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The Maslov index for Lagrangian pairs on $\mathbb{R}^{2n}$

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ABSTRACT

We discuss a definition of the Maslov index for Lagrangian pairs on $\mathbb{R}^{2n}$ based on spectral flow, and develop many of its salient properties. We provide two applications to illustrate how our approach leads to a straightforward analysis of the relationship between the Maslov index and the Morse index for Schrödinger operators on $[0, 1]$ and $\mathbb{R}$.

1. Introduction

With origins in the work of V.P. Maslov [16] and subsequent development by V.I. Arnol'd [1], the Maslov index on $\mathbb{R}^{2n}$ is a tool for determining the nature of intersections between two evolving Lagrangian subspaces (see Definition 1.1). As discussed in [6], several equivalent definitions are available, and we focus on a definition for Lagrangian pairs based on the development in [4] (using the definition of spectral flow introduced in [17]). We note at the outset that the theory associated with the Maslov index has now been extended well beyond the simple setting of our analysis (see, for example, [4,7,8,10]); nonetheless, the Maslov index for Lagrangian pairs on $\mathbb{R}^{2n}$ is a useful tool, and a systematic development of its properties is certainly warranted.

As a starting point, we define what we will mean by a *Lagrangian subspace* of $\mathbb{R}^{2n}$.

Definition 1.1. We say $\ell \subset \mathbb{R}^{2n}$ is a Lagrangian subspace if ℓ has dimension n and

$$(Jx, y)_{\mathbb{R}^{2n}} = 0,$$

for all $x, y \in \ell$. Here, $(\cdot, \cdot)_{\mathbb{R}^{2n}}$ denotes Euclidean inner product on $\mathbb{R}^{2n}$, and

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$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

with I_n the $n \times n$ identity matrix. We sometimes adopt standard notation for symplectic forms, $\omega(x, y) = (Jx, y)_{\mathbb{R}^{2n}}$. Finally, we denote by $\Lambda(n)$ the collection of all Lagrangian subspaces of $\mathbb{R}^{2n}$, and we will refer to this as the *Lagrangian Grassmannian*.

A simple example, important for intuition, is the case $n = 1$, for which $(Jx, y)_{\mathbb{R}^2} = 0$ if and only if x and y are linearly dependent. In this case, we see that any line through the origin is a Lagrangian subspace of $\mathbb{R}^2$. As a foreshadowing of further discussion, we note that each such Lagrangian subspace can be identified with precisely two points on the unit circle S^1.

More generally, any Lagrangian subspace of $\mathbb{R}^{2n}$ can be spanned by a choice of n linearly independent vectors in $\mathbb{R}^{2n}$. We will generally find it convenient to collect these n vectors as the columns of a $2n \times n$ matrix $\mathbf{X}$, which we will refer to as a *frame* for ℓ. Moreover, we will often write $\mathbf{X} = \binom{X}{Y}$, where X and Y are $n \times n$ matrices.

Given any two Lagrangian subspaces ℓ_1 and ℓ_2, with associated frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$, we can define the complex $n \times n$ matrix

$$\tilde{W} = -(X_1 + iY_1)(X_1 - iY_1)^{-1}(X_2 - iY_2)(X_2 + iY_2)^{-1}, \tag{1.1}$$

which we will see in Section 3 is unitary. (We will also verify in Section 3 that $(X_1 - iY_1)$ and $X_2 + iY_2$ are both invertible, and that $\tilde{W}$ is independent of the choice of frames we take for ℓ_1 and ℓ_2.) Notice that if we switch the roles of ℓ_1 and ℓ_2 then $\tilde{W}$ will be replaced by $\tilde{W}^{-1}$, and since $\tilde{W}$ is unitary this is $\tilde{W}^*$. We conclude that the eigenvalues in the switched case will be complex conjugates of those in the original case.

Remark 1.2. We use the tilde to distinguish the $n \times n$ complex-valued matrix $\tilde{W}$ from the Souriau map (see equation (3.8) below), which is a related $2n \times 2n$ matrix often—as here—denoted W. The general form of $\tilde{W}$ appears in a less general context in [9,12]. For the special case $\mathbf{X}_2 = \binom{0}{I}$ (associated, for example, with Dirichlet boundary conditions for a Sturm–Liouville eigenvalue problem) we see that

$$\tilde{W} = (X_1 + iY_1)(X_1 - iY_1)^{-1}, \tag{1.2}$$

which has been extensively studied, perhaps most systematically in [2] (particularly Chapter 10). If we let $\tilde{W}_D$ denote (1.2) for $\mathbf{X}_1 = \binom{0}{I}$ and for $j = 1, 2$ set

$$\tilde{W}_j = (X_j + iY_j)(X_j - iY_j)^{-1},$$

then our form for $\tilde{W}$ can be viewed as the composition map

$$-\tilde{W}_1\tilde{W}_D(\tilde{W}_2\tilde{W}_D)^{-1} = -\tilde{W}_1(\tilde{W}_2)^{-1}. \tag{1.3}$$

For a related observation regarding the Souriau map see Remark 3.3.

Combining observations from Sections 2 and 3, we will establish the following theorem (cf. Lemma 1.3 in [4]).

Theorem 1.3. *Suppose $\ell_1, \ell_2 \subset \mathbb{R}^{2n}$ are Lagrangian subspaces, with respective frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$, and let $\tilde{W}$ be as defined in (1.1). Then*

$$\dim \ker(\tilde{W} + I) = \dim(\ell_1 \cap \ell_2).$$

That is, the dimension of the eigenspace of $\tilde{W}$ associated with the eigenvalue -1 is precisely the dimension of the intersection of the Lagrangian subspaces ℓ_1 and ℓ_2.

Given a parameter interval $I = [a, b]$, which can be normalized to $[0, 1]$, we consider maps $\ell : I \to \Lambda(n)$, which will be expressed as $\ell(t)$. In order to specify a notion of continuity, we need to define a metric on $\Lambda(n)$, and following [10] (p. 274), we do this in terms of orthogonal projections onto elements $\ell \in \Lambda(n)$. Precisely, let $\mathcal{P}_i$ denote the orthogonal projection matrix onto $\ell_i \in \Lambda(n)$ for $i = 1, 2$. I.e., if $\mathbf{X}_i$ denotes a frame for ℓ_i, then $\mathcal{P}_i = \mathbf{X}_i(\mathbf{X}_i^t\mathbf{X}_i)^{-1}\mathbf{X}_i^t$. We take our metric d on $\Lambda(n)$ to be defined by

$$d(\ell_1, \ell_2) := \|\mathcal{P}_1 - \mathcal{P}_2\|,$$

where $\|\cdot\|$ can denote any matrix norm. We will say that $\ell : I \to \Lambda(n)$ is continuous provided it is continuous under the metric d. Likewise, for $\mathcal{L} = (\ell_1, \ell_2) \in \Lambda(n) \times \Lambda(n)$ and $\mathcal{M} = (m_1, m_2) \in \Lambda(n) \times \Lambda(n)$, we take

$$\rho(\mathcal{L}, \mathcal{M}) = (d(\ell_1, m_1)^2 + d(\ell_2, m_2)^2)^{1/2}. \tag{1.4}$$

Given two continuous maps $\ell_1(t), \ell_2(t)$ on a parameter interval I, we denote by $\mathcal{L}(t)$ the path

$$\mathcal{L}(t) = (\ell_1(t), \ell_2(t)).$$

In what follows, we will define the Maslov index for the path $\mathcal{L}(t)$, which will be a count, including both multiplicity and direction, of the number of times the Lagrangian paths ℓ_1 and ℓ_2 intersect. In order to be clear about what we mean by multiplicity and direction, we observe that associated with any path $\mathcal{L}(t)$ we will have a path of unitary complex matrices as described in (1.1). We have already noted that the Lagrangian subspaces ℓ_1 and ℓ_2 intersect at a value $t_0 \in I$ if and only if $\tilde{W}(t_0)$ has -1 as an eigenvalue. In the event of such an intersection, we define the multiplicity of the intersection to be the multiplicity of -1 as an eigenvalue (since $\tilde{W}$ is unitary the algebraic and geometric multiplicities are the same). When we talk about the direction of an intersection, we mean the direction the eigenvalues of $\tilde{W}$ are moving (as t varies) along the unit circle S^1 as they pass through -1 (we take counterclockwise as the positive direction). We note that the eigenvalues certainly do not all need to be moving in the same direction, and that we will need to take care with what we mean by a crossing in the following sense: we must decide whether to increment the Maslov index upon arrival or upon departure.

Following [4,10,17], we proceed by choosing a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $I = [a, b]$, along with numbers $\epsilon_j \in (0, \pi)$ so that $\ker\big(\tilde{W}(t) - e^{i(\pi \pm \epsilon_j)}I\big) = \{0\}$ for $t_{j-1} < t < t_j$; that is, $e^{i(\pi \pm \epsilon_j)} \in \mathbb{C} \setminus \sigma(\tilde{W}(t))$, for $t_{j-1} < t < t_j$ and $j = 1, \dots, n$. Moreover, for each $j = 1, \dots, n$ and any $t \in [t_{j-1}, t_j]$ there are only finitely many values $\theta \in [0, \epsilon_j]$ for which $e^{i(\pi+\theta)} \in \sigma(\tilde{W}(t))$.

Fix some $j \in \{1, 2, \dots, n\}$ and consider the value

$$k(t, \epsilon_j) := \sum_{0 \le \theta < \epsilon_j} \dim\ker\big(\tilde{W}(t) - e^{i(\pi+\theta)}I\big), \tag{1.5}$$

for $t_{j-1} \le t \le t_j$. This is precisely the sum, along with multiplicity, of the number of eigenvalues of $\tilde{W}(t)$ that lie on the arc

$$A_j := \{e^{it} : t \in [\pi, \pi + \epsilon_j)\}.$$

The stipulation that $e^{i(\pi \pm \epsilon_j)} \in \mathbb{C} \setminus \sigma(\tilde{W}(t))$, for $t_{j-1} < t < t_j$ asserts that no eigenvalue can enter A_j in the clockwise direction or exit in the counterclockwise direction during the interval $t_{j-1} < t < t_j$. In this way, we see that $k(t_j, \epsilon_j) - k(t_{j-1}, \epsilon_j)$ is a count of the number of eigenvalues that entered A_j in the counterclockwise direction minus the number that left in the clockwise direction during the interval (t_{j-1}, t_j).

In dealing with the catenation of paths, it's particularly important to understand this quantity if an eigenvalue resides at -1 at either $t = t_{j-1}$ or $t = t_j$ (i.e., if an eigenvalues begins or ends at a crossing). If an eigenvalue moving in the counterclockwise direction arrives at -1 at $t = t_j$, then we increment the difference forward, while if the eigenvalue arrives at -1 from the clockwise direction we do not. On the other hand, suppose an eigenvalue resides at -1 at $t = t_{j-1}$ and moves in the counterclockwise direction. There is no change, and so we do not increment the difference, but we decrement the difference if the eigenvalue leaves in the clockwise direction. In summary, the difference increments forward upon arrivals in the counterclockwise direction, but not upon arrivals in the clockwise direction, and it decrements upon departure in the clockwise direction, but not upon departure in the counterclockwise direction.

We are now ready to define the Maslov index.

Definition 1.4. Let $\mathcal{L}(t) = (\ell_1(t), \ell_2(t))$, where $\ell_1, \ell_2 : I \to \Lambda(n)$ are continuous paths in the Lagrangian–Grassmannian. The Maslov index $\mathrm{Mas}(\mathcal{L}; I)$ is defined by

$$\mathrm{Mas}(\mathcal{L}; I) = \sum_{j=1}^{n} (k(t_j, \epsilon_j) - k(t_{j-1}, \epsilon_j)). \tag{1.6}$$

Remark 1.5. In [6] the authors provide a list of six properties that entirely characterize the Maslov index for a pair of Lagrangian paths. Our definition satisfies their properties, except for the choice of normalization (their Property VI), which is reversed. In our notation, their normalization is specified for $n = 1$ with reference to Lagrangian subspaces ℓ_1 and ℓ_2 with respective frames $\mathbf{X}_1 = \binom{1}{0}$ and $\mathbf{X}_2 = \binom{\cos t}{\sin t}$. For this choice, we have

$$\tilde{W}(t) = -\frac{\cos t - i \sin t}{\cos t + i \sin t},$$

for which we see immediately that $\tilde{W}(-\frac{\pi}{4}) = -i$, $\tilde{W}(0) = -1$, and $\tilde{W}(\frac{\pi}{4}) = i$. This path is monotonic, so the following three values are immediate: $\mathrm{Mas}(\ell_1, \ell_2; [-\frac{\pi}{4}, \frac{\pi}{4}]) = -1$, $\mathrm{Mas}(\ell_1, \ell_2; [-\frac{\pi}{4}, 0]) = 0$, and $\mathrm{Mas}(\ell_1, \ell_2; [0, \frac{\pi}{4}]) = -1$. (Cf. equation (1.7) in [6].)

We also note two additional definitions of the Maslov index for paths. In Section 3 of [18] the authors give a definition based on crossing forms, and in Section 3.5 of [10] the author gives a definition based on a direct sum of the Lagrangian pairs. In Section 3 (of the current paper) we clarify how these two definitions are related to our Definition 1.4.

