A CONVERGENCE TO BROWNIAN MOTION ON SUB-RIEMANNIAN MANIFOLDS

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Abstract. This paper considers a classical question of approximation of Brownian motion by a random walk in the setting of a sub-Riemannian manifold M. To construct such a random walk we first address several issues related to the degeneracy of such a manifold. In particular, we define a family of sub-Laplacian operators naturally connected to the geometry of the underlying manifold. In the case when M is a Riemannian (non-degenerate) manifold, we recover the Laplace-Beltrami operator. We then construct the corresponding random walk, and under standard assumptions on the sub-Laplacian and M we show that this random walk converges (at the level of semigroups) to a process, horizontal Brownian motion, whose infinitesimal generator is the sub-Laplacian. An example of the Heisenberg group equipped with a standard sub-Riemannian metric is considered in detail, in which case the sub-Laplacian we introduced is shown to be the sum of squares (Hörmander's) operator.

Contents

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1. Introduction

This paper describes a geometrically natural piecewise Hamiltonian-flow random walk in a sub-Riemannian manifold, whose semi-group converges to that of a horizontal Brownian motion on the manifold. In this setting we define a sub-Laplacian by means of uniform averaging of second order derivatives. In particular, in the Riemannian case we recover the Laplace-Beltrami operator; in the case of the Heisenberg group equipped with a standard sub-Riemannian metric, we recover the sum of squares (Hörmander's) operator. While the current paper presents the probabilistic aspects of this construction, the geometric study of this sub-Laplacian can be found in [10]. As we will see in Section 4, the sub-Laplacian we study is the one that generates the horizontal Brownian motion.

Over the last half century, Brownian motion on Riemannian manifolds has developed into a well-understood and rich theory. Much of this development relies heavily on the Riemannian structure as one can see from the monographs [9, 11]. There are two major ingredients which are canonical in the Riemannian case: the Riemannian volume μ and the corresponding Laplace-Beltrami operator Δ_{LB}. Recall that the Laplace-Beltrami operator is usually defined as div grad, where div is defined with respect to the Riemannian volume μ. From here, a Brownian motion on a Riemannian manifold can be described as a stochastic process with the infinitesimal generator Δ_{LB}.

This approach is not easily available in the sub-Riemannian case. There are several measures which might be used in lieu of the Riemannian volume such as the Hausdorff measure, Popp's measure (see [1, 17]), left or right Haar measure in the case of Lie groups. Each choice of the measure will lead to a possibly different sub-Laplacian, and therefore to a different Brownian motion. A more detailed analysis of sub-Laplacians and natural choices of measures is presented in [10].

Instead of making this choice, we develop a more classical approach of constructing a Brownian motion as the limit of an appropriately-scaled random walk. Any complete list of references working in this direction on Riemannian manifolds would undoubtedly include the now-classic works [12, 16], and most relevant to our work, the *isotropic transport process* studied by M. Pinsky in [18]. Motivated by Pinsky's approach, the sub-Laplacian we construct is canonical *with respect to the limiting process of the random walk*. This sub-Laplacian $\mathcal{L}$ defined by (3.1) is elemental in the sub-Riemannian setting without some a priori canonical choice.

There are several fundamental issues in our construction which are not present in the Riemannian setting. Such issues prevent us from adopting a Pinsky-type process to a sub-Riemannian manifold. For example, one of the basic relations which has been exploited in the Riemannian setting is the duality between the tangent and cotangent spaces. This duality is not available in the sub-Riemannian setting, which led us to the realization that it seems to be more appropriate to construct the random walk in the cotangent space, rather than in the tangent space. Another major ingredient in the Riemannian case are solutions of the Hamilton-Jacobi equations which are degenerate in the sub-Riemannian case. We overcome the problem of the non-uniqueness of solutions to the Hamilton-Jacobi equations with given initial position and velocity (tangent) vector by using a compatible Riemannian metric in the definition of the sub-Laplacian $\mathcal{L}$. Even though this definition seems to depend on the choice of the compatible Riemannian metric, we show that $\mathcal{L}$ actually only depends on the corresponding "vertical" bundle.

We further mention that there is interest in seeing how work by Bakry, Baudoin, Garofalo *et al* [2,4–6] on generalized curvature-dimension inequalities for such manifolds is related to dissipation of horizontal diffusions. We expect further study of connections between diffusions on sub-Riemannian manifolds and corresponding generators, as well as of behavior of hypoelliptic heat kernels and corresponding functional inequalities such as in [3, 7, 8, 15].

2. Background and Notation

2.1. **Sub-Riemannian basics.**

We start by reviewing standard definitions of sub-Riemannian geometry that can be found e.g. in [17] and originally were introduced by R. Strichartz in [19, 20]. Let M be a d-dimensional connected smooth manifold, with tangent and cotangent bundles TM and T^*M respectively.

Definition 2.1. For $m \leqslant d$, let $\mathcal{H}$ be a smooth sub-bundle of TM where each fiber $\mathcal{H}_q$ has dimension m and is equipped with an inner product which smoothly varies between fibers. Then

(1) the triple $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ is called a *sub-Riemannian manifold of rank* m;
(2) $\mathcal{H}$ is called a *horizontal distribution* on M, and $\langle\cdot,\cdot\rangle$ a *sub-Riemannian metric*;
(3) sections of $\mathcal{H}$ are called *horizontal vector fields* and curves on M whose velocity vectors are always horizontal are called *horizontal curves.*

Assumption 2.2 (Hörmander's condition)**.** *Throughout this paper we assume that the distribution $\mathcal{H}$ satisfies Hörmander's (bracket generating) condition; that is, horizontal vector fields with their Lie brackets span the tangent space T_qM at every point $q \in M$.*

Under Hörmander's condition any two points on M can be connected by a horizontal curve by the Chow-Rachevski theorem. Thus there is a natural *sub-Riemannian distance* (*Carnot-Carathéodory distance*) on M defined as the infimum over the lengths of horizontal curves connecting two points. In turn, this affords us the notion of a *horizontal geodesic*, a horizontal curve whose length (locally) realizes the Carnot-Carathéodory distance.

Due to degeneracy of the sub-Riemannian metric on the tangent bundle, it is convenient to introduce the *cometric* on T^*M corresponding to the sub-Riemannian structure. This is a particular section of the bundle of symmetric bilinear forms on the cotangent bundle,

$$\langle\!\langle\cdot,\cdot\rangle\!\rangle_q : T_q^*M \times T_q^*M \to \mathbb{R}, \ q \in M.$$

We relate the cometric to the sub-Riemanian metric via the *sub-Riemannian bundle map* $\beta : T^*M \to TM$ with image $\mathcal{H}$ defined in the spirit of Riesz's theorem by

$$\langle \beta_q(p), v\rangle_q = p(v) \tag{2.1}$$

for all $q \in M, p \in T_q^*M,$ and $v \in \mathcal{H}_qM$. Hence the correspondence between the sub-Riemannian metric and cometric can be summarized as

$$\langle\!\langle \varphi, \psi\rangle\!\rangle_q = \langle \beta_q(\varphi), \beta_q(\psi)\rangle_q = \varphi\left(\beta_q(\psi)\right) = \psi\left(\beta_q(\varphi)\right), \tag{2.2}$$

for all $q \in M$, and $\varphi, \psi \in T_q^*M$.

