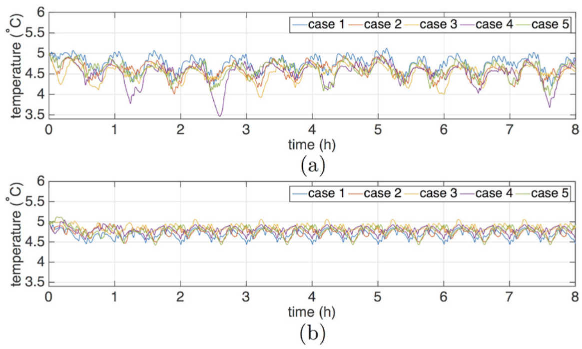

Fig. 4. The food temperatures (in 5 display cases out of 10) controlled by (a) the linear approximation-based algorithm (Algorithm 3), and (b) the submodular optimization algorithm (Algorithm 4).

5.1 *Energy Efficiency*

As opposed to the synchronized food temperature profiles controlled by the traditional method (see Fig. 3 (a)), the proposed controllers induce alternating patterns of the temperatures as shown in Fig. 4. Such patterns result from the explicit consideration of heat transfers between neighboring display cases in the optimization module through the constraint (6b), which represents the interconnected temperature dynamics. Using the spatial heat transfers, the proposed controllers do not turn ON or OFF all the expansion valves at the same time. Instead, they predict the temperature evolution for a short period and selects the valves to turn ON that are effective to minimize the deviation from the desirable temperature range during the period. As a result, the ON/OFF operation of expansion valves is desynchronized, unlike in the case of the traditional controller. This desynchronization maintains the temperatures near the upper-bound $T^{\max}$ reducing temperature fluctuations. Therefore, it intuitively improves energy efficiency. As summarized in Table 1, the proposed controllers save 7.5–8% of energy. Note that the outer optimization module minimizes the total energy consumption while the inner optimization module is responsible for maintaining the temperature profiles in a desirable range. When the bilevel combinatorial problem is exactly solved for all time, the average power consumption is 10.29kW. Therefore, the two proposed controllers' performances are 99.5% and 98.9% of the optimal controller although their theoretical suboptimality bounds are 39% and 63%.

Table 1: Energy savings by the proposed controllers

	PI	linear	submodular
average kW	11.24	10.34	10.40
energy saving	–	8.0%	7.5%
suboptimality	90.1%	99.5%	98.9%

5.2 *Reduced Compressor Switching*

Another advantage of the proposed controllers is the considerable reduction on the number of compressor switching instances. By desynchronizing the switching instances of expansion valves in the inner optimization module, the proposed controllers significantly reduce the variation of suction pressure. Our conservative compressor control approach presented in Section 4 also helps to minimize the deviation of the suction pressure from its reference. As a result, the controllers significantly reduce the fluctuations on the number of ON compressors as shown in Fig. 5. First, the maximum number of ON compressors is decreased from six to two. This reduction suggests that a mechanically more compact compressor or a smaller number of compressors in the rack may be enough if the proposed controllers are adopted. Second, the proposed controllers reduce the number of compressor switching instances by 54.0–71.6% as summarized in Table 2. These infrequent compressor operation strategies are beneficial to decelerate the mechanical degradation of compressors.

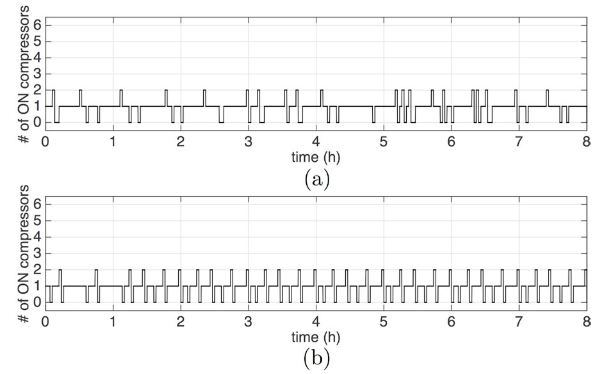

Fig. 5. The number of ON compressors controlled by (a) the linear approximation-based algorithm (Algorithm 3), and (b) the submodular optimization algorithm (Algorithm 4).

