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The Novikov conjecture for algebraic K-theory of the group algebra over the ring of Schatten class operators

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ABSTRACT

In this paper, we prove the algebraic K-theory Novikov conjecture for group algebras over the ring of Schatten class operators. The main technical tool in the proof is an explicit construction of the Connes–Chern character.

1. Introduction

Let Γ be a group and R be an H-unital ring. Let $R\Gamma$ be the group algebra of the group Γ over the ring R. The isomorphism conjecture of Farrell–Jones states that the following assembly map is an isomorphism:

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$$A : H_n^{Or\Gamma}(E_{\mathcal{VCY}}(\Gamma), \mathbb{K}(R)^{-\infty}) \longrightarrow K_n(R\Gamma),$$

where $\mathcal{VCY}$ is the family of virtually cyclic subgroups of Γ, $E_{\mathcal{VCY}}(\Gamma)$ is the universal Γ-space with isotropy in $\mathcal{VCY}$, $H_n^{Or\Gamma}(E_{\mathcal{VCY}}(\Gamma), \mathbb{K}(R)^{-\infty})$ is a generalized Γ-equivariant homology theory associated to the non-connective algebraic K-theory spectrum $\mathbb{K}(R)^{-\infty}$, and $K_n(R\Gamma)$ is the algebraic K-theory of $R\Gamma$.

The isomorphism conjecture provides an algorithm for computing the algebraic K-theory of $R\Gamma$ in terms of the algebraic K-theory of R. This conjecture was introduced in [18] for $R = \mathbb{Z}$ and for unital rings R in [1]. When R is H-unital, the isomorphism conjecture follows from the unital case by using the excision theorem in algebraic K-theory [32]. The algebraic K-theory isomorphism conjecture goes back to [21]. There are analogous conjectures in L-theory [26,27] and C^*-algebra K-theory [4]. Important cases of the isomorphism conjecture have been verified in [18,19,2].

The algebraic K-theoretic Novikov conjecture states that the assembly map:

$$H_n(B\Gamma, \mathbb{K}(R)^{-\infty}) \longrightarrow K_n(R\Gamma),$$

is rationally injective, where $B\Gamma$ is the classifying space of the group Γ. The algebraic K-theoretic Novikov conjecture follows from the (rational) injectivity part of the isomorphism conjecture. By a remarkable theorem of Bökstedt–Hsiang–Madsen [6], the algebraic K-theoretic Novikov conjecture holds for $R = \mathbb{Z}$ if the homology groups of Γ are finitely generated.

The main purpose of this paper is to prove the (rational) injectivity part of the algebraic K-theory isomorphism conjecture for group algebras over the ring of Schatten class operators. As a consequence, we obtain the algebraic K-theory Novikov conjecture for group algebras over the ring of Schatten class operators. The motivation for considering group algebras over the ring of Schatten class operators comes from the deep work of Connes–Moscovici on higher index theory of elliptic operators and its applications to the Novikov conjecture [11]. In Connes–Moscovici's higher index theory, the K-theory of the group algebra over the ring of Schatten class operators serves as the receptacle for the higher index of an elliptic operator.

For the convenience of readers we recall that, for any $p \geq 1$, an operator T on an infinite dimensional and separable Hilbert space H is said to be Schatten p-class if $tr((T^*T)^{p/2}) < \infty$, where tr is the standard trace defined by $tr(P) = \sum_n < Pe_n, e_n >$ for any bounded operator P acting on H and an orthonormal basis $\{e_n\}_n$ of H ($tr(P)$ is independent of the choice of the orthonormal basis). Let $\mathcal{S}_p$ be the ring of all Schatten p-class operators on an infinite dimensional and separable Hilbert space. We define the ring $\mathcal{S}$ of all Schatten class operators to be $\cup_{p\geq 1}\mathcal{S}_p$.

The following theorem is the main result of this paper.

Theorem 1.1. *Let $\mathcal{S}$ be the ring of all Schatten class operators on an infinite dimensional and separable Hilbert space. The assembly map*

$$A : H_n^{Or\Gamma}(E_{\mathcal{VCY}}(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty}) \longrightarrow K_n(\mathcal{S}\Gamma)$$

is rationally injective for any group Γ*, where* $\mathcal{S}\Gamma$ *is the group algebra of the group* Γ *over the ring* $\mathcal{S}$*.*

As a consequence, we obtain the algebraic K-theoretic Novikov conjecture for the group algebra $\mathcal{S}\Gamma$.

The main technical tool in the proof of Theorem 1.1 is an explicit construction of a Connes–Chern character using an equivariant cyclic simplicial homology theory. As a consequence of this explicit construction, we obtain a local property of the Connes–Chern character. This local property of the Connes–Chern character plays an important role in the proof.

This paper is organized as follows. In Section 2, we collect a few preliminary results which will be used later in the paper. In Section 3, we reduce our main theorem to the case of lower algebraic K-theory. In Section 4, we introduce a cyclic simplicial homology theory to construct a Connes–Chern character. The Connes–Chern character plays a crucial role in the proof of the main theorem. We use the explicit construction of the Connes–Chern character to prove an important local property of the Connes–Chern character for K-theory elements with small propagation. In Section 5, we prove the main theorem of this paper.

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2. Preliminaries

In this section, we collect a few concepts and results useful for this paper.

Let R be a ring and let R^+ be the unital ring obtained from R by adjoining a unit. The ring R is defined to be H-unital if $Tor_i^{R^+}(\mathbb{Z}, \mathbb{Z}) = 0$ for all i. The importance of H-unitality is that it guarantees excision in algebraic K-theory [32].

If R is a $\mathbb{Q}$-algebra and $R_{\mathbb{Q}}^+$ is the unital $\mathbb{Q}$-algebra obtained from R with the unit adjoined, then R is H-unital if and only if $Tor_i^{R_{\mathbb{Q}}^+}(\mathbb{Q}, \mathbb{Q}) = 0$ [32].

The following result follows from [35] and Theorem 8.2.1 of [14].

Theorem 2.1. $\mathcal{S}$ *is H-unital.*

By Theorem 7.10 in [32], we have:

Theorem 2.2. *If R is H-unital, then $R\Gamma$ is H-unital for any group Γ.*

As a consequence, we obtain that $\mathcal{S}\Gamma$ is H-unital.

Recall that a ring R is called K_n-regular if the natural map:

$$K_n(R) \to K_n(R[t_1, \cdots, t_m]),$$

is an isomorphism for each $m \geq 1$. We say that R is K-regular if R is K_n-regular for all n.

The following result is a special case of Theorem 8.2.5 in [14].

Theorem 2.3. *$\mathcal{S}$ is K-regular.*

The following result follows from the proof of Proposition 2.14 in [24].

Proposition 2.4. *If R is a K-regular $\mathbb{R}$-algebra, then the natural map:*

$$H_n^{Or\Gamma}(E_{\mathcal{FIN}}(\Gamma), \mathbb{K}(R)^{-\infty}) \to H_n^{Or\Gamma}(E_{\mathcal{VCY}}(\Gamma), \mathbb{K}(R)^{-\infty}),$$

is an isomorphism, where $\mathcal{FIN}$ is the family of finite subgroups of Γ and $E_{\mathcal{FIN}}(\Gamma)$ is the universal Γ-space with isotropy in $\mathcal{FIN}$.

The above proposition implies that the isomorphism conjecture for the ring $\mathcal{S}$ is equivalent to the statement that the assembly map:

$$A : H_n^{Or\Gamma}(E_{\mathcal{FIN}}(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty}) \to K_n(\mathcal{S}\Gamma),$$

is an isomorphism and Theorem 1.1 is equivalent to the statement that the above assembly map is rationally injective.

By Proposition 7.2.3, Remark 7.2.6 and Theorem 8.2.5 in [14], we know that $K_n(\mathcal{S})$ is 2-periodic and $K_0(\mathcal{S}) = \mathbb{Z}$ and $K_1(\mathcal{S}) = 0$. This implies that the domain of the assembly map A in Theorem 1.1 is rationally isomorphic to $\oplus_{k\ even} H_{n+k}^{Or\Gamma}(E_{\mathcal{FIN}}(\Gamma), \mathbb{Q})$.

3. Reduction to the lower algebraic K-theory case

In this section, we prove the following reduction result.

Proposition 3.1. *Theorem 1.1 follows from the following special case of the theorem for lower algebraic K-theory: i.e. the assembly map*

$$A : H_n^{Or\Gamma}(E_{\mathcal{VCY}}(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty}) \to K_n(\mathcal{S}\Gamma)$$

is rationally injective for $n \leq 0$.

Proof. By Proposition 7.2.3, Remark 7.2.6 and Theorem 8.2.5 in [14], we have $K_n(\mathcal{S}) = \mathbb{Z}$ when n is even and $K_n(\mathcal{S}) = 0$ when n is odd. It follows that

$$H_{-2}(pt, \mathbb{K}(\mathcal{S})^{-\infty}) = \mathbb{Z}.$$

By definition, the assembly map:

$$A : H_{-2}(pt, \mathbb{K}(\mathcal{S})^{-\infty}) \to K_{-2}(\mathcal{S})$$

is an isomorphism and it maps the generator z of $H_{-2}(pt, \mathbb{K}(\mathcal{S})^{-\infty})$ to the Bott element of $K_{-2}(\mathcal{S})$ (denoted by b). For any positive integer k, we can use the product operation to construct the Bott element b^k in $K_{-2k}(\mathcal{S})$, where the product is defined using a natural (injective) homomorphism $\mathcal{S} \otimes \mathcal{S} \to \mathcal{S}$ induced by the homomorphism $\mathcal{S}(\mathcal{H}) \otimes \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H} \otimes \mathcal{H})$ and a choice of isomorphism $\mathcal{S}(\mathcal{H} \otimes \mathcal{H}) \cong \mathcal{S}(\mathcal{H})$ (here $\mathcal{H}$ is an infinite dimensional and separable Hilbert space, $\mathcal{S}(\mathcal{H})$ and $\mathcal{S}(\mathcal{H} \otimes \mathcal{H})$ respectively denote the rings of Schatten class operators on $\mathcal{H}$ and $\mathcal{H} \otimes \mathcal{H}$, and $\mathcal{S} \otimes \mathcal{S}$ is the algebraic tensor product of $\mathcal{S}$ with $\mathcal{S}$).

