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STABLE FOLIATIONS NEAR A TRAVELING FRONT FOR REACTION DIFFUSION SYSTEMS

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Abstract. We establish the existence of a stable foliation in the vicinity of a traveling front solution for systems of reaction diffusion equations in one space dimension that arise in the study of chemical reactions models and solid fuel combustion. In this way we complement the orbital stability results from earlier papers by A. Ghazaryan, S. Schecter and Y. Latushkin. The essential spectrum of the differential operator obtained by linearization at the front touches the imaginary axis. In spaces with exponential weights, one can shift the spectrum to the left. We study the nonlinear equation on the intersection of the unweighted and weighted spaces. Small translations of the front form a center unstable manifold. For each small translation we prove the existence of a stable manifold containing the translated front and show that the stable manifolds foliate a small ball centered at the front.

1. **Introduction.** Traveling fronts are solutions to partial differential equations which move with constant speed without changing their shapes and are asymptotic to spatially constant steady states. Traveling fronts are important by many reasons and have intensively been studied. We refer to the books and review papers [7, 37, 38] and to more recent sources such as [14, 19, 26, 27, 28, 29, 35] that contain further bibliography.

In this paper we study the dynamics in the vicinity of traveling fronts for a class of reaction diffusion equations in one space dimension. A typical example arising in combustion theory for solid fuels, cf. [4, 10, 21], is given by

$$u_t = u_{xx} + vg(u), \quad v_t = \epsilon v_{xx} + \kappa vg(u), \tag{1.1}$$

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where $u, v \in \mathbb{R}$, $\epsilon \geq 0$, $\kappa \in \mathbb{R}$, and $g(u) = \mathrm{e}^{-1/u}$ for $u > 0$ and $g(u) = 0$ for $u \leq 0$. These and more general equations covered by our hypotheses often appear in the work on chemical reaction models and in combustion models, see, e.g., [13, 32, 33, 34, 36]. In such systems the spectrum of the linearization of the equation at the front touches the imaginary axis, cf. [29, 30]. To shift the spectrum to the left, one employs exponentially weighted spaces. This idea goes back to [31] and [24]. However, in weighted spaces one can lose the Lipschitz properties of the nonlinearity. We shall study reaction terms with a certain "product" structure as in (1.1) which allows one to overcome these difficulties. The investigation of this class of nonlinearities was initiated by A. Ghazaryan in [8] and then continued in [9, 10, 11], see also the review paper [12]. In particular, it was proved in [11] that under appropriate assumptions on the nonlinearity the traveling front is orbitally stable; that is, any solution originating in a small vicinity of the front converges exponentially in the weighted norm to a translation of the front.

In this paper we continue the work in [11] now utilizing the theory of invariant manifolds, cf. [2, 5, 20]. We analyze the dynamics in greater detail by proving in Theorem 4.1 the existence of a stable foliation near the front. Specifically, we observe that the set of all translations of the front serves as a local central unstable manifold consisting of fixed points. Next, using the Lyapunov-Perron method, cf. e.g. [16, 17, 18], we establish the existence and the fundamental properties of a locally invariant stable manifold going through each translation of the front. We also show that these manifolds foliate a small neighborhood of the front and therefore each point in the neighborhood belongs to one of them, cf. [3, 5]. Moreover, the orbit of the point converges to the translation of the front along the stable manifold as proved in [11].

In the construction of the local stable manifolds we have to face the problem that the linearization enjoys good decay properties only in weighted spaces on which the nonlinearity is not locally Lipschitz. To overcome this difficulty, we use both the product structure of the nonlinearity (cf. Hypothesis 2.2) and additional decay properties of the linearization at the limit of the traveling front as $\xi \to -\infty$, see Lemmas 3.1 and 3.2.

The paper is organized as follows. In Section 2 we formulate our assumptions and prove several preliminary results. In Section 3 we study the Lyapunov-Perron operator of which fixed points define the stable manifolds. In Section 4 we formulate and prove our main result on the existence of the stable manifolds and discuss two examples.

Notation. Throughout the paper, $|\cdot|$ and $\langle \cdot, \cdot \rangle$ are the Euclidean norm and the scalar product in $\mathbb{R}^n$. For a given map $f : \mathbb{R}^m \to \mathbb{R}^k$, its differential with respect to y is written as $\partial_y f : \mathbb{R}^m \to \mathcal{B}(\mathbb{R}^m, \mathbb{R}^k)$. We let $\mathcal{B}(\mathcal{E}, \mathcal{F})$ be the set of linear bounded operators between Banach spaces $\mathcal{E}$ and $\mathcal{F}$, and abbreviate $\mathcal{B}(\mathcal{E}) = \mathcal{B}(\mathcal{E}, \mathcal{E})$. We denote by C a generic constant that may change from one estimate to another, and use T to designate transposition. For a Banach space with norm $\|\cdot\|$, we write $\mathbb{B}_\delta(\|\cdot\|)$ for the closed ball of radius δ centered at 0.

We denote by $\mathcal{E}_0$ with norm $|\cdot|_0$ either the Sobolev space H^1 or the space BUC of bounded uniformly continuous functions on $\mathbb{R}$ with vector values, and by $\mathcal{E}_\alpha$ with norm $|\cdot|_\alpha$ the respective space of (exponentially) weighted functions, see (2.12). Let $|\cdot|_\beta$ be the norm on the intersection space $\mathcal{E}_\beta := \mathcal{E}_0 \cap \mathcal{E}_\alpha$; i.e., $|y|_\beta := \max\{|y|_0, |y|_\alpha\}$.

2. **The setting.** We consider the system of reaction diffusion equations

$$Y_t = DY_{xx} + R(Y), \qquad x \in \mathbb{R},\ t \geq 0, \tag{2.1}$$

where $D = \mathrm{diag}(d_1, \dots, d_n)$, $d_j \geq 0$, $Y(t,x) \in \mathbb{R}^n$, and $R : \mathbb{R}^n \to \mathbb{R}^n$ is a C^4 function satisfying additional properties listed below.

Passing in (2.1) to the moving coordinate frame $\xi = x - ct$ and redenoting ξ again by x, we arrive at the nonlinear equation

$$Y_t = DY_{xx} + cY_x + R(Y), \qquad x \in \mathbb{R},\ t \geq 0. \tag{2.2}$$

We discuss the wellposedness of this system in Remark 2.3.

Hypothesis 2.1. *We assume that for some velocity $c > 0$ the system (2.2) admits a stationary solution $Y_0 \in C^4(\mathbb{R})$; i.e, (2.1) possesses the traveling front solution $Y(t,x) = Y_0(x - ct)$. It is also required that $Y_0(x)$ converges to the end states $Y_\pm$ as $x \to \pm\infty$ exponentially; i.e.,*

$$\begin{aligned} |Y_0(x) - Y_-| &\leq C\mathrm{e}^{-\omega_- x}, \qquad x \leq 0, \\ |Y_0(x) - Y_+| &\leq C\mathrm{e}^{-\omega_+ x}, \qquad x \geq 0, \end{aligned} \tag{2.3}$$

for some $\omega_- < 0 < \omega_+$ and $C > 0$. Replacing R by $\tilde{R}(Y) := R(Y + Y_-)$, we can and will assume that $Y_- = 0$ (and we then drop the tilde).

We further assume that the nonlinear term R in (2.1) and (2.2) has the following product structure.

Hypothesis 2.2. *The nonlinear term R belongs to $C^4(\mathbb{R}^n, \mathbb{R}^n)$. In appropriate variables $Y = (U, V)^T$ with $U \in \mathbb{R}^{n_1}$, $V \in \mathbb{R}^{n_2}$ and $n_1 + n_2 = n$, we have*

$$R(U, 0) = (A_1 U, 0) \tag{2.4}$$

for a constant $n_1 \times n_1$ matrix A_1.

In other words, we suppose that

$$R(U,V) = \begin{pmatrix} A_1 U + R_1(U,V) \\ R_2(U,V) \end{pmatrix},$$

where the maps R_j belong to $C^3(\mathbb{R}^n, \mathbb{R}^{n_j})$ and satisfy $R_j(U, 0) = 0$ for $j \in \{1, 2\}$ and $U \in \mathbb{R}^{n_1}$. Note that condition (2.4) yields $R(0,0) = R(Y_-) = 0$. We also split

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \qquad \text{where} \quad D_1 = \mathrm{diag}(d_1, \dots, d_{n_1}), \quad D_2 = \mathrm{diag}(d_{n_1+1}, \dots, d_n).$$

Let $q \in \mathbb{R}$. We write $Y_q(x) = Y_0(x - q)$ for the shifted wave. Since (2.2) is translationally invariant, Y_q is again a steady state solution of (2.2) and thus yields a traveling wave solution for (2.1). Linearizing (2.2) at Y_q (that is, substituting $Y_q + Y$ instead of Y in (2.2)), we arrive at the equation

$$Y_t = L_q Y + F_q(Y), \qquad \text{where} \quad L_q Y = DY_{xx} + cY_x + \partial_Y R(Y_q) Y. \tag{2.5}$$

Here, ∂_Y is the differential with respect to $Y \in \mathbb{R}^n$ and the nonlinear term $F_q : \mathbb{R}^n \to \mathbb{R}^n$ is written as

$$F_q(Y) = \int_0^1 \left(\partial_Y R(Y_q + tY) - \partial_Y R(Y_q)\right) Y \, \mathrm{d}t. \tag{2.6}$$

The linearization of (2.2) at $Y_- = (0,0)^T$ is given by

$$Y_t = L^- Y + G(Y), \qquad \text{where} \quad L^- Y = DY_{xx} + cY_x + \partial_Y R(0) Y \tag{2.7}$$

and $G:\mathbb{R}^n\to\mathbb{R}^n$; $G(Y)=R(Y)-\partial_Y R(0)Y$. We remark that

$$(L_q-L^-)Y=B_qY \qquad \text{with} \quad B_q(x)=\partial_Y R(Y_q(x))-\partial_Y R(0). \tag{2.8}$$

Below we impose conditions on L_0 at $q=0$; i.e., on the linearization at the original traveling wave Y_0. We further consider L_q for $|q|\le q_0$ with some $q_0>0$, which will be fixed sufficiently small in the final theorem. The shifted wave Y_q decays as in Hypothesis 2.1 with the same exponents $\omega_\pm$ and constants C only depending on q_0. Assumption (2.4) also yields the formulas

$$\partial_Y R(0,0)=\begin{pmatrix} A_1 & \partial_V R_1(0,0)\\ 0 & \partial_V R_2(0,0)\end{pmatrix}, \qquad L^-=\begin{pmatrix} L^{(1)} & \partial_V R_1(0,0)\\ 0 & L^{(2)}\end{pmatrix} \tag{2.9}$$

with the differential expressions

$$\begin{aligned} L^{(1)}U &= D_1U_{xx}+cU_x+A_1U,\\ L^{(2)}V &= D_2V_{xx}+cV_x+\partial_V R_2(0,0)V. \end{aligned} \tag{2.10}$$

Remark 2.3. We consider the equations (2.2) and (2.5) on the space $\mathcal{E}_0$ which is either the Sobolev space $H^1(\mathbb{R})^n$ or the space of bounded uniformly continuous functions $BUC(\mathbb{R})^n$. It is straightforward to check that the nonlinearities R and F_q are Lipschitz on bounded subsets of $\mathcal{E}_0$.