Armed with the cometric, we conclude this section by defining the corresponding *sub-Riemannian Hamiltonian* $H : T^*M \to \mathbb{R}$ by

$$H(q,p) := \frac{1}{2}\langle\!\langle p, p\rangle\!\rangle_q, \ q \in M, p \in T_q^*M$$

from which we can recover the cometric via polarization. Again we note the following equivalent descriptions of the map

$$H(q,p) = \frac{1}{2}\langle\!\langle p, p\rangle\!\rangle_q = \frac{1}{2}\langle\!\langle \beta_q(p), \ \beta_q(p)\rangle\!\rangle_q = \frac{1}{2}p(\beta_q(p)). \tag{2.3}$$

The Hamiltonian is used to generate the dynamics of the system, where $H(q,p)$ gives the (kinetic) energy of a body located at q with momentum p.

2.2. **Canonical coordinates and compatible metrics.** In the non-degenerate (Riemannian) case, the metric and cometric are matrix inverses of each other when written in any given local frame. Indeed, these matrices are represented componentwise by the lowered and raised indices g_{ij} and g^{ij} respectively. The degeneracy in the sub-Riemannian case disallows for such a relationship, leaving us with a choice of Riemannian metrics which will be *compatible* with a given sub-Riemannian structure. The general non-canonical choice of compatible metrics will eventually lead us to defining a family of sub-Laplacians corresponding to the choice of compatible metric.

Definition 2.3. Let g be a Riemannian metric on M extending the sub-Riemannian metric; i.e., $g|_{\mathcal{H}_q\times\mathcal{H}_q} = \langle\cdot,\cdot\rangle_q$ for all $q \in M$. Then we say that g is *compatible* with the sub-Riemannian structure, or simply that g is a *compatible metric*.

Within this paper, the purpose of introducing a compatible metric $(\cdot,\cdot)$ is to take advantage of the induced bundle map $g : TM \to T^*M$ defined by $g(v) = (\cdot, v)$, the standard duality $TM \leftrightarrow T^*M$ described generally through Riesz's theorem. This is a tool that we lose in the sub-Riemannian setting as we can associate to each cotangent (momentum) vector a corresponding horizontal (velocity) vector via $T^*M \xrightarrow{\beta} \mathcal{H}$, but are unable to canonically map back $\mathcal{H} \xrightarrow{?} T^*M$. With a compatible metric g on hand, we then recover our return $\mathcal{H} \xrightarrow{g} T^*M$. However, as already mentioned, with the full strength of the Riemannian metric, we have a full bundle isomorphism $TM \to T^*M$, but this is more machinery than we need since we will only be considering the mapping on the horizontal distribution; something we explore presently through an observation from [10].

Proposition 2.4. *Let $(\cdot,\cdot)$ be a Riemannian metric on M and let $g : TM \to T^*M$ be the corresponding bundle map. Then $(\cdot,\cdot)$ is a compatible metric if and only if $\beta \circ g|_{\mathcal{H}} = \mathrm{Id}_{\mathcal{H}}$. Further, suppose $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ are compatible metrics with corresponding bundle maps $g_1, g_2 : TM \to T^*M$. For $i = 1, 2$, let $\mathcal{V}_i$ be the orthogonal compliment of $\mathcal{H}$ in TM with respect to $(\cdot,\cdot)_i$. Then $g_1(v) = g_2(v)$ for every $v \in \mathcal{H}$ if and only if $\mathcal{V}_1 = \mathcal{V}_2$.*

Idea of the Proof. For a Riemannian metric $(\cdot,\cdot)$, the corresponding bundle map $g : TM \to T^*M$ can be written as $g_{\mathcal{H}} \oplus g_{\mathcal{V}} : \mathcal{H} \oplus \mathcal{V} \to T^*M$, where $\mathcal{V}$ is the $(\cdot,\cdot)$-orthogonal compliment of $\mathcal{H}$ in TM. From here, noticing that $g(\mathcal{V}) = \mathrm{Null}(\beta)$, and thus $T^*M = g_{\mathcal{H}}(\mathcal{H}) \oplus \mathrm{Null}(\beta)$, we have $\beta = \beta_{\mathcal{H}} \oplus 0 : g_{\mathcal{H}}(\mathcal{H}) \oplus \mathrm{Null}(\beta) \to TM$. Moreover, g is compatible if and only if $g_{\mathcal{H}} = \beta_{\mathcal{H}}^{-1}$ which in turn happens if and only if $\beta \circ g = \mathrm{Id}_{\mathcal{H}} \oplus 0$. From here, it is easy enough to deduce that the mapping

$\mathcal{H} \ni v \mapsto g(v)$ depends only on $g_{\mathcal{H}}$ and $\mathcal{V}$, but not on the behavior of $g_{\mathcal{V}}$. Since, if g is compatible, then $g_{\mathcal{H}} = \beta_{\mathcal{H}}^{-1}$ is completely determined by $\mathcal{V}$ and the sub-Riemannian structure, the assertions of this proposition follow. □

With Proposition 2.4 understood, instead of introducing a compatible metric, we could build up the remaining work by selecting a smooth *vertical* sub-bundle $\mathcal{V} \subset TM$ such that $TM = \mathcal{H} \oplus \mathcal{V}$, use this to distinguish a compliment of $\mathrm{Null}(\beta)$, say H, in T^*M such that we have $\beta = \beta_{\mathcal{H}} \oplus 0 : H \oplus \mathrm{Null}(\beta) \to TM$ and hence recover a "return map" with $\mathcal{H} \xrightarrow{\beta_{\mathcal{H}}^{-1}} T^*M$. As for the theory that follows, the only role that a compatible metric g serves is to distinguish the vertical bundle. However, for some calculational purposes, it seems advantageous to keep working in terms of a compatible metric g.

Notation 2.5. *Let g be a compatible metric. For local coordinates $\mathbf{x} = (x^1, ..., x^d)$ on M, we define the local maps $\beta^{ij} : M \to \mathbb{R}$ and $g_{ij} : M \to \mathbb{R}$ by*

$$\beta^{ij}(q) := \langle\langle dx^i, dx^j \rangle\rangle_q \text{ and } g_{ij}(q) = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle_q$$

for all q in the domain of $\mathbf{x}$. The $d \times d$ matrices with entries β^{ij} and g_{ij} will be denoted by B and G respectively.

As B is the matrix representation of the bundle map $\beta : T^*M \to TM$ in local coordinates, G is the local coordinate matrix representation of the bundle map $TM \to T^*M$ defined by $v \mapsto g(\cdot, v)$.