When $n = 2k$, we have the following commutative diagram:

$$\begin{array}{ccc} H_n^{Or\Gamma}(E_{\mathcal{VCY}}(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty}) & \xrightarrow{A} & K_n(\mathcal{S}\Gamma) \\ \downarrow\times z^k & & \downarrow\times b^k \\ H_0^{Or\Gamma}(E_{\mathcal{VCY}}(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty}) & \xrightarrow{A} & K_0(\mathcal{S}\Gamma) \end{array},$$

where the vertical product maps are well defined with the help of a natural homomorphism $\mathcal{S} \otimes \mathcal{S} \to \mathcal{S}$ and the K-theory properties of $\mathcal{S}$ and $\mathcal{S}\Gamma$ being H-unital (Theorems 2.1 and 2.2). By Theorems 8.2.5 and 6.5.3 in [14] and Theorem 8.3 (the Bott periodicity) in [16], we know that the Bott element b^k is a generator of $K_{-2k}(\mathcal{S})$. It follows that the product map

$$H_i(pt, \mathbb{K}(\mathcal{S})^{-\infty}) \xrightarrow{\times z^k} H_{i-2k}(pt, \mathbb{K}(\mathcal{S})^{-\infty})$$

is an isomorphism for every integer i. This implies that the product map

$$H_i^{Or\Gamma}(E_{\mathcal{VCY}}(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty}) \xrightarrow{\times z^k} H_{i-2k}^{Or\Gamma}(E_{\mathcal{VCY}}(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty})$$

is an isomorphism by using the fact that both homology theories $\{H_i^{Or\Gamma}(\,\cdot\,, \mathbb{K}(\mathcal{S})^{-\infty})\}_{i\in\mathbb{Z}}$ and $\{H_{i-2k}^{Or\Gamma}(\,\cdot\,, \mathbb{K}(\mathcal{S})^{-\infty})\}_{i\in\mathbb{Z}}$ have a Mayer–Vietoris sequence and a five lemma argument.

When $n = 2k + 1$, we have the following commutative diagram:

$$\begin{array}{ccc} H_n^{Or\Gamma}(E_{\mathcal{VCY}}(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty}) & \xrightarrow{A} & K_n(\mathcal{S}\Gamma) \\ \downarrow\times z^{k+1} & & \downarrow\times b^{k+1} \\ H_{-1}^{Or\Gamma}(E_{\mathcal{VCY}}(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty}) & \xrightarrow{A} & K_{-1}(\mathcal{S}\Gamma) \end{array},$$

where the vertical product maps are well defined with the help of a natural homomorphism $\mathcal{S}\otimes\mathcal{S} \to \mathcal{S}$ and the K-theory properties of $\mathcal{S}$ and $\mathcal{S}\Gamma$ being H-unital (Theorems 2.1 and 2.2). By the same argument as in the even case, we know that the product map

$$H_i(pt, \mathbb{K}(\mathcal{S})^{-\infty}) \xrightarrow{\times z^{k+1}} H_{i-(2k+2)}(pt, \mathbb{K}(\mathcal{S})^{-\infty})$$

is an isomorphism for every integer i. This fact, together with a standard Mayer–Vietoris sequence and five lemma argument, implies that the product map

$$H_i^{Or\Gamma}(E_{\mathcal{VCY}}(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty}) \xrightarrow{\times z^{k+1}} H_{i-(2k+2)}^{Or\Gamma}(E_{\mathcal{VCY}}(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty})$$

is an isomorphism.

Now Proposition 3.1 follows from the above commutative diagrams and the fact that the left vertical maps in the diagrams are isomorphisms. $\square$

4. Cyclic simplicial homology theory and the Connes–Chern character

In this section, we introduce an equivariant cyclic simplicial homology theory to construct the Connes–Chern character for $K_n(\mathcal{S}\Gamma)$ when $n \leq 0$. The Connes–Chern character is a key tool in the proof of the main theorem. We use this explicit construction to prove an important local property of the Connes–Chern character for K-theory elements with small propagation. This local property will be useful in the proof of the main theorem.

Let X be a simplicial complex. Let σ be a simplex of X. Define two orderings of its vertex set to be equivalent if they differ from each other by an even permutation. Each of the equivalence classes is called an orientation of σ. If $\{v_0, ..., v_k\}$ is the set of all vertices of σ, we use the symbol $[v_0, \cdots, v_k]$ to denote the oriented simplex with the particular ordering $(v_0, \cdots, v_k)$.

A locally finite k-chain on X is a formal sum

$$\sum_{(v_0,\cdots,v_k)} c_{(v_0,\cdots,v_k)}[v_0, \cdots, v_k],$$

where

(1) the summation is taken over all orderings $(v_0, \cdots, v_k)$ of all k-simplices $\{v_0, \cdots, v_k\}$ of X and $c_{(v_0,\cdots,v_k)} \in \mathbb{C}$;
(2) $[v_0, \cdots, v_k]$ is identified with $-[v_0', \cdots, v_k']$ in the above sum if $(v_0, \cdots, v_k)$ and $(v_0', \cdots, v_k')$ are opposite orientations of the same simplex;
(3) for any compact subset K of X, there are at most finitely many ordered simplices $(v_0, \cdots, v_k)$ intersecting K such that $c_{(v_0,\cdots,v_k)} \neq 0$.

We remark that in the above definition the summation is taken over all $(v_0, \cdots, v_k)$ instead of $[v_0, \cdots, v_k]$ for the purpose to have consistent notations in the definitions of the Connes–Chern characters for lower algebraic K-groups of $\mathcal{S}_1\Gamma$ using simplicial homology groups and lower algebraic K-groups of $\mathcal{S}_p\Gamma$ using cyclic simplicial homology groups later in this section.

Let $C_k(X)$ be the abelian group of all locally finite k-chains on X.

Let

$$\partial_k : C_k(X) \to C_{k-1}(X)$$

be the standard simplicial boundary map. We define the locally finite simplicial homology group:

$$H_n(X) = Ker\ \partial_n / Im\ \partial_{n+1}.$$

If X has a proper simplicial action of Γ, let $C_k^\Gamma(X) \subset C_k(X)$ be the abelian group consisting of all Γ-invariant locally finite k-chains on X.

Let

$$\partial_k^\Gamma : C_k^\Gamma(X) \to C_{k-1}^\Gamma(X)$$

be the restriction of the standard simplicial boundary map. We define the locally finite Γ-equivariant simplicial homology group:

$$H_n^\Gamma(X) = Ker\ \partial_n^\Gamma / Im\ \partial_{n+1}^\Gamma.$$

Without loss of generality, in the proof of Theorem 1.1, we can assume that Γ is a countable group (this is because every group is an inductive limit of countable groups). We endow Γ with a proper left invariant length metric (here properness simply means that every ball with finite radius has finitely many elements). We remark that such a proper length metric always exists for any countable group. For each $d \geq 0$, the Rips complex $P_d(\Gamma)$ is the simplicial complex with Γ as its vertex set and where a finite subset $\{\gamma_0, \cdots, \gamma_n\}$ of Γ forms a simplex iff $d(\gamma_i, \gamma_j) \leq d$ for all $0 \leq i, j \leq n$. It is not difficult to show that $\cup_{d\geq 1} P_d(\Gamma)$ is a model for $E_{\mathcal{FIN}}(\Gamma)$. When Γ is torsion free, $\cup_{d\geq 1} P_d(\Gamma)$ is a universal space for free and proper action of Γ. In this case, $\lim_{d\to\infty} H_n^\Gamma(P_d(\Gamma))$ is the group homology of Γ defined using a standard resolution.

To motivate the general construction of the Connes–Chern character, we shall first consider the special case when Γ is torsion free.

Let $\mathcal{A}(\Gamma, \mathcal{S})$ be the algebra of all kernels

$$k : \Gamma \times \Gamma \to \mathcal{S}$$

such that

(1) for each k, there exists $r \geq 0$ such that $k(x, y) = 0$ if $d(x, y) > r$ (the smallest such r is called the propagation of k);

(2) k is Γ-invariant, i.e. $k(gx, gy) = k(x, y)$ for all $g \in \Gamma$ and $(x, y) \in \Gamma \times \Gamma$;
(3) the product in $\mathcal{A}(\Gamma, \mathcal{S})$ is defined by:

$$(k_1 k_2)(x, y) = \sum_{z \in \Gamma} k_1(x, z) k_2(z, y).$$

We identify $\mathcal{S}\Gamma$ with $\mathcal{A}(\Gamma, \mathcal{S})$ by the isomorphism:

$$\sum_{g \in \Gamma} s_g g \to k(x, y) = s_{x^{-1}y},$$

where $s_g \in \mathcal{S}$ for each $g \in \Gamma$. For each $p \geq 1$, we can naturally identify $\mathcal{S}_p\Gamma$ with $\mathcal{A}(\Gamma, \mathcal{S}_p)$, where $\mathcal{A}(\Gamma, \mathcal{S}_p)$ is defined by replacing $\mathcal{S}$ with $\mathcal{S}_p$ in the above definition of $\mathcal{A}(\Gamma, \mathcal{S})$.

For each non-negative even integer n, we shall first define the Connes–Chern character c_n for a countable torsion free group Γ

$$c_n : K_0(\mathcal{S}_1\Gamma) \to \lim_{d \to \infty} (\oplus_{k \ even, \ k \leq n} \ H_k^\Gamma(P_d(\Gamma)))$$

by:

$$[\tilde{q}] - [q_0] \to$$

$$\sum_{k \ even, \ k \leq n} \sum_{(x_0, \cdots, x_k)} tr(q(x_0, x_1) q(x_1, x_2) \cdots q(x_k, x_0))[x_0, \cdots, x_k],$$

where $P_d(\Gamma)$ is the Rips complex and $\tilde{q}$ is an idempotent in $M_m((\mathcal{S}_1\Gamma)^+)$, $\tilde{q} = q + q_0$ for some $q \in M_m(\mathcal{S}_1\Gamma)$ and idempotent $q_0 \in M_m(\mathbb{C})$, and the summation is taken over all orderings $(x_0, \cdots, x_k)$ of all n-simplices $\{x_0, \cdots, x_k\}$ of $P_d(\Gamma)$ for some d large enough such that $q(x, y) = 0$ if $d(x, y) > d/(n + 1)$.