For the differential expressions L_q and L^- defined in (2.5) and (2.7), respectively, we denote by $\mathcal{L}_q$ and $\mathcal{L}^-$ the differential operators on $\mathcal{E}_0$ on their natural domain $\mathcal{D}$ defined as follows. For $\mathcal{E}_0=H^1(\mathbb{R})^n$, the domain $\mathcal{D}$ of $\mathcal{L}_q$ and of $\mathcal{L}^-$ consists of the vector functions $Y=(Y_j)_{j=1}^n$ whose components Y_j belong to $H^3(\mathbb{R})$ if $d_j>0$ and to $H^2(\mathbb{R})$ if $d_j=0$. For $\mathcal{E}_0=BUC(\mathbb{R})^n$, we choose the domain analogously with $H^3(\mathbb{R})$ replaced by $BUC^2(\mathbb{R})$ and $H^2(\mathbb{R})$ replaced by $BUC^1(\mathbb{R})$, the spaces of differentiable functions which are bounded and have bounded, uniformly continuous derivatives. The operators $\mathcal{L}_q$ and $\mathcal{L}^-$ generate strongly continuous semigroups $\{T_q(t)\}_{t\ge0}$ and $\{S(t)\}_{t\ge0}$ on $\mathcal{E}_0$, respectively, cf. e.g. [9, §2.2].

Standard results then show the local wellposedness of (2.5) in $\mathcal{E}_0$ for initial values y_0 in the domain of $\mathcal{L}_q$, where the (classical) solutions belong to $C^1([0,t_0),\mathcal{E}_0)$ and take values in $\mathcal{D}$. They are given by Duhamel's formula

$$Y(t)=T_q(t)y_0+\int_0^t T_q(t-\tau)F_q(Y(\tau))\,\mathrm{d}\tau, \qquad t\ge 0. \tag{2.11}$$

See e.g. Theorems 6.1.4 and 6.1.6 in [23]. A function $Y\in C([0,t_0),\mathcal{E}_0)$ satisfying (2.11) is called a *mild solution* of (2.5). This concept is strictly weaker than that of classical solvability. We mostly work with mild solutions. Similar remarks apply to (2.2) and the differential expression $D\partial_{xx}+c\partial_x$ equipped with the same domain $\mathcal{D}$. Approximating a given initial value Y_q+y_0 with $y_0\in\mathcal{E}_0$ in $Y_q+\mathcal{E}_0$ by functions in $Y_q+\mathcal{D}$, we see that all mild solutions of (2.2) are given by $Y_q+Y(t)$ where $Y(t)$ solves (2.11). ◇

Let $\alpha=(\alpha_-,\alpha_+)\in\mathbb{R}^2$. We say that $\gamma_\alpha:\mathbb{R}\to\mathbb{R}$ is a weight function of class α if γ_α is C^3, $\gamma_\alpha(x)>0$ for all $x\in\mathbb{R}$, and $\gamma_\alpha(x)=\mathrm{e}^{\alpha_-x}$ for $x\le -x_0$ and $\gamma_\alpha(x)=\mathrm{e}^{\alpha_+x}$ for $x\ge x_0$ for some $x_0>0$. We shall always assume that

$$0<\alpha_-<-\omega_- \ \text{ and } \ 0\le\alpha_+<\omega_+, \tag{2.12}$$

where $\omega_\pm$ are the exponents mentioned in (2.3). Given such a pair $\alpha=(\alpha_-,\alpha_+)$, we introduce the weighted space $\mathcal{E}_\alpha=\{u:\mathbb{R}\to\mathbb{R}^n:\gamma_\alpha u\in\mathcal{E}_0\}$ with the norm $|u|_\alpha=|\gamma_\alpha u|_0$. (Recall that $\mathcal{E}_0$ with norm $|\cdot|_0$ is either $H^1(\mathbb{R})^n$ or $BUC(\mathbb{R})^n$.) The intersection space $\mathcal{E}_\beta = \mathcal{E}_0 \cap \mathcal{E}_\alpha$ is endowed with the norm $|u|_\beta = \max\{|u|_0, |u|_\alpha\}$. The differential expressions L_q, L^- etc. equipped with their natural domains define operators in $\mathcal{E}_\alpha$ which are denoted by $\mathcal{L}_{q,\alpha}$, $\mathcal{L}_\alpha^-$ etc. (cf. Remark 2.3). On the spectrum of $\mathcal{L}_{0,\alpha}$, we impose the following assumptions.

Hypothesis 2.4. *In addition to Hypotheses 2.1 and 2.2, we assume that there exists $\alpha = (\alpha_-, \alpha_+) \in \mathbb{R}^2$ such that (2.12) with $\omega_\pm$ from (2.3) and the following assertions hold.*

(a) $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}_{\text{ess}}(\mathcal{L}_{0,\alpha})\} < 0$ *for the differential expression L_0 defined in (2.5).*
(b) The only element of $\operatorname{Sp}(\mathcal{L}_{0,\alpha})$ *in* $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\}$ *is a simple eigenvalue at $\lambda = 0$ with Y_0' being the respective eigenfunction.*

Here the essential spectrum $\operatorname{Sp}_{\text{ess}}(A)$ of a closed densely defined operator contains all points in the spectrum $\operatorname{Sp}(A)$ which are not isolated eigenvalues of finite algebraic multiplicity. We discuss various consequences of the above hypothesis which are important for our proofs.

Lemma 2.5. *Let Hypotheses 2.1 and 2.2 hold. We claim that assertions (a) and (b) in Hypothesis 2.4 are satisfied for $\mathcal{E}_0 = H^1(\mathbb{R})^n$ or $\mathcal{E}_0 = BUC(\mathbb{R})^n$ if and only if they hold when $\mathcal{E}_0$ is replaced by the space $L_2(\mathbb{R})^n$ and $\mathcal{E}_\alpha$ by the space $L^2_\alpha(\mathbb{R})^n$ of functions u with $\gamma_\alpha u \in L^2(\mathbb{R})$ which is endowed with the norm $|u|_\alpha = |\gamma_\alpha u|_{L^2}$.*

Proof. The "if" part of the lemma is proved in Lemma 3.8 of [11]. So we assume Hypothesis 2.4 for $\mathcal{E}_0 = H^1(\mathbb{R})^n$ or $\mathcal{E}_0 = BUC(\mathbb{R})^n$. Then assertion (a) of this hypothesis for $\mathcal{E}_0 = L^2(\mathbb{R})^n$ is true since the right-hand boundary of the essential spectra of $\mathcal{L}_{0,\alpha}$ is the same for all three spaces by [11, Lemma 3.5].

To show assertion (b) for $\mathcal{E}_0 = L^2(\mathbb{R})^n$, we assume that $\mathcal{L}_{0,\alpha}$ on $L^2_\alpha(\mathbb{R})^n$ has an isolated eigenvalue λ of finite algebraic multiplicity with $\operatorname{Re}\lambda \geq 0$. By means of the isomorphism $u(\cdot) \mapsto \gamma(\cdot)u(\cdot)$ between $L^2_\alpha(\mathbb{R})^n$ and $L^2(\mathbb{R})^n$ we obtain a differential operator $\hat{\mathcal{L}}$ in $L^2(\mathbb{R})^n$ which is similar to $\mathcal{L}_{0,\alpha}$ in $L^2_\alpha(\mathbb{R})^n$, cf. [11, Eqn. (3.2)], and hence possesses the unstable isolated eigenvalue λ, too. Palmer's Dichotomy Theorem in [22] says that the first order system corresponding to the second order eigenvalue problem for $\hat{\mathcal{L}}$ admits exponential dichotomies on $\mathbb{R}_-$ and $\mathbb{R}_+$. Arguing as in the proof of Lemma 3.8 of [11], we see that the respective eigenfunction Z decays exponentially as $x \to \pm\infty$ since its value at zero belongs to the intersection of the dichotomy subspaces. The eigenfunction thus belongs to $BUC(\mathbb{R})^n$, and also to $H^1(\mathbb{R})^n$ since Z_x can be bounded by Z itself due to the eigenvalue equation, see (3.3) in [11]. As a result, $\hat{\mathcal{L}}$ in $H^1(\mathbb{R})^n$ or $BUC(\mathbb{R})^n$ has the unstable eigenvalue λ of the same multiplicity and therefore also $\mathcal{L}_{0,\alpha}$ in $\mathcal{E}_\alpha$. Hypothesis 2.4(b) for $\mathcal{E}_0 = H^1(\mathbb{R})^n$ or $\mathcal{E}_0 = BUC(\mathbb{R})^n$ now shows that $\lambda = 0$ and that it is simple, completing the proof of the lemma. □

The exponential decay of the eigenfunction in the proof of the previous lemma also follows from a general (even multidimensional) result of this type proved in [25, Theorem 3.1(iii)].

Lemma 2.6. *Assume that Hypothesis 2.4 holds. Then assertions (a) and (b) in Hypothesis 2.4 are satisfied by the operator $\mathcal{L}_{q,\alpha}$ instead of $\mathcal{L}_{0,\alpha}$ and by the function Y_q' instead of Y_0'.*

Proof. The operators $\mathcal{L}_{q,\alpha}$ and $\mathcal{L}_{0,\alpha}$ are similar via the transformation $Y \mapsto Y(\cdot - q)$ which also maps Y' into Y_q'. The assertions then easily follow. □

Lemma 2.6 says that $\lambda = 0$ is an isolated simple eigenvalue for $\mathcal{L}_{q,\alpha}$. We let P_q^c denote the spectral projection for $\mathcal{L}_{q,\alpha}$ in $\mathcal{E}_\alpha$ onto $\ker \mathcal{L}_{q,\alpha} = \operatorname{span}\{Y_q'\}$, and write $P_q^s = I - P_q^c$ for the complementary projection. The following lemma collects important properties of these operators.

Lemma 2.7. *Assume Hypothesis 2.4. Let $q_0 > 0$. The projection P_q^c is given by*

$$P_q^c Y = \zeta_q(Y)\, Y_q', \qquad \zeta_q(Y_q') = 1, \tag{2.13}$$

for an element ζ_q in the one dimensional kernel of $\mathcal{L}_{q,\alpha}^*$. It induces maps

$$P_q^c \in \mathcal{B}(\mathcal{E}_\alpha) \cap \mathcal{B}(\mathcal{E}_\beta, \mathcal{E}_\alpha) \cap \mathcal{B}(\mathcal{E}_\alpha, \mathcal{E}_\beta) \cap \mathcal{B}(\mathcal{E}_\beta) \cap \mathcal{B}(\mathcal{E}_\alpha, \mathcal{E}_0) \cap \mathcal{B}(\mathcal{E}_\beta, \mathcal{E}_0),$$
$$P_q^s \in \mathcal{B}(\mathcal{E}_\alpha) \cap \mathcal{B}(\mathcal{E}_\beta) \cap \mathcal{B}(\mathcal{E}_\beta, \mathcal{E}_\alpha) \cap \mathcal{B}(\mathcal{E}_\beta, \mathcal{E}_0),$$

whose norms are uniformly bounded for $|q| \le q_0$ (We use the same notation P_q^c and P_q^s on all these spaces.). The projections further satisfy

$$\|P_q^c - P_p^c\|_{\mathcal{B}(\mathcal{E}_\beta)} \le C|q-p|, \qquad \|P_q^c - P_p^c\|_{\mathcal{B}(\mathcal{E}_\alpha)} \le C|q-p| \tag{2.14}$$

for $|p|, |q| \le q_0$ and a constant independent of p and q.

