RATIONAL DIFFERENTIAL FORMS ON THE LINE AND SINGULAR VECTORS IN VERMA MODULES OVER $\widehat{\mathfrak{sl}}_2$

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Abstract. We construct a monomorphism of the De Rham complex of scalar multivalued meromorphic forms on the projective line, holomorphic on the complement to a finite set of points, to the chain complex of the Lie algebra of $\mathfrak{sl}_2$-valued algebraic functions on the same complement with coefficients in a tensor product of contragradient Verma modules over the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. We show that the existence of singular vectors in the Verma modules (the Malikov-Feigin-Fuchs singular vectors) is reflected in the new relations between the cohomology classes of logarithmic differential forms.

1. Introduction

1.1. We consider two complexes. The first is the De Rham complex of scalar multivalued meromorphic forms on the projective line that are holomorphic on the complement to a finite set of points. The second is the chain complex of the Lie algebra of $\mathfrak{sl}_2$-valued algebraic functions on the same complement with coefficients in a tensor product of contragradient Verma modules over the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. We construct a monomorphism of the first complex to the second and show that the existence of singular vectors in the Verma modules is reflected in the new relations between the cohomology classes of logarithmic differential forms.

This construction has two motivations.

The first motivation was to generalize the principal construction of [SV]. In [SV], we identified the tensor products of contragradient Verma modules over a semisimple Lie algebra and the spaces of the top degree logarithmic differential forms over certain configuration spaces. We also identified the logarithmic parts of the De Rham complexes over the configuration spaces with some standard Lie algebra chain complexes having coefficients in these tensor products, cf. in [KS] a $\mathcal{D}$-module explanation of this correspondence.

[1] Supported in part by NSF grant DMS-1362924 and Simons Foundation grant #336826.

The second idea was that the appearance of singular vectors in Verma modules over affine Lie algebras is reflected in the *new relations* between the cohomology classes of logarithmic differential forms; moreover, in some sense this correspondence should be one-to-one. This was proved in an important particular case in [FSV]; in [STV] a one-to-one correspondence was established "on the level of parameters". In the present work we construct (see Section 6.4) this correspondence for another non-trivial class of singular vectors, namely for (a part of) Malikov-Feigin-Fuchs (MFF) ones, cf. [MFF]. (The first examples, were worked out during the preparation of [FSV].) It turns out that MFF vectors (having quite complicated form) admit a very simple definition using certain *limiting procedure*, see Theorem 6.2.

Our paper is related to the recent paper [AFO] by M. Aganagic, E. Frenkel, A. Okounkov devoted to quantum q-Langlands correspondence. In Section 6 of [AFO] the authors discuss how conformal blocks of a WZW model are related to conformal blocks of the "dual" $\mathcal{W}$-algebra. If the conformal blocks are defined by one-dimensional integrals the problem is reduced to comparing multivalued meromorphic forms on the projective line in terms of representation theory of $\widehat{\mathfrak{sl}}_2$.

This paper had been prepared for publication in the fall of 2015 while the second author visited the MPI in Bonn. The second author thanks MPI for hospitality. The authors thank E. Mukhin for interest in this paper, E. Frenkel for interesting discussions, and the anonymous referee who helped to improve the exposition.

2. The De Rham complex of a hypergeometric function

2.1. Let $z_1, \dots, z_n$ be pairwise distinct complex numbers, $z_{n+1} = \infty$, $U = \mathbb{C} - \{z_1, \dots, z_n\}$,

$$\ell = \prod_{1 \le i < j \le n} (z_i - z_j)^{M^i M^j / 2\kappa} \prod_{i=1}^{n} (t - z_i)^{-M^i/\kappa} , \tag{1}$$

where t is a coordinate on $\mathbb{C}$ and $M^1, \dots, M^n, \kappa$ are complex parameters. The function ℓ is a multivalued holomorphic function on U with singularities at $z_1, \dots, z_n$ and infinity.

The function ℓ defines a hypergeometric function of $z_1, \dots, z_n$ by the formula

$$I(z_1, \dots, z_n) = \int_\gamma \ell \, dt. \tag{2}$$

Here γ is a suitable cycle on U, for example a path connecting two points z_i, z_j.

Consider the *twisted De Rham complex* associated with ℓ:

$$0 \longrightarrow \Omega^0 \xrightarrow{\tilde{d}} \Omega^1 \longrightarrow 0. \tag{3}$$

Here Ω^p is the space of rational differential forms on $\mathbb{C}$ regular on U. The differential $\tilde{d}$ is given by the formula

$$\tilde{d} = d_{DR} + \alpha \wedge \cdot , \tag{4}$$

where d_{DR} is the De Rham differential and the second summand is the left exterior multiplication by the form

$$\alpha = -\frac{1}{\kappa}\sum_{i=1}^{n} \frac{M^i}{t-z_i}dt. \tag{5}$$

Formula (4) is motivated by the computation

$$d_{DR}(\ell\omega) = \ell d_{DR}\omega + d_{DR}\ell \wedge \omega = \ell(d_{DR}\omega + \alpha\wedge\omega).$$

The complex $\Omega^\bullet$ is the complex of global *algebraic* sections of the De Rham complex of $(\mathcal{O}_U^{an}, \nabla)$ where $\nabla = d_{DR} + \alpha \wedge \cdot$ is the integrable connection on the sheaf $\mathcal{O}_U^{an}$ of holomorphic functions on U.

If $\mathcal{S}$ is the locally constant sheaf of horizontal sections then the cohomology $H^\bullet(U;\mathcal{S})$ is equal to $H^\bullet(\Omega^\bullet)$.

If the monodromy of ℓ is non-trivial, then

$$H^0(\Omega^\bullet) = 0, \ \dim\ H^1(\Omega^\bullet) = n-1. \tag{6}$$

2.2. The simplest elements of Ω^1 are *logarithmic forms*:

$$\omega_i = M^i \frac{d(t-z_i)}{t-z_i}\ , \qquad i = 1, \dots , n. \tag{7}$$

They are cohomologically dependent:

$$\omega_1 + \ldots + \omega_n = -\kappa d(1). \tag{8}$$

For generic $M^1, \dots , M^n, \kappa$ the forms $\omega_1, \dots , \omega_n$ generate the space H^1 and the relation $\sum\ \omega_i \sim 0$ is the only one, [STV].

2.3. Resonances. For special *resonance* values of parameters the forms $\omega_1, \dots , \omega_n$ span a proper subspace of H^1.