Proposition 4.1. *Let Γ be a countable torsion free group. For each non-negative even integer n, the Connes–Chern character c_n is a well defined homomorphism from $K_0(\mathcal{S}_1\Gamma)$ to $\lim_{d \to \infty} (\oplus_{k \ even, \ k \leq n} \ H_k^\Gamma(P_d(\Gamma)))$.*

Proof. We first observe that $c([\tilde{q}] - [q_0])$ is Γ-invariant by using the Γ-invariance of q.

For each even $k \leq n$, we shall prove that

$$\partial_k^\Gamma (\sum_{(x_0, \cdots, x_k)} tr(q(x_0, x_1) q(x_1, x_2) \cdots q(x_k, x_0))[x_0, \cdots, x_k]) = 0.$$

This implies that $c([\tilde{q}] - [q_0])$ is a cycle.

We leave to the reader the proof that the homology class of

$$\sum_{k \ even, \ k \leq n} \sum_{(x_0, \cdots, x_k)} tr(q(x_0, x_1) q(x_1, x_2) \cdots q(x_k, x_0))[x_0, \cdots, x_k]$$

depends only on the K-theory class $[\tilde{q}] - [q_0]$.

By the assumption that $\tilde{q}$ and q_0 are idempotents, we have

$$q^2 = q - q_0 q - q q_0.$$

It follows that

$$\begin{aligned}
&\partial_k\Big(\sum_{(x_0,\cdots,x_k)} tr(q(x_0,x_1)q(x_1,x_2)\cdots q(x_k,x_0))[x_0,\cdots,x_k]\Big) = \\
&\sum_{i=1}^{k}(-1)^i \sum_{(x_0,\cdots,\hat{x_i},\cdots,x_k)} tr(q(x_0,x_1)q(x_1,x_2)\cdots q^2(x_{i-1},x_{i+1})\cdots q(x_k,x_0)) \\
&\qquad \times[x_0,\cdots,\hat{x}_i,\cdots,x_k] \\
&+ \sum_{(x_1,\cdots,x_k)} tr(q(x_1,x_2)\cdots q^2(x_k,x_1))[x_1,\cdots,x_k] = \\
&\sum_{i=1}^{k}(-1)^i \sum_{(x_0,\cdots,\hat{x_i},\cdots,x_k)} tr(q(x_0,x_1)q(x_1,x_2)\cdots q(x_{i-1},x_{i+1})\cdots q(x_k,x_0)) \\
&\qquad \times[x_0,\cdots,\hat{x}_i,\cdots,x_k] \\
&+ \sum_{(x_1,\cdots,x_k)} tr(q(x_1,x_2)\cdots q(x_k,x_1))[x_1,\cdots,x_k] - \\
&\Big(\sum_{i=1}^{k}(-1)^i \sum_{(x_0,\cdots,\hat{x_i},\cdots,x_k)} tr(q(x_0,x_1)q(x_1,x_2)\cdots q_0 q(x_{i-1},x_{i+1})\cdots q(x_k,x_0)) \\
&\qquad \times[x_0,\cdots,\hat{x}_i,\cdots,x_k] \\
&+ \sum_{(x_1,\cdots,x_k)} tr(q(x_1,x_2)\cdots q_0 q(x_k,x_1))[x_1,\cdots,x_k] + \\
&\sum_{i=1}^{k}(-1)^i \sum_{(x_0,\cdots,\hat{x_i},\cdots,x_k)} tr(q(x_0,x_1)q(x_1,x_2)\cdots q(x_{i-1},x_{i+1})q_0\cdots q(x_k,x_0)) \\
&\qquad \times[x_0,\cdots,\hat{x}_i,\cdots,x_k] \\
&+ \sum_{(x_1,\cdots,x_k)} tr(q(x_1,x_2)\cdots q(x_k,x_1)q_0)[x_1,\cdots,x_k]\Big).
\end{aligned}$$

Using the trace property and the definition of oriented simplices and the assumption that k is even, we have

$$\sum_{(x_0,\cdots,\hat{x_i},\cdots,x_k)} tr(q(x_0,x_1)q(x_1,x_2)\cdots q(x_{i-1},x_{i+1})\cdots q(x_k,x_0))[x_0,\cdots,\hat{x}_i,\cdots,x_k] = 0,$$

$$\sum_{(x_1,\cdots,x_k)} tr(q(x_1,x_2)\cdots q(x_k,x_1))[x_1,\cdots,x_k] = 0,$$

$$\sum_{(x_0,\cdots,\hat{x}_i,\cdots,x_k)} (tr(q(x_0,x_1)q(x_1,x_2)\cdots q_0 q(x_{i-1},x_{i+1})\cdots q(x_k,x_0))[x_0,\cdots,\hat{x}_i,\cdots,x_k] +$$
$$tr(q(x_0,x_1)q(x_1,x_2)\cdots q(x_{i-1},x_{i+1})q_0\cdots q(x_k,x_0))[x_0,\cdots,\hat{x}_i,\cdots,x_k]) = 0,$$
$$\sum_{(x_1,\cdots,x_k)} (tr(q(x_1,x_2)\cdots q_0 q(x_k,x_1))[x_1,\cdots,x_k] +$$
$$tr(q(x_1,x_2)\cdots q(x_k,x_1)q_0)[x_1,\cdots,x_k]) = 0. \quad \square$$

By the definition of lower algebraic K-theory using the group algebra over the free abelian group $\mathbb{Z}^n$, we can similarly define the Connes–Chern character:

$$c_n : K_i(\mathcal{S}_1\Gamma) \to \lim_{d\to\infty} (\oplus_{k+i\ even,\ k\le n}\ H_k^\Gamma(P_d(\Gamma)))$$

for each non-negative integer n and $i < 0$.

Next we extend the construction of the Connes–Chern character c_n to the $\mathcal{S}_p$ case for each $p \ge 1$ and each non-negative integer n when Γ is torsion free:

$$c_n : K_0(\mathcal{S}_p\Gamma) \to \lim_{d\to\infty} (\oplus_{k\ even,\ k\le n}\ H_k^\Gamma(P_d(\Gamma))).$$

We need to introduce an equivariant cyclic simplicial homology group to define the Connes–Chern character.

Let X be a simplicial complex. An ordered k-simplex $(v_0,\cdots,v_k)$ is defined to be an ordered finite sequence of vertices in a simplex of X, where v_i is allowed to be equal to v_j for some distinct pair of i and j.

Recall that the following permutation is called a cyclic permutation

$$(v_0,...,v_k) \to (v_k,v_0,\cdots,v_{k-1}).$$

We define two ordered simplices $(v_0,\cdots,v_k)$ and $(v_0',\cdots,v_k')$ to be equivalent if one ordered simplex can be obtained from the other ordered simplex by any number of cyclic permutations when k is even and by an even number of cyclic permutations when k is odd. Each of the equivalence classes is called a cyclically oriented simplex. If $(v_0,...,v_k)$ is an ordered simplex of X, we use the symbol $[v_0,\cdots,v_k]_\lambda$ to denote the corresponding cyclically oriented simplex.

A locally finite cyclic k-chain on X is a formal sum

$$\sum_{(v_0,\cdots,v_k)} c_{(v_0,\cdots,v_k)}[v_0,\cdots,v_k]_\lambda,$$

where

(1) the summation is taken over all ordered simplices $(v_0,\cdots,v_k)$ of X and $c_{(v_0,\cdots,v_k)} \in \mathbb{C}$;
(2) $[v_0,\cdots,v_k]_\lambda$ is identified with $(-1)^k[v_k,v_0,\cdots,v_{k-1}]_\lambda$ in the above sum;

(3) for any compact subset K of X, there are at most finitely many ordered simplices $(v_0, \cdots, v_k)$ intersecting K such that $c_{(v_0,\cdots,v_k)} \neq 0$.

Let $C_k^\lambda(X)$ be the abelian group of all locally finite cyclic k-chains on X. Let

$$\partial_k^\lambda : C_k^\lambda(X) \to C_{k-1}^\lambda(X)$$

be the standard boundary map.

We define the cyclic simplicial homology group:

$$H_n^\lambda(X) = Ker\ \partial_n^\lambda / Im\ \partial_{n+1}^\lambda.$$

If X has a simplicial proper action of a group Γ, we can define $C_k^{\lambda,\Gamma}(X) \subseteq C_k^\lambda(X)$ to be the subspace of all Γ-invariant locally finite cyclic k-chains on X.

Let

$$\partial_k^{\lambda,\Gamma} : C_k^{\lambda,\Gamma}(X) \to C_{k-1}^{\lambda,\Gamma}(X)$$

be the restriction of the standard boundary map.

We define the Γ-equivariant cyclic simplicial homology group $H_n^{\lambda,\Gamma}(X)$ by:

$$H_n^{\lambda,\Gamma}(X) = Ker\ \partial_n^{\lambda,\Gamma} / Im\ \partial_{n+1}^{\lambda,\Gamma}.$$

The following result computes the Γ-equivariant cyclic simplicial homology group in terms of the Γ-equivariant simplicial homology groups.