Table 2: Compressor switching reductions by the proposed controllers

	PI	linear	submodular
# of switchings	324	92	149
reduction	–	71.6%	54.0%

6. CONCLUSIONS

The proposed controller explicitly takes into account the switched and interconnected dynamics, and is therefore is suitable for multi-case supermarket refrigeration systems. However, it has to solve a bilevel combinatorial optimization problem in near real time, which is a challenging task. To overcome this difficulty, we proposed two polynomial-time approximation algorithms that are based on the structural properties of this optimization problem. We demonstrated the performance of the proposed controllers through case studies using a benchmark refrigeration system model and found that (i) they improve energy efficiency by 7.5–8% and (ii) they reduce the number of compressor switchings by 54–71.6%.

Appendix A. PROOF OF PROPOSITION 1

We use the linear system representation (6b) of the food and air temperature dynamics (Equations (1) and (2)). We first notice that $A_{i,j} \geq 0 \quad \forall i \neq j$, where $A_{i,j}$ represents the (i,j)th entry of the matrix A. Furthermore, $k_{\text{air}-\text{evap}} T_{\text{evap}} \leq 0$ due to the non-positive evaporator temperature. Hence, we have $B_{i,j} \leq 0 \; \forall i, j$. Using Proposition III.2 in Angeli and Sontag [2003], we conclude that the system (6b) is *input-monotone* such that for any $\alpha, \beta \in \mathbb{R}^n$ with $\alpha_i \leq \beta_i$, $i = 1, \cdots, n$,

$$x_i^\alpha \geq x_i^\beta, \quad i = 1, \cdots, n,$$

where x^α denotes the solution of the system (6b) when its input is chosen as α.

Appendix B. PROOF OF THEOREM 1

In (6b), we notice that

$$x(t) = e^{A(t-t_k)}\mathbf{x}_{\text{meas}} + \int_{t_k}^{t} e^{A(t-s)} B\alpha ds,$$

which implies that $x(t)$ is linear in α. Therefore, V is concave with respect to α in a continuously relaxed space, $\mathbb{R}^n$. Then, the result follows from Theorem 2 in Yang et al. [2016].

Appendix C. PROOF OF THEOREM 2

Let $T_{\text{food},i}^X$ denote the temperature of the food product in display case i given that the expansion valves in X are open. Due to the linearity of the system dynamics (1) and (2), $T_{\text{food},i}^X$ is modular, i.e.,

$$T_{\text{food},i}^X = \sum_{\mathbf{a} \in X} T_{\text{food},i}^{\{\mathbf{a}\}}.$$

Therefore, for any $X \subseteq Y \subseteq \Omega$ and any $\mathbf{a} \in \Omega \setminus Y$

$$T_{\text{food},i}^{X \cup \{\mathbf{a}\}} - T_{\text{food},i}^X = T_{\text{food},i}^{Y \cup \{\mathbf{a}\}} - T_{\text{food},i}^Y.$$

Furthermore, Proposition 1 yields the following monotonicity result: for any $X \subseteq Y \subseteq \Omega$

$$T_{\text{food},i}^X \geq T_{\text{food},i}^Y,$$

i.e., as we open more expansion valves, the food temperature decreases. Lastly, the concavity of $\mathcal{V}(X) = \mathcal{V}(\emptyset) - \sum_{i=1}^n \int_{t_k}^{t_{k+1}} (T_{\text{food},i}^X - T_i^{\max})_+^2 dt$ in $T_{\text{food},i}^X$ implies that $X \subseteq Y \subseteq \Omega$ and any $\mathbf{a} \in \Omega \setminus Y$

$$\mathcal{V}(X \cup \{\mathbf{a}\}) - \mathcal{V}(X) \geq \mathcal{V}(Y \cup \{\mathbf{a}\}) - \mathcal{V}(Y).$$

Therefore, $\mathcal{V}$ is submodular. Its monotonicity follows from Proposition 1.

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