Example 2.1 (Contact manifolds)**.** Let M be a $2n$+1-dimensional manifold and ω a contact 1-form on M, that is, a 1-form such that $d\omega$ is non-degenerate on $\mathrm{Ker}(\omega)$. Let $\mathcal{H} := \mathrm{Ker}(\omega)$, which defines a $2n$-dimensional horizontal distribution on M, called a *contact distribution*, and we assume that $\mathcal{H}$ is equipped with inner product $\langle \cdot, \cdot \rangle$. The sub-Riemannian manifold $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ is called a *contact sub-Riemannian manifold.* With any contact form ω we can associate its Reeb vector field, which is the unique vector field X_0 satisfying the conditions $\omega(X_0) = 1$ and $d\omega(X_0, \cdot) = 0$. Hence for any local orthonormal frame $X_1, ..., X_{2n}$ for the distribution $\mathcal{H}$ we have that $X_0, X_1, ..., X_{2n}$ is a local frame, since X_0 is transversal to $\mathcal{H}$. Finally, if $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{H}$, we can extend it to X_0 by $g(X_0, X_0) = 1$ and setting $\mathcal{H} \perp X_0$. This g is then naturally compatible with the sub-Riemannian structure. Moreover, for contact sub-Riemannian manifolds there are no abnormal geodesics, that is, all geodesics are smooth and are projections of the trajectories of the Hamiltonian vector field in T^*M given by the Legendre transform of the inner product on $\mathcal{H}$. The Heisenberg group is an example of a contact manifold where ω is a standard symplectic form.

2.3. **Hamilton-Jacobi Equations.** We can now re-write the Hamiltonian $H : T^*M \to \mathbb{R}$ defined by (2.3) using canonical coordinates. By identifying the vector $(q^1, ..., q^d, p_1, ..., p_d)$ in $\mathbb{R}^{d \times d}$ with the point $(q, p) \in T^*M$ using local coordinates for the standard identification $q^i = x^i(q)$ and $p = \sum_{i=1}^{d} p_i dx^i$, then

$$H(q, p) = \frac{1}{2} \sum_{i,j=1}^{d} p_i p_j \beta^{ij}(q). \tag{2.4}$$

A curve $(q(t), p(t)) \in T^*M$ satisfies the *Hamilton-Jacobi equations* when

$$\dot{q}^i(t) = \frac{\partial H}{\partial p_i}(q(t), p(t)) = \sum_{j=1}^{d} p_j(t)\beta^{ij}(q(t)) \tag{2.5}$$

$$\dot{p}_i(t) = -\frac{\partial H}{\partial q^i}(q(t), p(t)) = -\frac{1}{2}\sum_{k,j=1}^{d} p_k(t)p_j(t)\frac{\partial \beta^{kj}}{\partial q^i}\Big|_{q(t)} \tag{2.6}$$

where we have slightly abused notation in the common way, conflating $\frac{\partial}{\partial p_i}$ with the partial derivative of (2.4) in terms of p_i, and $\frac{\partial}{\partial q^i}$ with $\frac{\partial}{\partial x^i}$ in (2.6). Equations (2.5) and (2.6) are collectively known as the Hamilton-Jacobi equations.

Taking a time derivative in (2.5) we get

$$\ddot{q}^k(t) = \sum_{i,j,l=1}^{d}\left\{\beta^{il}(q(t))\frac{\partial \beta^{kj}}{\partial q^l}\Big|_{q(t)} - \frac{1}{2}\beta^{kl}(q(t))\frac{\partial \beta^{ij}}{\partial q^l}\Big|_{q(t)}\right\}p_i(t)p_j(t). \tag{2.7}$$

Define the *raised Christoffel symbols* locally by

$$\Gamma^{ijk}(q) := -\frac{1}{2}\sum_{l=1}^{d}\left\{\beta^{il}(q)\frac{\partial \beta^{jk}}{\partial x^l}\Big|_q + \beta^{jl}(q)\frac{\partial \beta^{ik}}{\partial x^l}\Big|_q - \beta^{lk}(q)\frac{\partial \beta^{ij}}{\partial x^l}\Big|_q\right\}. \tag{2.8}$$

Rewriting (2.7) with (2.8) while suppressing the time dependence,

$$\ddot{q}^k = -\sum_{i,j=1}^{d}\Gamma^{ijk}(q)p_ip_j. \tag{2.9}$$

The negative signs in (2.8) and (2.9) are just by convention so that the acceleration term is consistent with standard Riemannian definitions.

Notation 2.6. *We let Φ be the Hamilton flow*

$$\Phi : [0, \tau) \times T^*M \longrightarrow T^*M,$$

where if $(x, p) \in T_x^*M$ then $t \mapsto \Phi_t(x, p)$ is the curve $(q(t), p(t))$ in T^*M satisfying the Hamilton-Jacobi equations with initial conditions $q(0) = x$ and $p(0) = p$ for t in some maximal interval $[0, \tau)$.

Remark 2.7. If $(q(t), p(t)) = \Phi_t(x, p)$, then $q(t)$ is a horizontal curve. Indeed, (2.5) guarantees that $\dot{q}(t) = \beta(p(t)) \in \mathcal{H}_{q(t)}$.

Assumption 1. *We henceforth assume that the sub-Riemannian manifold $(M, \mathcal{H}, \langle\cdot,\cdot,\rangle)$ is complete with respect to the Carnot-Carathéodory metric. Note that in this case this sub-Riemannian manifold is also geodesically complete by a sub-Riemannian Hopf-Rinow theorem (e.g. [19, Theorem 7.1]). In particular, for each $(x, p) \in T^*M$, $\Phi_t(x, p)$ is defined for all $t \geq 0$.*

3. Horizontal sub-Laplacians and the Heisenberg group

In this section we introduce a family of second order differential operators on M indexed by Riemannian metrics compatible with the sub-Riemannian structure $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$. In the Riemannian case when $\mathcal{H} = TM$, we recover the Laplace-Beltrami up to a constant scaling factor; in the Heisenberg case using the standard compatible metric introduced in Example 2.1, we get the familiar sums of squares Laplacian up to a constant scaling factor.

3.1. **Horizontal sub-Laplacians.** Definition 3.2 below introduces horizontal sub-Laplacian operators, but before we can give the definition, some notation is in order.

Notation 3.1. *We denote the unit sphere in $\mathcal{H}_x$ by $\mathcal{S}_x^{\mathcal{H}} := \{v \in \mathcal{H}_x : \langle v, v\rangle_x = 1\}$. The (unique) rotationally invariant measure on $\mathcal{S}_x$ will be denoted $\mathbb{U}_x$.*

Definition 3.2. Let $(\cdot,\cdot)$ be a compatible metric, and let g be the corresponding bundle map $TM \to T^*M$. We define $\mathcal{L}$ on $C^\infty(T^*M)$ by

$$\mathcal{L}f(x) := \int_{\mathcal{S}_x^{\mathcal{H}}} \left\{ \frac{d}{dt}\Big|_0 \frac{d}{ds}\Big|_0 f\big(\Phi_{t+s}(x, g(v))\big)\right\} \mathbb{U}_x(dv). \tag{3.1}$$

We will call $\mathcal{L}$ the *horizontal sub-Laplacian* corresponding to g.

As is now obvious from Proposition 2.4 and the remarks that followed, we have the following statement.