One of the most important features of the Maslov index is homotopy invariance, for which we need to consider continuously varying families of Lagrangian paths. To set some notation, we denote by $\mathcal{P}(I)$ the collection of all paths $\mathcal{L}(t) = (\ell_1(t), \ell_2(t))$, where $\ell_1, \ell_2 : I \to \Lambda(n)$ are continuous paths in the Lagrangian–Grassmannian. We say that two paths $\mathcal{L}, \mathcal{M} \in \mathcal{P}(I)$ are homotopic provided there exists a family $\mathcal{H}_s$ so that $\mathcal{H}_0 = \mathcal{L}$, $\mathcal{H}_1 = \mathcal{M}$, and $\mathcal{H}_s(t)$ is continuous as a map from $[a, b] \times [0, 1]$ into $\Lambda(n)$.

The Maslov index has the following properties (see, for example, Theorem 3.6 in [10]).

(P1) (Path Additivity) If $a < b < c$ then

$$\mathrm{Mas}(\mathcal{L}; [a, c]) = \mathrm{Mas}(\mathcal{L}; [a, b]) + \mathrm{Mas}(\mathcal{L}; [b, c]).$$

(P2) (Homotopy Invariance) If $\mathcal{L}, \mathcal{M} \in \mathcal{P}(I)$ are homotopic, with $\mathcal{L}(a) = \mathcal{M}(a)$ and $\mathcal{L}(b) = \mathcal{M}(b)$ (i.e., if $\mathcal{L}, \mathcal{M}$ are homotopic with fixed endpoints) then

$$\mathrm{Mas}(\mathcal{L}; [a, b]) = \mathrm{Mas}(\mathcal{M}; [a, b]).$$

Remark 1.6. For (P1), the only issue regards cases in which there is an intersection at $t = b$. For example, suppose the intersection is an arrival in the clockwise direction, followed by departure in the same direction. Then at this intersection, $\text{Mas}(\mathcal{L}; [a, c])$ decrements by 1, $\text{Mas}(\mathcal{L}; [a, b])$ is unaffected, and $\text{Mas}(\mathcal{L}; [b, c])$ decrements by 1. Other cases are similar.

Verification of (P2) requires more work, and we leave that discussion to an appendix.

2. Framework for W and $\tilde{W}$

In Section 3, we will use the formulation of [4,10] to derive our form of $\tilde{W}$, and in preparation for that we will briefly discuss the nature of this formulation. This material has all been covered in a much more general case in [4,10], and our motivation for including this section is simply to allow readers to understand this framework in the current setting.

We record at the outset an important property of Lagrangian frames.

Proposition 2.1. *A $2n \times n$ matrix $\mathbf{X} = \binom{X}{Y}$ is a frame for a Lagrangian subspace if and only if the columns of $\mathbf{X}$ are linearly independent, and additionally*

$$X^t Y - Y^t X = 0. \tag{2.1}$$

We refer to this relation as the Lagrangian property for frames.

Proof. To see this, we observe by definition that $\mathbf{X}$ is the frame of a Lagrangian subspace if and only if its columns are linearly independent, and each of its column pairs $\binom{x_i}{y_i}$, $\binom{x_j}{y_j}$ satisfies

$$(J\begin{pmatrix} x_i \\ y_i \end{pmatrix}, \begin{pmatrix} x_j \\ y_j \end{pmatrix})_{\mathbb{R}^{2n}} = 0; \quad \text{i.e.,} \ (\begin{pmatrix} -y_i \\ x_i \end{pmatrix}, \begin{pmatrix} x_j \\ y_j \end{pmatrix})_{\mathbb{R}^{2n}} = (x_i, y_j)_{\mathbb{R}^{2n}} - (x_j, y_i)_{\mathbb{R}^{2n}} = 0.$$

Observing that

$$(X^t Y - Y^t X)_{ij} = (x_i, y_j)_{\mathbb{R}^n} - (x_j, y_i)_{\mathbb{R}^n},$$

we obtain the claim. □

Remark 2.2. It is clear that the Lagrangian property can alternatively be expressed as

$$\mathbf{X}^t J \mathbf{X} = 0.$$

We next observe that for a given pair of Lagrangian subspaces $\mathcal{L} = (\ell_1, \ell_2) \in \Lambda(n) \times \Lambda(n)$ we can change our choice of frames without changing either the associated $\tilde{W}$ or the projection matrices $\mathcal{P}_1$ and $\mathcal{P}_2$.

Proposition 2.3. *Suppose $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$ are any two frames for the same Lagrangian subspace $\ell \subset \mathbb{R}^{2n}$. Then*

$$(X_1 + iY_1)(X_1 - iY_1)^{-1} = (X_2 + iY_2)(X_2 - iY_2)^{-1},$$

and likewise

$$\mathbf{X}_1(\mathbf{X}_1^t \mathbf{X}_1)^{-1}\mathbf{X}_1^t = \mathbf{X}_2(\mathbf{X}_2^t \mathbf{X}_2)^{-1}\mathbf{X}_2^t.$$

Proof. Under our assumptions, there exists an invertible $n \times n$ matrix M so that $\mathbf{X}_1 = \mathbf{X}_2 M$. In particular, we must have $X_1 = X_2 M$ and $Y_1 = Y_2 M$. But then

$$\begin{aligned}(X_1 + iY_1)(X_1 - iY_1)^{-1} &= (X_2 M + iY_2 M)(X_2 M - iY_2 M)^{-1} \\ &= (X_2 + iY_2)MM^{-1}(X_2 - iY_2)^{-1} = (X_2 + iY_2)(X_2 - iY_2)^{-1}.\end{aligned}$$

Likewise,

$$\begin{aligned}\mathbf{X}_1(\mathbf{X}_1^t\mathbf{X}_1)^{-1}\mathbf{X}_1^t &= \mathbf{X}_2 M((\mathbf{X}_2 M)^t \mathbf{X}_2 M)^{-1}(\mathbf{X}_2 M)^t \\ &= \mathbf{X}_2 M(M^t(\mathbf{X}_2^t\mathbf{X}_2)M)^{-1}M^t\mathbf{X}_2^t = \mathbf{X}_2 MM^{-1}(\mathbf{X}_2^t\mathbf{X}_2)^{-1}(M^t)^{-1}M^t\mathbf{X}_2^t \\ &= \mathbf{X}_2(\mathbf{X}_2^t\mathbf{X}_2)^{-1}\mathbf{X}_2^t. \qquad \square\end{aligned}$$

Next, we introduce a complex Hilbert space, which we will denote $\mathbb{R}^{2n}_J$. The elements of this space will continue to be real-valued vectors of length $2n$, but we will define multiplication by complex scalars $\gamma = \alpha + i\beta$ as

$$(\alpha + i\beta)u := \alpha u + \beta J u, \quad u \in \mathbb{R}^{2n}, \alpha + i\beta \in \mathbb{C},$$

and we will define a complex scalar product

$$(u, v)_{\mathbb{R}^{2n}_J} := (u, v)_{\mathbb{R}^{2n}} - i\omega(u, v), \quad u, v \in \mathbb{R}^{2n}$$

(recalling $\omega(u, v) = (Ju, v)_{\mathbb{R}^{2n}}$). It is important to note that, considered as a real vector space, $\mathbb{R}^{2n}_J$ is identical to $\mathbb{R}^{2n}$, and not its complexification $\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$. (In fact, $\mathbb{R}^{2n}_J \cong \mathbb{C}^n$ while $\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^{2n}$.) However, it is easy to see that $\mathbb{R}^{2n}_J \cong \ell \otimes_{\mathbb{R}} \mathbb{C}$ for any Lagrangian subspace $\ell \in \Lambda(n)$, and we'll take advantage of this correspondence.

For a matrix U acting on $\mathbb{R}^{2n}_J$, we denote the adjoint in $\mathbb{R}^{2n}_J$ by U^{J*} so that

$$(Uu, v)_{\mathbb{R}^{2n}_J} = (u, U^{J*}v)_{\mathbb{R}^{2n}_J},$$

for all $u, v \in \mathbb{R}^{2n}_J$. We denote by $\mathfrak{U}_J$ the space of unitary matrices acting on $\mathbb{R}^{2n}_J$ (i.e., the matrices so that $UU^{J*} = U^{J*}U = I$). In order to clarify the nature of $\mathfrak{U}_J$, we note that we have the identity

$$(Uu, Uv)_{\mathbb{R}^{2n}_J} = (u, v)_{\mathbb{R}^{2n}_J},$$

from which

$$(Uu, Uv)_{\mathbb{R}^{2n}} - i(JUu, Uv)_{\mathbb{R}^{2n}} = (u, v)_{\mathbb{R}^{2n}} - i(Ju, v)_{\mathbb{R}^{2n}}.$$

Equating real parts, we see that U must be unitary as a matrix on $\mathbb{R}^{2n}$, while by equating imaginary parts we see that $UJ = JU$. We have, then,

$$\mathfrak{U}_J = \{U \in \mathbb{R}^{2n \times 2n} \,|\, U^t U = UU^t = I_{2n},\, UJ = JU\}.$$

Fix some Lagrangian subspace $\ell_0 \subset \mathbb{R}^{2n}$, and notice that $J(\ell_0)$ is orthogonal to ℓ_0; i.e., if $\mathbf{X}_0 = \binom{X_0}{Y_0}$ is a frame for ℓ_0 then $J\mathbf{X}_0 = \binom{-Y_0}{X_0}$ is a frame for $J(\ell_0)$, and we have

$$\begin{pmatrix} X_0^t & Y_0^t \end{pmatrix} \begin{pmatrix} -Y_0 \\ X_0 \end{pmatrix} = -X_0^t Y_0 + Y_0^t X_0 = 0,$$

by the Lagrangian property. In this way, we see that

$$\mathbb{R}^{2n} = \ell_0 \oplus J(\ell_0),$$

so that given any $z \in \mathbb{R}^{2n}$ we can express z uniquely as $z = x + Jy$ for some $x, y \in \ell_0$. We define the conjugate of z in R^{2n}_J by

$$\tau_0 z := x - Jy.$$

Notice that we can compute $\tau_0 = 2\Pi_0 - I_{2n}$, where Π_0 is the orthogonal projection onto ℓ_0. For any $U \in \mathfrak{U}_J$, we define

$$U^T := \tau_0 U^t \tau_0, \tag{2.2}$$

which is also in $\mathfrak{U}_J$ (as follows easily from our next proposition).

Proposition 2.4. *Let $\mathbf{X}_0 = \binom{X_0}{Y_0}$ be a frame for a Lagrangian subspace $\ell_0 \subset \mathbb{R}^{2n}$. Then the matrix $X_0^t X_0 + Y_0^t Y_0$ is symmetric and positive definite, and if we set $M_0 := (X_0^t X_0 + Y_0^t Y_0)^{-1/2}$ we have*

$$\begin{aligned}
\Pi_0 &= \begin{pmatrix} X_0 M_0^2 X_0^t & X_0 M_0^2 Y_0^t \\ Y_0 M_0^2 X_0^t & Y_0 M_0^2 Y_0^t \end{pmatrix} \\
\tau_0 &= \begin{pmatrix} 2X_0 M_0^2 X_0^t - I & 2X_0 M_0^2 Y_0^t \\ 2Y_0 M_0^2 X_0^t & 2Y_0 M_0^2 Y_0^t - I \end{pmatrix},
\end{aligned}$$

with additionally $\tau_0^t = \tau_0$, $\tau_0^2 = I$, and $J\tau_0 = -\tau_0 J$.

Proof. These claims can all be proven in a straightforward manner, using the following identities, which are established in the proof of Lemma 3.3 in [12]:

$$\begin{aligned}
X_0 M_0^2 X_0^t + Y_0 M_0^2 Y_0^t &= I; \\
X_0 M_0^2 Y_0^t - Y_0 M_0^2 X_0^t &= 0.
\end{aligned} \tag{2.3}$$

Noting that

$$\mathbf{X}_0^t \mathbf{X}_0 = \begin{pmatrix} X_0^t & Y_0^t \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = X_0^t X_0 + Y_0^t Y_0,$$

we see that

$$\begin{aligned}
\Pi_0 = \mathbf{X}_0 (\mathbf{X}_0^t \mathbf{X}_0)^{-1} \mathbf{X}_0^t &= \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} M_0^2 \begin{pmatrix} X_0^t \\ Y_0^t \end{pmatrix} \\
&= \begin{pmatrix} X_0 M_0^2 Y_0^t & X_0 M_0^2 Y_0^t \\ Y_0 M_0^2 X_0^t & Y_0 M_0^2 Y_0^t \end{pmatrix}.
\end{aligned}$$

The remaining claims follow in a straightforward manner. □

Now, given a second Lagrangian subspace ℓ, let $U \in \mathfrak{U}_J$ satisfy

$$\ell = U(J(\ell_0)), \tag{2.4}$$

or equivalently

$$U^t(\ell) = J(\ell_0). \tag{2.5}$$

(Such a matrix U is not uniquely defined.) We define

$$W = UU^T = U\tau_0 U^t \tau_0,$$

and it follows from Proposition 2.4 that $W \in \mathfrak{U}_J$.