Here are the resonance conditions.

(a) $M^i = -a\kappa$ where $a = 0, 1, 2, \dots\ ;\ i = 1, \dots , n$.

(b) $M^{n+1} = -2 + a\kappa$ where $a = 1, 2, \dots$. Here $M^{n+1} := M^1 + \ldots + M^n - 2$.

Each resonance condition implies a new cohomological relation between the forms $\omega_1, \dots , \omega_n$.

2.4. Example. If $M^{n+1} = -2 + \kappa$, then $\sum_{i=1}^n z_i\omega_i \sim 0$.

2.5. Example. If $M^{n+1} = -2 + 2\kappa$, then

$$\sum_{i=1}^{n} z_i^2 \omega_i - \frac{1}{\kappa} \left(\sum_{j=1}^{n} z_j M^j \right) \left(\sum_{i=1}^{n} z_i \omega_i \right) \sim 0.$$

It turns out that there is a direct connection between the relations among the forms $\omega_1, \dots, \omega_n$ and singular vectors in Verma modules over the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. An explanation of this relation is the subject of this work.

2.6. For any $i = 1, \dots, n$, define a number a_i by the formula $a_i := -M^i/\kappa$, if $-M^i/\kappa$ is a non-negative integer, and set $a_i := \infty$ otherwise.

Set $a_{n+1} := (M^1 + \dots + M^n)/\kappa$ if the right-hand side is a positive integer and $a_{n+1} := \infty$ otherwise.

The number $\mathrm{card}\{i \in \{1, \dots, n+1\} |\ a_i < \infty\}$ will be called the *number of resonances.*

Introduce the *restricted De Rham complex*

$$0 \longrightarrow \Omega_R^0 \longrightarrow \Omega_R^1 \longrightarrow 0 \tag{9}$$

as the subcomplex of (3) where $\Omega_R^p \subset \Omega^p$ is the subspace of the forms ω such that

(**r**) for any $i = 1, \dots, n+1$, the degree of the pole of ω at the point $t = z_i$ is not greater than a_i if $a_i < \infty$.

2.7. Lemma.

(a) $\Omega_R^\bullet$ *is a subcomplex of* $\Omega^\bullet$.

(b) *The forms* $\omega_1, \dots, \omega_n$ *belong to* Ω_R^1 *and generate the space* $H^1(\Omega_R^\bullet)$.

(c) *The natural homomorphism* $H^1(\Omega_R^\bullet) \longrightarrow H^1(\Omega^\bullet)$ *is a monomorphism.*

(d) *The codimension of the subspace* $H^1(\Omega_R^\bullet) \subset H^1(\Omega^\bullet)$ *is equal to the number of resonances.*

(e) *The forms* $d(t - z_i)/(t - z_i)^{-a_i - 1}$ *for resonance points* $t = z_i$ *and the form* $t^{a_{n+1}-1}dt$, *if* $t = \infty$ *is a resonance point, give a basis of the space* $H^1(\Omega^\bullet)/H^1(\Omega_R^\bullet)$.

2.8. It is convenient to use the following basis in $\Omega^\bullet$.

(a) An *elementary function* is a function $(t - z_i)^{-b}$ $(b \in \mathbb{Z}_{>0})$ or t^b $(b \in \mathbb{Z}_{\geq 0})$.

(b) An *elementary differential form* is a form $d(t-z_i)/(t-z_i)^b$ $(b \in \mathbb{Z}_{>0})$ or $t^b dt$ $(b \in \mathbb{Z}_{\geq 0})$.

We have two basic formulas:

$$\kappa \, \tilde{d}((t - z_i)^{-b}) = -(M^i + b\kappa) d(t - z_i)/(t - z_i)^{b+1} + \tag{10}$$

$$\sum_{k=1}^{b} \sum_{j \neq i} M^j/(z_j - z_i)^k \cdot d(t - z_i)/(t - z_i)^{b+1-k} - \sum_{j \neq i} M^j/(z_j - z_i)^b \cdot d(t - z_j)/(t - z_j)$$

and

$$\kappa\,\tilde{d}(t^b) = \left(b\kappa - \sum_{j=1}^{n} M^j\right) t^{b-1}dt - \tag{11}$$

$$\sum_{k=1}^{b-1}\sum_{j=1}^{n} M^j z_j^k t^{b-1-k}dt - \sum_{j=1}^{n} M^j z_j^b d(t-z_j)/(t-z_j).$$

If the resonance condition 2.3(a) is satisfied and $b = a$, then the first term in the right-hand side of (10) disappears. Similarly, if the condition 2.3(b) is satisfied and $b = a$, then the first term in the right-hand side of (11) disappears.

In the next sections we will give an interpretation for the elementary functions (forms) and formulas (10), (11) in terms of $\widehat{\mathfrak{sl}}_2$-representations.

Lemma 2.7 follows easily from (10) and (11).

3. The Gauss-Manin connection

In this section we show the important fact that the subbundle with fiber $H^1(\Omega^\bullet_R(z_1,\dots,z_n)) \subset H^1(\Omega^\bullet(z_1,\dots,z_n))$ is invariant with respect to the Gauss-Manin connection on the bundle with fiver $H^1(\Omega^\bullet(z_1,\dots,z_n))$, which moves the points $z_1,\dots,z_n$. By Lemma 2.7 the classes of logarithmic forms $\omega_1,\dots,\omega_n$ generate $H^1(\Omega^\bullet_R(z_1,\dots,z_n))$ for every distinct $z_1,\dots,z_n$. Our goal will be to describe the relations between these classes in terms of $z_1,\dots,z_n$.

3.1. When the points $z_1,\dots,z_n$ are moving, the cohomology groups $H^\bullet(U;\mathcal{S})$ (as well as the dual homology groups $H_\bullet(U;\mathcal{S}^*)$) form a vector bundle with a flat *Gauss-Manin* connection.

Set $\mathbb{C}^{[n]} := \mathbb{C}^n - \bigcup_{i<j}\{z\in\mathbb{C}^n \mid z_i = z_j\}$; $\mathbb{C}^{[n+1]} := \mathbb{C}^{n+1} - \bigcup_{i<j}\{(z,t)\in\mathbb{C}^{n+1} \mid z_i = z_j\} - \bigcup_{i=1}^n\{(z,t)\in\mathbb{C}^{n+1} \mid t = z_i\}$. Let $\psi : \mathbb{C}^{[n+1]} \longrightarrow \mathbb{C}^{[n]},\ (z,t)\mapsto z$, be the projection. This projection is a locally trivial bundle with fiber $U(z) = \{t\in\mathbb{C} \mid t\neq z_i,\ i=1,\dots,n\}$.