Proof. Basic operator theory (see, e.g., [6, Lemma 2.13]) yields

$$\operatorname{ran}(I_{\mathcal{E}_\alpha} - P_q^c) = \ker P_q^c = \operatorname{ran}(\mathcal{L}_{q,\alpha}).$$

Assertion (2.13) then follows since the kernel of the adjoint operator is also one dimensional, cf. [15, Theorem IV.5.13]. As in the proof of Lemma 2.6, the operators $\mathcal{L}_{q,\alpha}^*$ and $\mathcal{L}_{0,\alpha}^*$ are similar and therefore the norms of $\zeta_q \in \mathcal{E}_\alpha^*$ are bounded uniformly for $|q| \le q_0$. Also, in view of Lemma 3.3 in [11], the first three derivatives of the shifted wave Y_q are bounded by $C\mathrm{e}^{-\omega_- \xi}$ for $\xi \le 0$ and by $C\mathrm{e}^{-\omega_+ \xi}$ for $\xi \ge 0$ with $\omega_\pm$ from Hypothesis 2.1 and constants C only depending on q_0. We conclude that

$$|P_q^c Y|_\alpha = |\zeta_q(Y)|\, |Y_q'|_\alpha \le C|Y|_\alpha\, |Y_q'|_\alpha \le C|Y|_\beta\, |Y_q'|_\alpha,$$
$$|P_q^c Y|_0 = |\zeta_q(Y)|\, |Y_q'|_0 \le C|Y|_\alpha\, |Y_q'|_0 \le C|Y|_\beta\, |Y_q'|_0,$$

which yields the asserted mapping properties of P_q^c and P_q^s.

To show (2.14), we note that (2.5) yields

$$\begin{aligned} L_q - L_p &= \partial_Y R(Y_0(\cdot - q)) - \partial_Y R(Y_0(\cdot - p)) \\ &= \int_0^1 \partial_{YY} R(sY_q + (1-s)Y_p)\,\mathrm{d}s\,[Y_0(\cdot - q) - Y_0(\cdot - p)], \end{aligned}$$

$$Y_0(x-q) - Y_0(x-p) = -\int_0^1 Y_0'(x - p - s(q-p))(q-p)\,\mathrm{d}s. \tag{2.15}$$

For $\mathcal{E}_0 = BUC$ we deduce

$$\begin{aligned} \|\mathcal{L}_{q,\alpha} - \mathcal{L}_{p,\alpha}\|_{\mathcal{B}(\mathcal{E}_\alpha)} &= \sup_{x\in\mathbb{R}} |\partial_Y R(Y_0(x-q)) - \partial_Y R(Y_0(x-p))| \le C|q-p|, \\ \|\mathcal{L}_q - \mathcal{L}_p\|_{\mathcal{B}(\mathcal{E}_0)} &= \sup_{x\in\mathbb{R}} |\partial_Y R(Y_0(x-q)) - \partial_Y R(Y_0(x-p))| \le C|q-p|, \end{aligned} \tag{2.16}$$

and similarly for $\mathcal{E}_0 = H^1$. These estimates can easily be transferred to the resolvents on a sufficiently small circle around 0 which implies the claim (2.14). □

Remark 2.8. To provide extra information, we now determine ζ_q from (2.13) as a solution of a differential equation. Lemma 2.5 yields that Hypothesis 2.4 is also true if we replace $\mathcal{E}_0$ by $L^2(\mathbb{R})$. We first determine ζ_q for the operator $\mathcal{L}_{q,\alpha}^*$ acting on the dual $L_\alpha^2(\mathbb{R})^*$ of the space $L_\alpha^2(\mathbb{R})$ of functions with the exponential weight γ_α. We recall that the operator $\gamma_\alpha : L_\alpha^2(\mathbb{R}) \to L^2(\mathbb{R})$; $Y(\cdot) \mapsto \gamma_\alpha(\cdot)Y(\cdot)$, is an isometric isomorphism. Moreover, $L^2_\alpha(\mathbb{R})^*$ can be identified with L^2-space with the weight $1/\gamma_\alpha$, where the duality map between $L^2_\alpha(\mathbb{R})$ and $L^2_\alpha(\mathbb{R})^*$ is given by the usual (real) L^2-scalar product. Hence, the adjoint operator $\gamma^*_\alpha : L^2(\mathbb{R}) \to L^2_\alpha(\mathbb{R})^*$ coincides with the multiplication operator by γ_α.

The operator $\gamma_\alpha \mathcal{L}_{q,\alpha}\gamma_\alpha^{-1}$ in $L^2(\mathbb{R})$ is Fredholm since it is similar to the Fredholm operator $\mathcal{L}_{q,\alpha}$ in $L^2_\alpha(\mathbb{R})$. The adjoint of $\gamma_\alpha \mathcal{L}_{q,\alpha}\gamma_\alpha^{-1}$ in $L^2(\mathbb{R})$ is also Fredholm, and it is equal to $\gamma_\alpha^{-1}\mathcal{L}^*_{q,\alpha}\gamma_\alpha$ since $\gamma^*_\alpha = \gamma_\alpha$. We note that the dimension of the kernels is preserved by similarity and duality. The functional $\zeta_q \in \ker \mathcal{L}^*_{q,\alpha}$ from (2.13) is then represented by $\zeta_q = \gamma_\alpha Z_q$ where $Z_q \in L^2(\mathbb{R})$ belongs to $\ker\big(\gamma_\alpha^{-1}\mathcal{L}^*_{q,\alpha}\gamma_\alpha\big)$. In other words, $Z_q \in L^2(\mathbb{R})$ is the unique (up to a normalization) solution on $\mathbb{R}$ of the differential equation $\big(\gamma_\alpha^{-1}\mathcal{L}^*_{q,\alpha}\gamma_\alpha\big)Z_q = 0$. Reasoning as in the proof of Lemma 3.8 in [11] (see also Lemma 2.5) we conclude that the solution Z_q decays exponentially to zero as $x \to \pm\infty$. Moreover, Z_q is the translation $Z_0(\cdot - q)$ of Z_0, and the decay of the function Z_q is thus uniform in q for $|q| \le q_0$. Formula (2.13) now yields

$$P^c_q Y = \pi_q(Y)Y'_q \qquad \text{with} \quad \pi_q(Y) = \int_{\mathbb{R}} \langle Z_q(x), \gamma_\alpha(x)Y(x)\rangle \,\mathrm{d}x \tag{2.17}$$

for all $Y \in L^2_\alpha(\mathbb{R})$, where Z_q is the exponentially decaying function normalized such that $\pi_q(Y'_q) = 1$.

The exponential decay of Z_q also follows from a general (even multidimensional) result of this type proved in [25, Theorem 3.1(iii)].

Finally, returning to the cases $\mathcal{E}_0 = H^1(\mathbb{R})^n$ or $\mathcal{E}_0 = BUC(\mathbb{R})^n$, we notice that $\pi_q(\cdot)$ is a bounded functional on $\mathcal{E}_\alpha$ in both cases. Using also the decay properties of Y'_q recalled in the proof of Lemma 2.7, we confirm from (2.17) once again that P^c_q is a bounded operator from both $\mathcal{E}_\beta$ and $\mathcal{E}_\alpha$ into $\mathcal{E}_\beta$, with uniform constants for $q \in [-q_0, q_0]$. $\diamond$

Let B_q be multiplication operator induced by the matrix valued function $B_q(\cdot)$ from (2.8). The next result follows from Lemma 8.2 of [11] and its proof.

Lemma 2.9. *Assume Hypothesis 2.4. The operator B_q belongs to $\mathcal{B}(\mathcal{E}_\alpha, \mathcal{E}_0)$ and satisfies $\|B_q - B_p\|_{\mathcal{B}(\mathcal{E}_\alpha,\mathcal{E}_0)} \le C|q-p|$ for $q, p \in [-q_0, q_0]$. Here the constant does not depend on p and q, but on q_0.*

The operators $\mathcal{L}_q$ and $\mathcal{L}_{q,\alpha}$ generate strongly continuous semigroups on $\mathcal{E}_0$ and $\mathcal{E}_\alpha$, respectively, which are both denoted by $\{T_q(t)\}_{t\ge 0}$, see e.g. [9, §2.2]. By Lemma 2.6, there are numbers

$$0 > -\nu > \sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_{q,\alpha}) \setminus \{0\}\},$$

Lemma 3.13 of [11] then yields the exponential decay

$$\|T_q(t)P^s_q\|_{\mathcal{B}(\mathcal{E}_\alpha)} \le C\mathrm{e}^{-\nu t}, \qquad t \ge 0, \tag{2.18}$$

see also [9]. The constant C can be chosen uniform in q because of the transformation used in the proof of Lemma 2.6.

Also the operators $\mathcal{L}^-$ and $\mathcal{L}^-_\alpha$ generate strongly continuous semigroups on $\mathcal{E}_0$ and $\mathcal{E}_\alpha$, designated by $\{S(t)\}_{t\ge 0}$. Since the multiplication operator B_q is bounded on these spaces, formula (2.8) implies the variation of constant formula

$$T_q(t-\tau) = S(t-\tau) + \int_\tau^t S(t-s)B_qT_q(s-\tau)\,\mathrm{d}s, \qquad t \ge \tau \ge 0,\ q \in \mathbb{R}. \tag{2.19}$$

The upper triangular structure of the operator $\mathcal{L}^-$ indicated in (2.9) implies an analogous representation of the semigroup

$$S(t) = \begin{pmatrix} S_1(t) & Q(t) \\ 0 & S_2(t) \end{pmatrix} \qquad \text{and} \qquad Q(t) = \int_0^t S_1(t-s)\partial_V R_1(0,0) S_2(s)\,\mathrm{d}s. \tag{2.20}$$

Here $\{S_1(t)\}_{t\geq 0}$ and $\{S_2(t)\}_{t\geq 0}$ are the semigroups generated by the operators $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ from (2.10), respectively. On these semigroups we impose the following assumptions.

Hypothesis 2.10. *The strongly continuous semigroup $\{S_1(t)\}_{t\geq 0}$ is bounded and the semigroup $\{S_2(t)\}_{t\geq 0}$ is uniformly exponentially stable on $\mathcal{E}_0$; that is,*

$$\|S_1(t)\|_{\mathcal{B}(\mathcal{E}_0)} \leq C, \qquad \|S_2(t)\|_{\mathcal{B}(\mathcal{E}_0)} \leq C\mathrm{e}^{-\rho t}$$

for some $\rho > 0$ and all $t \geq 0$.

In particular, $\mathrm{Re}(\mathrm{Sp}(A_1)) \leq 0$ and $\mathrm{Re}(\mathrm{Sp}(\partial_V R_2(0,0))) < 0$; see also [11, Appendix A] on further comments on the relations of the hypothesis and the spectrum of A_1.