Lemma 2.5. *For ℓ_0, ℓ, and W as above*

$$\ker(W + I) = (\ell \cap \ell_0) \oplus J(\ell \cap \ell_0).$$

Proof. As a start, take any $z \in (\ell \cap \ell_0) \oplus J(\ell \cap \ell_0)$, and write $z = x + Jy$ for some $x, y \in \ell \cap \ell_0$. We compute

$$\begin{aligned} Wz &= U\tau_0 U^t \tau_0 (x + Jy) \\ &= U\tau_0 U^t (x - Jy) \\ &= U\tau_0 (U^t x - JU^t y) \\ &\overset{*}{=} U(-U^t x - JU^t y) = -x - Jy = -z, \end{aligned}$$

where in obtaining the equality indicated with $*$ we have observed from (2.4) and (2.5) that $U^t x \in J(\ell_0)$ and $JU^t y \in \ell_0$.

On the other hand, suppose $z \in \mathbb{R}^{2n}$ satisfies $Wz = -z$. We can write $z = x + Jy$ for some $x, y \in \ell_0$, and we would like to show that $x, y \in \ell$ so that in fact $x, y \in \ell \cap \ell_0$. We compute

$$\begin{aligned} -(x + Jy) = U\tau_0 U^t \tau_0 (x + Jy) &= U\tau_0 U^t (x - Jy) \\ &= U\tau_0 (U^t x - U^t Jy), \end{aligned}$$

which implies

$$-(U^t x + U^t Jy) = \tau_0 (U^t x - U^t Jy).$$

It's straightforward to see that this can only hold if $U^t Jy \in \ell_0$ and $U^t x \in J(\ell_0)$, which according to (2.4) and (2.5) implies that $x, y \in \ell$. □

For a similar statement in a more general context, see equation (2.37) in [10].

The relationship between ℓ_0, ℓ, and $U \in \mathfrak{U}_J$ provides a natural and productive connection between the elements ℓ of the Lagrangian Grassmannian and elements $U \in \mathfrak{U}_J$. However, the associated unitary matrices are not uniquely specified, and consequently the spectrum of U contains redundant information. For example, in the simple case of $\mathbb{R}^2$ this redundant information corresponds with our previous observation that each element $\ell \in \Lambda(1)$ corresponds with two points on S^1. We overcome this difficulty by defining a new (uniquely specified) unitary matrix W in $\mathbb{R}^{2n}_J$ by $W = UU^T$.

We observe that the unitary condition $UJ = JU$ implies U must have the form

$$U = \begin{pmatrix} U_{11} & -U_{21} \\ U_{21} & U_{11} \end{pmatrix} = \begin{pmatrix} U_{11} & 0 \\ 0 & U_{11} \end{pmatrix} + J \begin{pmatrix} U_{21} & 0 \\ 0 & U_{21} \end{pmatrix}.$$

In addition, we have the scaling condition

$$\begin{aligned} U_{11}^t U_{11} + U_{21}^t U_{21} &= I \\ U_{11} U_{11}^t + U_{21} U_{21}^t &= I \\ U_{11}^t U_{21} - U_{21}^t U_{11} &= 0 \\ U_{11} U_{21}^t - U_{21} U_{11}^t &= 0 \end{aligned} \tag{2.6}$$

(from $UU^t = U^t U = I$). In this way, there is a natural one-to-one correspondence between matrices $U \in \mathfrak{U}_J$ and the $n \times n$ complex unitary matrices $\tilde{U} = U_{11} + iU_{21}$ (i.e., the $\tilde{U} \in \mathbb{C}^{n\times n}$ so that $\tilde{U}^*\tilde{U} = \tilde{U}\tilde{U}^* = I$). It follows that the matrix $W = UU^T$, which can be expressed as

$$W = \begin{pmatrix} W_{11} & -W_{21} \\ W_{21} & W_{11} \end{pmatrix},$$

has a natural corresponding matrix $\tilde{W} = W_{11} + iW_{21}$. We will see in section 3 that our matrix $\tilde{W}$ in (1.1) is constructed in precisely this way.

Proof of Theorem 1.3. Let W and $\tilde{W}$ be as in the preceding paragraph, and suppose $z = x + Jy$, $x, y \in \ell_0$, is an eigenvector for W, associated to the eigenvalue $\lambda = -1$. If we write $x = \binom{x_1}{x_2}$ and $y = \binom{y_1}{y_2}$ then the equation $Wz = -z$ becomes

$$\begin{aligned} W_{11}(x_1 - y_2) - W_{21}(x_2 + y_1) &= -(x_1 - y_2) \\ W_{21}(x_1 - y_2) + W_{11}(x_2 + y_1) &= -(x_2 + y_1). \end{aligned}$$

We see that if $w = u + iv$, with $u = x_1 - y_2$ and $v = x_2 + y_1$, then $\tilde{W}w = -w$. Moreover, w cannot be trivial, because if $w = 0$ then $x_1 = y_2$ and $x_2 = -y_1$, so that

$$0 = \omega(x, y) = (Jx, y) = |x_1|^2 + |x_2|^2,$$

which would imply $x = 0$, and consequently $y = 0$. This contradicts our assumption that z is an eigenvector of W.

On the other hand, notice that if $w = u + iv$ is any eigenvector of $\tilde{W}$ associated to the eigenvalue $\lambda = -1$, then

$$\begin{aligned} W_{11}u - W_{21}v &= -u \\ W_{11}v + W_{21}u &= -v. \end{aligned}$$

If we set $x = \binom{x_1}{x_2} = \binom{u}{v}$ then $Wx = -x$, and likewise if we set $y = \binom{y_1}{y_2} = \binom{v}{-u}$ then $WJy = -Jy$. We see that each eigenvector of $\tilde{W}$ associated to $\lambda = -1$ corresponds with precisely two eigenvectors of W associated to $\lambda = -1$. Since $\dim\ker(W + I) = 2\dim(\ell_0 \cap \ell)$ (from Lemma 2.5), the theorem follows immediately. $\square$

3. Derivation of W and $\tilde{W}$

In this section, we will use our general formulation from Section 2 to derive the form of $\tilde{W}$ expressed in (1.1). We begin by collecting some straightforward observations that will be used throughout our derivation.

Lemma 3.1. *If $\mathbf{X} = \binom{X}{Y}$ is a frame for a Lagrangian subspace $\ell \subset \mathbb{R}^{2n}$ then $X^tX + Y^tY$ is a symmetric positive definite matrix, and the matrices $X - iY$ and $X + iY$ are both invertible.*

Proof. First, if $\mathbf{X}$ is the frame for a Lagrangian subspace $\ell \subset \mathbb{R}^{2n}$ then the columns of $\mathbf{X}$ must be linearly independent. Positive definiteness (and hence invertibility) of $\mathbf{X}^t\mathbf{X} = X^tX + Y^tY$ follows (see, e.g., p. 28 in [14]; also, note that it's clear that this matrix is symmetric).

Turning to invertibility of $X \pm iY$, we focus on $X + iY$, noting that if this matrix has zero as an eigenvalue then there will be a vector $w = u + iv$ so that $(X + iY)(u + iv) = 0$, which means

$$\begin{aligned} Xu - Yv &= 0 \\ Yu + Xv &= 0. \end{aligned} \tag{3.1}$$

If we multiply the first of these equations by Y^t and the second by X^t and subtract the results (recalling the Lagrangian property of frames (2.1)) we obtain $(X^tX + Y^tY)v = 0$. But we've already seen that $(X^tX + Y^tY)$ is invertible, so we must have $v = 0$. Likewise, if we multiply the first equation in (3.1) by X^t and the second by Y^t we find that $u = 0$, which contradicts our assumption that $w = u + iv$ is an eigenvector associated with zero. □

To begin our construction of $\tilde{W}$, we let ℓ_1 and ℓ_2 denote two Lagrangian subspaces of $\mathbb{R}^{2n}$, with associated frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$. As discussed in Section 2, we proceed by associating this pair of Lagrangian subspaces with a matrix $U \in \mathfrak{U}_J$. In particular, U should map $\ell_2^\perp = J(\ell_2)$ to ℓ_1. In terms of frames, this asserts that

$$\mathbf{X}_1 = UJ\mathbf{X}_2,$$

where in order to ensure the unitary normalization $U_{11}^tU_{11} + U_{21}^tU_{21} = I$, we note that for each $i = 1, 2$ we can choose the frame $\mathbf{X}_i$ to be $\binom{X_iM_i}{Y_iM_i}$ for any $n \times n$ invertible matrix M_i. With this choice, we find that U should solve

$$\begin{pmatrix} X_1M_1 \\ Y_1M_1 \end{pmatrix} = \begin{pmatrix} U_{11} & -U_{21} \\ U_{21} & U_{11} \end{pmatrix} \begin{pmatrix} -Y_2M_2 \\ X_2M_2 \end{pmatrix}. \tag{3.2}$$

We will verify below that the choices

$$M_i = (X_i^tX_i + Y_i^tY_i)^{-1/2}$$

suffice. We can express (3.2) as

$$\begin{pmatrix} (X_1M_1)^t \\ (Y_1M_1)^t \end{pmatrix} = V \begin{pmatrix} U_{11}^t \\ U_{21}^t \end{pmatrix}; \quad V = \begin{pmatrix} -(Y_2M_2)^t & -(X_2M_2)^t \\ (X_2M_2)^t & -(Y_2M_2)^t \end{pmatrix}. \tag{3.3}$$

Using identities of the form (2.3), we can check that V is orthogonal, allowing us to solve for U and see that

$$U = \begin{pmatrix} X_1M_1 & -Y_1M_1 \\ Y_1M_1 & X_1M_1 \end{pmatrix} \begin{pmatrix} -M_2Y_2^t & M_2X_2^t \\ -M_2X_2^t & -M_2Y_2^t \end{pmatrix} =: U_1U_2.$$

We now compute

$$W = UU^T = U\tau_2U^t\tau_2 = U_1U_2\tau_2U_2^tU_1^t\tau_2,$$

where τ_2 denotes the conjugation operator obtained as in Section 2, with ℓ_0 replaced by ℓ_2. As in Proposition 2.4, we have

$$\tau_2 = \begin{pmatrix} 2X_2M_2^2X_2^t - I & 2X_2M_2^2Y_2^t \\ 2Y_2M_2^2X_2^t & 2Y_2M_2^2Y_2^t - I \end{pmatrix},$$

and computing directly we can show that

$$U_2\tau_2U_2^t = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$$

Using this intermediate step, and computing directly again we arrive at

$$\begin{aligned} U_1U_2\tau_2U_2^tU_1^t\tau_2 &= \begin{pmatrix} X_1M_1^2X_1^t - Y_1M_1^2Y_1^t & -2X_1M_1^2Y_1^t \\ 2X_1M_1^2Y_1^t & X_1M_1^2X_1^t - Y_1M_1^2Y_1^t \end{pmatrix} \\ &\quad \times \begin{pmatrix} Y_2M_2^2Y_2^t - X_2M_2^2X_2^t & -2X_2M_2^2Y_2 \\ 2X_2M_2^2Y_2^t & Y_2M_2^2Y_2^t - X_2M_2^2X_2^t \end{pmatrix} =: W_1W_2. \end{aligned}$$

Last, we identify the matrix $\tilde{W}$, which we can compute as $\tilde{W} = \tilde{W}_1\tilde{W}_2$. First, it's clear that

$$\begin{aligned} \tilde{W}_1 &= X_1M_1^2X_1^t - Y_1M_1^2Y_1^t + i2X_1M_1^2Y_1^t \\ &= (X_1 + iY_1)M_1^2(X_1^t + iY_1^t), \end{aligned} \tag{3.4}$$

where we've used the identity $X_1M_1^2Y_1^t = Y_1M_1^2X_1^t$ (see the proof of Proposition 2.4). Using the Lagrangian property (2.1), we see that

$$\begin{aligned} (X_1 - iY_1)^{-1}(X_1^t + iY_1^t)^{-1} &= \Big((X_1^t + iY_1^t)(X_1 - iY_1)\Big)^{-1} \\ &= \Big(X_1^tX_1 + Y_1^tY_1 + i(Y_1^tX_1 - X_1^tY_1)\Big)^{-1} = M_1^2. \end{aligned} \tag{3.5}$$

Continuing with our calculation of $\tilde{W}_1$, we conclude

$$\begin{aligned} \tilde{W}_1 &= (X_1 + iY_1)(X_1 - iY_1)^{-1}(X_1^t + iY_1^t)^{-1}(X_1^t + iY_1^t) \\ &= (X_1 + iY_1)(X_1 - iY_1)^{-1}. \end{aligned}$$

Proceeding similarly, we find

$$\tilde{W}_2 = -(X_2 - iY_2)(X_2 + iY_2)^{-1},$$

from which the form of $\tilde{W}$ in (1.1) is immediate.