Define an integrable connection $\nabla = d_{DR} + \beta\wedge\cdot$ on the sheaf $\mathcal{O}^{an}$ of holomorphic functions on $\mathbb{C}^{[n+1]}$, where $\beta\wedge\cdot$ is the left multiplication by the form

$$\beta = -\sum_{i=1}^{n}\frac{M^i}{\kappa}\,\frac{d(t-z_i)}{t-z_i} + \sum_{i<j}\frac{M^iM^j}{2\kappa}\,\frac{d(z_i-z_j)}{z_i-z_j}, \tag{12}$$

cf. (5). Let $\mathcal{S}$ be the locally constant sheaf of horizontal sections. The fiber bundle ψ together with the local system $\mathcal{S}$ on $\mathbb{C}^{[n+1]}$ defines a vector bundle $R^1\psi$ on $\mathbb{C}^{[n]}$ with the fiber $H^1(U(z);\mathcal{S}|_{U(z)})$ over $z\in\mathbb{C}^{[n]}$. We have an isomorphism

$$H^1(U(z);\mathcal{S}|_{U(z)}) = H^1(\Omega^\bullet(z)),$$

where $\Omega^\bullet(z)$ is the twisted De Rham complex of the fiber defined in 2.1.

This vector bundle has a canonical *Gauss-Manin connection*

$$\nabla = \sum \nabla_{z_i} dz_i, \tag{13}$$

which can be defined as follows. Let $A \subset \mathbb{C}^{[n+1]}$ be a Zariski open set, $\Omega^1(A\times\mathbb{C})$ the space of rational differential forms on $A\times\mathbb{C}$ whose poles are on the hyperplanes $t = z_i$. A form $\omega \in \Omega^1(A\times\mathbb{C})$ defines a section $[\omega]$ of $R^1\psi$ over A with the value $[\omega|_{U(z)}] \in H^1(\Omega^\bullet(z))$ at z. The form $\eta := d_{DR}\omega + \beta\wedge\omega$ can be written as

$$\eta = \sum_{i<j} \eta_{ij}(t,z) dz_i \wedge dz_j + \sum_i \eta_i(t,z) dz_i \wedge dt, \tag{14}$$

where η_{ij}, η_i are functions. By definition,

$$\nabla_{z_i}[\omega] := [\eta_i dt].$$

Elementary differential forms $d(t-z_i)/(t-z_i)^b$ and $t^b dt$ generate $\Omega^1(A\times\mathbb{C})$ over the ring of rational functions on $\mathbb{C}^{[n]}$ regular on A. Hence, to compute the Gauss-Manin connection it is sufficient to compute (14) for elementary differential forms.

The following two formulas give a description of the Gauss-Manin connection:

$$\kappa\beta \wedge \frac{d(t-z_i)}{(t-z_i)^b} = \tag{15}$$

$$\sum_{j<k;\ i\notin\{j,k\}} \frac{M^j M^k}{2}\ \frac{d(z_j-z_k)}{z_j-z_k} \wedge \frac{d(t-z_i)}{(t-z_i)^b} + \sum_{j\neq i} \frac{M^j(M^i-2)}{2}\ \frac{d(z_j-z_i)}{(z_j-z_i)} \wedge \frac{d(t-z_i)}{(t-z_i)^b} +$$

$$\sum_{j\neq i} M^j \frac{d(z_j-z_i)}{(z_j-z_i)^b} \wedge \frac{d(t-z_j)}{t-z_j} - \sum_{j\neq i}\sum_{m=1}^{b-1} M^j \frac{d(z_i-z_j)}{(z_j-z_i)^{b-m+1}} \wedge \frac{d(t-z_i)}{(t-z_i)^m},$$

$$\kappa\beta \wedge t^b dt = \sum_{j<k} \frac{M^j M^k}{2}\ \frac{d(z_j-z_k)}{(z_j-z_k)} \wedge t^b dt + \tag{16}$$

$$\sum_{a=0}^{b-1}\sum_{i=1}^{n} M^i z_i^{b-a+1} dz_i \wedge t^a dt + \sum_{i=1}^{n} M^i z_i^b dz_i \wedge \frac{d(t-z_i)}{t-z_i}.$$

The first formula has an important special case. For the logarithmic forms defined in (7), we have

$$\kappa\beta \wedge \omega_i = \sum_{j<k;\ i\notin\{j,k\}} \frac{M^j M^k}{2}\ \frac{d(z_j-z_k)}{z_j-z_k} \wedge \omega_i + \tag{17}$$

$$\sum_{j\neq i} \frac{M^j(M^i-2)}{2}\ \frac{d(z_j-z_i)}{z_j-z_i} \wedge \omega_i + \sum_{j\neq i} M^i \frac{d(z_j-z_i)}{z_j-z_i} \wedge \omega_j.$$

3.2. Corollary. *Consider the subbundle $R^1\psi_R \subset R^1\psi$ with fiber $H^1(\Omega^\bullet_R(z)) \subset H^1(\Omega^\bullet(z))$. This subbundle is invariant under the Gauss-Manin connection.*

Here $\Omega^\bullet_R(z)$ is the restricited De Rham complex introduced in 2.6.

In fact, by Lemma 2.7(b), the logarithmic forms generate the fibers of $R^1\psi_R$ and by (17), their covariant derivatives are expressed through logarithmic forms.

4. Representations of $\widehat{\mathfrak{sl}}_2$

4.1. Let $\mathfrak{sl}_2$ be the Lie algebra of complex 2×2-matrices with the zero trace; let e, f, h be its standard generators, subject to the relations

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

Let $\widehat{\mathfrak{sl}}_2$ be the corresponding affine Lie algebra $\widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2[T, T^{-1}] \oplus \mathbb{C}c$ with the bracket

$$[aT^i, bT^j] = [a, b]T^{i+j} + i\langle a, b\rangle\delta_{i+j,0}c,$$

where c is the central element, $\langle a, b\rangle := \mathrm{tr}(ab)$.

Set $f_1 = f,\ e_1 = e,\ h_1 = h,\ f_2 = eT^{-1},\ e_2 = fT,\ h_2 = c - h$. These elements are the standard Chevalley generators defining $\widehat{\mathfrak{sl}}_2$ as the Kac-Moody algebra corresponding to the Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

4.2. Remark. The algebra $\widehat{\mathfrak{sl}}_2$ has an automorphism π sending c, eT^i, fT^i, hT^i to c, fT^i, eT^i, $-hT^i$ respectively.