Proposition 3.3. *Suppose $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ are compatible metrics giving rise to orthogonal compliments $\mathcal{V}_1$ and $\mathcal{V}_2$ of $\mathcal{H}$, respectively. For $i = 1, 2$, if $\mathcal{L}_i$ is defined by (3.1) with respect to $(\cdot,\cdot)_i$, then $\mathcal{L}_1 = \mathcal{L}_2$ whenever $\mathcal{V}_1 = \mathcal{V}_2$.*

3.2. **A formula for $\mathcal{L}$ in local coordinates.** Working in local coordinates, we set $q(t) := \pi(\Phi_t(x,p))$, where π is the projection onto M. Defining $v = \beta(p)$, we get

$$\begin{aligned}
&\frac{d}{dt}\Big|_0 \frac{d}{ds}\Big|_0 f(q(t+s)) = \frac{d}{dt}\Big|_0 \left\{ \sum_{i=1}^d \dot{q}^i(t) \frac{\partial f}{\partial x^i}\Big|_{q(t)} \right\} \\
&= \sum_{i=1}^d \left\{ \ddot{q}^i(0)\frac{\partial f}{\partial x^i}\Big|_x + \sum_{j=1}^d \dot{q}^i(0)\dot{q}^j(0)\frac{\partial^2 f}{\partial x^i \partial x^j}\Big|_x \right\} \\
&= \sum_{i=1}^d \left\{ -\sum_{k,l=1}^d \Gamma^{kli}(x)\, p_k p_l \frac{\partial f}{\partial x^i}\Big|_x + \sum_{j=1}^d v^i v^j \frac{\partial^2 f}{\partial x^i \partial x^j}\Big|_x \right\} \\
&= \sum_{i,j=1}^d \left\{ v^i v^j \frac{\partial^2 f}{\partial x^i \partial x^j}\Big|_x - \sum_{k=1}^d \Gamma^{ijk}(x) p_i\, p_j \frac{\partial f}{\partial x^k}\Big|_x \right\}
\end{aligned} \tag{3.2}$$

Proposition 3.4. *Let $(\cdot,\cdot)$ be a compatible metric with corresponding bundle map $g : TM \to T^*M$. For $1 \leqslant i, j \leqslant d$,*

$$\int_{\mathcal{S}_x^{\mathcal{H}}} v^i v^j \mathbb{U}_x(dv) = \frac{1}{m}\beta^{ij}(x) \tag{3.3}$$

and

$$\int_{\mathcal{S}_x^{\mathcal{H}}} p_i p_j \mathbb{U}_x(dv) = \frac{1}{m}\sum_{a,b=1}^d g_{ia}\beta^{ab} g_{bj}(x). \tag{3.4}$$

Here $p = g(v)$.

Proof. Rewrite (3.3) as

$$\begin{aligned}
\int_{\mathcal{S}_x^{\mathcal{H}}} dx^i(v) dx^j(v) \mathbb{U}_x(dv) &= \int_{\mathcal{S}_x^{\mathcal{H}}} \langle \beta_x(dx^i), v\rangle \langle \beta_x(dx^j), v\rangle d\mathbb{U}_x(v) \\
&= \frac{1}{m}\langle \beta_x(dx^i), \beta_x(dx^j)\rangle = \frac{1}{m}\beta^{ij}(x)
\end{aligned}$$

The second equality follows from Corollary 5.4 below. From here (3.4) follows by a similar argument after realization that $p_i = \sum_{a=1}^{d} g_{ia} v^a$ and $p_j = \sum_{b=1}^{d} g_{jb} v^b$. □

Combining Proposition 3.4 with (3.2) leads immediately to

Theorem 3.5. *The horizontal sub-Laplacian indexed by g can be locally written as*

$$\begin{aligned}\mathcal{L} &= \frac{1}{m}\sum_{i,j=1}^{d}\left\{\beta^{ij}\frac{\partial^2}{\partial x^i \partial x^j} - \sum_{a,b,k=1}^{d}\Gamma^{ijk} g_{ia}\beta^{ab} g_{bj}\frac{\partial}{\partial x^k}\right\} \\ &= \frac{1}{m}\sum_{i,j=1}^{d}\left\{\beta^{ij}\frac{\partial^2}{\partial x^i \partial x^j} - \sum_{k=1}^{d}\Gamma^{ijk}\left[GBG\right]_{ij}\frac{\partial}{\partial x^k}\right\}\end{aligned} \tag{3.5}$$

where $[GBG]_{ij}$ is the ijth entry of the matrix GBG and G and B are defined in Notation 2.5.

Remark 3.6. In the case that $\mathcal{H} = TM$, $B = G^{-1}$ and hence

$$\mathcal{L} = \frac{1}{m}\sum_{i,j=1}^{d}\left\{\beta^{ij}\frac{\partial^2}{\partial x^i \partial x^j} - \sum_{k=1}^{d}\Gamma^{ijk} g_{ij}\frac{\partial}{\partial x^k}\right\},$$

which is the ($\frac{1}{m}$ scaled) local formula for the Laplace-Beltrami operator on the Riemannian manifold (M, g).

With Proposition 3.3 in mind, (3.5) appears deceivingly dependent on the structure of the compatible metric with the repeat appearance of its corresponding matrix G. However, using the notation in the proof of Proposition 2.4, we have $g \circ \beta \circ g = g_{\mathcal{H}} \circ 0$, which as the proof of and remarks following Proposition 2.4 indicate, $g_{\mathcal{H}}$ is determined by the sub-Riemannian structure once the vertical bundle $\mathcal{V}$ is fixed. The following example in the Heisenberg case illustrates this.

3.3. **An example: the Heisenberg group.** Let $\mathbb{H}$ be the Heisenberg group; that is, $\mathbb{H} \cong \mathbb{R}^3$ with the multiplication defined by

$$(x_1, y_1, z_1) \star (x_2, y_2, z_2) := \left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}\omega\left(x_1, y_1; x_2, y_2\right)\right),$$

where ω is the standard symplectic form

$$\omega\left(x_1, y_1; x_2, y_2\right) := x_1 y_2 - y_1 x_2.$$

Left multiplication by $(x, 0, 0)$ and $(0, y, 0)$ induce two left-invariant vector fields

$$\begin{aligned} X(q) &:= \left.\frac{\partial}{\partial x}\right|_q - \frac{1}{2} y \left.\frac{\partial}{\partial z}\right|_q \\ Y(q) &:= \left.\frac{\partial}{\partial y}\right|_q + \frac{1}{2} x \left.\frac{\partial}{\partial z}\right|_q \end{aligned} \tag{3.6}$$

for any $q \in \mathbb{H}$. At each point $q \in \mathbb{H}$ the globally defined vector fields $X(q)$ and $Y(q)$ span a two-dimensional subspace of $T_q\mathbb{H}$; set $\mathcal{H}_q := \operatorname{Span}\{X(q), Y(q)\}$ and then

$$\mathcal{H} := \bigcup_{q \in \mathbb{H}} \mathcal{H}_q$$

can be taken as the horizontal distribution. Moreover, at each $q \in \mathbb{H}$ we have

$$[X(q), Y(q)] = \frac{\partial}{\partial z}\Big|_q =: Z(q),$$

and so Hörmander's condition is satisfied. Consider $M = \mathbb{H}$, the horizontal distribution $\mathcal{H}$ defined as above, and the inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{H}_q$ defined so that $\{X(q), Y(q)\}$ is an orthonormal basis for $\mathcal{H}_q$. Recall also that in Example 2.1 we described $(\mathbb{H}, \mathcal{H}, \langle\cdot,\cdot\rangle)$ as a contact manifold with Z as a Reeb vector field.