Using the argument leading to (3.5), we obtain the identities

$$\begin{aligned} (X_j - iY_j)^{-1} &= M_j^2(X_j^t + iY_j^t) \\ (X_j + iY_j)^{-1} &= M_j^2(X_j^t - iY_j^t), \end{aligned} \tag{3.6}$$

for $j = 1, 2$. This provides us with the alternative form

$$\tilde{W} = -(X_1 + iY_1)M_1^2(X_1^t + iY_1^t)(X_2 - iY_2)M_2^2(X_2^t - iY_2^t).$$

Using (3.4) (and the fact that M_1^2 is self-adjoint), we compute

$$\begin{aligned} \tilde{W}_1\tilde{W}_1^* &= (X_1 + iY_1)M_1^2(X_1^t + iY_1^t)(X_1 - iY_1)M_1^2(X_1^t - iY_1^t) \\ &= (X_1 + iY_1)M_1^2(X_1^t - iY_1^t) = I, \end{aligned}$$

verifying that $\tilde{W}_1$ is unitary. Likewise, $\tilde{W}_2$ is unitary, and so $\tilde{W}$ is unitary.

Remark 3.2. We can now extend Arnol'd's Det^2 map to the current setting (see, for example, Section 1.3 in [1]). We define a map $\mathrm{Det}^2 : \Lambda(n) \times \Lambda(n) \to S^1$ as follows: given any Lagrangian pair $\ell_1, \ell_2 \in \Lambda(n)$ and respectively any frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$, $\mathbf{X}_2 = \binom{X_2}{Y_2}$, we set

$$\begin{aligned}\mathrm{Det}^2(\ell_1, \ell_2) &:= \det \tilde{W} = -\det\Big\{\Big((X_1 + iY_1)M_1^2(X_1^t + iY_1^t)\Big) \cdot \Big((X_2 - iY_2)M_2^2(X_2^t - iY_2^t)\Big)\Big\} \\ &= -\frac{\det^2(X_1 + iY_1)}{\det(X_1^t X_1 + Y_1^t Y_1)} \cdot \frac{\det^2(X_2 - iY_2)}{\det(X_2^t X_2 + Y_2^t Y_2)}.\end{aligned} \tag{3.7}$$

We have already seen that $\tilde{W}$ does not depend on the choice of frames, and so the map Det^2 is well-defined.

For some calculations, it's productive to observe that we can express our matrix W in the coordinate-free form

$$W = -(2\mathcal{P}_1 - I)(2\mathcal{P}_2 - I), \tag{3.8}$$

sometimes referred to as the *Souriau map*. Here, $\mathcal{P}_1$ and $\mathcal{P}_2$ are respectively orthogonal projections onto ℓ_1 and ℓ_2, and given particular frames $\mathbf{X}_i = \binom{X_i}{Y_i}$ we can express these as

$$\mathcal{P}_i = \mathbf{X}_i(\mathbf{X}_i^t \mathbf{X}_i)^{-1}\mathbf{X}_i^t = \begin{pmatrix} X_i \\ Y_i \end{pmatrix} M_i^2 \begin{pmatrix} X_i^t & Y_i^t \end{pmatrix} = \begin{pmatrix} X_i M_i^2 X_i^t & X_i M_i^2 Y_i^t \\ Y_i M_i^2 X_i^t & Y_i M_i^2 Y_i^t \end{pmatrix},$$

where $M_i = (X_i^t X_i + Y_i^t Y_i)^{-1/2}$. We see that

$$2\mathcal{P}_i - I_{2n} = \begin{pmatrix} 2X_i M_i^2 X_i^t - I_n & 2X_i M_i^2 Y_i^t \\ 2Y_i M_i^2 X_i^t & 2Y_i M_i^2 Y_i^t - I_n \end{pmatrix}.$$

Using the relations

$$\begin{aligned} X_i M_i^2 X_i^t + Y_i M_i^2 Y_i^t &= I_n \\ X_i M_i^2 X_i^t - Y_i M_i^2 Y_i^t &= 0, \end{aligned}$$

and temporarily setting

$$\begin{aligned} A_i &= X_i M_i^2 X_i^t - Y_i M_i^2 Y_i^t \\ B_i &= 2X_i M_i^2 Y_i^t, \end{aligned}$$

we can check that

$$\begin{aligned}(2\mathcal{P}_1 - I_n)(2\mathcal{P}_2 - I_n) &= \begin{pmatrix} A_1 & B_1 \\ B_1 & -A_1 \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ B_2 & -A_2 \end{pmatrix} \\ &= -\begin{pmatrix} A_1 & -B_1 \\ B_1 & A_1 \end{pmatrix}\begin{pmatrix} -A_2 & -B_2 \\ B_2 & -A_2 \end{pmatrix} = -W.\end{aligned}$$

In order to clarify the relationship between W and $\tilde{W}$, we recall that since $W \in \mathfrak{U}_J$ we have the correspondence

$$W = \begin{pmatrix} W_{11} & -W_{21} \\ W_{21} & W_{11} \end{pmatrix} \iff \tilde{W} = W_{11} + iW_{21}.$$

We can easily check that W and $\tilde{W}$ have precisely the same eigenvalues, and indeed we have

$$\tilde{W}(u+iv) = e^{i\theta}(u+iv)$$

if and only if

$$W\begin{pmatrix} u+iv \\ v-iu \end{pmatrix} = e^{i\theta}\begin{pmatrix} u+iv \\ v-iu \end{pmatrix} \quad \text{and} \quad W\begin{pmatrix} -v+iu \\ u+iv \end{pmatrix} = e^{i\theta}\begin{pmatrix} -v+iu \\ u+iv \end{pmatrix}.$$

I.e., $e^{i\theta}$ is an eigenvalue of $\tilde{W}$ with multiplicity k if and only if it is an eigenvalue of W with multiplicity $2k$. Notice that this simply generalizes our observations from the proof of Theorem 1.3.

Remark 3.3. We are now in a position to observe that our composition relation from Remark 1.2 corresponds with Corollary 2.45 in [10]. In particular, if we let $\mathcal{P}_D$ denote projection onto the Dirichlet Lagrangian subspace (i.e., the Lagrangian subspace with frame $\binom{0}{I}$), and we set

$$W_{1D} = -(2\mathcal{P}_1 - I)(2\mathcal{P}_D - I)$$
$$W_{D2} = -(2\mathcal{P}_D - I)(2\mathcal{P}_2 - I),$$

then Corollary 2.45 in [10] asserts

$$W = -W_{1D}W_{D2},$$

which corresponds with the composition (1.3). (Here, W is from (3.8).)

3.1. Relation to Furutani's development

In [10] (Section 3.5), the author takes a different approach to computing the Maslov index for a pair of evolving Lagrangian subspaces, and we verify here that the two approaches are equivalent in the current setting. As a starting point, we denote by H_ω the symplectic Hilbert space obtained by equipping $\mathbb{R}^{2n}$ with the symplectic form $\omega(x,y) = (Jx,y)_{\mathbb{R}^{2n}}$, and likewise we denote by $H_{-\omega}$ the symplectic Hilbert space obtained by equipping $\mathbb{R}^{2n}$ with the symplectic form $-\omega(x,y) = (-Jx,y)_{\mathbb{R}^{2n}}$. Following [10], we denote the direct sum of these spaces

$$\mathbb{H} = H_\omega \boxplus H_{-\omega}.$$

Now let $\ell_1, \ell_2 \subset \mathbb{R}^{2n}$ denote two Lagrangian subspaces with associated frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$. We can identify the direct sum $\ell_1 \oplus \ell_2$ with a subspace of $\mathbb{R}^{4n}$. For $z_1, z_2 \in \mathbb{R}^{4n}$, we set

$$\omega_{\mathbb{J}}(z_1,z_2) = (\mathbb{J}z_1,z_2)_{\mathbb{R}^{4n}}; \quad \mathbb{J} = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}.$$

It follows immediately from the assumption that ℓ_1 and ℓ_2 are Lagrangian subspaces in $\mathbb{R}^{2n}$ that

$$\mathbf{Z} = \begin{pmatrix} \mathbf{X}_1 & 0_{2n\times n} \\ 0_{2n\times n} & \mathbf{X}_2 \end{pmatrix}$$

is a frame for a Lagrangian subspace in $\mathbb{R}^{4n}$. We denote this Lagrangian subspace ℓ, and note that we can associate it with $\ell_1 \oplus \ell_2$.

In [10], the author detects intersections between ℓ_1 and ℓ_2 by identifying intersections between ℓ and the diagonal in $\mathbb{H}$: i.e., the Lagrangian subspace $\Delta \subset \mathbb{R}^{4n}$ with frame $\mathbf{Z}_\Delta = \binom{I_{2n}}{I_{2n}}$. The orthogonal projection associated with ℓ can be expressed as

$$\mathcal{P}_{\mathbf{Z}} = \mathbf{Z}(\mathbf{Z}^t\mathbf{Z})\mathbf{Z}^t = \begin{pmatrix} \mathcal{P}_1 & 0 \\ 0 & \mathcal{P}_2 \end{pmatrix},$$

and likewise the orthogonal projection associated with Δ can be expressed as

$$\mathcal{P}_\Delta = \frac{1}{2}\begin{pmatrix} I_{2n} & I_{2n} \\ I_{2n} & I_{2n} \end{pmatrix}.$$

We can now compute the Souriau map for ℓ and Δ as

$$\mathcal{W} = -(2\mathcal{P}_{\mathbf{Z}} - I_{4n})(2\mathcal{P}_\Delta - I_{4n}) = \begin{pmatrix} 0 & I_{2n} - 2\mathcal{P}_1 \\ I_{2n} - 2\mathcal{P}_2 & 0 \end{pmatrix}.$$

We see that the eigenvalues of $\mathcal{W}$ will satisfy

$$\det\begin{pmatrix} -\lambda I_{2n} & I_{2n} - 2\mathcal{P}_1 \\ I_{2n} - 2\mathcal{P}_2 & -\lambda I_{2n} \end{pmatrix} = \det\Big(\lambda^2 I - (I_{2n} - 2\mathcal{P}_1)(I_{2n} - 2\mathcal{P}_2)\Big).$$

We see that the values $-\lambda^2$ will be the eigenvalues of the Souriau map (3.8).

According to Lemma 2.5 we have an intersection of ℓ_1 and ℓ_2 if and only if -1 is an eigenvalue of W, and the multiplicity of -1 as an eigenvalue of W is twice the dimension of the intersection. In this case, we will have eigenvalues λ of $\mathcal{W}$ satisfying $-\lambda^2 = -1$. We see that $\mathcal{W}$ has two corresponding eigenvalues $\lambda = -1, +1$, each with the same multiplicity for $\mathcal{W}$ as -1 has for W. Reversing the argument, we conclude that -1 is an eigenvalue of W if and only if it is an eigenvalue of $\mathcal{W}$, and its multiplicity as an eigenvalue of these two matrices agrees.

Finally, we will be able to conclude that the spectral flow through -1 is the same for W and $\mathcal{W}$ if the directions associated with crossings agree. Suppose $e^{i(\pi-\epsilon)}$ is an eigenvalue of W for some small $\epsilon > 0$ (i.e., an eigenvalue rotated slightly clockwise from -1). If λ is the associated eigenvalue of $\mathcal{W}$ then we will have $-\lambda^2 = e^{i(\pi-\epsilon)}$, and so $\lambda = e^{i(\pi-\frac{\epsilon}{2})}, e^{i(2\pi-\frac{\epsilon}{2})}$. If the eigenvalue of W rotates through -1 then its counterpart $e^{i(\pi-\frac{\epsilon}{2})}$ will rotate through -1 in the same direction. Other cases are similar, and we see that indeed the directions associated with the crossings agree.

4. Monotonicity

For many applications, such as the ones discussed in Section 5, we have monotonicity in the following sense: as the parameter $t \in I$ varies in a fixed direction, the eigenvalues of $\tilde{W}(t)$ move monotonically around S^1. In this section, we develop a general framework for checking monotonicity in specific cases.