4.3. Fix $k \in \mathbb{C}$, the value of the central charge. We assume that the action of c on all our representations is the multiplication by k.

For $M \in \mathbb{C}$, let $V(M, k - M)$ be the *Verma module* over $\widehat{\mathfrak{sl}}_2$ generated by a vector v subject to the relations $e_1 v = e_2 v = 0,\ h_1 v = Mv,\ h_2 v = (k - M)v$.

Let $\mathrm{U}\,\hat{\mathfrak{n}}_-$ be the enveloping algebra of the Lie subalgebra $\hat{\mathfrak{n}}_- \subset \widehat{\mathfrak{sl}}_2$ generated by f_1, f_2. The map $\mathrm{U}\,\hat{\mathfrak{n}}_- \longrightarrow V(M, k - M)$, $F \mapsto Fv$, is an isomorphism of $\mathrm{U}\,\hat{\mathfrak{n}}_-$-modules. The space $V(M, k - M)$ has a $\Gamma = \mathbb{Z}^2_{\geq 0}$-grading: a vector $f_{i_1} \cdot \ldots \cdot f_{i_p} v$ with $i_j \in \{1, 2\}$ has grading (p_1, p_2), where p_i is the number of i's in the sequence $i_1, \ldots, i_p$.

For $\gamma \in \Gamma$, denote by $V(M, k - M)_\gamma \subset V(M, k - M)$ the corresponding γ-homogeneous component. A homogeneous vector ω in $V(M, k - M)$, non-proportional to v, is called a *singular vector* if $e_1\omega = e_2\omega = 0$. The Verma module $V(M, k - M)$ is reducible if and only if it contains a singular vector. For generic M, $V(M, k - M)$ is irreducible.

4.4. Reducibility conditions. (See Kac-Kazhdan [KK]). Set $\kappa := k+2$. The Verma module $V(M, k-M)$ is reducible if and only if at least one of the equations (a)-(c) below is satisfied.

(a) $M = l - 1 - (a-1)\kappa$.

(b) $M = -l - 1 + a\kappa$.

(c) $\kappa = 0$.

Here $l, a = 1, 2, 3, \ldots$. If (M, κ) satisfies exactly one of the conditions (a), (b), then the module $V(M, k-M)$ contains a unique proper submodule and this submodule is generated by a singular vector of degree $(la, l(a-1))$ for condition (a) and of degree $(l(a-1), la)$ for condition (b).

The singular vectors are highly nontrivial and are given by the following theorem.

4.5. Theorem. (Malikov-Feigin-Fuchs, [MFF]) *For any positive integers a, l and $\kappa \in \mathbb{C}$, the monomial*

(a) $$F_{12}(l, a, \kappa) = f_1^{l+(a-1)\kappa} f_2^{l+(a-2)\kappa} f_1^{l+(a-3)\kappa} \cdot \ldots \cdot f_2^{l-(a-2)\kappa} f_1^{l-(a-1)\kappa}$$

lies in $\mathrm{U}\,\hat{\mathfrak{n}}_-$. *If $M = l-1-(a-1)\kappa$, then $F_{12}(l, a, \kappa)v$ is a singular vector of $V(M, k-M)$ of degree $(la, l(a-1))$. Similarly, the monomial*

(b) $F_{21}(l, a, \kappa) = f_2^{l+(a-1)\kappa} f_1^{l+(a-2)\kappa} f_2^{l+(a-3)\kappa} \cdot \ldots \cdot f_1^{l-(a-2)\kappa} f_2^{l-(a-1)\kappa}$

lies in $\mathrm{U}\,\hat{\mathfrak{n}}_-$. *If $M = -l-1+a\kappa$, then $F_{21}(l, a, \kappa)v$ is a singular vector of $V(M, k-M)$ of degree $(l(a-1), la)$.*

The explanation of the meaning of complex powers in these formulas see in [MFF].

4.6. Examples. **1.** $M = -2 + \kappa$, $F_{21}(1, 1, \kappa)v = f_2 v = \frac{e}{T} v$.

2. $M = -2 + 2\kappa$,

$$F_{21}(1, 2, \kappa)v = f_2^{1+\kappa} f_1 f_2^{1-\kappa} v = f \left(\frac{e}{T}\right)^2 v + (1+\kappa)\frac{h}{T}\,\frac{e}{T} v - (1+\kappa)\kappa \frac{e}{T^2} v.$$

4.7. Claim. The reducibility conditions for $l = 1$ (the dimension of the complex line is 1) correspond to the resonance conditions for $H^1(\Omega^\bullet_R) \subset H^1(\Omega^\bullet)$, and the singular vectors correspond to the relations between the forms $\omega_1, \ldots, \omega_n$, cf. examples 2.4, 2.5 and 4.6.

We will make this statement precise in the next section.

4.8. The maximal proper submodule of a Verma module V coincides with the kernel of the *Shapovalov form*, which is the unique symmetric bilinear form $S(\cdot, \cdot)$ on V characterized by the conditions $S(v, v) = 1$, $S(f_i x, y) = S(x, e_i y)$ for all $i = 1, 2$; $x, y \in V$.

One can regard S as a map $S : V \longrightarrow V^*$ where $V^* := \oplus_{\gamma \in \Gamma} V^*_\gamma$ and V^*_γ being the dual space to V_γ. There is a unique $\widehat{\mathfrak{sl}}_2$-module structure on V^* such that $\langle f_i \phi, x \rangle =$

$\langle \phi, e_i x\rangle$; $\langle e_i\phi, x\rangle = \langle \phi, f_i x\rangle$, where $\phi \in V^*$, $x \in V$, $i = 1, 2$. We call this $\widehat{\mathfrak{sl}}_2$-module V^* the *contragradient dual* of V. The map S is a morphism of $\widehat{\mathfrak{sl}}_2$-modules. The quotient $L := V/\operatorname{Ker}\ S$ is irreducible.

5. The main homomorphism

Conformal block construction.

5.1. In this section let $z = (z_1, \dots, z_n, z_{n+1} = \infty)$ be pairwise distinct points of the complex projective line $\mathbb{P}^1$. Fix local coordinates $t - z_1, \dots, t - z_n, 1/t$ at these points. Set $U(z) := \mathbb{P}^1 - \{z_1, \dots, z_{n+1}\}$. Notice that this $U(z)$ is the same $U(z)$ as in Section 3.