A covector $\varphi \in T_p^*M$ will be identified with the triple $(\varphi_1, \varphi_2, \varphi_3) \in \mathbb{R}^3$ via $\varphi = \varphi_1 dx + \varphi_2 dy + \varphi_3 dz$. We have that for each $q = (x, y, z) \in \mathbb{H}$, the sub-Riemannian bundle map $\beta : T^*M \to TM$ is defined by

$$(\varphi_1, \varphi_2, \varphi_3) \xrightarrow{\beta_q} \left(\varphi_1 - \frac{1}{2}y\varphi_3, \varphi_2 + \frac{1}{2}x\varphi_3, \frac{1}{2}(x\varphi_2 - y\varphi_1) + \frac{1}{4}(y^2 + x^2)\varphi_3\right). \tag{3.7}$$

The matrix representation of β with entries $\beta^{ij} = dx^i(\beta(dx^j))$ is

$$B(x,y,z) = \begin{pmatrix} 1 & 0 & -\frac{y}{2} \\ 0 & 1 & \frac{x}{2} \\ -\frac{y}{2} & \frac{x}{2} & \frac{x^2+y^2}{4} \end{pmatrix} \tag{3.8}$$

Using the fact that $(\mathbb{H}, \mathcal{H}, \langle\cdot,\cdot\rangle)$ is a contact sub-Riemannian manifold, we can extend the sub-Riemannian metric $\langle\cdot,\cdot\rangle$ to the Riemannian metric g which makes $\{X, Y, Z\}$ a global orthogonal frame with $g(Z, Z) = \lambda > 0$. The matrix representation of g with entries $g_{ij} = g(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i})$ is

$$G(x,y,z) = \begin{pmatrix} 1 + \frac{\lambda y^2}{4} & -\frac{\lambda xy}{4} & \frac{\lambda y}{2} \\ -\frac{\lambda xy}{4} & 1 + \frac{\lambda x^2}{4} & -\frac{\lambda x}{2} \\ \frac{\lambda y}{2} & -\frac{\lambda x}{2} & \lambda \end{pmatrix}, \tag{3.9}$$

and therefore,

$$GBG = \begin{pmatrix} 1 + \frac{\lambda y^2}{4} & -\frac{\lambda xy}{4} & \frac{\lambda y}{2} \\ -\frac{\lambda xy}{4} & 1 + \frac{\lambda x^2}{4} & -\frac{\lambda x}{2} \\ \frac{\lambda y}{2} & -\frac{\lambda x}{2} & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{y}{2} \\ 0 & 1 & \frac{x}{2} \\ -\frac{y}{2} & \frac{x}{2} & \frac{x^2+y^2}{4} \end{pmatrix} \begin{pmatrix} 1 + \frac{\lambda y^2}{4} & -\frac{\lambda xy}{4} & \frac{\lambda y}{2} \\ -\frac{\lambda xy}{4} & 1 + \frac{\lambda x^2}{4} & -\frac{\lambda x}{2} \\ \frac{\lambda y}{2} & -\frac{\lambda x}{2} & \lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here you can see the manifestation of Proposition 3.3 through the independence of GBG on any choice of λ. Using (2.8) and (3.8), for any $k = 1, 2, 3$, $\Gamma^{11k} = \Gamma^{22k} = 0$, which gives us all values needed to explicitly find (3.5) in this context.

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\sum_{i,j=1}^{3}\left\{\beta^{ij}\frac{\partial^2}{\partial x^i \partial x^j}\right\} - \frac{1}{2}\sum_{i,j,k=1}^{3}\left\{\Gamma^{ijk}\,[GBG]_{ij}\,\frac{\partial}{\partial x^k}\right\} \\ &= \frac{1}{2}\sum_{i,j=1}^{3}\left\{\beta^{ij}\frac{\partial^2}{\partial x^i \partial x^j}\right\} - 0 \\ &= \frac{1}{2}\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{4}(x^2 + y^2)\frac{\partial^2}{\partial z^2} - y\frac{\partial^2}{\partial x \partial z} + x\frac{\partial^2}{\partial y \partial z}\right\} \end{aligned}$$

Thus we can rewrite $\mathcal{L}$ as

$$\mathcal{L} = \frac{1}{2}\left(X^2 + Y^2\right).$$

4. Convergence and Random Walks

The first part of this section discusses the semi-group convergence results necessary to prove the convergence of the random walk developed in Section 4.2 to a horizontal Brownian motion. The main result is Theorem 4.11.

4.1. Convergence of semigroups.

Let $\mathscr{C}(T^*M)$ be a subspace of the smooth bounded functions on T^*M, $C_b^\infty(T^*M)$, closed with respect to the sup norm. We identify $C_b^\infty(M)$ as a closed subspace of $C_b^\infty(T^*M)$, where $f \in C_b^\infty(M)$ is identified with $\tilde{f} \in C_b^\infty(T^*M)$ when $\tilde{f}(x,p) = f(x)$ for every $(x,p) \in T^*M$. Let $\mathscr{C}(M) = \mathscr{C}(T^*M) \cap C_b^\infty(M)$.

Remark 4.1. We will require further assumptions to be made on the space $\mathscr{C}(T^*M)$ below in Assumptions 2 and 3. With these in mind, a reasonable example is the space of functions $f \in C_b^\infty(T^*M)$ such that $f(x,p) \to 0$ whenever $d_{CC}(x,o) \to \infty$ for some fixed $o \in M$; note in this case $\mathscr{C}(M) = C_0^\infty(M)$. Another example is the space of functions $f \in C_b^\infty(T^*M)$ such that $f(x,p) = 0$ whenever x lands outside some compact subset in M; in this case $\mathscr{C}(M) = C_c^\infty(M)$.

Definition 4.2. For $f \in C_b^\infty(T^*M)$, we define the *Hamiltonian vector field* by

$$\mathscr{D}_H f(x,p) = \frac{d}{dt}\Big|_{t=0} f\left(\Phi_t(x,p)\right). \tag{4.1}$$

Remark 4.3. If $f \in C_b^\infty(M)$, then $\mathscr{D}_H f(x,p) = v(f)$ where $v = \beta(p)$.

Remark 4.4. The semigroup property of flows implies that if $f \in \mathscr{C}(T^*M)$, then

$$\mathscr{D}_H\left(\mathscr{D}_H f\right)(x,p) = \frac{d}{ds}\Big|_{s=0}\frac{d}{dt}\Big|_{t=0} f\left(\Phi_{t+s}(x,p)\right). \tag{4.2}$$

Definition 4.5. For $f \in C_b^\infty(T^*M)$, the *horizontally averaged projection*

$$\mathcal{P} : C_b^\infty(T^*M) \to C_b^\infty(M),$$

is defined by

$$\mathcal{P}f(x) = \int_{\mathcal{S}_x^{\mathcal{H}}} f(x, g(v))\mathbb{U}_x(dv). \tag{4.3}$$

Here, as before, $\mathbb{U}_x$ is the rotationally invariant (uniform) probability measure on the unit sphere $\mathcal{S}_x^{\mathcal{H}}$.