Hypothesis 2.10 and (2.20) imply the boundedness of $\{S(t)\}_{t\geq 0}$ on $\mathcal{E}_0$; i.e.,

$$\|S(t)\|_{\mathcal{B}(\mathcal{E}_0)} \leq C, \qquad t \geq 0. \tag{2.21}$$

We next show that the semigroup $\{T_q(t)\}_{t\geq 0}$ is bounded on the space $\mathcal{E}_\beta$, too

Lemma 2.11. *Assume Hypotheses 2.4 and 2.10. Take $q_0 > 0$ and let $\alpha = (\alpha_-, \alpha_+)$ satisfy (2.12). Then we have*

$$\sup_{|q|\leq q_0} \sup_{t\geq 0} \|T_q(t)\|_{\mathcal{B}(\mathcal{E}_\beta)} < \infty. \tag{2.22}$$

Proof. The variation of constant formula (2.19) yields on $\mathcal{E}_\beta$

$$T_q(t)P_q^s = S(t)P_q^s + \int_0^t S(t-s)B_q T_q(s)P_q^s\,\mathrm{d}s. \tag{2.23}$$

As noted in Lemma 2.7 and (2.21), the projection P_q^s belongs to $\mathcal{B}(\mathcal{E}_\beta, \mathcal{E}_0)$ and to $\mathcal{B}(\mathcal{E}_\beta, \mathcal{E}_\alpha)$ while the semigroup $S(t)$ is uniformly bounded in $\mathcal{E}_0$ for $|q| \leq q_0$ and $t \geq 0$, respectively. Using (2.23), these facts, Lemma 2.9 and the exponential decay in (2.18), we can estimate

$$\begin{aligned}
\|T_q(t)P_q^s\|_{\mathcal{B}(\mathcal{E}_\beta,\mathcal{E}_0)} &\leq C\|P_q^s\|_{\mathcal{B}(\mathcal{E}_\beta,\mathcal{E}_0)} \\
&\qquad + C\int_0^t \|B_q\|_{\mathcal{B}(\mathcal{E}_\alpha,\mathcal{E}_0)}\|T_q(s)P_q^s\|_{\mathcal{B}(\mathcal{E}_\alpha)}\|P_q^s\|_{\mathcal{B}(\mathcal{E}_\beta,\mathcal{E}_\alpha)}\,\mathrm{d}s \\
&\leq C + C\int_0^t \mathrm{e}^{-\nu t}\,\mathrm{d}s \leq C
\end{aligned}$$

for all $t \geq 0$ and $|q| \leq q_0$, with uniform constants. In view of the inequality $\|T_q(t)P_q^s\|_{\mathcal{B}(\mathcal{E}_\alpha)} \leq C\mathrm{e}^{-\nu t}$ from (2.18), we have proved (2.22) with $T_q(t)$ replaced by $T_q(t)P_q^s$. Writing the semigroup as $T_q(t) = T_q(t)P_q^s + T_q(t)P_q^c$ on $\mathcal{E}_\beta$, it remains to show (2.22) with $T_q(t)$ replaced by $T_q(t)P_q^c$. Recall from Lemma 2.7 that $P_q^c = I - P_q^s \in \mathcal{B}(\mathcal{E}_\beta)$ projects $\mathcal{E}_\beta$ onto the kernel of the generators $\mathcal{L}_{q,0}$ and $\mathcal{L}_{q,\alpha}$ of the semigroup $\{T_q(t)\}_{t\geq 0}$ on $\mathcal{E}_0$ and $\mathcal{E}_\alpha$. We conclude that $T_q(t)P_q^c Y = P_q^c Y$ for all $Y \in \mathcal{E}_\beta$ and $t \geq 0$. Therefore, $\|T_q(t)P_q^c\|_{\mathcal{B}(\mathcal{E}_\beta)} \leq C$ for $t \geq 0$, completing the proof of (2.22). □

3. **The Lyapunov-Perron operator.** In this section we introduce the Lyapunov-Perron operator associated with the nonlinear equation (2.5) and show that it is a contraction of a small ball in a certain space of functions $u : \mathbb{R} \to \mathcal{E}_0 \cap \mathcal{E}_\alpha$. First, we establish the main technical estimates for the nonlinearity $F_q : \mathbb{R}^n \to \mathbb{R}^n$ defined in (2.6).

Lemma 3.1. *Assume that $\alpha = (\alpha_-, \alpha_+)$ satisfies (2.12) and that the nonlinearity $R \in C^4(\mathbb{R}^n, \mathbb{R}^n)$ fulfills (2.4). Let $\delta_1 > 0$ and choose a radius $\delta \in (0, \delta_1]$. Then for all functions $y = (u, v)$ and $\bar{y} = (\bar{u}, \bar{v})$ from $\mathcal{E}_\beta$ with $|y|_\beta, |\bar{y}|_\beta \le \delta$ the estimates*

$$|F_q(y)|_0 \le C|y|_0 \,(|y|_\alpha + |v|_0), \tag{3.1}$$

$$|F_q(y)|_\alpha \le C|y|_0 \,|y|_\alpha, \tag{3.2}$$

$$|F_q(y) - F_q(\bar{y})|_0 \le C\big(|y - \bar{y}|_0 \,(|y|_\alpha + |\bar{y}|_\alpha) + |y - \bar{y}|_0 \,|v|_0 + |\bar{y}|_0 \,|v - \bar{v}|_0\big), \tag{3.3}$$

$$|F_q(y) - F_q(\bar{y})|_\alpha \le C|y - \bar{y}|_\alpha \,(|y|_0 + |\bar{y}|_0) \tag{3.4}$$

are true, where $C = C(\delta_1, q_0)$ and $|q| \le q_0$.

Proof. Let $|y|_\beta, |\bar{y}|_\beta \le \delta \le \delta_1$. From the proof of Lemma 8.3 in [11] we recall the representation

$$F_q(y) = I_1(y) + I_2(y) + I_3(y) + I_4(y) + I_5(y),$$

where $Y_q = (U_q, V_q)$, $y = (u, v)$,

$$I_1(y) = \int_0^1 (\partial_u r(Y_q + ty) - \partial_u r(Y_q))\, uV_q \,\mathrm{d}t,$$

$$I_2(y) = \int_0^1 (\partial_u r(Y_q + ty)u)\, tv \,\mathrm{d}t,$$

$$I_3(y) = \int_0^1 (\partial_v r(Y_q + ty) - \partial_v r(Y_q))\, vV_q \,\mathrm{d}t,$$

$$I_4(y) = \int_0^1 (\partial_v r(Y_q + ty)v)\, tv \,\mathrm{d}t,$$

$$I_5(y) = \int_0^1 (r(Y_q + ty) - r(Y_q))\, v \,\mathrm{d}t,$$

and the function $r \in C^3(\mathbb{R}^n, \mathbb{R}^{n\times n})$ is given by

$$r(u, v) = \int_0^1 \partial_v R(u, tv) \,\mathrm{d}t.$$

We note that r is only applied to functions which are uniformly bounded by $C(1 + \delta_1)$. It is then straightforward to check the inequalities $|I_j(y)|_0 \le C|y|_0 \,|v|_0$ for $j \in \{2, \ldots, 5\}$ and $|I_j(y)|_\alpha \le C|y|_0 \,|y|_\alpha$ for $j \in \{1, 2, \ldots, 5\}$. Since $uV_q = (\gamma_\alpha u)(\gamma_\alpha^{-1} V_q)$ and $(\gamma_\alpha^{-1} V_q) \in BUC^1(\mathbb{R})^{n_1}$ by Lemma 3.7 of [11], we can further estimate $|I_1(y)|_0 \le C|y|_0 \,|y|_\alpha$, finishing the proof of (3.1) and (3.2). Here and below the constants only depend on δ_1 and q_0.

To show (3.3) and (3.4), we deal with each integral I_j separately. The terms $|y - \bar{y}|_0 \,(|y|_\alpha + |\bar{y}|_\alpha)$ and $|y - \bar{y}|_\alpha \,(|y|_0 + |\bar{y}|_0)$ come from I_1 while the remaining ones originate from I_2 through I_5. We first represent $I_1(y) - I_1(\bar{y})$ as

$$I_1(y) - I_1(\bar{y}) = \int_0^1 \int_0^1 \partial_y \partial_u r(Y_q + st(y - \bar{y}) + t\bar{y}) uV_q t(y - \bar{y}) \,\mathrm{d}s \,\mathrm{d}t$$

$$+\int_0^1\int_0^1 \partial_y\partial_u r(Y_q+st\bar y)(u-\bar u)V_q t\bar y\,\mathrm{d}s\,\mathrm{d}t. \tag{3.5}$$

Using $uV_q(y-\bar y) = (\gamma_\alpha u)(\gamma_\alpha^{-1}V_q)(y-\bar y)$ and $(u-\bar u)V_q\bar y = (u-\bar u)\,\gamma_\alpha^{-1}V_q\,\gamma_\alpha y$ as above, we conclude that $|I_1(y)-I_1(\bar y)|_0 \le C|y-\bar y|_0\,(|y|_\alpha + |\bar y|_\alpha)$. If we multiply (3.5) by γ_α, we directly estimate $|I_1(y)-I_1(\bar y)|_\alpha \le C(|y|_0+|\bar y|_0)\,|y-\bar y|_\alpha$ since $|u|\le|y|$. Likewise, we write $I_5(y)-I_5(\bar y)$ as

$$\begin{aligned} I_5(y)-I_5(\bar y) &= \int_0^1\int_0^1 (\partial_y r(Y_q+sty)-\partial_y r(Y_q+st\bar y))\,tyv\,\mathrm{d}s\,\mathrm{d}t\\ &\quad+\int_0^1\int_0^1 \partial_y r(Y_q+ts\bar y)tv(y-\bar y)\,\mathrm{d}s\,\mathrm{d}t\\ &\quad+\int_0^1\int_0^1 \partial_y r(Y_q+st\bar y)t\bar y(v-\bar v)\,\mathrm{d}s\,\mathrm{d}t \end{aligned} \tag{3.6}$$

and obtain the bound $|I_5(y)-I_5(\bar y)|_0 \le C(|y-\bar y|_0\,|v|_0 + |\bar y|_0\,|v-\bar v|_0)$, recalling that $|y|_0\le\delta_1$ by assumption. After multiplying (3.6) by γ_α, it also follows that $|I_5(y)-I_5(\bar y)|_\alpha \le C(|y|_0\,|y-\bar y|_\alpha + |\bar y|_0\,|v-\bar v|_\alpha)$ since $|v|\le|y|$. Similarly, the formulas

$$\begin{aligned} I_2(y)-I_2(\bar y) &= \int_0^1 \big(\partial_u r(Y_q+ty)-\partial_u r(Y_q+t\bar y)\big)utv\,\mathrm{d}t &&(3.7)\\ &\quad+\int_0^1 \partial_u r(Y_q+t\bar y)(u-\bar u)tv\,\mathrm{d}t + \int_0^1 \partial_u r(Y_q+t\bar y)\bar u t(v-\bar v)\,\mathrm{d}t,\\ I_4(y)-I_4(\bar y) &= \int_0^1 \big(\partial_v r(Y_q+ty)-\partial_v r(Y_q+t\bar y)\big)vtv\,\mathrm{d}t &&(3.8)\\ &\quad+\int_0^1 \partial_v r(Y_q+t\bar y)(v-\bar v)tv\,\mathrm{d}t + \int_0^1 \partial_v r(Y_q+t\bar y)\bar v t(v-\bar v)\,\mathrm{d}t \end{aligned}$$

imply the inequalities

$$\begin{aligned} |I_2(y)-I_2(\bar y)|_0 &\le C(|y-\bar y|_0\,|v|_0+|\bar y|_0\,|v-\bar v|_0),\\ |I_4(y)-I_4(\bar y)|_0 &\le C(|y-\bar y|_0\,|v|_0+|\bar y|_0\,|v-\bar v|_0). \end{aligned}$$

Multiplying (3.7) and (3.8) by γ_α, we also derive

$$\begin{aligned} |I_2(y)-I_2(\bar y)|_\alpha &\le C(|y|_0\,|y-\bar y|_\alpha+|\bar y|_0\,|v-\bar v|_\alpha),\\ |I_4(y)-I_4(\bar y)|_\alpha &\le C(|y|_0\,|y-\bar y|_\alpha+|\bar y|_0\,|v-\bar v|_\alpha). \end{aligned}$$

We finally compute

$$\begin{aligned} I_3(y)-I_3(\bar y) &= \int_0^1\int_0^1 \partial_y\partial_v r(Y_q+st(y-\bar y)+t\bar y)vV_q t(y-\bar y)\,\mathrm{d}s\,\mathrm{d}t\\ &\quad+\int_0^1\int_0^1 \partial_y\partial_v r(Y_q+st\bar y)(v-\bar v)V_q t\bar y\,\mathrm{d}s\,\mathrm{d}t. \end{aligned}$$

Again we infer that

$$\begin{aligned} |I_3(y)-I_3(\bar y)|_0 &\le C(|y-\bar y|_0\,|v|_0+|\bar y|_0\,|v-\bar v|_0),\\ |I_3(y)-I_3(\bar y)|_\alpha &\le C(|y|_0\,|y-\bar y|_\alpha+|\bar y|_0\,|v-\bar v|_\alpha). \end{aligned}$$

This completes the proof of the lemma. □

It follows from the observations after Lemma 2.9 that the realization of L_q in $\mathcal{E}_\beta = \mathcal{E}_0 \cap \mathcal{E}_\alpha$ generates a strongly continuous semigroup. The Lipschitz properties proved in the above lemma thus imply that the semilinear equation (2.5) is locally wellposed also in $\mathcal{E}_\beta$, cf. Remark 2.3. $\diamond$

We next establish basic properties of the Lyapunov-Perron operator $\Phi_q(y, z_0)$ defined for $t \geq 0$ by

$$\Phi_q(y,z_0)(t) = T_q(t)P_q^{\mathrm{s}} z_0 + \int_0^t T_q(t-\tau)P_q^{\mathrm{s}} F_q(y(\tau))\,\mathrm{d}\tau - \int_t^\infty P_q^{\mathrm{c}} F_q(y(\tau))\,\mathrm{d}\tau, \quad (3.9)$$

where $|q| \leq q_0$ and $z_0 \in \mathcal{E}_0 \cap \mathcal{E}_\alpha = \mathcal{E}_\beta$ satisfies

$$|z_0|_\beta = \max\{|z_0|_0, |z_0|_\alpha\} \leq \delta_0, \quad (3.10)$$

for some $\delta_0 > 0$. Here we use that P_q^c maps into the kernel of the generator of $\{T_q(t)\}_{t\geq 0}$, see Lemma 2.7, so that the semigroup is just the identity on the range of P_q^c and we can omit it in the second integral in (3.9).