Let $\mathfrak{sl}_2(U(z))$ be the Lie algebra of $\mathfrak{sl}_2$-valued rational functions on $\mathbb{P}^1$ regular on $U(z)$, with the pointwise bracket. Let $W_1, \dots, W_{n+1}$ be representations of $\widehat{\mathfrak{sl}}_2$. The algebra $\mathfrak{sl}_2(U(z))$ acts on the space $W_1 \otimes \dots \otimes W_{n+1}$:

$$a(t)\cdot(w_1\otimes\dots\otimes w_{n+1}) \mapsto [a(t+z_1)]w_1\otimes w_2\otimes\dots\otimes w_{n+1}+\dots+ \tag{18}$$
$$w_1\otimes\dots\otimes w_{n-1}\otimes[a(t+z_n)]w_n\otimes w_{n+1}+w_1\otimes\dots\otimes w_n\otimes\pi([a(1/t)])w_{n+1},$$

where $[b(t)]$ denotes the Laurent expansion of a function $b(t)$ at $t = 0$ and the letter π denotes the automorphism of $\widehat{\mathfrak{sl}}_2$ introduced in 4.2. We assume that all representations W_i have the following finiteness property:

(**Fin**) given $w \in W_i$ and $a \in \mathfrak{sl}_2$, we have $aT^j \cdot w = 0$ for all $j \gg 0$,

so that the action of Laurent power series is well defined. The action of c adds up to zero due to the residue formula. Thus, we have the multiplication map

$$\mu(z):\ \mathfrak{sl}_2(U(z))\otimes(\otimes_{i=1}^{n+1} W_i)\longrightarrow \otimes_{i=1}^{n+1} W_i\ . \tag{19}$$

The space

$$(\otimes_{i=1}^{n+1} W_i)_{\mathfrak{sl}_2(U(z))} := \operatorname{Coker}\ \mu(z) \tag{20}$$

is called the *space of conformal blocks* at z.

The Knizhnik-Zamolodchikov connection, see [KZ], [F].

5.2. Let $\{X^a\}$, $a = 1, 2, 3$, be an orthonormal basis of $\mathfrak{sl}_2$. Set

$$L_{-1} := \frac{1}{\kappa}\sum_{i=0}^{\infty}\sum_{a=1}^{3}(X^aT^{-i-1})(X^aT^i).$$

It is a well defined operator on a representation satisfying the property (Fin) above.

We have

$$[L_{-1}, XT^i] = -iXT^{i-1} \tag{21}$$

for any $X \in \mathfrak{sl}_2$ and any i.

5.3. The following notation will be used: the action of an element X of an algebra on the i-th factor of a tensor product of modules will be denoted by $X^{(i)}$.

Recall the notations of 3.1. For an open subspace $A \subset \mathbb{C}^{[n]}$, set $U_A := (A \times \mathbb{P}^1) \cap \mathbb{C}^{[n+1]} \subset \mathbb{C}^{[n]} \times \mathbb{P}^1$. Denote by $\mathfrak{sl}_2(U_A)$ the Lie algebra of algebraic $\mathfrak{sl}_2$-valued functions on U_A.

Let $W_1, \dots, W_{n+1}$ be representations of $\widehat{\mathfrak{sl}}_2$ satisfying (Fin). Consider the trivial vector bundle $\mathcal{W}_A := A \times (W_1 \otimes \dots \otimes W_{n+1}) \longrightarrow A$. The Lie algebra $\mathfrak{sl}_2(U_A)$ acts on its holomorphic sections by formula (18).

Consider the flat connection on the bundle $\mathcal{W}_A$: $\nabla = \sum_{i=1}^n \nabla_{z_i} dz_i$,

$$\nabla_{z_i} G(z) = \partial_{z_i} G(z) + L_{-1}^{(i)} G(z), \tag{22}$$

where $G(z) \in \Gamma(A; \mathcal{W}_A)$.

5.4. Lemma. (Cf. [F]) *For any $X \in \mathfrak{sl}_2(U_A)$, $G \in \Gamma(A; \mathcal{W}_A)$, we have $\nabla_{z_i}(XG) = (\nabla_{z_i} X)G + X(\nabla_{z_i} G)$.*

This is proved by a direct computation.

5.5. Let $\mathrm{Im}\ \mu \subset \mathcal{W}_{\mathbb{C}^{[n]}}$ denote the subspace whose intersection with the fiber at a point z is equal to the image of $\mu(z)$. According to the lemma this subspace is invariant with respect to the connection. Consider the quotient bundle $\overline{\mathcal{W}}$ over $\mathbb{C}^{[n]}$ with fiber Coker $\mu(z)$. This bundle is called *the bundle of conformal blocks.*

5.6. Corollary. *The connection defined in (22) induces a connection on the bundle of conformal blocks.*

The induced integrable connection on $\overline{\mathcal{W}}$ is called the *Knizhnik-Zamolodchikov* (KZ) connection. The *KZ equation* on G is the horizontality condition, $\nabla G = 0$.

The main construction.

5.7. Let $M^1, \dots, M^n, k$ be complex numbers, $k \neq -2$. Set $M^{n+1} := M^1 + \dots + M^n - 2$. Let V_i denote the Verma module $V(M^i, k - M^i)$ and V_i^* the contragradient dual, cf. 4.8.

According to 5.1, the Lie algebra $\mathfrak{sl}_2(U(z))$ acts on $\otimes_{i=1}^{n+1} V_i^*$, so that we can consider the standard chain complex $C_\bullet(\mathfrak{sl}_2(U(z)); \otimes_{i=1}^{n+1} V_i^*)$. Its right end looks as in (19):

$$C_\bullet(\mathfrak{sl}_2(U(z)); \otimes_{i=1}^{n} V_i^*) : \ \dots \longrightarrow \mathfrak{sl}_2(U(z)) \otimes \ (\otimes_{i=1}^{n+1} V_i^*) \xrightarrow{d} \otimes_{i=1}^{n+1} V_i^* \longrightarrow 0, \tag{23}$$

with $d = \mu(z)$, where $\mu(z)$ is defined in 5.1. We assign to the last term degree 0 and agree that d has degree 1, so that the whole complex sits in the nonpositive area.