Proposition 4.2. *Let Γ be a group. Let X be a simplicial complex with a proper simplicial action of Γ. We have*

$$H_n^{\lambda,\Gamma}(X) \cong (\oplus_{k\leq n,\ k=n\ mod\ 2}\ H_k^\Gamma(X)).$$

Proof. Given an ordering $(v_0, \cdots, v_k)$ of $k+1$ number of vertices in X, we use the same notation $(v_0, \cdots, v_k)$ to denote the corresponding ordered k-simplex. Let $C_k^{ord,\Gamma}(X)$ be the abelian group of all Γ-invariant locally finite ordered k-chains

$$\sum_{(v_0,\cdots,v_k)} c_{(v_0,\cdots,v_k)}(v_0, \cdots, v_k),$$

where the sum is taken over all ordered k-simplices of X and $c_{(v_0,\cdots,v_k)} \in \mathbb{C}$. We remark that, in the above definition, a pair of vertices in $(v_0, \cdots, v_k)$ are allowed to be the same.

Let $C_{k,0}^{ord,\Gamma}(X)$ be the abelian subgroup of $C_k^{ord,\Gamma}(X)$ consisting of all Γ-invariant locally finite ordered k-chains with the following special form:

$$\sum_v c_v \overbrace{(v, \cdots, v)}^{\text{k+1}},$$

where the sum is taken over all vertices of X and $c_v \in \mathbb{C}$.

We define an abelian group $C^{ord,\Gamma}_{k,red}(X)$ by:

$$C^{ord,\Gamma}_{k,red}(X) := \begin{cases} C^{ord,\Gamma}_{k}(X) & \text{if } k \text{ is even,} \\ C^{ord,\Gamma}_{k}(X)/C^{ord,\Gamma}_{k,0}(X) & \text{if } k \text{ is odd,} \end{cases}$$

where $C^{ord,\Gamma}_{k}(X)/C^{ord,\Gamma}_{k,0}(X)$ is the quotient group of $C^{ord,\Gamma}_{k}(X)$ over $C^{ord,\Gamma}_{k,0}(X)$.

The standard boundary map on $C^{ord,\Gamma}_{k}(X)$ induces a boundary map:

$$\partial^{ord,\Gamma}_{k,red} : C^{ord,\Gamma}_{k,red}(X) \longrightarrow C^{ord,\Gamma}_{k-1,red}(X).$$

We define a new homology group:

$$H^{ord,\Gamma}_{n,red}(X) = Ker\ \partial^{ord,\Gamma}_{n,red}/Im\ \partial^{ord,\Gamma}_{n+1,red}.$$

Let χ_k be the natural chain map from $C^{ord,\Gamma}_{k,red}(X)$ to $C^{\lambda,\Gamma}_{k}(X)$ defined by:

$$[\sum_{(v_0,\cdots,v_k)} c_{(v_0,\cdots,v_k)}(v_0,\cdots,v_k)] \longrightarrow \sum_{(v_0,\cdots,v_k)} c_{(v_0,\cdots,v_k)}[v_0,\cdots,v_k]_\lambda,$$

for every $[\sum_{(v_0,\cdots,v_k)} c_{(v_0,\cdots,v_k)}(v_0,\cdots,v_k)] \in C^{ord,\Gamma}_{k,red}(X)$. This map is well defined because $\overbrace{[v,\cdots,v]}^{\text{k+1}}{}_\lambda = 0$ when k is odd.

By a standard Mayer–Vietoris and five lemma argument, it is not difficult to prove that χ induces an isomorphism χ_* from $H^{ord,\Gamma}_{n,red}(X)$ to $H^{\lambda,\Gamma}_{n}(X)$. This is because both homology theories satisfy the Mayer–Vietoris sequences and χ_* is an isomorphism when $X = \Gamma/F$ as Γ-spaces for some finite subgroup F of Γ.

We define a natural chain map

$$\phi_{k,k} : C^{ord,\Gamma}_{k,red}(X) \to C^{\Gamma}_{k}(X)$$

by:

$$[\sum_{(v_0,\cdots,v_k)} c_{(v_0,\cdots,v_k)}(v_0,\cdots,v_k)] \longrightarrow \sum_{(v_0,\cdots,v_k)} c_{(v_0,\cdots,v_k)}[v_0,\cdots,v_k]$$

for every $[\sum_{(v_0,\cdots,v_k)} c_{(v_0,\cdots,v_k)}(v_0,\cdots,v_k)] \in C^{ord,\Gamma}_{k,red}(X)$. This map is well defined because $\overbrace{[v,\cdots,v]}^{\text{k+1}} = 0$ when k is odd (more generally when $k \geq 1$).

For each ordered k-simplex $(v_0, \cdots, v_k)$ of X, we let

$$\varphi_{k,k-2}((v_0, \cdots, v_k)) :=$$
$$\begin{cases} (-1)^{j-i+1}[v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_k] & \text{if } (i,j) \text{ is the smallest pair} \\ & \text{such that } i < j,\ v_i = v_j, \\ 0 & \text{if there exists no pair } i < j \text{ such that } v_i = v_j, \end{cases}$$

where the smallest (i, j) is taken with respect to the dictionary order of $\{(m, l) : 0 \leq m < l \leq k\}$ given by: $(m, l) < (m', l')$ iff either (1) $m < m'$, or (2) $m = m'$ and $l < l'$.

We define a linear map

$$\phi_{k,k-2} : C^{ord,\Gamma}_{k,red}(X) \to C^{\Gamma}_{k-2}(X)$$

by:

$$[\sum_{(v_0,\cdots,v_k)} c_{(v_0,\cdots,v_k)}(v_0, \cdots, v_k)] \longrightarrow \sum_{(v_0,\cdots,v_k)} c_{(v_0,\cdots,v_k)}\varphi_{k,k-2}((v_0, \cdots, v_k))$$

for every $[\sum_{(v_0,\cdots,v_k)} c_{(v_0,\cdots,v_k)}(v_0, \cdots, v_k)] \in C^{ord,\Gamma}_{k,red}(X)$.

Note that $\phi_{k,k-2}$ is well defined. Elementary computations show that $\phi_{k,k-2}$ is a chain map.

Similarly we can construct a chain map

$$\phi_{k,l} : C^{ord,\Gamma}_{k,red}(X) \to C^{\Gamma}_{l}(X)$$

if $0 \leq l \leq k$ and $k - l$ is even.

Using the above chain maps, we construct a chain map:

$$\psi_n = \oplus_{l \leq n,\ n-l\ even}\ \phi_{n,l} : C^{ord,\Gamma}_{n,red}(X) \to (\oplus_{l \leq n,\ n-l\ even}\ C^{\Gamma}_{l}(X)).$$

The chain map ψ_n induces an isomorphism ψ_* on the homology groups since both homology theories satisfy the Mayer–Vietoris sequence and the chain map induces an isomorphism at the homology level if $X = \Gamma/F$ as Γ-spaces for some finite subgroup F of Γ.

Finally Proposition 4.2 follows from the facts that χ_* and ψ_* are isomorphisms. $\square$

For any positive even integer $n \geq p$, we now define the Connes–Chern character for a countable torsion free group Γ

$$c_n : K_0(\mathcal{S}_p\Gamma) \to H^{\lambda,\Gamma}_n(P_d(\Gamma)),$$

where Γ is endowed with a proper length metric.

Let $\tilde{q}$ be an idempotent in $M_m((\mathcal{S}_p\Gamma)^+)$ and $\tilde{q} = q + q_0$ for some $q \in M_m(\mathcal{S}_p\Gamma)$ and idempotent $q_0 \in M_m(\mathbb{C})$. We identify q with an element in $\mathcal{A}(\Gamma, \mathcal{S}_p)$ (note that $\mathcal{A}(\Gamma, M_m(\mathcal{S}_p))$ is isomorphic to $\mathcal{A}(\Gamma, \mathcal{S}_p)$). Let d be greater than or equal to $n+1$ times the propagation of q, i.e. $q(x, y) = 0$ if $d(x, y) > d/(n + 1)$.

For each positive even integer $n \geq p$, the Connes–Chern character c_n of $[\tilde{q}] - [q_0]$ is defined to be homology class of

$$\sum_{(x_0,\cdots,x_n)} tr(q(x_0, x_1)\cdots q(x_n, x_0))[x_0, \cdots, x_n]_\lambda \in H_n^{\lambda,\Gamma}(P_d(\Gamma)),$$

where the summation is taken over all ordered n-simplices $(x_0, \cdots, x_n)$ of $P_d(\Gamma)$.

We remark that the choice of n guarantees that the trace in the above definition of the Connes–Chern character is finite.

By Proposition 4.2, the above Connes–Chern character induces a Connes–Chern character:

$$c_n : K_0(\mathcal{S}_p\Gamma) \to \lim_{d\to\infty} (\oplus_{k\ even,\ k\leq n}\ H_k^\Gamma(P_d(\Gamma)))$$

for any non-negative integer $n \geq p$.

For an arbitrary non-negative even integer n, let n' be a positive even integer satisfying $n' \geq max\{n, p\}$. Let $\pi_{n',n}$ be the natural projection from $\lim_{d\to\infty}(\oplus_{k\ even,\ k\leq n'}\ H_k^\Gamma(P_d(\Gamma)))$ to $\lim_{d\to\infty}(\oplus_{k\ even,\ k\leq n}\ H_k^\Gamma(P_d(\Gamma)))$. We define the Connes–Chern character c_n from $K_0(\mathcal{S}_p\Gamma)$ to $\lim_{d\to\infty}(\oplus_{k\ even,\ k\leq n}\ H_k^\Gamma(P_d(\Gamma)))$ to be $\pi_{n',n} \circ c_{n'}$. It is not difficult to verify that the definition of c_n is independent of the choice of n'.

Proposition 4.3. *Let Γ be a countable torsion free group. For any non-negative even integer n, the Connes–Chern character c_n is a well defined homomorphism from $K_0(\mathcal{S}_p\Gamma)$ to $\lim_{d\to\infty}(\oplus_{k\ even,\ k\leq n}\ H_k^\Gamma(P_d(\Gamma)))$.*

The proof of the above proposition is similar to the proof of Proposition 4.1 and is therefore omitted. Note that when $p = 1$, the above definition of the Connes–Chern character coincides with the prior definition of the Connes–Chern character.

Next we shall construct the Connes–Chern character for a general group Γ.