As a starting point, we take the following lemma from [12] (see also Theorem V.6.1 in [2]):

Lemma 4.1 *([12], Lemma 3.11). Let $\tilde{W}(t)$ be a smooth family of unitary $n \times n$ matrices on some interval I, satisfying the differential equation $\frac{d}{dt}\tilde{W}(t) = i\tilde{W}(t)\tilde{\Omega}(t)$, where $\tilde{\Omega}(t)$ is a continuous, self-adjoint and negative-definite $n \times n$ matrix. Then the eigenvalues of $\tilde{W}(t)$ move (strictly) monotonically clockwise on the unit circle as τ increases.*

In order to employ Lemma 4.1 we need to obtain a convenient form for $\frac{d\tilde{W}}{dt}$. For this, we begin by writing $\tilde{W}(t) = -\tilde{W}_1(t)\tilde{W}_2(t)$, where

$$\tilde{W}_1(t) = (X_1(t) + iY_1(t))(X_1(t) - iY_1(t))^{-1}$$
$$\tilde{W}_2(t) = (X_2(t) - iY_2(t))(X_2(t) + iY_2(t))^{-1}.$$

For $\tilde{W}_1(t)$ we have

$$\begin{aligned}\frac{d\tilde{W}_1}{dt} &= (X_1'(t) + iY_1'(t))(X_1(t) - iY_1(t))^{-1} \\ &\quad - (X_1(t) + iY_1(t))(X_1(t) - iY_1(t))^{-1}(X_1'(t) - iY_1'(t))(X_1(t) - iY_1(t))^{-1} \\ &= (X_1'(t) + iY_1'(t))(X_1(t) - iY_1(t))^{-1} \\ &\quad - \tilde{W}_1(X_1'(t) - iY_1'(t))(X_1(t) - iY_1(t))^{-1} \\ &= \tilde{W}_1\tilde{W}_1^*(X_1'(t) + iY_1'(t))(X_1(t) - iY_1(t))^{-1} \\ &\quad - \tilde{W}_1(X_1'(t) - iY_1'(t))(X_1(t) - iY_1(t))^{-1} \\ &= \tilde{W}_1\Big\{\tilde{W}_1^*(X_1'(t) + iY_1'(t)) - (X_1'(t) - iY_1'(t))\Big\}(X_1(t) - iY_1(t))^{-1},\end{aligned}$$

where we have liberally taken advantage of the fact that $\tilde{W}$ is unitary. Here,

$$\begin{aligned}\{\cdots\} &= (X_1(t)^t + iY_1(t)^t)^{-1}(X_1(t)^t - iY_1(t)^t)(X_1'(t) + iY_1'(t)) - (X_1'(t) - iY_1'(t)) \\ &= (X_1(t)^t + iY_1(t)^t)^{-1}\Big[(X_1(t)^t - iY_1(t)^t)(X_1'(t) + iY_1'(t)) \\ &\quad - (X_1(t)^t + iY_1(t)^t)(X_1'(t) - iY_1'(t))\Big],\end{aligned}$$

and

$$[\cdots] = 2i(X_1(t)^tY_1'(t) - Y_1(t)^tX_1'(t)).$$

We conclude that

$$\frac{d\tilde{W}_1}{dt} = i\tilde{W}_1(t)\tilde{\Omega}_1(t),$$

where

$$\tilde{\Omega}_1(t) = 2\Big((X_1(t) - iY_1(t))^{-1}\Big)^*\Big(X_1(t)^tY_1'(t) - Y_1(t)^tX_1'(t)\Big)\Big((X_1(t) - iY_1(t))^{-1}\Big).$$

Proceeding similarly for $\tilde{W}_2(t)$ we find

$$\frac{d\tilde{W}_2}{dt} = i\tilde{W}_2(t)\tilde{\Omega}_2(t),$$

where

$$\tilde{\Omega}_2(t) = -2\Big((X_2(t) + iY_2(t))^{-1}\Big)^*\Big(X_2(t)^tY_2'(t) - Y_2(t)^tX_2'(t)\Big)\Big((X_2(t) + iY_2(t))^{-1}\Big).$$

Combining these observations, we compute

$$\begin{aligned}\frac{d\tilde{W}}{dt} &= -\frac{d\tilde{W}_1}{dt}\tilde{W}_2 - \tilde{W}_1\frac{d\tilde{W}_2}{dt} \\ &= -i\tilde{W}_1(t)\tilde{\Omega}_1(t)\tilde{W}_2(t) - i\tilde{W}_1(t)\tilde{W}_2(t)\tilde{\Omega}_2(t)\end{aligned}$$

$$= i(-\tilde{W}_1(t)\tilde{W}_2(t))\tilde{W}_2(t)^*\tilde{\Omega}_1(t)\tilde{W}_2(t) + i(-\tilde{W}_1(t)\tilde{W}_2(t))\tilde{\Omega}_2(t)$$
$$= i\tilde{W}(t)\Big\{\tilde{W}_2(t)^*\tilde{\Omega}_1(t)\tilde{W}_2(t) + \tilde{\Omega}_2(t)\Big\}.$$

That is, we have

$$\frac{d\tilde{W}}{dt} = i\tilde{W}(t)\tilde{\Omega}(t),$$

where

$$\tilde{\Omega}(t) = \tilde{W}_2(t)^*\tilde{\Omega}_1(t)\tilde{W}_2(t) + \tilde{\Omega}_2(t).$$

We notice particularly that we can write

$$\tilde{W}_2^*\tilde{\Omega}_1\tilde{W}_2 = 2\Big((X_1 - iY_1)^{-1}\tilde{W}_2\Big)^*(X_1^tY_1' - Y_1^tX_1')\Big((X_1 - iY_1)^{-1}\tilde{W}_2\Big).$$

We see that the nature of $\tilde{\Omega}(t)$ will be determined by the matrices $(X_1(t)^tY_1'(t) - Y_1(t)^tX_1'(t))$ and $(X_2(t)^tY_2'(t) - Y_2(t)^tX_2'(t))$. In order to check that these matrices are symmetric, we differentiate the Lagrangian property

$$X_1(t)^tY_1(t) - Y_1(t)^tX_1(t) = 0$$

to see that

$$X_1(t)^tY_1'(t) - Y_1(t)^tX_1'(t) = Y_1'(t)^tX_1(t) - X_1'(t)^tY_1(t).$$

Symmetry of $(X_1(t)^tY_1'(t) - Y_1(t)^tX_1'(t))$ is immediate, and we proceed similarly for $(X_2(t)^tY_2'(t) - Y_2(t)^tX_2'(t))$. We conclude that $\tilde{\Omega}(t)$ is self-adjoint.

Finally, for monotonicity, we need to check that $\tilde{\Omega}(t)$ is definite. We show how to do this in certain cases in Section 5. For convenient reference, we summarize these observations into a lemma.

Lemma 4.2. *Suppose $\ell_1, \ell_2 : I \to \Lambda(n)$ denote paths of Lagrangian subspaces with C^1 frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$ (respectively). If the matrices*

$$-\mathbf{X}_1^tJ\mathbf{X}_1' = X_1(t)^tY_1'(t) - Y_1(t)^tX_1'(t)$$

and (noting the sign change)

$$\mathbf{X}_2^tJ\mathbf{X}_2' = -(X_2(t)^tY_2'(t) - Y_2(t)^tX_2'(t))$$

are both non-negative and at least one is positive definite then the eigenvalues of $\tilde{W}(t)$ rotate in the counterclockwise direction as t increases. Likewise, if both of these matrices are non-positive, and at least one is negative definite then the eigenvalues of $\tilde{W}(t)$ rotate in the clockwise direction as t increases.

4.1. Monotonicity at crossings

We are often interested in the rotation of eigenvalues of $\tilde{W}$ through -1; i.e., the rotation associated with an intersection of our Lagrangian subspaces. Let t_* denote the time of intersection. As discussed in [12], if we let $\tilde{\mathcal{P}}$ denote projection onto $\ker(\tilde{W} + I)$, then the rotation of eigenvalues through -1 is determined by the eigenvalues of the matrix $\tilde{\mathcal{P}}\tilde{\Omega}(t_*)\tilde{\mathcal{P}}$. Notice that if $\tilde{v} \in \ker(\tilde{W} + I)$ we will have

$$-(X_1(t_*)+iY_1(t_*))(X_1(t_*)-iY_1(t_*))^{-1}(X_2(t_*)-iY_2(t_*))(X_2(t_*)+iY_2(t_*))^{-1}\tilde{v}=-\tilde{v},$$

and correspondingly

$$(X_1(t_*)-iY_1(t_*))^{-1}\tilde{W}_2(t_*)\tilde{v}=(X_1(t_*)+iY_1(t_*))^{-1}\tilde{v}.$$

Recalling relations (3.6), we find that

$$(X_1(t_*)+iY_1(t_*))^{-1}\tilde{v}=M_1(t_*)^2(X_1(t_*)^t-iY_1(t_*)^t)\tilde{v}.$$

We see that if $\tilde{\Omega}(t_*)$ acts on $\ker(\tilde{W}+I)$ we can replace it with

$$\begin{aligned}\tilde{\Omega}_{\mathcal{P}}(t_*):=&\,2\Big(M_1(t_*)^2(X_1(t_*)^t-Y_1(t_*)^t)\Big)^*\Big(X_1(t_*)^tY_1'(t_*)-Y_1^t(t_*)X_1'(t_*)\Big)\\&\times M_1(t_*)^2(X_1(t_*)^t-Y_1(t_*)^t)\\&-2\Big(M_2(t_*)^2(X_2(t_*)^t-Y_2(t_*)^t)\Big)^*\Big(X_2(t_*)^tY_2'(t_*)-Y_2^t(t_*)X_2'(t_*)\Big)\\&\times M_2(t_*)^2(X_2(t_*)^t-Y_2(t_*)^t).\end{aligned}$$

If we express $\tilde{v}=v_1+iv_2$, we can write

$$\begin{aligned}(X_1(t_*)-iY_1(t_*))^{-1}\tilde{W}_2(t_*)\tilde{v}&=M_1(t_*)^2(X_1(t_*)^t-iY_1(t_*)^t)(v_1+iv_2)\\&=M_1(t_*)^2\Big\{X_1(t_*)^tv_1+Y_1(t_*)^tv_2+i(X_1(t_*)^tv_2-Y_1(t_*)^tv_1)\Big\}\\&=M_1(t_*)^2\Big\{X_1(t_*)^tv_1+Y_1(t_*)^tv_2\Big\}.\end{aligned}$$

Here, we have observed that it follows from the Lagrangian property that $X_1(t_*)^tv_2-Y_1(t_*)^tv_1=0$. Likewise,

$$M_2(t_*)^2(X_2(t_*)^t-Y_2(t_*)^t)(\tilde{v})=M_2(t_*)^2\Big\{X_2(t_*)^tv_1+Y_2(t_*)^tv_2\Big\}.$$

If we now write

$$\tilde{\Omega}_{\mathcal{P}}(t_*)=\tilde{\Omega}_{\mathcal{P}}^{(1)}(t_*)+\tilde{\Omega}_{\mathcal{P}}^{(2)}(t_*),$$

then the quadratic form associated with $\tilde{\Omega}_{\mathcal{P}}^{(1)}(t_*)$ will take the form

$$\begin{aligned}\Big(\tilde{\Omega}_{\mathcal{P}}^{(1)}(t_*)\tilde{v},\tilde{v}\Big)_{\mathbb{C}^n}=&\,2\Big((X_1(t_*)^tY_1'(t_*)-Y_1^t(t_*)X_1'(t_*))M_1(t_*)^2\Big\{X_1(t_*)^tv_1+Y_1(t_*)^tv_2\Big\},\\&M_1(t_*)^2\Big\{X_1(t_*)^tv_1+Y_1(t_*)^tv_2\Big\}\Big)_{\mathbb{C}^n},\end{aligned}\tag{4.1}$$

and likewise the quadratic form associated with $\tilde{\Omega}_{\mathcal{P}}^{(2)}(t_*)$ will take the form

$$\begin{aligned}\Big(\tilde{\Omega}_{\mathcal{P}}^{(2)}(t_*)\tilde{v},\tilde{v}\Big)_{\mathbb{C}^n}=&\,2\Big((X_2(t_*)^tY_2'(t_*)-Y_2^t(t_*)X_2'(t_*))M_2(t_*)^2\Big\{X_2(t_*)^tv_1+Y_2(t_*)^tv_2\Big\},\\&M_2(t_*)^2\Big\{X_2(t_*)^tv_1+Y_2(t_*)^tv_2\Big\}\Big)_{\mathbb{C}^n}.\end{aligned}\tag{4.2}$$

We will use (4.1) and (4.2) in our next section in which we relate our approach to the development of [18], based on crossing forms.

4.2. Relation to crossing forms

In this section, we discuss the relation between our development and the crossing forms of [18]. As a starting point, let $\ell_1(t)$ denote a path of Lagrangian subspaces, and let ℓ_2 denote a fixed *target* Lagrangian subspace. Let the respective frames be

$$\mathbf{X}_1(t) = \begin{pmatrix} X_1(t) \\ Y_1(t) \end{pmatrix}; \quad \mathbf{X}_2 = \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix},$$

and let t_* denote the time of a crossing; i.e.,

$$\ell_1(t_*) \cap \ell_2 \neq \{0\}.$$

The corresponding matrix $\tilde{W}(t)$ will be

$$\tilde{W}(t) = -(X_1(t) + iY_1(t))(X_1(t) - iY_1(t))^{-1}(X_2 - iY_2)(X_2 + iY_2)^{-1}.$$

Our goal is to compare the information obtained by computing $\tilde{W}'(t_*)$ with the information we get from the crossing form at t_*.

Following [18], we construct the crossing form at t_* as a map

$$\Gamma(\ell_1, \ell_2; t_*) : \ell_1(t_*) \cap \ell_2 \to \mathbb{R}$$

defined as follows: given $v \in \ell_1(t_*) \cap \ell_2$, we find $u \in \mathbb{R}^n$ so that $v = \mathbf{X}_1(t_*)u$, and compute

$$\begin{aligned} \Gamma(\ell_1, \ell_2; t_*)(v) &= (X_1(t_*)u, Y_1'(t_*)u)_{\mathbb{R}^n} - (X_1(t_*)u, Y_1'(t_*)u)_{\mathbb{R}^n} \\ &= \Big((X_1(t_*)^t Y_1'(t_*) - Y_1(t_*)^t X_1'(t_*))u, u\Big). \end{aligned}$$

Since $v \in \ell_1(t_*) \cap \ell_2 \subset \ell_1(t_*)$ the vector u is uniquely defined and we can compute it in terms of the Moore–Penrose pseudo-inverse of $\mathbf{X}_1$,

$$u = (\mathbf{X}_1^t \mathbf{X}_1)^{-1} \mathbf{X}_1^t v = M_1^2 (X_1^t v_1 + Y_1^t v_2),$$

where $v = \binom{v_1}{v_2}$.