Let us now make the following observation.

Proposition 4.6. *For every $f \in C_b^\infty(M)$, $\mathcal{L}f = \mathcal{P}\mathscr{D}_H\mathscr{D}_H f$.*

Assumption 2. *We henceforth assume that $\mathscr{D}_H(\mathscr{C}(T^*M)) \subset \mathscr{C}(T^*M)$.*

Set $\mathcal{I}$ as the identity operator on $C_b^\infty(T^*M)$. We denote by $e^{t(\mathcal{P}-\mathcal{I})}$ the strongly continuous contraction semigroup on $C_b^\infty(T^*M)$ whose bounded generator is $\mathcal{P}-\mathcal{I}$. We denote by $e^{t\mathscr{D}_H}$ the strongly continuous contraction semigroup on $\mathscr{C}(T^*M)$ whose generator is $\mathscr{D}_H$. Using Notation 2.6 we have

$$e^{t\mathscr{D}_H} f(x,p) = f\left(\Phi_t(x,p)\right).$$

Finally, for any $\alpha > 0$ we denote by $T_\alpha(t)$ the strongly continuous contraction semigroup on $\mathscr{C}(T^*M)$ whose generator is $\mathscr{D}_H + \alpha(\mathcal{P}-\mathcal{I})$. This is possible since $\mathcal{P}-\mathcal{I}$ is bounded.

For more generalized notions of summing together generators we refer to [21]. Our aim is to prove a limit theorem of $T_\alpha(\alpha t)$ as $\alpha \to \infty$ using [14, Theorem 2.2]. To this end, we first state some prerequisites which follow easily from the definitions.

Lemma 4.7. *The following hold.*
1) $\mathcal{P}(\mathscr{C}(T^*M)) = \mathscr{C}(M)$.
2) $\mathcal{P}\mathscr{D}_H f = 0$ *for* $f \in \mathscr{C}(M)$.

Following the notation of T. Kurtz in [14], define

$$D_0 := \{f \in \mathscr{C}(T^*M) \cap \mathcal{P}(\mathscr{C}(T^*M)) : \text{there exists } h \in \operatorname{Dom}(\mathscr{D}_H) \text{ such that } (\mathcal{P} - \mathcal{I})h = -\mathscr{D}_H f\}. \tag{4.4}$$

Using the first claim of Lemma 4.7, we see that $\mathscr{C}(T^*M) \cap \mathcal{P}(\mathscr{C}(T^*M)) = \mathscr{C}(M)$. Moreover, for $f \in \mathscr{C}(M)$, define $h := \mathscr{D}_H f$. By the second claim of Lemma 4.7, $(\mathcal{P} - \mathcal{I})h = -\mathscr{D}_H f$. We conclude that $D_0 = \mathscr{C}(M)$.

Before getting to Theorem 4.9, the main result regarding convergence to a sub-Riemannian Brownian motion, we first make an assumption necessary to apply the result of Kurtz we wish to use.

Assumption 3. *We henceforth assume the semigroup* $e^{t\mathcal{L}}$ *is Feller in the following sense: for every* $t \geqslant 0$, $\lambda > 0$, *and* $h \in \mathscr{C}(M)$

$$x \longmapsto \int_0^\infty e^{-\lambda t} e^{t\mathcal{L}} h(x)dt \in \mathscr{C}(M).$$

Remark 4.8. For example, both Assumptions 2 and 3 hold if we let M be the Heisenberg group, $\mathcal{L}$ be as in Example 3.3, and set $\mathscr{C}(T^*M)$ to be those $f \in C_b^\infty(T^*M)$ such that $f(x,p) \to 0$ as $d_{CC}(x,0) \to \infty$.

We can now formulate the main result of this section.

Theorem 4.9. $\lim_{\alpha\to\infty} T_\alpha(\alpha t)f = e^{t\mathcal{L}}f$ *uniformly for every* $f \in \mathscr{C}(M)$.

Proof. By Assumption 3, for any $h \in \mathscr{C}(M) = D_0$ and $\lambda > 0$, the function $k(x) = \int_0^\infty e^{-\lambda t} e^{t\mathcal{L}} h(x)dt$ is in $\mathscr{C}(M)$; moreover, $(\lambda - \mathcal{L})k = h$. This shows that $\mathscr{C}(M) \subset \operatorname{Ran}(\lambda - \mathcal{L})$. Hence by [14, Theorem 2.2], the closure of $\mathcal{P}\mathscr{D}_H\mathscr{D}_H$ is the generator of a strongly continuous contraction semigroup $e^{t\mathcal{P}\mathscr{D}_H\mathscr{D}_H}$ such that

$$\lim_{\alpha\to\infty} T_\alpha(\alpha t)f = e^{t\mathcal{P}\mathscr{D}_H\mathscr{D}_H}f$$

for every $f \in \mathscr{C}(M)$, where the limit is in the sup norm. As noted in Proposition 4.6, $\mathcal{P}\mathscr{D}_H\mathscr{D}_H = \mathcal{L}$ on $\mathscr{C}(M)$. This concludes the proof. □

4.2. **A sub-Riemannian random walk.** Let $\varepsilon > 0$ be a parameter that we eventually take to zero. Let $\{e_i\}_{i=1}^\infty$ be i.i.d. exponential random variables with parameter 1 and define $e_0 := 0$. Let us fix $(x,p) \in T^*M$ as our initial position and momentum and let $v = \beta(p)$. Define $(\xi_t^\varepsilon, p_t^\varepsilon) = \Phi_{\varepsilon t}(x, g(v))$ for $0 \le t < e_1$. Given e_1, let $x_1^\varepsilon = \pi \circ \Phi_{\varepsilon e_1}(x, g(v)) \in T^*M$ where $\pi : T^*M \to M$ is the canonical projection, and take v_1^ε randomly from $\mathcal{S}_{x_1^\varepsilon}^{\mathcal{H}}$ such that the law of v_1^ε is $\mathbb{U}_{x_1^\varepsilon}$. From here, for $0 \le t < e_2$, define $(\xi_{t+e_1}^\varepsilon, p_{t+e_1}^\varepsilon) = \Phi_{\varepsilon t}(x_1^\varepsilon, g(v_1^\varepsilon))$. Continuing recursively, for each $k \ge 0$, once given $\{(x_0, v_0), (x_1^\varepsilon, v_1^\varepsilon), ..., (x_k^\varepsilon, v_k^\varepsilon)\}$ and $\{e_i\}_{i=1}^{k+1}$, define $x_{k+1}^\varepsilon = \pi\big(\Phi_{\varepsilon e_{k+1}}(x_k^\varepsilon, g(v_k^\varepsilon))\big)$ and take v_{k+1}^ε randomly from $\mathcal{S}_{x_{k+1}^\varepsilon}^{\mathcal{H}}$ such that the law of v_{k+1}^ε is $\mathbb{U}_{x_{k+1}^\varepsilon}$. From here, for $0 \le t < e_{k+2}$ define $(\xi_{t+\tau_{k+1}}^\varepsilon, p_{t+\tau_{k+1}}^\varepsilon) = \Phi_{\varepsilon t}(x_{k+1}^\varepsilon, g(v_{k+1}^\varepsilon))$ where $\tau_{k+1} := e_1 + \cdots + e_{k+1}$.