For a continuous map $y = (u,v) : \mathbb{R}_+ \to \mathcal{E}_\beta = \mathcal{E}_0 \cap \mathcal{E}_\alpha$ we define the norms

$$\|y\|_{\omega,\alpha} = \sup_{t\geq 0} \mathrm{e}^{\omega t}|y(t)|_\alpha, \quad \|y\|_{0,0} = \sup_{t\geq 0}|y(t)|_0, \quad \|v\|_{\omega,0} = \sup_{t\geq 0} \mathrm{e}^{\omega t}|v(t)|_0,$$

where $\omega > 0$ is specified below and $\alpha = (\alpha_-, \alpha_+)$ is given by (2.12). Let $\delta > 0$. Then $(\mathbb{B}_\delta, \|\cdot\|)$ is the set of continuous functions $y : \mathbb{R}_+ \to \mathcal{E}_0 \cap \mathcal{E}_\alpha$ such that

$$\|y\| := \max\left(\|y\|_{\omega,\alpha}, \|y\|_{0,0}, \|v\|_{\omega,0}\right) \leq \delta. \quad (3.11)$$

We recall from Hypothesis 2.10 and (2.18) the exponential estimates

$$\|S_2(t)\|_{\mathcal{B}(\mathcal{E}_0)} \leq C\mathrm{e}^{-\rho t}, \qquad \|T_q(t)P_q^{\mathrm{s}}\|_{\mathcal{B}(\mathcal{E}_\alpha)} \leq C\mathrm{e}^{-\nu t} \quad (3.12)$$

for $t \geq 0$. For technical reasons (see the next proof), if necessary we have to modify these exponents such that

$$0 < \omega < \rho < \nu. \quad (3.13)$$

This is always possible, though one may lose information here. By Lemma 2.11, the semigroup $\{T_q(t)\}_{t\geq 0}$ is bounded in $\mathcal{E}_\beta$. The above constants do not depend on q.

Lemma 3.2. *Take $q_0 > 0$. Let $\delta > 0$ and $\delta_0 = \delta_0(\delta, q_0) > 0$ be small enough. For each $z_0 \in \mathbb{B}_{\delta_0}(|\cdot|_\beta)$ the Lyapunov-Perron operator $y \mapsto \Phi_q(y, z_0)$ leaves $\mathbb{B}_\delta(\|\cdot\|)$ invariant and is a strict contraction on this ball for all $|q| \leq q_0$. Moreover, for the norm $\|\cdot\|$ defined in (3.11) one has*

$$\|\Phi_q(y,z_0) - \Phi_q(\bar{y},\bar{z}_0)\| \leq C|z_0 - \bar{z}_0|_\beta + C\delta\|y - \bar{y}\| \quad (3.14)$$

for some $C > 0$ and all $z_0, \bar{z}_0 \in \mathbb{B}_{\delta_0}(|\cdot|_\beta)$, $y, \bar{y} \in \mathbb{B}_\delta(\|\cdot\|)$, and $|q| \leq q_0$.

Proof. Let $t \geq 0$, $\delta, \delta_0 > 0$, $z_0, \bar{z}_0 \in \mathbb{B}_{\delta_0}(|\cdot|_\beta)$, $y, \bar{y} \in \mathbb{B}_\delta(\|\cdot\|)$, and $|q| \leq q_0$. Below the constants are uniform for δ, δ_0 and q in bounded subsets. By $\pi_1 y = u$ and $\pi_2 y = v$, we denote the projection of $y = (u,v)$ onto its first and second components. Formulas (2.19) and (2.20) yield

$$\pi_2 T_q(t-\tau) = S_2(t-\tau)\pi_2 + \int_\tau^t S_2(t-s)\pi_2 B_q T_q(s-\tau)\,\mathrm{d}s, \quad 0 \leq \tau \leq t. \quad (3.15)$$

1a) Using (3.15), (3.12), Lemmas 2.7 and 2.9, the second component of the first integral in (3.9) can be estimated by

$$\mathrm{e}^{\omega t}\left|\pi_2 \int_0^t T_q(t-\tau)P_q^{\mathrm{s}} F_q(y(\tau))\,\mathrm{d}\tau\right|_0 \quad (3.16)$$

$$\leq C\mathrm{e}^{\omega t}\int_0^t\left(\mathrm{e}^{-\rho(t-\tau)}|F_q(y(\tau))|_\beta+\int_\tau^t\mathrm{e}^{-\rho(t-s)}\mathrm{e}^{-\nu(s-t)}|F_q(y(\tau))|_\alpha\,\mathrm{d}s\right)\mathrm{d}\tau,$$

since

$$|S_2(t-\tau)\pi_2 P_q^s F_q(y(\tau))|_0\leq\|S_2(t-\tau)\pi_2\|_{\mathcal{B}(\mathcal{E}_0)}\|P_q^s\|_{\mathcal{B}(\mathcal{E}_\beta,\mathcal{E}_0)}|F_q(y(\tau))|_\beta, \tag{3.17}$$

$$\begin{aligned}&|S_2(t-s)\pi_2 B_q T_q(s-\tau)P_q^s F_q(y(\tau))|_0 \\ &\qquad\leq\|S_2(t-s)\pi_2\|_{\mathcal{B}(\mathcal{E}_0)}\|B_q\|_{\mathcal{B}(\mathcal{E}_\alpha,\mathcal{E}_0)}\|T_q(s-\tau)P_q^s\|_{\mathcal{B}(\mathcal{E}_\alpha)}|F_q(y(\tau))|_\alpha.\end{aligned} \tag{3.18}$$

Because of (3.1) and (3.2), the formulas (3.16) and (3.13) yield

$$\begin{aligned}&\mathrm{e}^{\omega t}\left|\pi_2\int_0^t T_q(t-\tau)P_q^{\mathrm{s}}F_q(y(\tau))\,\mathrm{d}\tau\right|_0\\
&\leq C\mathrm{e}^{\omega t}\int_0^t\Big(\mathrm{e}^{-\rho(t-\tau)}\mathrm{e}^{-\omega\tau}\mathrm{e}^{\omega\tau}(|y(\tau)|_\alpha+|v(\tau)|_0)\,|y(\tau)|_0\\
&\qquad+\int_\tau^t\mathrm{e}^{-\rho(t-s)}\mathrm{e}^{-\nu(s-t)}\mathrm{e}^{-\omega\tau}\mathrm{e}^{\omega\tau}|y(\tau)|_\alpha\,|y(\tau)|_0\,\mathrm{d}s\Big)\mathrm{d}\tau\\
&\leq C(\|y\|_{\omega,\alpha}+\|v\|_{\omega,0})\,\|y\|_{0,0}\int_0^t\mathrm{e}^{(\omega-\rho)(t-\tau)}\,\mathrm{d}\tau\\
&\quad+C\|y\|_{\omega,\alpha}\,\|y\|_{0,0}\int_0^t\mathrm{e}^{\omega(t-\tau)}\left(\int_\tau^t\mathrm{e}^{-\rho(t-s)}\mathrm{e}^{-\nu(s-\tau)}\,\mathrm{d}s\right)\mathrm{d}\tau\\
&\leq C\|y\|^2\leq C\delta^2.\end{aligned}$$

We next employ (3.12), (3.2) and (3.13) to bound

$$\begin{aligned}\mathrm{e}^{\omega t}\left|\int_0^t T_q(t-\tau)P_q^{\mathrm{s}}F_q(y(\tau))\,\mathrm{d}\tau\right|_\alpha&\leq C\int_0^t\mathrm{e}^{\omega t}\mathrm{e}^{-\nu(t-\tau)}\mathrm{e}^{-\omega\tau}\mathrm{e}^{\omega\tau}|y(\tau)|_0\,|y(\tau)|_\alpha\,\mathrm{d}\tau\\
&\leq C\|y\|_{0,0}\,\|y\|_{\omega,\alpha}\leq C\delta^2.\end{aligned}$$

To finish with the first integral in (3.9), it remains to control the $|\cdot|_0$ norm of its first component. Here (2.19), (2.21), Lemma 2.7 (in particular, that $P_q^s\in\mathcal{B}(\mathcal{E}_\beta,\mathcal{E}_0)$), Lemma 2.9, (3.1), (3.12), (3.2) and (3.13) imply the inequalities

$$\begin{aligned}&\left|\pi_1\int_0^t T_q(t-\tau)P_q^{\mathrm{s}}F_q(y(\tau))\,\mathrm{d}\tau\right|_0\\
&=\left|\pi_1\int_0^t S(t-\tau)P_q^{\mathrm{s}}F_q(y(\tau))\,\mathrm{d}\tau+\pi_1\int_0^t\int_\tau^t S(t-s)B_qT_q(s-\tau)P_q^{\mathrm{s}}F_q(y(\tau))\,\mathrm{d}s\,\mathrm{d}\tau\right|_0\\
&\leq C\int_0^t|F_q(y(\tau))|_\beta\,\mathrm{d}\tau+C\|B_q\|_{\mathcal{B}(\mathcal{E}_\alpha,\mathcal{E}_0)}\int_0^t\int_\tau^t|T_q(s-\tau)P_q^{\mathrm{s}}F_q(y(\tau))|_\alpha\,\mathrm{d}s\,\mathrm{d}\tau\\
&\leq C\int_0^t\mathrm{e}^{-\omega\tau}|y(\tau)|_0\,\mathrm{e}^{\omega\tau}(|y(\tau)|_\alpha+|v(\tau)|_0)\,\mathrm{d}\tau+C\int_0^t\int_\tau^t\mathrm{e}^{-\nu(s-\tau)}|y(\tau)|_0\,|y(\tau)|_\alpha\,\mathrm{d}s\,\mathrm{d}\tau\\
&\leq C\|y\|_{0,0}\,(\|y\|_{\omega,\alpha}+\|v\|_{\omega,0})\int_0^t\mathrm{e}^{-\omega\tau}\mathrm{d}\tau+C\|y\|_{0,0}\|y\|_{\omega,\alpha}\int_0^t\int_\tau^t\mathrm{e}^{-\nu(s-\tau)}\mathrm{d}s\,\mathrm{e}^{-\omega\tau}\,\mathrm{d}\tau\\
&\leq C\delta^2.\end{aligned}$$