Let Γ_{fin} be the set of all elements with finite order in Γ. The group Γ acts on Γ_{fin} by conjugations:

$$\gamma \cdot x = \gamma x \gamma^{-1}$$

for all $\gamma \in \Gamma$ and $x \in \Gamma_{fin}$.

Let X be a simplicial complex with a proper simplicial action of Γ. Equip the vertex set $V(X)$ of X with a Γ-invariant proper pseudo-metric d_V. Let Γ act on $\Gamma_{fin} \times X$ diagonally.

Let $r \geq 0$. For each $g \in \Gamma_{fin}$, we define $X_{g,r}$ to be the simplicial subcomplex of X consisting all simplices $\{v_0, \cdots, v_p\}$ satisfying $d_V(v_i, gv_i) \leq r$ for all $0 \leq i \leq p$.

For each ordered simplex $(v_0, ..., v_k)$ of $X_{g,r}$, we define the following transformation to be a g-cyclic permutation

$$(v_0, ..., v_k) \to (gv_k, v_0, \cdots, v_{k-1}).$$

We define two ordered simplices $(v_0, \cdots, v_k)$ and $(v_0', \cdots, v_k')$ of $X_{g,r}$ to be g-equivalent if one ordered simplex can be obtained from the other ordered simplex by any number of g-cyclic permutations of ordered simplices in $X_{g,r}$ when k is even and by an even number of g-cyclic permutations when k is odd. Each of the equivalence classes is called a g-cyclically oriented simplex. If $(v_0, ..., v_k)$ is an ordered simplex of $X_{g,r}$, we use the symbol $[v_0, \cdots, v_k]_{\lambda,g}$ to denote the corresponding g-cyclically oriented simplex.

We define $\mathbb{C}^{\lambda}_{k,r}(X)$ to be the abelian group of all locally finite k-chains:

$$\sum_{g \in \Gamma_{fin}} (g, \sum_{(v_0, \cdots, v_k)} c_{(v_0, \cdots, v_k), g}[v_0, \cdots, v_k]_{\lambda,g}),$$

where

(1) the second summation is taken over all ordered simplices $(v_0, \cdots, v_k)$ of $X_{g,r}$ and $c_{(v_0, \cdots, v_k), g} \in \mathbb{C}$;
(2) $[v_0, \cdots, v_k]_{\lambda,g}$ is identified with $(-1)^k[gv_k, v_0, \cdots, v_{k-1}]_{\lambda,g}$ in the above sum;
(3) for each $g \in \Gamma_{fin}$ and any compact subset K of X, there are at most finitely many ordered simplices $(v_0, \cdots, v_k)$ intersecting K such that $c_{(v_0, \cdots, v_k), g} \neq 0$.

The diagonal action of Γ on $\Gamma_{fin} \times X$ induces a natural Γ-action on $\mathbb{C}^{\lambda}_{k,r}(X)$. Let $\mathbb{C}^{\lambda,\Gamma}_{k,r}(X) \subseteq \mathbb{C}^{\lambda}_{k,r}(X)$ be the abelian group consisting of all Γ-invariant k-chains in $\mathbb{C}^{\lambda}_{k,r}(X)$.

We have a natural boundary map:

$$\partial^{\lambda,\Gamma}_{k,r} : \mathbb{C}^{\lambda,\Gamma}_{k,r}(X) \longrightarrow \mathbb{C}^{\lambda,\Gamma}_{k-1,r}(X).$$

We define the following equivariant homology theory by:

$$\mathbb{H}^{\lambda,\Gamma}_{n,r}(X) = Ker\ \partial^{\lambda,\Gamma}_{n,r} / Im\ \partial^{\lambda,\Gamma}_{n+1,r}.$$

When Γ is torsion free, Γ_{fin} consists of the identity element and we have

$$\mathbb{H}^{\lambda,\Gamma}_{n,r}(X) = H^{\lambda,\Gamma}_n(X).$$

For each $r \geq 0$, let $\hat{X}_r$ be the simplicial subspace of $\Gamma_{fin} \times X$ defined by:

$$\hat{X}_r = \{(g, x) \in \Gamma_{fin} \times X : x \in X_{g,r}\}.$$

The diagonal action of Γ on $\Gamma_{fin} \times X$ induces a natural Γ-action on $\hat{X}_r$.

We define

$$\mathbb{H}^{\Gamma}_{n,r}(X) = H^{\Gamma}_n(\hat{X}_r).$$

The following result computes our new equivariant homology theory of the Rips complex in terms of the (locally finite) equivariant simplicial homology theory.

Proposition 4.4. *Let Γ be a countable group with a proper length metric. We have*

$$\lim_{d\to\infty}\lim_{r\to\infty}\mathbb{H}^{\lambda,\Gamma}_{n,r}(P_d(\Gamma)) \cong \lim_{d\to\infty}\lim_{r\to\infty}(\oplus_{k\leq n,\ k=n\ mod\ 2}\ \mathbb{H}^{\Gamma}_{n,r}(P_d(\Gamma))).$$

Proof. Let X be a simplicial complex with a proper and cocompact action of Γ. We define an equivalence relation $\sim$ on the chain group $\mathbb{C}^{\lambda,\Gamma}_{k,r}(X)$ as follows. Two chains z and z' in $\mathbb{C}^{\lambda,\Gamma}_{k,r}(X)$ are said to be equivalent if

$$z = \sum_{g\in\Gamma_{fin}}(g, \sum_{(v_0,\cdots,v_k)} c_{(v_0,\cdots,v_k),g}[v_0,\cdots,v_k]_{\lambda,g}),$$
$$z' = \sum_{g\in\Gamma_{fin}}(g, \sum_{(v_0,\cdots,v_k)} c_{(v_0,\cdots,v_k),g}[v'_0,\cdots,v'_k]_{\lambda,g}),$$

and for each $0\leq i\leq k$ there exists an integer j such that $v'_i = g^j v_i$.

Let $C^{\lambda,\Gamma}_{k,r}(X)$ be the chain group $\mathbb{C}^{\lambda,\Gamma}_{k,r}(X)/\sim$. We define $\tilde{\mathbb{H}}^{\lambda,\Gamma}_{n,r}(X)$ to be the n-th homology group of $C^{\lambda,\Gamma}_{k,r}(X)$.

The quotient chain map ϕ from $\mathbb{C}^{\lambda,\Gamma}_{k,r}(X)$ to $C^{\lambda,\Gamma}_{k,r}(X)$ induces a homomorphism

$$\phi_* : \mathbb{H}^{\lambda,\Gamma}_{n,r}(X) \to \tilde{\mathbb{H}}^{\lambda,\Gamma}_{n,r}(X).$$

We observe that the properness and the cocompactness of the Γ action on X imply that, for each $r\geq 0$, there exists $N>0$ such that if $g\in\Gamma_{fin}$ and $X_{g,r}$ is nonempty, then the order of the group element g is bounded by N. As a consequence, for any $d\geq 0$ and $r\geq 0$, there exist $d'\geq d$ and $r'\geq r$ such that, for any $g\in\Gamma_{fin}$ and any simplex in $(P_d(\Gamma))_{g,r}$ with vertices $\{v_0,\cdots,v_k\}$, the simplex with vertices $\{g^{i_0}v_0,\cdots,g^{i_k}v_k : 1\leq i_j\leq N, 0\leq j\leq k\}$ is a simplex in $(P_{d'}(\Gamma))_{g,r'}$. This implies that ϕ is a chain homotopy equivalence from the chain complex $\lim_{d\to\infty}\lim_{r\to\infty}\mathbb{C}^{\lambda,\Gamma}_{k,r}(P_d(\Gamma))$ to the chain complex $\lim_{d\to\infty}\lim_{r\to\infty}C^{\lambda,\Gamma}_{k,r}(P_d(\Gamma))$ with a homotopy inverse chain map ψ from $\lim_{d\to\infty}\lim_{r\to\infty}C^{\lambda,\Gamma}_{k,r}(P_d(\Gamma))$ to $\lim_{d\to\infty}\lim_{r\to\infty}\mathbb{C}^{\lambda,\Gamma}_{k,r}(P_d(\Gamma))$ defined by

$$\psi([\sum_{g\in\Gamma_{fin}}(g, \sum_{(v_0,\cdots,v_k)} c_{(v_0,\cdots,v_k),g}[v_0,\cdots,v_k]_{\lambda,g})]) =$$
$$\sum_{g\in\Gamma_{fin}}(g, \sum_{(v_0,\cdots,v_k)}\frac{1}{n_g^{k+1}}\sum_{i_0,\cdots,i_k=1}^{n_g} c_{(v_0,\cdots,v_k),g}[g^{i_0}v_0,\cdots,g^{i_k}v_k]_{\lambda,g})$$

for each $[\sum_{g\in\Gamma_{fin}}(g,\sum_{(v_0,\cdots,v_k)}c_{(v_0,\cdots,v_k),g}[v_0,\cdots,v_k]_{\lambda,g})]$ in $\lim_{d\to\infty}\lim_{r\to\infty}C^{\lambda,\Gamma}_{k,r}(P_d(\Gamma))$, where n_g is the order of the group element g. It follows that the homomorphism ϕ_* is an isomorphism from $\lim_{d\to\infty}\lim_{r\to\infty}\mathbb{H}^{\lambda,\Gamma}_{n,r}(P_d(\Gamma))$ to $\lim_{d\to\infty}\lim_{r\to\infty}\tilde{\mathbb{H}}^{\lambda,\Gamma}_{n,r}(P_d(\Gamma))$.

Two vertices v and v' of $X_{g,r}$ are defined to be equivalent if $v=g^jv'$ for some j. We denote the equivalence class of v by $[v]$.

We define $\tilde{X}_{g,r}$ to be the simplicial complex consisting of simplices $\{[v_0],\cdots,[v_k]\}$ for all simplices $\{v_0,\cdots,v_k\}$ in $X_{g,r}$.