On the other hand, consider the shifted De Rham complex (3) corresponding to $\kappa = k+2$:

$$\Omega^\bullet(U(z))[1] : \ 0 \longrightarrow \Omega^0(U(z)) \longrightarrow \Omega^1(U(z)) \longrightarrow 0. \tag{24}$$

Here the shift [1] means simply that we assign to Ω^j degree $j-1$.

In the rest of the section we construct a monomorphism of complexes $\Omega^\bullet(U(z))[1] \hookrightarrow C_\bullet(\mathfrak{sl}_2(U(z)); \otimes_{i=1}^{n+1} V_i^*)$.

5.8. First we need a basis in the complex (23). Let $\gamma = (p_1, p_2) \in \mathbb{Z}^2_{\geq 0}$ and $p_1 > p_2$. We fix the following bases of homogeneous components V_γ of all Verma modules V:

$$\frac{f}{T^{i_1}} \cdot \ldots \cdot \frac{f}{T^{i_a}} \frac{h}{T^{j_1}} \cdot \ldots \cdot \frac{h}{T^{j_b}} \frac{e}{T^{l_1}} \cdot \ldots \cdot \frac{e}{T^{l_c}} v \tag{25}$$

where

$$0 \leq i_a \leq i_{a-1} \leq \ldots \leq i_1,\ 1 \leq j_b \leq j_{b-1} \leq \ldots \leq j_1,\ 1 \leq l_c \leq l_{c-1} \leq \ldots \leq l_1; \tag{26}$$

$$\sum_{s=1}^{a} i_s + \sum_{s=1}^{b} j_s + \sum_{s=1}^{c} l_s + a - c = p_1, \qquad \sum_{s=1}^{a} i_s + \sum_{s=1}^{b} j_s + \sum_{s=1}^{c} l_s = p_2.$$

For $p_1 < p_2$, we fix a basis of the form

$$\frac{e}{T^{l_1}} \cdot \ldots \cdot \frac{e}{T^{l_c}} \frac{h}{T^{j_1}} \cdot \ldots \cdot \frac{h}{T^{j_b}} \frac{f}{T^{i_1}} \cdot \ldots \cdot \frac{f}{T^{i_a}} v, \tag{27}$$

with the indices satisfying (26). These are bases by the Poincaré-Birkhoff-Witt theorem.

Notice that the elements X/T^i and X/T^j ($X \in \mathfrak{sl}_2$) commute.

We fix the bases in the contragradient Verma modules V^* which are dual to the bases distinguished above in the Verma modules.

If $\{v_i\}$ is a basis in V, then we denote the dual basis by $\{(v_i)^*\}$.

5.9. Define a map

$$\eta^1 :\ \Omega^1(U(z)) \longrightarrow \otimes_{i=1}^{n+1} V_i^*$$

by the formulas

$$\frac{d(t-z_m)}{(t-z_m)^{b+1}} \mapsto -\kappa v_1^* \otimes \ldots \otimes \left(\frac{f}{T^b} v_m\right)^* \otimes \ldots \otimes v_{n+1}^*, \tag{28}$$

$$t^b dt \mapsto \kappa v_1^* \otimes \ldots \otimes v_n^* \otimes \left(\frac{e}{T^{b+1}} v_{n+1}\right)^*, \tag{29}$$

for $b \geq 0$.

Define a map

$$\eta^0 :\ \Omega^0(U(z)) \longrightarrow \mathfrak{sl}_2(U(z)) \otimes (\otimes_{i=1}^{n+1} V_i^*)$$

by the formulas

$$\frac{1}{(t-z_m)^b} \ \mapsto\ \frac{f}{(t-z_m)^b} \otimes v_1^* \otimes \ldots \otimes v_{n+1}^* - \tag{30}$$

$$\sum_{l=1}^{b}\ \Big[\frac{e}{(t-z_m)^l}\otimes v_1^*\otimes\ldots\otimes 2\sum_{i+j=b-l,\ i\geq j\geq 0}\left(\frac{f}{T^i}\frac{f}{T^j}v_m\right)^*\otimes\ldots\otimes v_{n+1}^*+$$

$$\frac{h}{(t-z_m)^l}\otimes v_1^*\otimes\ldots\otimes\left(\frac{f}{T^{b-l}}v_m\right)^*\otimes\ldots\otimes v_{n+1}^*\Big],$$

for $b\geq 1$;

$$1\ \mapsto\ f\otimes v_1^*\otimes\ldots\otimes v_{n+1}^*,\qquad t\ \mapsto\ ft\otimes v_1^*\otimes\ldots\otimes v_{n+1}^*,\tag{31}$$

$$t^b\ \mapsto\ ft^b\otimes v_1^*\otimes\ldots\otimes v_{n+1}^*-\sum_{l=0}^{b-2}\ \Big[et^l\otimes v_1^*\otimes\ldots\otimes v_n^*\otimes 2\sum_{i+j=b-l,\, i\geq j\geq 1}\left(\frac{e}{T^i}\frac{e}{T^j}v_{n+1}\right)^*+$$

$$ht^{l+1}\otimes v_1^*\otimes\ldots\otimes v_n^*\otimes\left(\frac{e}{T^{b-l-1}}v_{n+1}\right)^*\Big],$$

for $b\geq 2$.

5.10. Theorem. *Formulas (28)-(31) define a monomorphism of complexes*

$$\eta:\ \Omega^\bullet(U(z))[1]\longrightarrow C_\bullet(\mathfrak{sl}_2(U(z));\otimes_{i=1}^{n+1}\ V_i^*).$$

5.11. Beginning of the proof of Theorem 5.10. We should check that

$$\eta^1(\tilde d(x))=\mu(z)(\eta^0(x))\tag{32}$$

for any $x\in\Omega^0(U(z))$. We have

$$1\ \overset{\eta}{\mapsto}\ f\otimes v_1^*\otimes\ldots\otimes v_{n+1}^*\ \overset{\mu}{\mapsto}\ \sum_{i=1}^n\ M^i v_1^*\otimes\ldots\otimes(fv_i)^*\otimes\ldots\otimes v_{n+1}^*,\tag{33}$$

$$1\ \overset{\tilde d}{\mapsto}\ -\frac{1}{\kappa}\sum_{i=1}^n\ M^i\frac{d(t-z_i)}{t-z_i}\ \overset{\eta}{\mapsto}\ \sum_{i=1}^n\ M^i v_1^*\otimes\ldots\otimes(fv_i)^*\otimes\ldots\otimes v_{n+1}^*,$$