1b) We now treat the term $T_q(t)P_q^{\mathrm{s}}z_0$ in (3.9). From (3.12) and (3.13) we infer

$$\mathrm{e}^{\omega t}|T_q(t)P_q^{\mathrm{s}}z_0|_\alpha\leq C\mathrm{e}^{\omega t}\mathrm{e}^{-\nu t}|z_0|_\alpha\leq C|z_0|_\beta\leq C\delta_0.$$

By means of (2.19), (2.21), Lemma 2.7 (in particular, that $P_q^s \in \mathcal{B}(\mathcal{E}_\beta, \mathcal{E}_0)$) and Lemma 2.9, as well as (3.12), we next compute

$$\begin{aligned} |\pi_1 T_q(t) P_q^{\mathrm{s}} z_0|_0 &\le |S(t) P_q^{\mathrm{s}} z_0|_0 + \int_0^t |S(t-s) B_q T_q(s) P_q^{\mathrm{s}} z_0|_0 \,\mathrm{d}s \\ &\le C|z_0|_\beta + C\int_0^t \|B_q\|_{\mathcal{B}(\mathcal{E}_\alpha,\mathcal{E}_0)} \mathrm{e}^{-\nu s} |z_0|_\alpha \,\mathrm{d}s \\ &\le C|z_0|_\beta \le C\delta_0. \end{aligned}$$

Finally, formulas (3.15), (3.12), Lemmas 2.7 and 2.9, as well as inequality (3.13) imply

$$\begin{aligned} \mathrm{e}^{\omega t}|\pi_2 T_q(t) P_q^{\mathrm{s}} z_0|_0 &\le \mathrm{e}^{\omega t}|S_2(t)\pi_2 P_q^{\mathrm{s}} z_0|_0 + \int_0^t |S_2(t-s)\pi_2 B_q T_q(s) P_q^{\mathrm{s}} z_0|_0 \,\mathrm{d}s \\ &\le C\mathrm{e}^{(\omega-\rho)t}|z_0|_\beta + C\int_0^t \mathrm{e}^{\omega t}\mathrm{e}^{-\rho(t-s)}\mathrm{e}^{-\nu s}|z_0|_\alpha \,\mathrm{d}s \\ &\le C|z_0|_\beta \le C\delta_0. \end{aligned}$$

1c) To show the invariance, it remains to bound the norms of the last integral in (3.9). Lemma 2.7 (in particular, that $P_q^c \in \mathcal{B}(\mathcal{E}_\alpha, \mathcal{E}_\beta)$) and estimate (3.2) yield

$$\begin{aligned} \mathrm{e}^{\omega t}\left|\int_t^\infty P_q^{\mathrm{c}} F_q(y(\tau))\,\mathrm{d}\tau\right|_\beta &\le C\int_t^\infty \mathrm{e}^{\omega t}|F_q(y(\tau))|_\alpha \,\mathrm{d}\tau \\ &\le C\int_t^\infty \mathrm{e}^{\omega t}\mathrm{e}^{-\omega\tau}\mathrm{e}^{\omega\tau}|y(\tau)|_0\,|y(\tau)|_\alpha \,\mathrm{d}\tau \\ &\le C\|y\|_{0,0}\,\|y\|_{\omega,\alpha} \le C\delta^2. \end{aligned}$$

We thus have shown that $\Phi_q(\cdot, z_0)$ leaves the ball $\mathbb{B}_\delta(\|\cdot\|)$ invariant if first $\delta > 0$ and then $\delta_0 > 0$ are chosen small enough.

2) For the contractivity we have to estimate the difference

$$\begin{aligned} \Phi_q(y, z_0) - \Phi_q(\bar{y}, z_0) = &\int_0^t T_q(t-\tau) P_q^{\mathrm{s}}\big(F_q(y(\tau)) - F_q(\bar{y}(\tau))\big)\,\mathrm{d}\tau \\ &- \int_t^\infty P_q^{\mathrm{c}}\big(F_q(y(\tau)) - F_q(\bar{y}(\tau))\big)\,\mathrm{d}\tau. \end{aligned} \tag{3.19}$$

2a) Using (2.19), (2.21), (3.12), Lemma 2.7 (in particular, that $P_q^s \in \mathcal{B}(\mathcal{E}_\beta, \mathcal{E}_0)$) and Lemma 2.9, we bound the first integral by

$$\begin{aligned} &\Big|\int_0^t T_q(t-\tau) P_q^{\mathrm{s}}\left(F_q(y(\tau)) - F_q(\bar{y}(\tau))\right)\mathrm{d}\tau\Big|_0 \\ &\quad\le \left|\int_0^t S(t-\tau) P_q^{\mathrm{s}}\big(F_q(y(\tau)) - F_q(\bar{y}(\tau))\big)\,\mathrm{d}\tau\right|_0 \\ &\qquad + \left|\int_0^t \int_\tau^t S(t-s) B_q T_q(s-\tau) P_q^{\mathrm{s}}\big(F_q(y(\tau)) - F_q(\bar{y}(\tau))\big)\,\mathrm{d}s\,\mathrm{d}\tau\right|_0 \\ &\quad\le C\int_0^t |F_q(y(\tau)) - F_q(\bar{y}(\tau))|_\beta \,\mathrm{d}\tau \\ &\qquad + C\int_0^t \int_\tau^t \|B_q\|_{\mathcal{B}(\mathcal{E}_\alpha,\mathcal{E}_0)} \mathrm{e}^{-\nu(s-\tau)}\,|F_q(y(\tau)) - F_q(\bar{y}(\tau))|_\alpha \,\mathrm{d}s\,\mathrm{d}\tau. \end{aligned}$$

The inequalities (3.3) and (3.4) then lead to

$$\begin{aligned}
&\Big|\int_0^t T_q(t-\tau)P_q^{\mathrm{s}}\left(F_q(y(\tau))-F_q(\bar y(\tau))\right)\mathrm{d}\tau\Big|_0\\
&\le C\int_0^t \mathrm{e}^{-\omega\tau}\Big[|y(\tau)-\bar y(\tau)|_0\,\mathrm{e}^{\omega\tau}(|y(\tau)|_\alpha+|\bar y(\tau)|_\alpha+|v(\tau)|_0)\\
&\qquad\qquad +|\bar y(\tau)|_0\,\mathrm{e}^{\omega\tau}|v(\tau)-\bar v(\tau)|_0+\mathrm{e}^{\omega\tau}\,|y(\tau)-\bar y(\tau)|_\alpha\,(|y(\tau)|_0+|\bar y(\tau)|_0)\Big]\,\mathrm{d}\tau\\
&\quad + C\int_0^t \mathrm{e}^{-\omega\tau}\mathrm{e}^{\omega\tau}|y(\tau)-\bar y(\tau)|_\alpha\,(|y(\tau)|_0+|\bar y(\tau)|_0)\,\mathrm{d}\tau\\
&\le C\|y-\bar y\|_{0,0}\left(\|y\|_{\omega,\alpha}+\|\bar y\|_{\omega,\alpha}+\|v\|_{\omega,0}\right)+C\|\bar y\|_{0,0}\,\|v-\bar v\|_{\omega,0}\\
&\quad + C\|y-\bar y\|_{\omega,\alpha}\,(\|y\|_{0,0}+\|\bar y\|_{0,0})\\
&\le C\delta\,\|y-\bar y\|.
\end{aligned}$$

The $|\cdot|_\alpha$-norm of the first integral in (3.19) is estimated by

$$\begin{aligned}
&\mathrm{e}^{\omega t}\Big|\int_0^t T_q(t-\tau)P_q^{\mathrm{s}}(F_q(y(\tau)-F_q(\bar y(\tau)))\,\mathrm{d}\tau\Big|_\alpha\\
&\quad\le C\int_0^t \mathrm{e}^{\omega t}\mathrm{e}^{-\nu(t-\tau)}\mathrm{e}^{-\omega\tau}\mathrm{e}^{\omega\tau}|y(\tau)-\bar y(\tau)|_\alpha\,(|y(\tau)|_0+|\bar y(\tau)|_0)\,\mathrm{d}\tau\\
&\quad\le C\int_0^t \mathrm{e}^{(\omega-\nu)(t-\tau)}\,\mathrm{d}\tau\,\|y-\bar y\|_{\omega,\alpha}\,(\|y\|_{0,0}+\|\bar y\|_{0,0})\\
&\quad\le C\delta\,\|y-\bar y\|,
\end{aligned}$$

employing (3.12), (3.4), and (3.13). As in (3.17) and (3.18), for the second component we use formulas (3.15) and (3.12), Lemma 2.7 (in particular, that $P_q^s\in\mathcal{B}(\mathcal{E}_\beta,\mathcal{E}_0)$) and Lemma 2.9, as well as inequalities (3.3), (3.4) and (3.13) to derive the estimates

$$\begin{aligned}
&\mathrm{e}^{\omega t}\Big|\pi_2\int_0^t T_q(t-\tau)P_q^{\mathrm{s}}(F_q(y(\tau)-F_q(\bar y(\tau)))\,\mathrm{d}\tau\Big|_0\\
&\le \mathrm{e}^{\omega t}\Big|\int_0^t S_2(t-\tau)\pi_2P_q^{\mathrm{s}}(F_q(y(\tau)-F_q(\bar y(\tau)))\,\mathrm{d}\tau\Big|_0\\
&\quad+\mathrm{e}^{\omega t}\Big|\int_0^t\int_\tau^t S_2(t-s)\pi_2B_qT_q(s-\tau)P_q^{\mathrm{s}}(F_q(y(\tau)-F_q(\bar y(\tau)))\,\mathrm{d}s\,\mathrm{d}\tau\Big|_0\\
&\le C\int_0^t \mathrm{e}^{\omega(t-\tau)}\mathrm{e}^{-\rho(t-\tau)}\mathrm{e}^{\omega\tau}\Big[|y(\tau)-\bar y(\tau)|_0\,(|y(\tau)|_\alpha+|\bar y(\tau)|_\alpha+|v(\tau)|_0)\\
&\quad+|\bar y(\tau)|_0\,|v(\tau)-\bar v(\tau)|_0+|y(\tau)-\bar y(\tau)|_\alpha\,(|y(\tau)|_0+|\bar y(\tau)|_0)\Big]\,\mathrm{d}\tau\\
&\quad+C\int_0^t\int_\tau^t \mathrm{e}^{\omega t-\rho(t-s)}\|B_q\|_{\mathcal{B}(\mathcal{E}_\alpha,\mathcal{E}_0)}\mathrm{e}^{-\nu(s-\tau)-\omega\tau}\mathrm{e}^{\omega\tau}|y(\tau)-\bar y(\tau)|_\alpha(|y(\tau)|_0+|\bar y(\tau)|_0)\mathrm{d}\tau\\
&\le C\Big(\|y-\bar y\|_{0,0}(\|y\|_{\omega,\alpha}+\|\bar y\|_{\omega,\alpha}+\|v\|_{\omega,0})+\|\bar y\|_{0,0}\,\|v-\bar v\|_{\omega,0}\\
&\quad+\|y-\bar y\|_{\omega,\alpha}\,(\|y\|_{0,0}+\|\bar y\|_{0,0})\Big)\\
&\le C\delta\,\|y-\bar y\|.
\end{aligned}$$

As a result, the $\|\cdot\|$-norm of the first integral in (3.19) is dominated by $C\delta\,\|y-\bar y\|$.

2b) For the second integral in (3.19), Lemma 2.7 and inequality (3.4) yield

$$
\begin{aligned}
&\mathrm{e}^{\omega t}\Big|\int_t^\infty P_q^c(F_q(y(\tau))-F_q(\bar y(\tau)))\,\mathrm{d}\tau\Big|_\beta\\
&\le C\int_t^\infty \mathrm{e}^{\omega(t-\tau)}\mathrm{e}^{\omega\tau}|y(\tau)-\bar y(\tau)|_\alpha\,(|y(\tau)|_0+|\bar y(\tau)|_0)\,\mathrm{d}\tau\\
&\le C\|y-\bar y\|_{\omega,\alpha}(\|y\|_{0,0}+\|\bar y\|_{0,0})\\
&\le C\delta\,\|y-\bar y\|.
\end{aligned}
$$

We have thus established

$$\|\Phi_q(y,z_0)-\Phi_q(\bar y,z_0)\|\le C\delta\,\|y-\bar y\|, \tag{3.20}$$

finishing the proof of the first part of Lemma 3.2.