Let

$$\tilde{X}_r=\{(g,x):g\in\Gamma_{fin},x\in\tilde{X}_{g,r}\}.$$

By an argument similar to the proof of Proposition 4.2, we have the following isomorphism:

$$\lim_{d\to\infty}\lim_{r\to\infty}\tilde{\mathbb{H}}^{\lambda,\Gamma}_{n,r}(P_d(\Gamma))\cong\lim_{d\to\infty}\lim_{r\to\infty}(\oplus_{k\le n,\ k=n\ mod\ 2}\ H^\Gamma_n((\widetilde{P_d(\Gamma)})_r)).$$

Finally we observe that the natural homomorphism

$$\lim_{d\to\infty}\lim_{r\to\infty}(\oplus_{k\le n,\ k=n\ mod\ 2}\ \mathbb{H}^\Gamma_{n,r}(P_d(\Gamma)))\to\lim_{d\to\infty}\lim_{r\to\infty}(\oplus_{k\le n,\ k=n\ mod\ 2}\ H^\Gamma_n((\widetilde{P_d(\Gamma)})_r))$$

is an isomorphism. $\square$

Let Γ be a countable group with a proper length metric. Let X be a simplicial complex with a proper and cocompact action of Γ. Let $\hat{X}$ be the subspace of $\Gamma_{fin}\times X$ defined by:

$$\hat{X}=\{(g,x)\in\Gamma_{fin}\times X:gx=x\}.$$

The diagonal action of Γ on $\Gamma_{fin}\times X$ induces a natural Γ-action on $\hat{X}$. Note that $\hat{X}$ is a simplicial complex with a simplicial action of Γ.

We define

$$\mathbb{H}^\Gamma_k(X)=H^\Gamma_k(\hat{X}).$$

We remark that $\mathbb{H}^\Gamma_k(X)$ is the equivariant homology theory of Baum–Connes [3].

Proposition 4.5. *Let Γ be a countable group with a proper length metric. We have*

$$\lim_{d\to\infty}\lim_{r\to\infty}\mathbb{H}^\Gamma_{n,r}(P_d(\Gamma))\cong\lim_{d\to\infty}\mathbb{H}^\Gamma_n(P_d(\Gamma)).$$

Proof. Let X be a simplicial complex with a proper and cocompact action of Γ. For each finite subset $F\subset\Gamma_{fin}$, let

$$F'=\{\gamma f\gamma^{-1}:\gamma\in\Gamma,f\in F\}.$$

For each $g \in \Gamma_{fin}$ and $r \geq 0$, we define $\check{X}_{g,r}$ to be the simplicial subcomplex of X consisting of simplices with vertices

$$\{g^{j_0}v_0, \cdots, g^{j_k}v_k : j_i \in \mathbb{Z}\}$$

for all simplices $\{v_0, \cdots, v_k\}$ in $X_{g,r}$.

We let

$$\hat{X}_F = \{(g,x) \in F' \times X : gx = x\}, \qquad \check{X}_{F,r} = \{(g,x) \in F' \times X : x \in \check{X}_{g,r}\}.$$

We have an inclusion map

$$i : P_d \hat{(}\Gamma)_F \to (P_d \check{(}\Gamma))_{F,r}.$$

The map i induces a homomorphism

$$i_* : \lim_{d\to\infty} \mathbb{H}_n^\Gamma(P_d(\Gamma)) \to \lim_{d\to\infty} \lim_{r\to\infty} \mathbb{H}_{n,r}^\Gamma(P_d(\Gamma)).$$

By the definition of $(P_d \check{(}\Gamma))_{F,r}$, for each $d \geq 0$ and $r \geq 0$, there exists $c > 0$ such that, for every point (g,x) in $(P_d \check{(}\Gamma))_{F,r}$, x is within distance c from a fixed point of g. It follows that, for each $d \geq 0$ and $r \geq 0$, there exist $d' \geq d$ and a continuous map

$$\psi : (P_d \check{(}\Gamma))_{F,r} \to (P_{d'} \hat{(}\Gamma))_F$$

such that if we write $\psi(g,x) = (g, \psi'(x))$, then we have

$$\sup\{d(\psi'(x), x) : (g,x) \in (P_{d'} \hat{(}\Gamma))_F\} < \infty,$$

where d is the restriction of the simplicial metric on $P_d(\Gamma)$.

The map ψ induces a homomorphism

$$\psi_* : \lim_{d\to\infty} \lim_{r\to\infty} \mathbb{H}_{n,r}^\Gamma(P_d(\Gamma)) \to \lim_{d\to\infty} \mathbb{H}_n^\Gamma(P_d(\Gamma)).$$

Using linear homotopies, it is not difficult to check that i_* and ψ_* are inverses to each other. $\square$

For each non-negative integer n, we are now ready to define the Connes–Chern character c_n for a general group Γ:

$$c_n : \quad K_0(\mathcal{S}\Gamma) \longrightarrow \lim_{d\to\infty} (\oplus_{k \ even,\ k\leq n} \ \mathbb{H}_n^\Gamma(P_d(\Gamma))).$$

For each $p \geq 1$ and even integer $n \geq p$, we shall first define the Connes–Chern character c_m:

$$c_n : \quad K_0(\mathcal{S}_p\Gamma) \longrightarrow \lim_{d\to\infty} (\oplus_{k\ even,\ k\leq n}\ \ \mathbb{H}_k^\Gamma(P_d(\Gamma)))$$

Let $\tilde{q}$ be an idempotent in $M_m((\mathcal{S}_p\Gamma)^+)$ and $\tilde{q} = q + q_0$ for some $q \in M_m(\mathcal{S}_p\Gamma)$ and idempotent $q_0 \in M_m(\mathbb{C})$. Let d be greater than or equal to $n+1$ times the propagation of q, i.e. $q(x, y) = 0$ if $d(x, y) > d/(n + 1)$.

The Connes–Chern character c_n of $[\tilde{q}] - [q_0]$ is defined to be the homology class of

$$\sum_{g\in\Gamma_{fin}} (g, \sum_{(x_0,\cdots,x_n)} tr(q(x_0, x_1)\cdots q(x_n, g^{-1}x_0))[x_0, \cdots , x_n]_{\lambda,g}) \in \mathbb{H}_{n,d}^{\lambda,\Gamma}(P_d(\Gamma)),$$

where the summation $\sum_{(x_0,\cdots,x_n)}$ is taken over all ordered n-simplices of $(P_d(\Gamma))_{g,d}$. We remark that the choice of n guarantees that the trace in the above definition of the Connes–Chern character is finite.

By Propositions 4.4 and 4.5, the above Connes–Chern character induces a Connes–Chern character:

$$c_n : K_0(\mathcal{S}_p\Gamma) \to \lim_{d\to\infty} (\oplus_{k\ even,\ k\leq n}\ \mathbb{H}_k^\Gamma(P_d(\Gamma))).$$

For an arbitrary non-negative integer n, let n' be a positive even integer satisfying $n' \geq max\{n, p\}$. Let $\pi_{n',n}$ be the natural projection from $\lim_{d\to\infty}(\oplus_{k\ even,\ k\leq n'}\ \mathbb{H}_k^\Gamma(P_d(\Gamma)))$ to $\lim_{d\to\infty}(\oplus_{k\ even,\ k\leq n}\ \mathbb{H}_k^\Gamma(P_d(\Gamma)))$. We define the Connes–Chern character c_n from $K_0(\mathcal{S}_p\Gamma)$ to $\lim_{d\to\infty}(\oplus_{k\ even,\ k\leq n}\ \mathbb{H}_k^\Gamma(P_d(\Gamma)))$ to be $\pi_{n',n} \circ c_{n'}$. It is not difficult to verify that the definition of c_n is independent of the choice of n'.

The proof of the following proposition is similar to the proof of Proposition 4.1 and is therefore omitted.

Proposition 4.6. *Let Γ be a countable group. For any non-negative integer n, the Connes–Chern character c_n is a well defined homomorphism from $K_0(\mathcal{S}_p\Gamma)$ to $\lim_{d\to\infty}(\oplus_{k\ even,\ k\leq n}\ \mathbb{H}_k^\Gamma(P_d(\Gamma)))$.*

Using the definition of lower algebraic K-theory, for each $p \geq 1$ and any non-negative integer n, we can similarly define

$$c_n : K_i(\mathcal{S}_p\Gamma) \to \lim_{d\to\infty} (\oplus_{k+i\ even, k\leq n}\ \ \mathbb{H}_k^\Gamma(P_d(\Gamma)))$$

for each $i < 0$.

Finally, with the help of the equality $\mathcal{S} = \cup_{p\geq 1}\mathcal{S}_p$, we obtain a Connes–Chern character

$$c_n : K_i(\mathcal{S}\Gamma) \to \lim_{d\to\infty} (\oplus_{k+i\ even,\ k\leq n}\ \mathbb{H}_k^\Gamma(P_d(\Gamma)))$$

for each non-negative integer n and $i \leq 0$.

Notice that when Γ is torsion free, Γ_{fin} consists only of the identity element and the above definition coincides with the prior definition for the torsion free case.

In the rest of this section, we study a local property of the Connes–Chern character for K-theory elements with small propagations. This local property will play an important role in the proof of the main theorem of this paper.

We shall need a few preparations to explain the concept of propagation in a continuous setting. Let X be a Γ-invariant simplicial subspace of $P_{d_0}(\Gamma)$ for some $d_0 \geq 0$. Endow $P_{d_0}(\Gamma)$ with a metric d such that its restriction to each simplex is the standard metric and $d(\gamma_1, \gamma_2) \leq d_\Gamma(\gamma_1, \gamma_2)$ for all γ_1 and γ_2 in $\Gamma \subseteq P_{d_0}(\Gamma)$, where d_Γ is the proper length metric on Γ. Let X be given the simplicial metric of $P_{d_0}(\Gamma)$. Let H be a Hilbert space with a Γ-action and let ϕ be a $*$-homomorphism from $C_0(X)$ to $B(H)$ which is covariant in the sense that $\phi(\gamma f)h = (\gamma(\phi(f))\gamma^{-1})h$ for all $\gamma \in \Gamma$, $f \in C_0(X)$ and $h \in H$. Such a triple $(C_0(X), \Gamma, \phi)$ is called a covariant system.

The following definition is due to John Roe [28].