Comparing with (4.1), and taking $\mathbf{X}_2$ in this setting to be $\mathbf{X}_2(t_*)$ in the setting of (4.1), we see that

$$\Gamma(\ell_1, \ell_2; t_*)(v) = \frac{1}{2}\Big(\tilde{\Omega}_{\mathcal{P}}^{(1)}(t_*)\tilde{v}, \tilde{v}\Big)_{\mathbb{C}^n}. \tag{4.3}$$

When computing the Maslov index with crossing forms, the rotation of eigenvalues of $\tilde{W}$ through -1 is determined by the signature of the crossing form. We see from (4.3) that this information is encoded in the eigenvalues of $\tilde{\Omega}_{\mathcal{P}}^{(1)}(t_*)$.

Turning now to path pairs, we recall that in [18] the crossing form for a pair of Lagrangian paths $\ell_1(t)$ and $\ell_2(t)$ is defined as

$$\Gamma(\ell_1, \ell_2; t_*) = \Gamma(\ell_1, \ell_2(t_*); t_*) - \Gamma(\ell_2, \ell_1(t_*); t_*).$$

Here, $\ell_2(t_*)$ is viewed as a constant Lagrangian subspace, so that our previous development can be applied to $\Gamma(\ell_1, \ell_2(t_*); t_*)$, and similarly for $\Gamma(\ell_2, \ell_1(t_*); t_*)$, in which case $\ell_1(t_*)$ is viewed as a constant Lagrangian subspace. In the previous calculations, we have already checked that

$$\Gamma(\ell_1, \ell_2(t_*); t_*)(v) = \frac{1}{2}\Big(\tilde{\Omega}^{(1)}_{\mathcal{P}}(t_*)\tilde{v}, \tilde{v}\Big)_{\mathbb{C}^n},$$

and we similarly find that

$$\Gamma(\ell_2, \ell_1(t_*); t_*)(v) = \frac{1}{2}\Big(\tilde{\Omega}^{(2)}_{\mathcal{P}}(t_*)\tilde{v}, \tilde{v}\Big)_{\mathbb{C}^n}.$$

Combining these expressions, we see that the crossing form for the Lagrangian pair $(\ell_1(t), \ell_2(t))$ at a crossing point t_* is

$$\Gamma(\ell_1, \ell_2; t_*) = \frac{1}{2}\Big(\tilde{\Omega}_{\mathcal{P}}(t_*)\tilde{v}, \tilde{v}\Big)_{\mathbb{C}^n}.$$

5. Applications

Although full applications will be carried out in separate papers, we indicate two motivating applications for completeness.

Application 1. In [12], the authors consider Schrödinger equations

$$\begin{aligned} -y'' + V(x)y &= \lambda y \\ \alpha_1 y(0) + \alpha_2 y'(0) &= 0 \\ \beta_1 y(1) + \beta_2 y'(1) &= 0, \end{aligned} \tag{5.1}$$

where $V \in C([0,1])$ is a real-valued symmetric matrix,

$$\operatorname{rank}[\alpha_1 \quad \alpha_2] = n; \quad \operatorname{rank}[\beta_1 \quad \beta_2] = n, \tag{5.2}$$

and we assume separated, self-adjoint boundary conditions, for which we have

$$\begin{aligned} \alpha_1\alpha_2^t - \alpha_2\alpha_1^t &= 0; \\ \beta_1\beta_2^t - \beta_2\beta_1^t &= 0. \end{aligned} \tag{5.3}$$

By a choice of scaling we can take, without loss of generality,

$$\begin{aligned} \alpha_1\alpha_1^t + \alpha_2\alpha_2^t &= I; \\ \beta_1\beta_1^t + \beta_2\beta_2^t &= I. \end{aligned}$$

In order to place this system in the current framework, we set $p = y$, $q = y'$, and $\mathbf{p} = \binom{p}{q}$, so that it can be expressed as a first-order system

$$\frac{d\mathbf{p}}{dx} = \mathbb{A}(x;\lambda)\mathbf{p}; \quad \mathbb{A}(x;\lambda) = \begin{pmatrix} 0 & I \\ V(x) - \lambda I & 0 \end{pmatrix}. \tag{5.4}$$

Since $\operatorname{rank}[\alpha_1 \quad \alpha_2] = n$, there exists an n-dimensional space of solutions to the left boundary condition

$$[\alpha_1 \quad \alpha_2]\,\mathbf{p}(0) = 0$$

(i.e., the kernel of $[\alpha_1 \quad \alpha_2]$). In particular, we see from (5.3) that we can take

$$\mathbf{X}_1(0,\lambda) = \begin{pmatrix} \alpha_2^t \\ -\alpha_1^t \end{pmatrix}.$$

By virtue of the Lagrangian property, we see that $\mathbf{X}_1(0;\lambda)$ is the frame for a Lagrangian subspace.

Let $\mathbf{X}_1(x,\lambda)$ be a path of frames created by starting with $\mathbf{X}_1(0,\lambda)$ and evolving according to (5.4). In order to see that $\mathbf{X}_1(x,\lambda)$ continues to be a frame for a Lagrangian subspace for all $x \in [0,1]$, we begin by setting

$$Z(x,\lambda) = X_1(x,\lambda)^t Y_1(x,\lambda) - Y_1(x,\lambda)^t X_1(x,\lambda),$$

and noting that $Z(0,\lambda) = 0$. Also (using prime to denote differentiation with respect to x),

$$\begin{aligned} Z' &= (X_1')^t Y_1 + X_1^t Y_1' - (Y_1')^t X_1 - Y_1^t X_1' \\ &= Y_1^t Y_1 + X_1^t (V(x) X_1 - \lambda X_1) - (V(x) X_1 - \lambda X_1)^t X_1 - Y_1^t Y_1 \\ &= 0, \end{aligned}$$

where we have observed $X_1' = Y_1$, $Y_1' = V(x)X_1 - \lambda X_1$, and have used our assumption that V is symmetric. We see that $Z(x,\lambda)$ is constant in x, and since $Z(0,\lambda) = 0$ this means $Z(x,\lambda) = 0$ for all $x \in [0,1]$. We conclude from Lemma 2.1 that $\mathbf{X}_1(x,\lambda)$ is the frame for a Lagrangian subspace for all $x \in [0,1]$. As usual, we denote the Lagrangian subspace associated with $\mathbf{X}_1$ by ℓ_1.

In this case, the second ("target") Lagrangian subspace is the one associated with the boundary conditions at $x = 1$. I.e.,

$$\mathbf{X}_2 = \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \beta_2^t \\ -\beta_1^t \end{pmatrix},$$

which is Lagrangian due to our boundary condition and the Lagrangian property. We denote the Lagrangian subspace associated with $\mathbf{X}_2$ by ℓ_2. We find that

$$\tilde{W}(x,\lambda) = -(X_1(x,\lambda) + iY_1(x,\lambda))(X_1(x,\lambda) - iY_1(x,\lambda))^{-1}(\beta_2^t + i\beta_1^t)(\beta_2^t - i\beta_1^t)^{-1}.$$

For comparison with [12], we observe that

$$(\beta_2^t + i\beta_1^t)(\beta_2^t - i\beta_1^t)^{-1} = \beta_2^t\beta_2 - \beta_1^t\beta_1 + 2i(\beta_2^t\beta_1), \tag{5.5}$$

and this right-hand side, along with the negative sign, is the form that appears in [12] (see p. 4517). In order to verify (5.5), we directly compute

$$(\beta_2 + i\beta_1)(\beta_2^t - i\beta_1^t) = \beta_2\beta_2^t + \beta_1\beta_1^t + i(\beta_1\beta_2^t - \beta_2\beta_1^t) = I,$$

showing that

$$(\beta_2^t - i\beta_1^t)^{-1} = (\beta_2 + i\beta_1).$$

But then

$$\begin{aligned} (\beta_2^t + i\beta_1^t)(\beta_2^t - i\beta_1^t)^{-1} &= (\beta_2^t + i\beta_1^t)(\beta_2 + i\beta_1) \\ &= \beta_2^t\beta_2 - \beta_1^t\beta_1 + i(\beta_2^t\beta_1 + \beta_1^t\beta_2) \\ &= \beta_2^t\beta_2 - \beta_1^t\beta_1 + 2i(\beta_2^t\beta_1). \end{aligned}$$

(These are the same considerations that led to (3.6).)

Turning to the important property of monotonicity, we see that we can consider monotonicity as x varies or as λ varies (or, in principle, we could consider any other path in the x–λ plane). We find that while monotonicity doesn't generally hold as x varies (except in special cases, such as Dirichlet boundary conditions), it does hold generally as λ varies. In order to see this, we observe that in light of Section 4 we can write

$$\frac{\partial \tilde{W}}{\partial \lambda} = i\tilde{W}\tilde{\Omega},$$

where

$$\tilde{\Omega} = 2\Big((X_1 - iY_1)^{-1}\tilde{W}_2\Big)^* \Big(X_1^t \partial_\lambda Y_1 - Y_1^t \partial_\lambda X_1\Big)\Big((X_1 - iY_1)^{-1}\tilde{W}_2\Big),$$

and

$$\tilde{W}_2 = (\beta_2^t + i\beta_1^t)(\beta_2^t - i\beta_1^t)^{-1}.$$

We see that monotonicity is determined by the matrix

$$A(x,\lambda) = X_1(x,\lambda)^t \partial_\lambda Y_1(x,\lambda) - Y_1(x,\lambda)^t \partial_\lambda X_1(x,\lambda),$$

where our introduction of the notation $A(x,\lambda)$ is simply for the convenience of the next calculation. Differentiating with respect to x, we find

$$\begin{aligned} A' &= (X_1')^t \partial_\lambda Y_1 + X_1^t \partial_\lambda Y_1' - (Y_1')^t \partial_\lambda X_1 - Y_1^t \partial_\lambda X_1' \\ &= Y_1^t \partial_\lambda Y_1 + X_1^t \partial_\lambda (V(x)X_1 - \lambda X_1) - (V(x)X_1 - \lambda X_1)^t \partial_\lambda X_1 - Y_1^t \partial_\lambda Y_1 \\ &= -X_1^t X_1. \end{aligned}$$

Integrating on $[0,x]$, we find

$$A(x,\lambda) = X_1(0,\lambda)^t \partial_\lambda Y_1(0,\lambda) - Y_1(0,\lambda)^t \partial_\lambda X_1(0,\lambda) - \int_0^x X_1(y,\lambda)^t X_1(y,\lambda) dy.$$

We observe that since $X_1(0,\lambda) = \alpha_2^t$ and $Y_1(0,\lambda) = -\alpha_1^t$, we have $\partial_\lambda X_1(0,\lambda) = 0$ and $\partial_\lambda Y_1(0,\lambda) = 0$, and so

$$A(x,\lambda) = -\int_0^x X_1(y,\lambda)^t X_1(y,\lambda) dy,$$

which is negative definite. We conclude that $\tilde{\Omega}$ is negative definite, and so for any $x \in [0,1]$, as λ increases the eigenvalues of $\tilde{W}$ rotate monotonically in the clockwise direction.

In order to summarize the result that these observations lead to, we will find it productive to fix $s_0 > 0$ (taken sufficiently small during the analysis) and $\lambda_\infty > 0$ (taken sufficiently large during the analysis), and to consider the rectangular path

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4,$$

where the paths $\{\Gamma_i\}_{i=1}^4$ are depicted in Fig. 1 (taken from [12]).

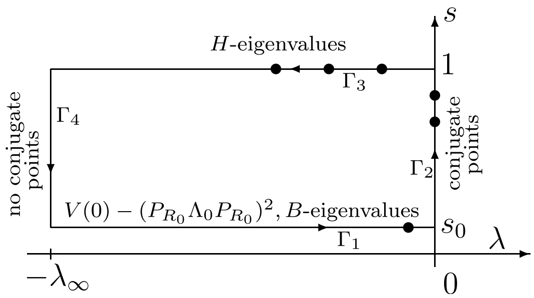

Fig. 1. Schematic of the path $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.

Due to path additivity,

$$\mathrm{Mas}(\ell_1, \ell_2; \Gamma) = \mathrm{Mas}(\ell_1, \ell_2; \Gamma_1) + \mathrm{Mas}(\ell_1, \ell_2; \Gamma_2) + \mathrm{Mas}(\ell_1, \ell_2; \Gamma_3) + \mathrm{Mas}(\ell_1, \ell_2; \Gamma_4),$$

and by homotopy invariance the Maslov index around any closed path will be 0, so that

$$\mathrm{Mas}(\ell_1, \ell_2; \Gamma) = 0.$$

In order to deal efficiently with our self-adjoint boundary conditions, we adapt an elegant theorem from [3] (see also an earlier version in [15]).