3) The remaining estimate

$$\|\Phi_q(y,z_0)-\Phi_q(y,\bar z_0)\|=\|T_q(\cdot)P_q^s(z_0-\bar z_0)\|\le C|z-\bar z_0|_\beta \tag{3.21}$$

was already shown in step 1c). □

4. **Stable manifolds.** For a small $q_0>0$ and each $q\in[-q_0,q_0]$, we now construct a function $\phi_q:\operatorname{ran}(P_q^s)\to P_q^c$ whose graph contains Y_q and it is a stable manifold $\mathcal{M}_q^s$ for the system (2.2). We further prove that the sets $\mathcal{M}_q^s$ satisfy the standard properties of stable manifolds and that they foliate a small neighborhood of Y_0.

Let $\delta,\delta_0>0$ be sufficiently small and $q_0>0$. Take $|q|\le q_0$ and $z_0\in\operatorname{ran}(P_q^s)\cap\mathbb{B}_{\delta_0}(|\cdot|_\beta)$. Lemma 3.2 then yields a unique function $y_{z_0}^q:\mathbb{R}_+\to\mathcal{E}_\beta$ which belongs to $\mathbb{B}_\delta(\|\cdot\|)$ and is a fixed point of the Lyapunov-Perron operator $\Phi_q(\cdot,z_0)$; that is,

$$y_{z_0}^q(t)=T_q(t)z_0+\int_0^t T_q(t-\tau)P_q^sF_q(y_{z_0}^q(\tau))\,\mathrm{d}\tau-\int_t^\infty P_q^cF_q(y_{z_0}^q(\tau))\,\mathrm{d}\tau \tag{4.1}$$

for $t\ge0$. At $t=0$ we obtain the identity

$$y_{z_0}^q(0)=z_0-\int_0^\infty P_q^cF_q(y_{z_0}^q(\tau))\,\mathrm{d}\tau$$

for all $z_0\in\operatorname{ran}(P_q^s)\cap\mathbb{B}_{\delta_0}(|\cdot|_\beta)$. We define the function $\phi_q:\operatorname{ran}(P_q^s)\cap\mathbb{B}_{\delta_0}(|\cdot|_\beta)\to\operatorname{ran}(P_q^c)$ by

$$\phi_q(z_0)=-\int_0^\infty P_q^cF_q(y_{z_0}^q(\tau))\,\mathrm{d}\tau. \tag{4.2}$$

In this notation, we have $y_{z_0}^q(0)=z_0+\phi_q(z_0)$ so that $y_{z_0}^q(0)$ belongs to the graph $\operatorname{graph}_{\delta_0}\phi_q$ of ϕ_q over the small neighborhood $\operatorname{ran}(P_q^s)\cap\mathbb{B}_{\delta_0}(|\cdot|_\beta)$ of 0. Adding and subtracting the term $\int_0^t P_q^cF_q(y_{z_0}^q(\tau))\,\mathrm{d}\tau$, we deduce from (4.1) that the fixed point $y=y_{z_0}^q$ of the Lyapunov-Perron operator satisfies the equation

$$y(t)=T_q(t)y(0)+\int_0^t T_q(t-\tau)F_q(y(\tau))\,\mathrm{d}\tau,\qquad t\ge0. \tag{4.3}$$

Consequently, $y=y_{z_0}^q$ is the mild solution of the nonlinear equation (2.5) in $\mathbb{B}_\delta(\|\cdot\|)$, and the function Y_q+y solves (2.2) in the mild sense, cf. Remark 2.3. By uniqueness, y_0^q is the 0 function. Let also $\bar z_0$ belong to $\operatorname{ran}(P_q^s)\cap\mathbb{B}_{\delta_0}(|\cdot|_\beta)$. Taking a sufficiently small $\delta>0$ in (3.14), we deduce the estimates

$$\|y_{z_0}^q-y_{\bar z_0}^q\|\le C|z_0-\bar z_0|_\beta,\qquad \|y_{z_0}^q\|\le C|z_0|_\beta. \tag{4.4}$$

For a number $\eta>0$ to be fixed below, the stable manifold $\mathcal{M}_q^s$ is then defined by

$$\mathcal{M}_q^s=\{Y_q+z_0+\phi_q(z_0):z_0\in\operatorname{ran}(P_q^s)\cap\mathbb{B}_{\delta_0}(|\cdot|_\beta)\}\cap(Y_0+\mathbb{B}_\eta(|\cdot|_\beta)), \tag{4.5}$$

where $|q| \le q_0$ and $Y_0 + \mathbb{B}_\eta(|\cdot|_\beta)$ is the closed ball in $\mathcal{E}_\beta = \mathcal{E}_\alpha \cap \mathcal{E}_0$ with radius η and centered at the original traveling wave Y_0.

Theorem 4.1. *Assume Hypotheses 2.4 and 2.10. Let $q_0 > 0$, $\delta > 0$, $\delta_0 = \delta_0(\delta, q_0) > 0$, $\eta = \eta(\delta_0) > 0$ be all sufficiently small, $|q| \le q_0$, and $\omega > 0$ be given by (3.13). Then the ball $Y_0 + \mathbb{B}_\eta(|\cdot|_\beta)$ is foliated by the stable manifolds $\mathcal{M}_q^s$ from (4.5) for the nonlinear equation (2.2) and the following assertions hold.*

(i) Each $\mathcal{M}_q^s$ is a Lipschitz manifold in $\mathcal{E}_\beta$. If $Y(0) \in \mathcal{M}_q^s$ and the mild solution $Y(t; Y(0))$ of (2.2) belongs to $Y_0 + \mathbb{B}_\eta(|\cdot|_\beta)$ for some $t \ge 0$, then $Y(t; Y(0))$ is contained in $\mathcal{M}_q^s$.

(ii) For each $Y(0) \in \mathcal{M}_q^s$ there exists a solution $Y(t; Y(0))$ of (2.2) which exists for all $t \ge 0$ and satisfies $|Y(t; Y(0)) - Y_q|_\beta \le \delta$ as well as
(a) $|Y(t; Y(0)) - Y_q|_\alpha \le C\mathrm{e}^{-\omega t}\,|Y(0) - Y_q|_\beta$,
(b) $|\pi_1(Y(t; Y(0)) - Y_q) - U_q|_0 \le C\,|Y(0) - Y_q|_\beta$,
(c) $|\pi_2(Y(t; Y(0)) - Y_q) - V_q|_0 \le C\mathrm{e}^{-\omega t}\,|Y(0) - Y_q|_\beta$
for all $t \ge 0$. Here, $Y_q = (U_q, V_q) = Y_0(\cdot - q)$ is the shifted traveling wave, $\pi_1 : Y = (U, V) \to U$, and $\pi_2 : Y = (U, V) \to V$.

(iii) If $Y(t; Y(0))$, $t \ge 0$, is a mild solution of (2.2) with $Y(0) \in Y_0 + \mathbb{B}_\eta(|\cdot|_\beta)$ that satisfies properties (a)–(c) in item (ii), then $Y(0)$ belongs to $\mathcal{M}_q^s$.

(iv) For $q \ne \bar{q}$, we have $\mathcal{M}_q^s \cap \mathcal{M}_{\bar{q}}^s = \emptyset$. Moreover, $Y_0 + \mathbb{B}_\eta(|\cdot|_\beta) = \bigcup_{|q| \le q_0} \mathcal{M}_q^s$.

(v) The map $[-q_0, q_0] \to \operatorname{ran}(P_q^c)$; $q \mapsto \phi_q(P_q^s z_0)$, is Lipschitz for each $z_0 \in \mathbb{B}_{\delta_0}(|\cdot|_\beta)$.

As a result, for each $Y(0) \in Y_0 + \mathbb{B}_\eta(|\cdot|_\beta)$ there exists exactly one shift $q \in [-q_0, q_0]$ such that $Y(0) \in \mathcal{M}_q^s$.

The following lemma will be used in the proof of Theorem 4.1. Recall the definition of the ball $\mathbb{B}_\delta(\|\cdot\|)$ in (3.11).

Lemma 4.2. *Assume Hypotheses 2.4 and 2.10. Let $q_0 > 0$, $\delta > 0$, $\delta_0 = \delta_0(\delta, q_0) > 0$ be chosen small enough, and let $|q| \le q_0$. Take $y_0 \in \mathcal{E}_\beta = \mathcal{E}_\alpha \cap \mathcal{E}_0$. Let $y = Y(\cdot; y_0) \in C([0, t_0), \mathcal{E}_0 \cap \mathcal{E}_\alpha)$ be the mild solution of the nonlinear equation (2.5) with the initial value $y(0) = y_0$, where $t_0 \in (0, \infty]$. Set $z_0 = P_q^s y_0$ and assume that $|z_0|_\beta \le \delta_0$. Then the following assertions are equivalent.*

(a) $y_0 = z_0 + \phi_q(z_0) \in \operatorname{graph}_{\delta_0}\phi_q$.

(b) y can be extended to a global mild solution of (2.5) in $\mathbb{B}_\delta(\|\cdot\|)$, and it is the fixed point $y_{z_0}^q$ of the Lyapunov-Perron operator $\Phi_q(\cdot, z_0)$ from (3.9).

(c) y can be extended to a global mild solution of (2.5) in $\mathbb{B}_\delta(\|\cdot\|)$.

Proof. (a)⇒(b): Assertion (a) and the equations (4.2) and (4.1) yield

$$y_0 = z_0 + \phi_q(z_0) = z_0 - \int_0^\infty P_q^c F_q(y_{z_0}^q(\tau))\,\mathrm{d}\tau = y_{z_0}^q(0),$$

where $y_{z_0}^q \in \mathbb{B}_\delta(\|\cdot\|)$ is the fixed point of $\Phi_q(\cdot, z_0)$. Since their initial values are the same, the mild solutions y and $y_{z_0}^q$ coincide by uniqueness of (4.3); i.e., (b) holds.

(b)⇒(c): This implication is obvious.

(c)⇒(a): In view of (c), Lemma 3.2 shows that the integral

$$z_c := P_q^c y_0 + \int_0^\infty P_q^c F_q(y(\tau))\,\mathrm{d}\tau \in \operatorname{ran}(P_q^c).$$

exists. Since y solves (4.3) and $T_q(t-\tau)$ is the identity on $\operatorname{ran}(P_q^c)$, we can write

$$\begin{aligned} y(t) &= T_q(t)y_0 + \int_0^t T_q(t-\tau)F_q(y(\tau))\,\mathrm{d}\tau \\ &= T_q(t)P_q^s y_0 + \int_0^t T_q(t-\tau)P_q^s F(y(\tau))\,\mathrm{d}\tau - \int_t^\infty P_q^c F(y(\tau))\,\mathrm{d}\tau \\ &\quad + P_q^c y_0 + \int_t^\infty P_q^c F(y(\tau))\,\mathrm{d}\tau + \int_0^t P_q^c F(y(\tau))\,\mathrm{d}\tau, \end{aligned}$$

using again Lemma 3.2 and (c). The definition of $\Phi_q(y, z_0)$ in (3.9) then yields

$$y(t) = \Phi_q(y, z_0)(t) + z_c, \qquad t \ge 0. \tag{4.6}$$

Due to the invariance of $\mathbb{B}_\delta(\|\cdot\|)$ with respect to the Lyapunov-Perron operator, (c) and (3.11), the functions y and $\Phi_q(y, z_0)$ tend to 0 in $\mathcal{E}_\alpha$ as $t \to \infty$, and hence $z_c = 0$. Equation (4.6) thus implies $y = \Phi_q(y, z_0)$ so that (a) is a consequence of the observations after (4.2). $\square$

Proof of Theorem 4.1. Recall from Remark 2.3 that all mild solutions of (2.2) are given by $y + Y_q$ for a mild solution y of (2.5).