We now have a (ε-scaled) random walk $B_t^\varepsilon(x,p) := (\xi_t^\varepsilon, p_t^\varepsilon)$ in T^*M. Here, the notation $B_t^\varepsilon(x,p)$ emphasizes that (x,p) are the initial conditions (and $\beta(p) = v$ is the initial horizontal velocity). Define $T_t^\varepsilon : \mathscr{C}(T^*M) \to \mathscr{C}(T^*M)$ by

$$T_t^\varepsilon f(x,p) = \mathbb{E}\left[f(B_t^\varepsilon(x,p))\right].$$

With this we are ready to present the final piece needed, Theorem 4.10, before the statement of convergence, Theorem 4.11.

Theorem 4.10. *For every $f \in \mathscr{C}(M)$,*

$$T_t^\varepsilon f = e^{t(\varepsilon \mathscr{D}_H + \mathcal{P} - \mathcal{I})} f. \tag{4.5}$$

We postpone the proof of Theorem 4.10 until Section 5.1. Note that both T_t^ε and $e^{t(\varepsilon \mathscr{D}_H + \mathcal{P} - \mathcal{I})}$ are defined on $\mathscr{C}(T^*M)$, however only agree on $\mathscr{C}(M)$. This is due to the fact that $B_t^\varepsilon(x,p_1) = B_t^\varepsilon(x,p_2)$ when $\beta(p_1) = \beta(p_2)$, even though $\Phi_t(x,p_1)$ need not be equal to $\Phi_t(x,p_2)$. This is in contrast with the Riemannian case (e.g., see [18, Proposition 3.3]) where these semigroups would agree on the entirety of $\mathscr{C}(T^*M)$ and allow us to ignore some of the difficulties and subtleties apparent in the sub-Riemannian setting.

As a corollary to Theorem 4.10, we arrive at the convergence result which is our main theorem.

Theorem 4.11. $\lim_{\varepsilon \to 0} T_{t/\varepsilon^2}^\varepsilon f = e^{t\mathcal{L}} f$ *uniformly for every $f \in \mathscr{C}(M)$.*

Proof. From Theorem 4.9, it follows that if $f \in \mathscr{C}(M)$ then $\lim_{\varepsilon \to 0} e^{(t/\varepsilon^2)(\varepsilon \mathscr{D}_H + \mathcal{P} - \mathcal{I})} f = e^{t\mathcal{L}} f$. Since Theorem 4.10 shows that T_t^ε and $e^{t(\varepsilon \mathscr{D}_H + \mathcal{P} - \mathcal{I})}$ agree on $\mathscr{C}(M)$, the result follows. □

Remark 4.12. Let $B = (B_t)_{t \geq 0}$ be the *horizontal Brownian motion* on M with generator $\mathcal{L}$; that is, $(B_t)_{t \geq 0}$ is a $\mathcal{L}$-diffusion on M. Theorem 4.11 shows that the $T_{t/\epsilon^2}^\varepsilon \to T_t$ strongly on $\mathscr{C}(M)$, where $T_t = e^{t\mathcal{L}}$ is the semi-group associated with $(B_t)_{t \geq 0}$. One might be interested in what we can infer about other convergences with respect to these processes $B^\varepsilon = (B_{t/\varepsilon^2}^\varepsilon)_{t \geq 0}$ to B, a question that will likely depend on the specifics of a particular sub-Riemannian manifold and the function space $\mathscr{C}$ chosen. However, a result in this vein is given in, for example, [13, Theorem 19.25].

5. Proofs

5.1. **The Proof of Theorem 4.10.** We continue with the notation introduced in Section 4.2. For the i.i.d. exponential random variables $\{e_i\}_{i=1}^\infty$ and for $k \geq 0$, let $\tau_k = e_0 + e_1 + \cdots + e_k$; recall that $e_0 := 0$. We denote by R_λ^ε the resolvent of $e^{\varepsilon t \mathscr{D}_H}$; that is,

$$R_\lambda^\varepsilon f(x,p) = \int_0^\infty e^{-\lambda t} e^{\varepsilon t \mathscr{D}_H} f(x,p)\, dt = \int_0^\infty e^{-\lambda t} f(\Phi_{\varepsilon t}(x,p))\, dt.$$

We denote by S_λ^ε the resolvent of T_t^ε; that is,

$$S_\lambda^\varepsilon f(x,p) = \int_0^\infty e^{-\lambda t} \mathbb{E}\left[f(B_t^\varepsilon(x,p))\right] dt. \tag{5.1}$$

Lemma 5.1. *For any $f \in \mathscr{C}(T^*M)$,*

$$\mathbb{E}\left[\int_0^{\tau_1} e^{-\lambda t} f(B_t^\varepsilon(x,p))\, dt\right] = R_{1+\lambda}^\varepsilon f(x, g \circ \beta(p)).$$

Proof. If the initial conditions of B_t^ε are (x,p), then or $0 \leqslant t < \tau_1$, $B_t^\varepsilon = \Phi_{\varepsilon t}(x, g \circ \beta(p))$. Thusly

$$\begin{aligned}
&\mathbb{E}_{(x,p)}\left[\int_0^{\tau_1} e^{-\lambda t} f(B_t^\varepsilon)dt\right] = \mathbb{E}_{(x,p)}\left[\int_0^{\tau_1} e^{-\lambda t} f(\Phi_{\varepsilon t}(x, g\circ\beta(p)))dt\right] \\
&= \int_0^\infty \int_0^t e^{-s} e^{-\lambda t} f(\Phi_{\varepsilon t}(x, g\circ\beta(p)))\, dt\, ds = \int_0^\infty e^{-(\lambda+1)t} f(\Phi_{\varepsilon t}(x, g\circ\beta(p)))\, dt \\
&= R_{1+\lambda}^\varepsilon f(x, g\circ\beta(p)).
\end{aligned}$$

This concludes the proof. □

Lemma 5.2. *For any $f \in \mathscr{C}(T^*M)$,*

$$\mathbb{E}\left[\int_{\tau_1}^\infty e^{-\lambda t} f(B_t^\varepsilon(x,p))dt\right] = R_{1+\lambda}^\varepsilon \mathcal{P} S_\lambda^\varepsilon f(x, g\circ\beta(p)).$$