Theorem 5.1 *(Adapted from [3]). Let α_1 and α_2 be as described in (5.2)–(5.3). Then there exist three orthogonal (and mutually orthogonal) projection matrices P_D (the Dirichlet projection), P_N (the Neumann projection), and $P_R = I - P_D - P_N$ (the Robin projection), and an invertible self-adjoint operator Λ acting on the space $P_R\mathbb{R}^n$ such that the boundary condition*

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0$$

can be expressed as

$$\begin{aligned} P_D y(0) &= 0 \\ P_N y'(0) &= 0 \\ P_R y'(0) &= \Lambda P_R y(0). \end{aligned}$$

Moreover, P_D can be constructed as the projection onto the kernel of α_2 and P_N can be constructed as the projection onto the kernel of α_1. Construction of the operator Λ is discussed in more detail in [3], and also in [12]. Precisely the same statement holds for β_1 and β_2 for the boundary condition at $x = 1$.

We also take the following from [12].

Definition 5.2. Let $(P_{D_0}, P_{N_0}, P_{R_0}, \Lambda_0)$ denote the projection quadruplet associated with our boundary conditions at $x = 0$, and let $(P_{D_1}, P_{N_1}, P_{R_1}, \Lambda_1)$ denote the projection quadruplet associated with our boundary conditions at $x = 1$. We denote by B the self-adjoint operator obtained by restricting $(P_{R_0}\Lambda_0 P_{R_0} - P_{R_1}\Lambda_1 P_{R_1})$ to the space $(\ker P_{D_0}) \cap (\ker P_{D_1})$.

The main result of [12] is the following theorem.

Theorem 5.3. *For system (5.1), let $V \in C([0,1])$ be a symmetric matrix in $\mathbb{R}^{n\times n}$, and let α_1, α_2, β_1, and β_2 be as in (5.2)–(5.3). In addition, let Q denote projection onto the kernel of B, and make the non-degeneracy assumption $0 \notin \sigma(Q(V(0) - (P_{R_0}\Lambda_0 P_{R_0})^2)Q)$. Then we have*

$$\operatorname{Mor}(H) = -\operatorname{Mas}(\ell, \ell_1; \Gamma_2) + \operatorname{Mor}(B) + \operatorname{Mor}(Q(V(0) - (P_{R_0}\Lambda_0 P_{R_0})^2)Q).$$

In order to clarify the nature of the terms $\operatorname{Mor}(B) + \operatorname{Mor}(Q(V(0) - (P_{R_0}\Lambda_0 P_{R_0})^2)Q)$, we show here how they easily arise from a naive perturbation argument; for a rigorous treatment, the reader is referred to [12].

First, we observe that a crossing at a point (s, λ) corresponds with a solution to the system

$$\begin{aligned} -y'' + V(x)y &= \lambda y \\ \alpha_1 y(0) + \alpha_2 y'(0) &= 0 \\ \beta_1 y(s) + \beta_2 y'(s) &= 0. \end{aligned} \tag{5.6}$$

Setting $\xi = x/s$ and $u(\xi) = y(x)$, we obtain the system

$$\begin{aligned} H(s)u := -u'' + s^2 V(s\xi)y &= s^2\lambda u \\ \alpha_1 u(0) + \frac{1}{s}\alpha_2 u'(0) &= 0 \\ \beta_1 u(1) + \frac{1}{s}\beta_2 u'(1) &= 0. \end{aligned} \tag{5.7}$$

Employing a straightforward energy estimate similar to the proof of Lemma 3.12 in [12], we find that there exists a constant c so that any eigenvalue of (5.6) satisfies

$$\lambda(s) \geq -\frac{c}{s} - \|V\|_{L^\infty(0,1)}.$$

This means that by taking λ_∞ sufficiently large we can ensure that there are no crossings along the left shelf. In order to understand crossings along the bottom shelf we set $\tilde{\lambda} = s^2\lambda(s)$ and take the naive expansions

$$\begin{aligned} \tilde{\lambda}(s) &= \tilde{\lambda}_0 + \tilde{\lambda}_1 s + \tilde{\lambda}_2 s^2 + \cdots \\ \phi(\xi; s) &= \phi_0(\xi) + \phi_1(\xi)s + \phi_2(\xi)s^2 + \cdots, \end{aligned} \tag{5.8}$$

where $\phi(\xi; s)$ is an eigenfunction corresponding with eigenvalue $\tilde{\lambda}(s)$. We emphasize that the spectral curves we are looking for will have the corresponding form

$$\lambda(s) = \frac{\tilde{\lambda}_0}{s^2} + \frac{\tilde{\lambda}_1}{s} + \tilde{\lambda}_2 + \ldots. \tag{5.9}$$

Using Theorem 5.1, we can express the boundary conditions for (5.7) as

$$\begin{aligned} &P_{D_0}u(0) = 0; && P_{D_1}u(1) = 0; \\ &P_{N_0}u'(0) = 0; && P_{N_1}u'(1) = 0; \\ &P_{R_0}u'(0) = s\Lambda_0 P_{R_0}u(0); && P_{R_1}u'(1) = s\Lambda_1 P_{R_1}u(1). \end{aligned}$$

Upon substitution of (5.8) into (5.7) with projection boundary conditions, we find that the zeroth order equation is $-\phi_0'' = \tilde{\lambda}_0\phi_0$ with boundary conditions

$$P_{D_0}\phi_0(0) = 0; \qquad P_{D_1}\phi_0(1) = 0;$$
$$P_{N_0}\phi_0'(0) = 0; \qquad P_{N_1}\phi_0'(1) = 0;$$
$$P_{R_0}\phi_0'(0) = 0; \qquad P_{R_1}\phi_0'(1) = 0.$$

Taking an $L^2(0,1)$ inner product of this equation with ϕ_0 we obtain

$$\begin{aligned}\tilde{\lambda}_0 \|\phi_0\|^2_{L^2(0,1)} &= \langle \phi_0'', \phi_0 \rangle \\ &= \|\phi_0'\|^2_{L^2(0,1)} - (\phi_0'(1), \phi_0(1))_{\mathbb{R}^n} + (\phi_0'(0), \phi_0(0))_{\mathbb{R}^n}.\end{aligned}$$

Observing that

$$\begin{aligned}(\phi_0'(1), \phi_0(1))_{\mathbb{R}^n} &= (\phi_0'(1), P_{D_1}\phi_0(1) + P_{N_1}\phi_0(1) + P_{R_1}\phi_0(1))_{\mathbb{R}^n} \\ &= (P_{N_1}\phi_0'(1) + P_{R_1}\phi_0'(1), \phi_0(1))_{\mathbb{R}^n} = 0,\end{aligned} \tag{5.10}$$

and noting that similarly $(\phi_0'(0), \phi_0(0))_{\mathbb{R}^n} = 0$, we see that

$$\tilde{\lambda}_0 \|\phi_0\|^2_{L^2(0,1)} = \|\phi_0'\|^2_{L^2(0,1)}.$$

Clearly, we must have $\tilde{\lambda}_0 \geq 0$, and if $\tilde{\lambda}_0 > 0$ the associated spectral curve will lie in the right quarter-plane and will not cross into the Maslov Box. On the other hand, if $\tilde{\lambda}_0 = 0$ then $\|\phi_0'\|_{L^2(0,1)} = 0$ and ϕ_0 will be a constant function. In this case, the only requirement on the constant vector ϕ_0 is (from the projection boundary conditions)

$$\phi_0 \in (\ker P_{D_0}) \cap (\ker P_{D_1}).$$

Let P denote the orthogonal projection onto the space $(\ker P_{D_0}) \cap (\ker P_{D_1})$ and set

$$B = P(P_{R_0}\Lambda_0 P_{R_0} - P_{R_1}\Lambda_1 P_{R_1})P$$

(i.e., B is the matrix defined in (5.2)). Since B is symmetric and maps $(\ker P_{D_0}) \cap (\ker P_{D_1})$ to itself, we can create an orthonormal basis for $(\ker P_{D_0}) \cap (\ker P_{D_1})$ from the eigenvectors of B. Moreover, let Q denote the orthogonal projection onto $\ker B$ (as in the statement of Theorem 5.3) and create an orthonormal basis for $\ker B$ from the eigenvectors of $Q(V(0) - (P_{R_0}\Lambda_0 P_{R_0})^2)Q$.

Now, we are ready for the order 1 equation, assuming already that $\tilde{\lambda}_0 = 0$. For any ϕ_0 selected from our chosen basis for $(\ker P_{D_0}) \cap (\ker P_{D_1})$, we obtain the equation $-\phi_1'' = \tilde{\lambda}_1 \phi_0$, with projection boundary conditions

$$P_{D_0}\phi_1(0) = 0; \qquad P_{D_1}\phi_1(1) = 0;$$
$$P_{N_0}\phi_1'(0) = 0; \qquad P_{N_1}\phi_1'(1) = 0; \tag{5.11}$$
$$P_{R_0}\phi_1'(0) = \Lambda_0 P_{R_0}\phi_0; \qquad P_{R_1}\phi_1'(1) = \Lambda_1 P_{R_1}\phi_0. \tag{5.12}$$

Upon taking an $L^2(0,1)$ inner product with ϕ_0, we find

$$\begin{aligned}\tilde{\lambda}_1 |\phi_0|^2_{\mathbb{R}^n} &= -\langle \phi_1'', \phi_0 \rangle \\ &= \Big((P_{R_0}\Lambda_0 P_{R_0} - P_{R_1}\Lambda_1 P_{R_1})\phi_0, \phi_0 \Big)_{\mathbb{R}^n} = \Big(B\phi_0, \phi_0 \Big)_{\mathbb{R}^n},\end{aligned}$$

using a calculation similar to (5.10). Since ϕ_0 is an eigenvector for B, $\tilde{\lambda}_1$ will be an eigenvalue of B. If $\tilde{\lambda}_1 > 0$ this eigenvalue will be in the right half-plane for s small and so won't cross into the Maslov Box. On the other hand, if $\tilde{\lambda}_1 < 0$ we will obtain a spectral curve with the asymptotic form $\lambda(s) \sim \frac{\tilde{\lambda}_1}{s}$, and (for λ_∞ chosen sufficiently large) this will enter the Maslov Box through the bottom shelf. These crossings are precisely counted by the term $\text{Mor}(B)$ in Theorem 5.3.

Finally, if $\tilde{\lambda}_1 = 0$ we need to proceed with the next order of our perturbation argument. For this step, we note that we have $\tilde{\lambda}_0 = 0$ and $\tilde{\lambda}_1 = 0$, and that we now restrict to $\phi_0 \in \ker B$. Our second order perturbation equation is $-\phi_2'' + V(0)\phi_0 = \tilde{\lambda}_2\phi_0$ subject to the conditions

$$\begin{aligned} &P_{D_0}\phi_2(0) = 0; && P_{D_1}\phi_2(1) = 0; \\ &P_{N_0}\phi_2'(0) = 0; && P_{N_1}\phi_2'(1) = 0; \\ &P_{R_0}\phi_2'(0) = \Lambda_0 P_{R_0}\phi_1(0); && P_{R_1}\phi_2'(1) = \Lambda_1 P_{R_1}\phi_1(1). \end{aligned}$$

We take an $L^2(0,1)$ inner product of this equation with ϕ_0 and compute

$$\begin{aligned} \tilde{\lambda}_2|\phi_0|^2_{\mathbb{R}^n} - (V(0)\phi_0, \phi_0)_{\mathbb{R}^n} &= -\langle \phi_2'', \phi_0\rangle = -(\phi_2'(1), \phi_0)_{\mathbb{R}^n} + (\phi_2'(0), \phi_0)_{\mathbb{R}^n} \\ &= (P_{R_0}\Lambda_0 P_{R_0}\phi_1(0) - P_{R_1}\Lambda_1 P_{R_1}\phi_1(1), \phi_0)_{\mathbb{R}^n}. \end{aligned}$$

In order to understand this last inner product, we note that for $\tilde{\lambda}_1 = 0$ we have $\phi_1'' = 0$ with boundary conditions (5.11). We can write $\phi_1(x) = ax + b$ for constant vectors $a, b \in \mathbb{R}^n$, and the conditions $P_{R_0}\phi_1'(0) = \Lambda_0 P_{R_0}\phi_0$ and $P_{R_1}\phi_1'(1) = \Lambda_1 P_{R_1}\phi_0$ imply $P_{R_0}a = P_{R_0}\Lambda_0 P_{R_0}\phi_0$ and likewise $P_{R_1}a = P_{R_1}\Lambda_1 P_{R_1}\phi_0$. Noting also that $\phi_1(1) - \phi_1(0) = a$, we compute

$$\begin{aligned} &(P_{R_0}\Lambda_0 P_{R_0}\phi_1(0) - P_{R_1}\Lambda_1 P_{R_1}\phi_1(1), \phi_0)_{\mathbb{R}^n} = (\phi_1(0), P_{R_0}\Lambda_0 P_{R_0}\phi_0)_{\mathbb{R}^n} - (\phi_1(1), P_{R_1}\Lambda_1 P_{R_1}\phi_0)_{\mathbb{R}^n} \\ &\quad = (\phi_1(0) - \phi_1(1), P_{R_0}\Lambda_0 P_{R_0}\phi_0)_{\mathbb{R}^n} = -(a, P_{R_0}\Lambda_0 P_{R_0}\phi_0)_{\mathbb{R}^n} \\ &\quad = -(P_{R_0}a, P_{R_0}\Lambda_0 P_{R_0}\phi_0)_{\mathbb{R}^n} = -(P_{R_0}\Lambda_0 P_{R_0}\phi_0, P_{R_0}\Lambda_0 P_{R_0}\phi_0)_{\mathbb{R}^n} \\ &\quad = -((P_{R_0}\Lambda_0 P_{R_0})^2\phi_0, \phi_0)_{\mathbb{R}^n}. \end{aligned}$$

We see that

$$\tilde{\lambda}_2|\phi_0|^2_{\mathbb{R}^n} = \Big((V(0) - (P_{R_0}\Lambda_0 P_{R_0})^2)\phi_0, \phi_0\Big)_{\mathbb{R}^n}.$$

Recalling that we have selected the vectors ϕ_0 to be orthonormal eigenvectors for the matrix $Q(V(0) - (P_{R_0}\Lambda_0 P_{R_0})^2)Q$, we see that we have a spectral curve entering the Maslov Box if and only if $\tilde{\lambda}_2$ is a negative eigenvalue of this matrix.