and these formulas agree with (32). Next we have

$$t\ \overset{\eta}{\mapsto}\ ft\otimes v_1^*\otimes\ldots\otimes v_{n+1}^*\ \overset{\mu}{\mapsto}\tag{34}$$

$$\sum_{i=1}^n\ M^iz_i\,v_1^*\otimes\ldots\otimes(fv_i)^*\otimes\ldots\otimes v_{n+1}^*\ +\ (k-M^{n+1})v_1^*\otimes\ldots\otimes\left(\frac{e}{T}v_{n+1}\right)^*,$$

$$t\ \overset{\tilde d}{\mapsto}\ \frac{1}{\kappa}\Big[\left(\kappa-\sum_{i=1}^n\ M^i\right)dt-\sum_{i=1}^n\ M^iz_i\frac{d(t-z_i)}{t-z_i}\Big]\ \overset{\mu}{\mapsto}$$

$$\left(k-\left(\sum_{i=1}^n\ M^i-2\right)\right)v_1^*\otimes\ldots\otimes\left(\frac{e}{T}v_{n+1}\right)^*\ +\sum_{i=1}^n M^iz_i\,v_1^*\otimes\ldots\otimes(fv_i)^*\otimes\ldots\otimes v_{n+1}^*,$$

and these formulas also agree with (32). Notice that calculating the action on M_{n+1}^* we use the automorphism π, see formula (18).

Similarly, to prove (32) for $x = t^2$, one needs the identity

$$\left(2\kappa - \sum_{i=1}^{n} M^i\right)\left(\frac{e}{T^2}v_{n+1}\right)^* = \frac{e}{T^2}\cdot v_{n+1}^* - 2f\left(\left(\frac{e}{T}\right)^2 v_{n+1}\right)^* + \frac{h}{T}\left(\frac{e}{T}v_{n+1}\right)^*, \tag{35}$$

and to prove (32) for $x = \frac{1}{t-z_i}$, one needs the identity

$$(M^i+\kappa)\left(\frac{f}{T}v_i\right)^* = \frac{f}{T}\cdot v_i^* - \frac{h}{T}(fv_i)^* - 2\frac{e}{T}(f^2v_i)^*. \tag{36}$$

5.12. Theorem. *For $M,\ k \in \mathbb{C}$, the following identities hold in the contragradient Verma module $V(M, k-M)^*$:*

(a) *for $b \geq 1$, we have*

$$(M+b(k+2))\left(\frac{f}{T^b}v\right)^* = \frac{f}{T^b}\cdot v^* - \sum_{l=1}^{b}\left[2\frac{e}{T^l}\sum_{i+j=b-l,\ i\geq j\geq 0}\left(\frac{f}{T^i}\frac{f}{T^j}v\right)^* + \frac{h}{T^l}\left(\frac{f}{T^{b-l}}v\right)^*\right];$$

(b) *for $b \geq 2$, we have*

$$(k-M+(b-1)(k+2))\left(\frac{e}{T^b}v\right)^* = \frac{e}{T^b}\cdot v^* + \sum_{l=0}^{b-2}\left[-2\frac{f}{T^l}\sum_{i+j=b-l,\ i\geq j\geq 1}\left(\frac{e}{T^i}\frac{e}{T^j}v\right)^* + \frac{h}{T^{l+1}}\left(\frac{e}{T^{b-l-1}}v\right)^*\right].$$

5.13. Proof of the theorem. The theorem is proved by direct verification. Each term of the expression in (a) is of degree $(b+1,b)$. The basis of $V(M,k-M)_{(b+1,b)}$ is described in Section 5.8. This gives us the dual basis of $V(M,k-M)^*_{(b+1,b)}$. One calculates in straightforward way the right-hand side of (a) in that basis and obtains the left-hand side. For example, for $b=1$, the space $V(M,k-M)^*_{(2,1)}$ has the basis $(\frac{f}{T}v)^*$, $(f\frac{h}{T}v)^*$, $(f^2\frac{e}{T}v)^*$, and we have $\frac{f}{T}\cdot v^* = (M+k)(\frac{f}{T}v)^* + (2M-2k)(f^2\frac{e}{T}v)^*$, $\frac{h}{T}\cdot(fv)^* = -2(\frac{f}{T}v)^* + 4(f\frac{h}{T}v)^* + (4M-4k)(f^2\frac{e}{T}v)^*$, $\frac{e}{T}(f^2v_i)^* = -2(f\frac{h}{T}v)^* - (2M-2k)(f^2\frac{e}{T}v)^*$. By adding these expressions we get the formula $(M+\kappa)\left(\frac{f}{T}v\right)^* = \frac{f}{T}\cdot v^* - \frac{h}{T}(fv)^* - 2\frac{e}{T}(f^2v)^*$ which gives statement (a) for $b=1$ and formula (36).

Similarly each term of the expression in (b) is of degree $(b-1,b)$. The basis of $V(M,k-M)_{(b-1,b)}$ is described in Section 5.8. This gives us the dual basis of $V(M,k-M)^*_{(b-1,b)}$. One calculates in straightforward way the right-hand side of (b) in that basis and obtains the left-hand side. For example, for $b=2$ the space $V(M,k-M)^*_{(1,2)}$ has the basis $((\frac{e}{T})^2fv)^*, (\frac{e}{T}\frac{h}{T}v)^*, (\frac{e}{T^2}v)^*$ and we have $\frac{e}{T^2}\cdot v^* = -2M((\frac{e}{T})^2fv)^* - 2k(\frac{e}{T}\frac{h}{T}v)^* + (2k-M)(\frac{e}{T^2}v)^*$, $f\cdot((\frac{e}{T})^2v)^* = M((\frac{e}{T})^2fv)^* - 2(\frac{e}{T}\frac{h}{T}v)^*$, $\frac{h}{T}\cdot(\frac{e}{T}v)^* = 4M((\frac{e}{T})^2fv)^* + (2k-4)(\frac{e}{T}\frac{h}{T}v)^* + 2(\frac{e}{T^2}v)^*$. By adding these expressions we get the formula $(2k+2-M)(\frac{e}{T^2}v)^* = \frac{e}{T^2}\cdot v^* - 2f\cdot((\frac{e}{T})^2v)^* + \frac{h}{T}\cdot(\frac{e}{T}v)^*$ which gives statement (b) for $b=2$ and formula (35).