Definition 4.7. Let H be a Hilbert space and let ϕ be a $*$-homomorphism from $C_0(X)$ to $B(H)$, the C^*-algebra of all bounded operators on H. Let T be a bounded linear operator acting on H.

(1) The support of T is defined to be the complement (in $X \times X$) of the set of all points $(x, y) \in X \times X$ for which there exists $f \in C_0(X)$ and $g \in C_0(X)$ satisfying $\phi(f)T\phi(g) = 0$ and $f(x) \neq 0$ and $g(y) \neq 0$;
(2) The propagation of T is defined to be:

$$\sup\ \{d(x, y) : (x, y) \in \operatorname{Supp}(T)\};$$

(3) Given $p \geq 1$, T is said to be locally Schatten p-class if $\phi(f)T$ and $T\phi(f)$ are Schatten p-class operators for each $f \in C_c(X)$, the algebra of all compactly supported continuous functions on X.

Definition 4.8. We define the covariant system $(C_0(X), \Gamma, \phi)$ to be admissible if

(1) the Γ-action on X is proper and cocompact;
(2) ϕ is nondegenerate in the sense that $\phi(C_0(X))H$ is dense in H;
(3) $\phi(f)$ is noncompact for any nonzero function $f \in C_0(X)$;
(4) for each $x \in X$, the action of the stabilizer group Γ_x on H is regular in the sense that it is isomorphic to the action of Γ_x on $l^2(\Gamma_x) \otimes W$ for some infinite dimensional Hilbert space W, where the Γ_x action on $l^2(\Gamma_x)$ is regular and the Γ_x action on W is trivial.

We remark that condition (4) in the above definition is unnecessary if Γ acts on X freely. In particular, if M is a compact manifold and $\Gamma = \pi_1(M)$, then $(C_0(\widetilde{M}), \Gamma, \phi)$ is an admissible covariant system, where $\widetilde{M}$ is the universal cover of M and $\phi(f)\xi = f\xi$ for each $f \in C_0(\widetilde{M})$ and all $\xi \in L^2(\widetilde{M})$. In general, for each locally compact metric space with a proper and cocompact isometric action of Γ, there exists an admissible covariant system $(C_0(X), \Gamma, \phi)$.

Definition 4.9. For any $p \geq 1$, let $(C_0(X), \Gamma, \phi)$ be an admissible covariant system. We define $\mathbb{C}_p(\Gamma, X, H)$ to be the ring of Γ-invariant locally Schatten p-class operators acting on H with finite propagation.

The proof of the following useful result is straightforward and is therefore omitted.

Proposition 4.10. *Let Γ be a countable group. Let X be a simplicial complex with a simplicial proper and cocompact action of Γ. If $(C_0(X), \Gamma, \phi)$ is an admissible covariant system, then the ring $\mathbb{C}_p(\Gamma, X, H)$ is isomorphic to the ring $\mathcal{S}_p\Gamma$.*

For each $r > 0$, let X_r be a Γ-invariant discrete subset of X such that

(1) X_r has bounded geometry, i.e. for each $R > 0$, there exists $N > 0$ such that any ball in X_r with radius R has at most N elements;
(2) X_r is r-dense in X, i.e. $d(x, X_r) < r$ for every $x \in X$;
(3) X_r is uniformly discrete, i.e. there exists $k_r > 0$ such that $d(z, z') \geq k_r$ for all distinct pairs of elements z and z' in X_r.

Let $\{U_z\}_{z \in X_r}$ be a Γ-equivariant disjoint Borel cover of X such that $z \in U_z$ and *diameter*$(U_z) < r$ for all z. Let χ_z be the characteristic function of U_z. Extend the $*$-representation ϕ to the algebra of all bounded Borel functions. If $k \in \mathbb{C}_p(\Gamma, X, H)$, let $k(x, y) = \phi(\chi_x) k \phi(\chi_y)$ for all x and y in X_r.

For any $r > 0$, let U'_z be the $10r$-neighborhood of U_z for each $z \in X_r$, i.e.

$$U'_z = \{x \in X : d(x, U_z) < 10r\}.$$

Let $O_r(X) = \{U'_z\}_{z \in X_r}$. Note that $O_r(X)$ is an open cover of X.

Let $N(O_r(X))$ be the nerve space of the open cover $O_r(X)$. We equip the vertex set V of the simplicial complex $N(O_r(X))$ with the pseudo-metric d_V defined by:

$$d_V(W, W') = \sup\{d(x, y) : x \in W, y \in W'\}$$

for any pair of vertices W and W' in $N(O_r(X))$.

For each non-negative even integer $n \geq p$, we define the Connes–Chern character

$$c_n : K_0(\mathbb{C}_p(\Gamma, X, H)) \to \oplus_{k\ even, k \leq n}\ \mathbb{H}^{\Gamma}_{k,r}(N(O_r(X)))$$

as follows.

Let $\tilde{q}$ be an idempotent in $M_m(\mathbb{C}_p(\Gamma, X, H))^+)$ and $\tilde{q} = q + q_0$ for some $q \in M_m(\mathbb{C}_p(\Gamma, X, H))$ and idempotent $q_0 \in M_m(\mathbb{C})$. Let r be the propagation of q. Let n be an even integer satisfying $n \geq p$.

The Connes–Chern character of $[\tilde{q}] - [q_0]$ is defined to be homology class of

$$\sum_{g \in \Gamma_{fin}} (g, \sum_{(x_0, \cdots, x_n)} tr(q(x_0, x_1) \cdots q(x_n, g^{-1}x_0))[x_0, \cdots, x_n]_{\lambda,g})$$

in the homology group $\mathbb{H}^{\lambda,\Gamma}_{n,(n+1)r}(N(O_{(n+1)r}(X)))$, where, for each g and r, $(x_0, \cdots, x_n)$ denotes the ordered simplex $(U'_{x_0}, \cdots, U'_{x_n})$ in the space $(N(O_{(n+1)r}(X)))_{g,r}$, $[x_0, \cdots, x_n]_{\lambda,g}$ denotes the g-cyclically oriented simplex $[U'_{x_0}, \cdots, U'_{x_n}]_{\lambda,g}$ in $(N(O_{(n+1)r}(X)))_{g,r}$, and the summation $\sum_{(x_0,\cdots,x_n)}$ in the above formula is taken over all ordered simplices in $(N(O_{(n+1)r}(X)))_{g,r}$.

The following proposition follows from the above definition and the proof of Proposition 4.1.

Proposition 4.11. *Let Γ be a countable group. Let X be a simplicial complex with a simplicial proper and cocompact action of Γ. For each $r > 0$, let n be an even integer satisfying $n \geq p$. If $(C_0(X), \Gamma, \phi)$ is an admissible covariant system, then the Connes–Chern character c_n of an element in $K_0(\mathbb{C}_p(\Gamma, X, H))$ with propagation less than or equal to $r > 0$ is a homology class in $\mathbb{H}^{\lambda,\Gamma}_{n,r}(N(O_{(n+1)r}(X)))$.*

We identify $K_0(\mathcal{S}_p\Gamma)$ with $\lim_{d\to\infty} \lim_X K_0(\mathbb{C}_p(\Gamma, X, H))$ using Proposition 4.10, where the direct limit $\lim_X$ is taken over the directed system of all Γ-invariant, Γ-compact subsets of $P_d(\Gamma)$. We also identify $\lim_{d\to\infty} \lim_X \ \mathbb{H}^{\lambda,\Gamma}_{n,r}(N(O_r(X))))$ with $\lim_{d\to\infty}(\oplus_{k\ even, k\leq n}\ \mathbb{H}^{\Gamma}_k(P_d(\Gamma)))$ using Propositions 4.4 and 4.5, where the direct limit $\lim_X$ is again taken over the directed system of all Γ-invariant, Γ-compact subsets of $P_d(\Gamma)$. Using the projection $\pi_{n',n}$ from $\lim_{d\to\infty}(\oplus_{k\ even, k\leq n'}\ \mathbb{H}^{\Gamma}_k(P_d(\Gamma)))$ to $\lim_{d\to\infty}(\oplus_{k\ even, k\leq n}\ \mathbb{H}^{\Gamma}_k(P_d(\Gamma)))$ for $n' = max\{n, p\}$, the above Connes–Chern character induces a Connes–Chern character:

$$c_n = \pi_{n',n} \circ c_{n'} : K_0(\mathcal{S}_p\Gamma) \to \lim_{d\to\infty}(\oplus_{k\ even,\ k\leq n}\ \ \mathbb{H}^{\Gamma}_k(P_d(\Gamma)))$$

for any non-negative even integer n. This construction gives back the Connes–Chern character in Proposition 4.6.

In the following corollary, we demonstrate a local property of the Connes–Chern character. This local property of the Connes–Chern character plays an important role in the proof of Theorem 1.1.

Corollary 4.12. *Let Γ be a countable group. Let X be a simplicial complex with a simplicial proper and cocompact action of Γ. Let $\tilde{q} = q + q_0$ be an element in $M_m(\mathbb{C}_p(\Gamma, X, H)^+)$ such that $q \in M_m(\mathbb{C}_p(\Gamma, X, H))$, $q_0 \in M_m(\mathbb{C})$, $\tilde{q}$ and q_0 are idempotents. For any non-*

negative integer n, when the propagation of q is sufficiently small, $c_n([\tilde{q}]-[q_0])$ *can be represented by a homology class in* $\oplus_{k\ even,\ k\leq n}\ \mathbb{H}_k^{\Gamma}(X)$*. More generally for each* $i \leq 0$ *and* $p \geq 1$*, the Connes–Chern character* c_n *of an element in* $K_i(\mathbb{C}_p(\Gamma, X, H))$ *with sufficiently small propagation can be represented by a homology class in* $\oplus_{k\ even,\ k\leq n}\ \mathbb{H}_{k+i}^{\Gamma}(X)$*.*

Proof. Let $\tilde{q} = q + q_0$ be an element in $M_m(\mathbb{C}_p(\Gamma, X, H)^+)$ such that $q \in M_m(\mathbb{C}_p(\Gamma, X, H))$, $q_0 \in M_m(\mathbb{C})$, $\tilde{q}$ and q_0 are idempotents. If $n \geq p$ and the propagation of q is less than or equal to $r > 0$, then by Proposition 4.11 we know that $c_n([\tilde{q}] - [q_0])$ can be represented by a homology class in $\mathbb{H}_{n,r}^{\lambda,\Gamma}(N(O_{(n+1)r}(X)))$.