In principle, if $\tilde{\lambda}_2 = 0$ we can proceed to the next step in the perturbation argument, but this is the case that we have eliminated by our non-degeneracy assumption.

Application 2. In [11], the authors consider Schrödinger equations on $\mathbb{R}$,

$$\begin{aligned} Hy &:= -y'' + V(x)y = \lambda y, \\ \text{dom}(H) &= H^1(\mathbb{R}), \end{aligned} \tag{5.13}$$

where $y \in \mathbb{R}^n$ and $V \in C(\mathbb{R})$ is a symmetric matrix satisfying the following asymptotic conditions:

(A1) The limits $\lim_{x\to\pm\infty} V(x) = V_\pm$ exist, and for all $M \in \mathbb{R}$,

$$\int_{-M}^{\infty} (1+|x|)|V(x)-V_+|dx < \infty; \quad \int_{-\infty}^{M} (1+|x|)|V(x)-V_-|dx < \infty.$$

(A2) The eigenvalues of $V_\pm$ are all positive.

As verified in [11], if $\lambda < 0$ then (5.13) will have n linearly independent solutions that decay as $x \to -\infty$ and n linearly independent solutions that decay as $x \to +\infty$. We express these respectively as

$$\begin{aligned}\phi^-_{n+j}(x;\lambda) &= e^{\mu^-_{n+j}(\lambda)x}(r^-_j + \mathcal{E}^-_j(x;\lambda))\\ \phi^+_j(x;\lambda) &= e^{\mu^+_j(\lambda)x}(r^+_{n+1-j} + \mathcal{E}^+_j(x;\lambda)),\end{aligned}$$

with also

$$\begin{aligned}\partial_x\phi^-_{n+j}(x;\lambda) &= e^{\mu^-_{n+j}(\lambda)x}(\mu^-_{n+j}r^-_j + \tilde{\mathcal{E}}^-_j(x;\lambda))\\ \partial_x\phi^+_j(x;\lambda) &= e^{\mu^+_j(\lambda)x}(\mu^+_j r^+_{n+1-j} + \tilde{\mathcal{E}}^+_j(x;\lambda)),\end{aligned}$$

for $j = 1, 2, \ldots, n$, where the nature of the $\mu^\pm_j$, $r^\pm_j$, and $\mathcal{E}^\pm_j(x;\lambda), \tilde{\mathcal{E}}^\pm_j(x;\lambda)$ are developed in [11], but won't be necessary for this brief discussion, except for the observation that under assumptions (A1) and (A2)

$$\lim_{x\to\pm\infty} \mathcal{E}^\pm_j(x;\lambda) = 0; \quad \lim_{x\to\pm\infty} \tilde{\mathcal{E}}^\pm_j(x;\lambda) = 0. \tag{5.14}$$

If we create a frame $\mathbf{X}^-(x;\lambda) = \binom{X^-(x;\lambda)}{Y^-(x;\lambda)}$ by taking $\{\phi^-_{n+j}\}_{j=1}^n$ as the columns of X^- and $\{\phi^{-\prime}_{n+j}\}_{j=1}^n$ as the respective columns of Y^- then it is straightforward to verify that $\mathbf{X}^-$ is a frame for a Lagrangian subspace, which we will denote ℓ^- (see [11]). Likewise, we can create a frame $\mathbf{X}^+(x;\lambda) = \binom{X^+(x;\lambda)}{Y^+(x;\lambda)}$ by taking $\{\phi^+_j\}_{j=1}^n$ as the columns of X^+ and $\{\phi^{+\prime}_j\}_{j=1}^n$ as the respective columns of Y^+. Then $\mathbf{X}^+$ is a frame for a Lagrangian subspace, which we will denote ℓ^+.

In either case, we can view the exponential multipliers $e^{\mu^\pm_j x}$ as expansion coefficients, and if we drop these off we retain frames for the same spaces. That is, we can create an alternative frame for ℓ^- by taking the expressions $r^-_j + \mathcal{E}^-_j(x;\lambda)$ as the columns of X^- and the expressions $\mu^-_{n+j}r^-_j + \tilde{\mathcal{E}}^-_j(x;\lambda)$ as the corresponding columns for Y^-. Using (5.14) we see that in the limit as x tends to $-\infty$ we obtain the frame $\mathbf{R}^-(\lambda) = \binom{R^-}{S^-(\lambda)}$, where

$$\begin{aligned}R^- &= \begin{pmatrix} r^-_1 & r^-_2 & \ldots & r^-_n \end{pmatrix}\\ S^-(\lambda) &= \begin{pmatrix} \mu^-_{n+1}r^-_1 & \mu^-_{n+2}r^-_2 & \ldots & \mu^-_{2n}r^-_n \end{pmatrix}.\end{aligned}$$

As discussed in [11], $\mathbf{R}^-$ is the frame for a Lagrangian subspace, which we will denote ℓ^-_∞. Proceeding similarly with ℓ^+, we obtain the asymptotic Lagrangian subspace ℓ^+_∞ with frame $\mathbf{R}^+(\lambda) = \binom{R^+}{S^+(\lambda)}$, where

$$\begin{aligned}R^+ &= \begin{pmatrix} r^+_n & r^+_{n-1} & \ldots & r^+_1 \end{pmatrix}\\ S^+(\lambda) &= \begin{pmatrix} \mu^+_1 r^+_n & \mu^+_2 r^+_{n-1} & \ldots & \mu^+_n r^+_1 \end{pmatrix}.\end{aligned} \tag{5.15}$$

We can now construct $\tilde{W}(x,\lambda)$ in this case as

$$\tilde{W}(x;\lambda) = -(X^-(x;\lambda) + iY^-(x;\lambda))(X^-(x;\lambda) - iY^-(x;\lambda))^{-1}(R^+ - iS^+(\lambda))(R^+ + iS^+(\lambda))^{-1}. \tag{5.16}$$

We will be interested in a closed path in the x–λ plane, determined by a sufficiently large value λ_∞. First, if we fix $\lambda = 0$ and let x run from $-\infty$ to $+\infty$, we denote the resulting path Γ_0 (the *right shelf*). Next, we let Γ_+ denote a path in which λ decreases from 0 to $-\lambda_\infty$. (We can view this as a path corresponding with the limit $x \to +\infty$, but the limiting behavior will be captured by the nature of the Lagrangian subspaces; we refer to this path as the *top shelf.*) Continuing counterclockwise along our path, we denote by Γ_∞ the path obtained by fixing $\lambda = -\lambda_\infty$ and letting x run from $+\infty$ to $-\infty$ (the *left shelf*). Finally, we close the path in an asymptotic sense by taking a final path, Γ_-, with λ running from $-\lambda_\infty$ to 0 (viewed as the asymptotic limit as $x \to +\infty$; we refer to this as the *bottom shelf*).

The principal result of [11] is as follows.

Theorem 5.4. *Let $V \in C(\mathbb{R})$ be a symmetric real-valued matrix, and suppose (A1) and (A2) hold. Then*

$$\mathrm{Mor}(H) = -\,\mathrm{Mas}(\ell^-, \ell^+_\infty; \Gamma_0).$$

Remark 5.5. As discussed in Section 5 of [11], Theorem 5.4 can be extended to the case

$$H_s y := -y'' + sy' + V(x)y = \lambda y, \tag{5.17}$$

for any $s \in \mathbb{R}$. This observation—for which the authors are indebted to [5]—allows the application of these methods in the study of spectral stability for traveling wave solutions in Allen–Cahn equations.

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Appendix A

In this brief appendix, we verify (P2) (homotopy invariance) for our definition of the Maslov index. We assume $\mathcal{L}(s,t) = (\ell_1(s,t), \ell_2(s,t))$ is continuous on a cartesian product of closed, bounded intervals $I \times J = [0,1] \times [a,b]$, and that $\mathcal{L}(s,a) = \mathcal{L}_a$ for all $s \in I$ and likewise $\mathcal{L}(s,b) = \mathcal{L}_b$ for all $s \in I$, for some fixed $\mathcal{L}_a, \mathcal{L}_b \in \Lambda(n) \times \Lambda(n)$. We denote by $\tilde{W}(s,t)$ the matrix (1.1) associated with $\mathcal{L}(s,t)$. It's straightforward to see from our metric (1.4) that continuity of $\mathcal{L}$ implies continuity of the associated frame $\mathbf{X}(s,t)$, which in turn (and along with non-degeneracy) implies continuity of $\tilde{W}(s,t)$. We know from Theorem II.5.1 in [13] that the eigenvalues of $\tilde{W}(s,t)$ must vary continuously with s and t. Moreover, we see from Theorem II.5.2 in the same reference that these eigenvalues can be tracked as n continuous paths $\{\mu^k(s,t)\}_{k=1}^n$, which in our case will be restricted to S^1.

For notational convenience, let's fix $s_1, s_2 \in I$ suitably close together (in a manner that we make precise below) and set $\tilde{W}_1(t) := \tilde{W}(s_1,t)$ and $\tilde{W}_2(t) := \tilde{W}(s_2,t)$.

Claim A.1. *Suppose $\mu(t)$ and $\nu(t)$ are any two continuous eigenvalue paths of $\tilde{W}_1(t)$ and $\tilde{W}_2(t)$ respectively, with $\mu(a) = \nu(a)$ and $\mu(b) = \nu(b)$. Then there exists $\epsilon > 0$ sufficiently small so that if*

$$\max_{t \in J} |\mu(t) - \nu(t)| < \epsilon$$

then the spectral flow of $\mu(t)$ is the same as the spectral flow of $\nu(t)$.

Proof. First, suppose neither $\mu(a)$ nor $\mu(b)$ is -1 (and so the same is true for $\nu(a)$ and $\nu(b)$). Take ϵ small enough so that $B_\epsilon(\mu(a))$ (the ball in $\mathbb{C}$ centered at $\mu(a)$ with radius ϵ) does not contain -1, and similarly for $\mu(b)$. According to our hypothesis, we will have $\mu(t), \nu(t) \in B_\epsilon(\mu(t))$ for all $t \in J$, and so the spectral flows for $\mu(t)$ and $\nu(t)$ will both match the flow for $B_\epsilon(\mu(t))$.

Suppose next that $\mu(a) = -1$, but $\mu(b)$ does not. In this case, there must be a first time, t_*, at which $B_\epsilon(\mu(t_*))$ does not contain -1. By assumption, we must have $\nu(t_*) \in B_\epsilon(\mu(t_*))$, and this allows us to apply an argument on $[t_*, b]$ similar to our argument on $[a, b]$ in the previous paragraph. A similar argument holds if $\mu(b) = -1$, but $\mu(a)$ does not.

Last, suppose $\mu(a) = -1$ and $\mu(b) = -1$. If $\mu(t)$ and $\nu(t)$ are both -1 for all $t \in J$ then we're finished. If not, i.e., if there exists a time t_* at which one or both $\mu(t_*)$ and $\nu(t_*)$ is not -1, then we can apply one of the first two cases to complete the proof. □

Since $I \times J$ is closed and bounded, the matrices $\tilde{W}(s,t)$ are uniformly continuous on $I \times J$. This means that given any $\tilde{\epsilon} > 0$ we can find $\delta > 0$ sufficiently small so that

$$|s_1 - s_2| < \delta \implies \max_{t \in J} \|\tilde{W}_1(t) - \tilde{W}_2(t)\| < \tilde{\epsilon}.$$

Fix any $k \in \{1, 2, \dots, n\}$, and set $\mu_1^k(t) = \mu^k(s_1, t)$ and $\mu_2^k(t) = \mu^k(s_2, t)$. By eigenvalue continuity, this means we can take δ small enough to ensure that

$$\max_{t \in J} |\mu_1^k(t) - \mu_2^k(t)| < \epsilon$$

for all $k \in \{1, 2, \dots, n\}$. But since ϵ is arbitrary, we see from our claim that the flow associated with each of these eigenvalue pairs must be the same, and so the spectral flow for $\tilde{W}_1(t)$ must agree with that of $\tilde{W}_2(t)$.

Finally, then, by starting with $s_1 = 0$, and proceeding to $s_2 = \frac{\delta}{2}$, $s_3 = \delta$ etc., we see that the Maslov index will be the same at each step, and that since the steps have fixed length we eventually arrive at $s = 1$. This concludes the proof of property (P2).

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