Proof. Notice that

$$\mathbb{E}\left[\int_{\tau_1}^\infty e^{-\lambda t} f(B_t^\varepsilon(x,p))dt\right] = \mathbb{E}\left[e^{-\lambda\tau_1}\int_0^\infty e^{-\lambda t} f(B_t^\varepsilon(x_1^\varepsilon, g(v_1^\varepsilon)))dt\right],$$

$$\mathbb{E}\left[\int_0^\infty e^{-\lambda t} f(B_t^\varepsilon(x_1^\varepsilon, g(v_1^\varepsilon)))dt \;\middle|\; (x_1^\varepsilon, v_1^\varepsilon)\right] = S_\lambda^\varepsilon f(x_1^\varepsilon, g(v_1^\varepsilon)),$$

and

$$\begin{aligned}
\mathbb{E}\left[S_\lambda^\varepsilon f(x_1^\varepsilon, g(v_1^\varepsilon)) \;\middle|\; \tau_1 = t\right] &= \mathbb{E}\left[S_\lambda^\varepsilon f(x_t, g(U))\right] \\
&= \int_{\mathcal{S}_{x_t}^{\mathcal{H}}} S_\lambda^\varepsilon f(x_t, g(v))\, \mathbb{U}_{x_t}(dv) = \mathcal{P} S_\lambda^\varepsilon f(x_t).
\end{aligned}$$

where $x_t = \pi \circ \Phi_{\varepsilon t}(x, g\circ\beta(p))$ (as before, $\pi : T^*M \to M$ is the canonical projection) and U is a uniform random variable on $\mathcal{S}_{x_t}^{\mathcal{H}}$. Putting these pieces together,

$$\begin{aligned}
&\mathbb{E}\left[\int_{\tau_1}^\infty e^{-\lambda t} f(B_t^\varepsilon(x,p))dt\right] = \mathbb{E}\left[e^{-\lambda\tau_1} S_\lambda^\varepsilon f(x_1^\varepsilon, v_1^\varepsilon)\right] = \mathbb{E}\left[e^{-\lambda\tau_1}\mathcal{P} S_\lambda^\varepsilon f(x_{\tau_1})\right] \\
&= \int_0^\infty e^{-\lambda t} e^{-t} \mathcal{P} S_\lambda^\varepsilon f(\Phi_{\varepsilon t}(x, g\circ\beta(p)))dt = R_{\lambda+1}^\varepsilon \mathcal{P} S_\lambda^\varepsilon f(x, g\circ\beta(p)).
\end{aligned}$$

Note that the third equality used $\mathcal{P} S_\lambda^\varepsilon f(x_{\tau_1}) = \mathcal{P} S_\lambda^\varepsilon f(\Phi_{\varepsilon t}(x, g\circ\beta(p)))$ by the identification of $\mathscr{C}(M)$ as a subset of $\mathscr{C}(T^*M)$. □

Proof of Theorem 4.10. Using Lemmas 5.1 and 5.2, we have

$$\begin{aligned}
S_\lambda^\varepsilon f(x,p) &= \mathbb{E}\left[\int_0^{\tau_1} e^{-\lambda t} f(B_t^\varepsilon(x,p))dt\right] + \mathbb{E}\left[\int_{\tau_1}^\infty e^{-\lambda t} f(B_t^\varepsilon(x,p))dt\right] \\
&= R_{1+\lambda}^\varepsilon f(x, g\circ\beta(p)) + R_{1+\lambda}^\varepsilon \mathcal{P}^g S_\lambda^\varepsilon f(x, g\circ\beta(p)).
\end{aligned}$$

Multiplying on the left by $1 + \lambda - \varepsilon\mathscr{D}_H$ yields

$$(1 + \lambda - \varepsilon\mathscr{D}_H) S_\lambda^\varepsilon f(x, g\circ\beta(p)) = f(x, g\circ\beta(p)) + \mathcal{P}^g S_\lambda^\varepsilon f(x, g\circ\beta(p)).$$

That is,

$$(\lambda - [\varepsilon\mathscr{D}_H + \mathcal{P}^g - \mathcal{I}]) S_\lambda^\varepsilon f(x, g\circ\beta(p)) = f(x, g\circ\beta(p)).$$

In particular, for any $f \in \mathscr{C}(M)$,

$$(\lambda - [\varepsilon\mathscr{D}_H + \mathcal{P}^g - \mathcal{I}]) S_\lambda^\varepsilon f = f.$$

From here we can now conclude the result. □

5.2. **Averaging over the unit sphere in an inner product space.** Here we provide details of the proof of Proposition 3.4 which are solely properties of finite-dimensional inner product spaces.

Proposition 5.3. *Let $\mathcal{X}$ be an n-dimensional real inner product space with inner product $\langle \cdot, \cdot \rangle$. Let S be the unit sphere in $\mathcal{X}$ with respect to this inner product and set μ as the rotationally invariant probability measure on S. Given any $X \in \mathcal{X}$,*

$$\int_S (X, \xi)^2 \mu(d\xi) = \frac{|X|^2}{n}.$$

Proof. It suffices to show that if $X \in S$, then $\int_S (X, \xi)^2 \mu(d\xi) = 1/n$. To this end, suppose $X, Y \in S$ and $l : S \to S$ is any rotation such that $l(Y) = X$. Since the adjoint of a rotation is again a rotation, we have,

$$\int_S (X, \xi)^2 \mu(d\xi) = \int_S (l(Y), \xi)^2 \mu(d\xi) = \int_S (Y, l^*(\xi))^2 \mu(d\xi) = \int_S (Y, \xi)^2 \mu(d\xi)$$

where the final identity follows from the rotational invariance of μ. This shows that the value of the integral is constant for any choice of $X \in S$. Set

$$a := \int_S (X, \xi)^2 \mu(d\xi).$$

Take $\{X_i : 1 \leqslant i \leqslant n\} \subset S$ to be an orthonormal basis for V, then for any $\xi \in S$

$$1 = \|\xi\|^2 = \sum_{i=1}^n (X_i, \xi)^2.$$

Therefore,

$$1 = \int_S \|\xi\|^2 \mu(d\xi) = \sum_{i=1}^n \int_S (X_i, \xi)^2 \mu(d\xi) = na$$

which then implies $a = 1/n$. □

Corollary 5.4. *Let $\mathcal{X}$, S, and μ be as in the previous proposition. Take $X, Y \in \mathcal{X}$. Then*

$$\int_S (X, \xi)(Y, \xi) \mu(d\xi) = \frac{(X, Y)}{n}.$$

Proof. By the previous proposition,

$$\int_S (X + Y, \xi)^2 \, \mu(d\xi) = \frac{|X + Y|^2}{n} = \frac{|X|^2}{n} + \frac{|Y|^2}{n} + 2\frac{(X, Y)}{n}.$$

On the other hand, $(X + Y, \xi)^2 = (X, \xi)^2 + (Y, \xi)^2 + 2(X, \xi)(Y, \xi)$. Hence another application of the previous proposition yields,

$$\begin{aligned} \int_S (X + Y, \xi)^2 \, \mu(d\xi) &= \int_S \left\{(X, \xi)^2 + (Y, \xi)^2 + 2(X, \xi)(Y, \xi)\right\} \mu(d\xi) \\ &= \frac{|X|^2}{n} + \frac{|Y|^2}{n} + 2 \int_S (X, \xi)(Y, \xi) \, \mu(d\xi). \end{aligned}$$

Comparing terms, the result now follows. □

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