Let

$$r_0 = \inf\ \{\ d(gU_z', U_z') :\ z \in X_r, g \in \Gamma, gU_z' \neq U_z'\},$$

where $\{U_z'\}_{z\in X_r}$ is the open cover $O_r(X)$ in the definition of the Connes–Chern character.

Assume that r is a sufficiently small positive number for the rest of this proof. We have $r_0 > 0$. We choose $\{U_z\}_{z\in X_r}$ such that $\{U_z'\}_{z\in X_r}$ is a good cover.

If $r < r_0$, then we have the following:

[1] by the definition of $\mathbb{H}_{n,r}^{\lambda,\Gamma}(N(O_r(X)))$, the homology group $\mathbb{H}_{n,r}^{\lambda,\Gamma}(N(O_r(X)))$ is equal to $H_n^{\lambda,\Gamma}((N(\widehat{O_r(X}))))$;

[2] by Proposition 4.2, the homology group $H_n^{\lambda,\Gamma}((N(\widehat{O_r(X)})))$ can be identified with $\oplus_{k\ even,k\leq n}\ H_k^{\Gamma}((N(\widehat{O_r(X)}))$;

[3] by the choice of $\{U_z'\}_{z\in X_r}$, the homology group $H_k^{\Gamma}((N(\widehat{O_r(X)})))$ is equal to $\mathbb{H}_k^{\Gamma}(X)$ for each k.

When $n \geq p$, Corollary 4.12 follows from the above statements and the definition of the lower algebraic K-groups. The case of an arbitrary non-negative integer n can be reduced to this special case by considering $n' = max\{n, p\}$ and using the identity $c_n = \pi_{n',n} \circ c_{n'}$, where $\pi_{n',n}$ is the projection from $\oplus_{k\ even,\ k\leq n'}\ \mathbb{H}_{k+i}^{\Gamma}(X)$ to $\oplus_{k\ even,\ k\leq n}\ \mathbb{H}_{k+i}^{\Gamma}(X)$. □

5. Proof of the main result

In this section, we give a proof of Theorem 1.1.

By Proposition 2.4, Theorem 1.1 follows from the following result.

Theorem 5.1. *Let* $\mathcal{S}$ *be the ring of all Schatten class operators on an infinite dimensional and separable Hilbert space. The assembly map*

$$A : H_n^{Or\Gamma}(E_{\mathcal{FIN}}(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty}) \to K_n(\mathcal{S}\Gamma)$$

is rationally injective for any group Γ*.*

Proof. Without loss of generality, we can assume that Γ is countable (this is because every group is an inductive limit of countable groups). We recall that $E_{\mathcal{FIN}}(\Gamma)$ can be identified with $\cup_{d\geq 1} P_d(\Gamma)$. For each $i \leq 0$ and non-negative integer n, composing the assembly map

$$A : H_i^{Or\Gamma}(E_{\mathcal{FIN}}(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty}) \to K_i(\mathcal{S}\Gamma)$$

with the Connes–Chern character

$$c_n : \quad K_i(\mathcal{S}\Gamma) \longrightarrow \lim_{d\to\infty} (\oplus_{k\ even,\ k\leq n}\ \mathbb{H}^{\Gamma}_{k+i}(P_d(\Gamma))),$$

we obtain a homomorphism

$$\psi_{i,n} : \quad H_i^{Or\Gamma}(E_{\mathcal{FIN}}(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty}) \to \lim_{d\to\infty} (\oplus_{k\ even, k\leq n}\ \mathbb{H}^{\Gamma}_{k+i}(P_d(\Gamma))).$$

By using the fact that $H_i^{Or\Gamma}(E_{\mathcal{FIN}}(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty})$ is $\lim_{d\to\infty} H_i^{Or\Gamma}(P_d(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty})$, we obtain a homomorphism

$$\psi_i : \quad H_i^{Or\Gamma}(E_{\mathcal{FIN}}(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty}) \to \lim_{d\to\infty} (\oplus_{k\ even}\ \mathbb{H}^{\Gamma}_{k+i}(P_d(\Gamma)))$$

such that ψ_i coincides with $\psi_{i,n}$ on the image of the natural map from $H_i^{Or\Gamma}(P_d(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty})$ to $H_i^{Or\Gamma}(E_{\mathcal{FIN}}(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty})$ when $n+i$ is greater than or equal to the dimension of $P_d(\Gamma)$. Such a map ψ_i is unique and is independent of the choice of n.

For each simplicial Γ-invariant and Γ-cocompact subspace X of $E_{\mathcal{FIN}}(\Gamma)$, let $\mathbb{C}(\Gamma, X, H) = \cup_{p\geq 1}\mathbb{C}_p(\Gamma, X, H)$, where $\mathbb{C}_p(\Gamma, X, H)$ is as in Definition 4.9. By the definition of the assembly map in [1] and the fact that this assembly map coincides with the classic assembly map (Corollary 6.3 in [1]), K-theory elements in the image of the assembly map

$$A : H_i^{Or\Gamma}(X, \mathbb{K}(\mathcal{S})^{-\infty}) \to K_i(\mathbb{C}(\Gamma, X, H))$$

can be represented by elements with arbitrarily small propagation for $i \leq 0$.

This, together with Corollary 4.12, implies that there exists a map (still denoted by ψ_i)

$$\psi_i : \quad H_i^{Or\Gamma}(X, \mathbb{K}(\mathcal{S})^{-\infty}) \to (\oplus_{k\ even}\ \mathbb{H}^{\Gamma}_{k+i}(X))$$

for each non-positive integer i such that the following diagram commutes:

$$\begin{array}{ccc} H_i^{Or\Gamma}(X, \mathbb{K}(\mathcal{S})^{-\infty}) & \xrightarrow{\psi_i} & (\oplus_{k\ even}\ \mathbb{H}^{\Gamma}_{k+i}(X)) \\ \downarrow j_* & & \downarrow j'_* \\ H_i^{Or\Gamma}(E_{\mathcal{FIN}}(\Gamma), \mathbb{K}(\mathcal{S})^{-\infty}) & \xrightarrow{\psi_i} & \lim_{d\to\infty}(\oplus_{k\ even}\ \mathbb{H}^{\Gamma}_{k+i}(P_d(\Gamma))) \end{array},$$

where j_* and j'_* are respectively induced by the inclusion maps.

If $X = \Gamma/F$ as Γ-spaces for some finite subgroup F of Γ, then it is straightforward to verify that ψ_i is an isomorphism after tensoring with $\mathbb{C}$. In fact, both sides are naturally isomorphic to the group $R(F)\otimes\mathbb{C}$, where $R(F)$ is the representation ring of F viewed as an additive group. Recall that, by Proposition 7.2.3, Remark 7.2.6 and Theorem 8.2.5 in [14], we have $K_n(\mathcal{S}) = \mathbb{Z}$ when n is even and $K_n(\mathcal{S}) = 0$ when n is odd. As a consequence, the homology theory $H_i^{Or\Gamma}(X, \mathbb{K}(\mathcal{S})^{-\infty})$ is 2-periodic. Note that the homology theory $\oplus_{k\ even}\, \mathbb{H}_{k+i}^{\Gamma}(X)$ is 2-periodic by definition. By the proof of the Mayer–Vietoris sequence using the mapping cone and the definition of the Connes–Chern character, we know the homomorphisms ψ_i commute with the Mayer–Vietoris sequences (up to scalars).

Using the above results, the fact that both homology theories satisfy the Mayer–Vietoris sequence and a five lemma argument, we can prove that the map

$$\psi_i : \quad H_i^{Or\Gamma}(X, \mathbb{K}(\mathcal{S})^{-\infty}) \to (\oplus_{k\ even}\, \mathbb{H}_{k+i}^{\Gamma}(X))$$

is an isomorphism after tensoring with $\mathbb{C}$ for $i \leq 0$. This implies that the assembly map A is rationally injective for $i \leq 0$. Now Theorem 5.1 follows from Proposition 3.1. $\square$

We comment that the algebraic K-theory isomorphism conjecture for the ring $\mathcal{S}\Gamma$ can be viewed as an algebraic counterpart of the Baum–Connes conjecture for the K-theory of the reduced group C^*-algebra of Γ [4]. The Farrell–Jones isomorphism conjecture and the Baum–Connes conjecture imply the following conjecture.

Conjecture 5.1. *Let K be the C^*-algebra of all compact operators on an infinite dimensional and separable Hilbert space, let $C_r^*(\Gamma)$ be the reduced group C^*-algebra of Γ. The natural homomorphism*

$$i_* : K_n(\mathcal{S}\Gamma) \to K_n(C_r^*(\Gamma) \otimes K)$$

is an isomorphism, where $C_r^*(\Gamma) \otimes K$ is the C^*-algebraic tensor product of $C_r^*(\Gamma)$ with K and i is the inclusion map from $\mathcal{S}\Gamma$ to $C_r^*(\Gamma) \otimes K$.

We remark that, by a theorem of Suslin–Wodzicki [32], the algebraic K-theory $K_n(C_r^*(\Gamma) \otimes K)$ is isomorphic to the topological K-theory $K_n^{top}(C_r^*(\Gamma))$. Theorem 1.1 implies that the Novikov higher signature conjecture follows from the (rational) injectivity of i_* in the above conjecture.

We speculate that the (algebraic) bivariant K-theory of Cuntz, Cuntz–Thom and Cortiñas–Thom should be useful in studying the algebraic K-theory isomorphism conjecture for $\mathcal{S}\Gamma$ [15–17,13]. Finally we mention several other related works [5,7–10,20,22,23,25,29–31,33,34,36,37].

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