(i) and (ii). Equations (4.1) and (4.2) show that $z_0 + \phi_q(z_0)$ is the value of $\Phi(z_0, \phi_q(z_0))$ at $t = 0$. From (4.4) we then deduce that ϕ_q and hence $\mathcal{M}_q^s$ are Lipschitz in $\mathcal{E}_\beta = \mathcal{E}_0 \cap \mathcal{E}_\alpha$.

Let $y_0 + Y_q$ belong to $\mathcal{M}_q^s$, where $z_0 = P_q^s y_0 \in \operatorname{ran}(P_q^s) \cap \mathbb{B}_{\delta_0}(|\cdot|_\beta)$. By Lemma 4.2, the fixed point $y_{z_0}^q$ is the mild solution $Y(\cdot; y_0)$ of (2.5) in $\mathbb{B}_\delta(\|\cdot\|)$ with the initial value y_0. Combined with (4.4) and (3.11), these facts imply (ii).

Take $t_0 > 0$ such that $|y(t_0) + Y_q - Y_0|_\beta \le \eta$. It is easy to see that $y(t_0 + \cdot)$ still belongs to $\mathbb{B}_\delta(\|\cdot\|)$ and that it is the mild solution of (2.5) with the initial value $y(t_0)$. Moreover, Remark 2.7 (in particular, that $P_q^s \in \mathcal{B}(\mathcal{E}_\beta)$) and (2.15) yield

$$|P_q^s y(t_0)|_\beta \le C\left(|y(t_0) + Y_q - Y_0|_\beta + |Y_0 - Y_q|_\beta\right) \le C(\eta + |q|) < \delta_0, \tag{4.7}$$

if we choose $\eta > 0$ and q_0 small enough. (Note that the constants are uniform for q in compact intervals and independent of η.) Therefore, $y(t_0) + Y_q$ is contained in $\mathcal{M}_q^s$ thanks to Lemma 4.2. So (i) is shown.

(iii). Take $Y(0) \in Y_0 + \mathbb{B}_\eta(|\cdot|_\beta)$ that satisfies properties (a)–(c) in item (ii). The function $y(t) = Y(t; Y(0)) - Y_q$ is a mild solution of (2.5) with initial value $Y(0) - Y_q$. Using again (2.15), we can estimate

$$|Y(0) - Y_q|_\beta \le |Y(0) - Y_0|_\beta + |Y_q - Y_0|_\beta \le \eta + C|q|.$$

Possibly decreasing $\eta, q_0 > 0$, we deduce from conditions (a)–(c) the inequality (3.11) for y and from Lemma 2.7 the estimate $|P_q^s(Y(0) - Y_q)|_\beta \le \delta_0$. Lemma 4.2 now yields that $y(0) \in \operatorname{graph}_{\delta_0} \phi_q$, proving (iii).

(iv). By Theorem 3.14 in [11], we can fix a sufficiently small radius $\eta > 0$ such that for each point $Y(0)$ in the ball $Y_0 + \mathbb{B}_\eta(|\cdot|_\beta; Y_0)$ there exists a shift $q = q(Y(0))$ such that the solution $Y(\cdot; Y(0))$ of (2.2) satisfies properties (a)–(c) of item (ii). We remark that in Theorem 3.14 we can choose the same number $\delta > 0$ as in the current proof and exponents[1] $\nu, \rho > \omega$ which are different from our exponents ν and ρ in (3.13). Item (iii) then implies that $Y(0)$ is contained in $\mathcal{M}_q^s$. If $Y(0)$ is also an element of $\mathcal{M}_{\bar{q}}^s$ for some $\bar{q} \in [-q_0, q_0]$, then the corresponding solution y would converge both to Y_q and $Y_{\bar{q}}$ as $t \to \infty$, and so $q = \bar{q}$. Hence, (iv) holds.

[1] In (7) of Theorem 3.14 there is a misprint, one has to replace ν by ρ.

(v). Let $|q|, |\bar q| \le q_0$ and $z_0 \in \mathbb{B}_{\delta_0}(|\cdot|_\beta)$. The maps $q \mapsto P_q^c \in \mathcal{B}(\mathcal{E}_\kappa, \mathcal{E}_\beta)$, $q \mapsto P_q^s \in \mathcal{B}(\mathcal{E}_\kappa)$ and $q \mapsto B_q \in \mathcal{B}(\mathcal{E}_\alpha, \mathcal{E}_0)$ are Lipschitz for $\kappa \in \{\beta, \alpha\}$ due to (2.14) and Lemma 2.9. Lemma 3.7 of [11] implies that $\gamma_\alpha Y_0'$ and $\gamma_\alpha^{-1} Y_0'$ are bounded. Using (2.6) and (2.15), we then deduce the estimate

$$|F_q(Y) - F_{\bar q}(Y)|_\beta \le C|Y|_\kappa\, |q - \bar q|$$

for all $Y \in \mathcal{E}_\kappa$ and $\kappa \in \{0, \alpha\}$. In view of (4.2), for (v) it remains to check that the map $q \mapsto y_{z_0}^q =: y_q$ is Lipschitz for $\|\cdot\|$. Since y_q is the fixed point, we infer from (3.9) the identity

$$y_q - y_{\bar q} = \Phi_q(y_q, z_0) - \Phi_{\bar q}(y_q, z_0) + \Phi_{\bar q}(y_q, z_0) - \Phi_{\bar q}(y_{\bar q}, z_0).$$

By (3.20), the second difference on the right hand side is bounded by $C\delta\, \|y_q - y_{\bar q}\|$ and can thus be absorbed by the left hand side possibly after decreasing $\delta > 0$ once more. To control the other difference, we note that the bounded perturbation theorem and (2.16) imply that $q \mapsto T_q(t) \in \mathcal{B}(\mathcal{E}_\kappa)$ is Lipschitz for $\kappa \in \{0, \alpha\}$ and uniformly for $t \ge 0$ in compact sets, see Corollary 3.1.3 of [23]. To extend this property to $\mathbb{R}_+$, let $t \in (n, n+1]$. We write

$$\begin{aligned} T_q(t)P_q^s - T_{\bar q}(t)P_{\bar q}^s = {}& (T_q(t-n) - T_{\bar q}(t-n))T_q(n)P_q^s + T_{\bar q}(t-n)T_q(n)P_q^s(P_q^s - P_{\bar q}^s) \\ & + T_{\bar q}(t-n)\sum_{k=0}^{n-1} T_q(n-k-1)P_q^s(T_q(1) - T_{\bar q}(1))T_{\bar q}(k)P_{\bar q}^s \\ & + T_{\bar q}(t-n)(P_q^s - P_{\bar q}^s)T_{\bar q}(n)P_{\bar q}^s. \end{aligned}$$

In the exponential decay estimate (2.18) for $T_q(t)P_q^s$ we can replace ν by a slightly larger number, see Lemma 3.13 of [11]. This and the above mentioned facts lead to the inequality

$$\|T_q(t)P_q^s - T_{\bar q}(t)P_{\bar q}^s\|_{\mathcal{B}(\mathcal{E}_\alpha)} \le C\mathrm{e}^{-\nu t}\, |q - \bar q|, \qquad t \ge 0.$$

As in Lemma 3.2 one can now show that

$$\|\Phi_q(y_q, z_0) - \Phi_{\bar q}(y_q, z_0)\| \le C|q - \bar q|.$$

Summing up, (v) is true. □

To conclude, we briefly mention two motivating examples borrowed from [12] that fit our setting. More details can be found in the papers [10] and [11], respectively. We stress, however, that for this type of examples Hypotheses 2.1, 2.2 and 2.4 (a) can rigorously be verified not in all cases while the absence of the unstable eigenvalues required in Hypothesis 2.4 (b) is usually checked only numerically for certain ranges of the parameter values.

Example 4.3. *Gasless combustion.* A simple combustion model in one space dimension has been mentioned in the Introduction and is given by the system

$$\partial_t u = \partial_{xx} u + v g(u), \quad \partial_t v = -\beta v g(u),$$

where $g(u) = e^{-\frac{1}{u}}$ if $u > 0$ and $g(u) = 0$ if $u \le 0$. In this system, u is the temperature, v is the concentration of unburned fuel, g is the unit reaction rate, and $\beta > 0$ is a constant parameter. This system was a primary guiding example in [8, 9, 10, 11, 12]. One motivation for looking at this well-studied problem, in which the reactant does not diffuse, was heat-enhanced methods of oil recovery in which the reactant is coke contained in the rock formation, see [1]. The value

$u = 0$ represents the ignition temperature and is also taken to be the background temperature, at which the reaction does not take place.

Clearly, Hypothesis 2.2 is satisfied here. One looks for traveling waves $Y_0 = (u_0, v_0)$ such that $Y_- = (u_-, 0)$ with $u_- > 0$, $Y_+ = (0, 1)$, and $(u_0(x), v_0(x))$ approaches these end states exponentially as $x \to \pm\infty$. For each $\beta > 0$ there is a unique $c > 0$ for which such a wave exists, cf. [12, §3.2]. This wave represents a combustion front that leaves behind of it high temperature $u_- = 1/\beta$ and no fuel, while in front of it temperature is 0 and there is fuel, with concentration normalized to 1. As discussed in Paragraph 3.2 of [12], Hypothesis 2.10 is true and Hypothesis 2.4 can be verified (partly numerically) for small $\beta > 0$.

We note the lack of diffusion in the second equation which inspired the linear Lemma 3.13 in [11] used to derive the exponential decay (2.18) from the spectral assumptions in Hypothesis (2.4), and the form of the nonlinear term in this and related problems which inspired the triangular and product structure of the nonlinearity in the current paper that follows from Hypothesis 2.2.

Example 4.4. *Exothermic-endothermic chemical reactions.* A model in which two chemical reactions occur at rates determined by temperature was studied in [32, 33], see also [11]. One reaction is exothermic (produces heat), the other is endothermic (absorbs heat). The system reads

$$\partial_t y_1 = \partial_{xx} y_1 + y_2 f_2(y_1) - \sigma y_3 f_3(y_1), \tag{4.8}$$

$$\partial_t y_2 = d_2 \partial_{xx} y_2 - y_2 f_2(y_1), \tag{4.9}$$

$$\partial_t y_3 = d_3 \partial_{xx} y_3 - \tau y_3 f_3(y_1). \tag{4.10}$$

Here y_1 is the temperature, y_2 is the quantity of an exothermic reactant, and y_3 is the quantity of an endothermic reactant. The parameters σ and τ are positive, and there are positive constants a_i and b_i such that $f_i(u) = a_i e^{-\frac{b_i}{u}}$ for $u > 0$ and $f_i(u) = 0$ for $u \leq 0$. In [32, 33] it is shown numerically that in certain parameter regimes there exist traveling wave solutions Y_0 of (4.8)–(4.10) with speed $c > 0$ and the end states $Y_- = (1 - \frac{\sigma}{\tau}, 0, 0)$ and $Y_+ = (0, 1, 1)$. Moreover, both end states are approached at an exponential rate, the zero eigenvalue of the linearization is simple, and there are no other eigenvalues in the right half plane. A rigorous motivation for the existence of such traveling wave is also given in [11, Section 9.2]. Assuming the existence of the traveling wave with these properties, the remaining hypotheses of the current paper are easy to verify.

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