The complete proofs of the theorem will be published elsewhere. □

5.14. End of the proof of Theorem 5.10. Theorem 5.10 is a direct corollary of Theorem 5.12, cf. (10), (11) and (28)-(31). □

6. Singular vectors in Verma modules

6.1. Let $S: \ V(M,k-M) \longrightarrow V(M,k-M)^*$ be the Shapovalov form. Set

$$X_b(M,k-M) := S^{-1}\left((M+b(k+2))\left(\frac{f}{T^b}v\right)^*\right), \tag{37}$$

$$Y_b(M,k-M) := S^{-1}\left((k-M+(b-1)(k+2))\left(\frac{e}{T^b}v\right)^*\right).$$

For generic values of M and k, the Shapovalov form S is non-degenerate and X_b and Y_b are well defined elements of the Verma module $V(M,k-M)$. The basis in $V(M,k-M)$ allows us to compare these vectors for different values of k, M. Obviously, $X_b(M,k-M)$, $Y_b(M,k-M)$ are holomorphic functions of k, M for generic k, M.

Consider the resonance lines

$$M = l-1-(a-1)(k+2), \qquad M = -l-1+a(k+2), \qquad k+2=0, \tag{38}$$

$(l, a \in \mathbb{Z}_{>0})$ on the (M,k)-plane, cf. 4.4.

6.2. Theorem.

(a) *Let $b \geq 0$ and let (M_0,k_0) be a point of the line $\{M=-b(k+2)\}$ which does not belong to other resonance lines. Then the vector-valued function $X_b(M,k-M)$ can be analytically continued to the point (M_0,k_0) and the vector $X_b(M_0,k_0-M_0)$ is a (nonzero) singular vector of $V(M,k-M)$.*

(b) *Let $b > 0$ and let (M_0,k_0) be a point of the line $\{M=-2+b(k+2)\}$ which does not belong to other resonance lines. Then the vector-valued function $Y_b(M,k-M)$ can be analytically continued to the point (M_0,k_0) and the vector $Y_b(M_0,k_0-M_0)$ is a (nonzero) singular vector of $V(M,k-M)$.*

Proof of (a). According to Theorem 5.12,

$$X_b(M,k-M) = \frac{f}{T^b}v - \sum_{l=1}^{b}\left[2\frac{e}{T^l}\sum_{i+j=b-l,\ i\geq j\geq 0} S^{-1}\left(\frac{f}{T^i}\frac{f}{T^j}v\right)^* + \frac{h}{T^l}S^{-1}\left(\frac{f}{T^{b-l}}v\right)^*\right]. \tag{39}$$

The right-hand side can be analytically continued to (M_0,k_0) since the elements $S^{-1}\left(\frac{f}{T^i}\frac{f}{T^j}v\right)^*$ and $S^{-1}\left(\frac{f}{T^{b-l}}v\right)^*$ are well defined at (M_0,k_0) by the results in Section 4.4. We have $S(X_b(M_0,k_0-M_0))=0$ by definition. Let us check that $X_b(M_0,k_0-M_0)\neq 0$. In fact, consider the basis $\{\frac{e}{T^{l_1}}\cdot\ldots\cdot\frac{e}{T^{l_\alpha}}\frac{h}{T^{j_1}}\cdot\ldots\cdot\frac{h}{T^{j_\beta}}\frac{f}{T^{i_1}}\cdot\ldots\cdot\frac{f}{T^{i_\gamma}}\}$ in $V(M_0,k_0-M_0)$. Formula (39) shows that the basis vector $\frac{f}{T^b}v$ comes to X_b with coefficient 1. This proves part (a) of the theorem. Part (b) is proved similarly. □

6.3. Corollary of Theorem 6.2. *If $b \geq 0$ and (M_0, k_0) is a point of the line $\{M = -b(k+2)\}$, which does not belong to other resonance lines, then $X_b(M_0, k_0 - M_0)$ is proportional to the Malikov-Feigin-Fuchs vectore $F_{12}(1, b+1, k_0+2)$. Similarly, if $b > 0$ and (M_0, k_0) is a point of the line $\{M = -2 + b(k+2)\}$ which does not belong to other resonance lines, then $Y_b(M_0, k_0 - M_0)$ is proportional to the Malikov-Feigin-Fuchs vectore $F_{21}(1, b, k_0+2)$.* □

6.4. Corollary of formulas (10) and (11). *If the resonance condition $k - M^{n+1} + (b-1)\kappa = 0$ of the identity 5.12.b holds, then formula (11) gives the following cohomological relation between the logarithmic forms ω_j introduced in 2.2:*

$$\sum_{m\geq 0} \sum_{\substack{l_0+\dots+l_m=b,\\ l_0,\dots,l_m>0}} (-\kappa)^{-m} \Big(\prod_{i=1}^{m} \Big(\frac{1}{l_i} \sum_{j=1}^{n} z_j^{l_i} M^j \Big) \Big) \sum_{j=1}^{n} z_j^{l_0} \omega_j \sim 0\,, \tag{40}$$

where $\kappa = k+2$. Similarly if the resonance conditiion $M^p + b\kappa = 0$ of the identity 5.12.a holds for some $p \leq n$, then formula (10) induces the following cohomological relation between the logarithmic forms ω_j:

$$\sum_{m\geq 0} \Big[\sum_{\substack{l_0+\dots+l_m=b,\\ l_0,\dots,l_m>0}} (-\kappa)^{-m} \Big(\prod_{i=1}^{m} \Big(\frac{1}{l_i} \sum_{\substack{j=1,\dots,n,\\ j\neq p}} \frac{M^j}{(z_j - z_p)^{l_i}} \Big) \Big) \sum_{\substack{j=1,\dots,n,\\ j\neq p}} \frac{\omega_j}{(z_j - z_p)^{l_0}} -$$

$$\sum_{\substack{l_1+\dots+l_m=b,\\ l_1,\dots,l_m>0}} (-\kappa)^{-m} \Big(\prod_{i=1}^{m} \Big(\frac{1}{l_i} \sum_{\substack{j=1,\dots n,\\ j\neq p}} \frac{M^j}{(z_j - z_p)^{l_i}} \Big) \Big) \omega_p \Big] \sim 0, \tag{41}$$

cf. Examples 2.4, 2.5. □

For instance, if $M^1 + \kappa = 0$, then

$$\sum_{j>1} \frac{\omega_j}{z_j - z_1} + \frac{1}{\kappa} \left(\sum_{j>1} \frac{M^j}{z_j - z_1} \right) \omega_1 \sim 0\ . \tag{42}$$

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