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Hamiltonian and Lagrangian formalisms of mutations in cluster algebras and application to dilogarithm identities

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We introduce and study a Hamiltonian formalism of mutations in cluster algebras using canonical variables, where the Hamiltonian is given by the Euler dilogarithm. The corresponding Lagrangian, restricted to a certain subspace of the phase space, coincides with the Rogers dilogarithm. As an application, we show how the dilogarithm identity associated with a period of mutations in a cluster algebra arises from the Hamiltonian/Lagrangian point of view.

Keywords: cluster algebras; dilogarithm; Hamiltonian; Lagrangian.

1. Introduction

1.1 *Background and motivation*

The *Euler dilogarithm*

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{\log(1-y)}{y} dy \tag{1.1}$$

appears in several areas of mathematics [1]. One of the main features of this function is that it satisfies a wide variety of functional relations (*dilogarithm identities*), many of which look miraculous and mysterious. *Cluster algebras* are a class of commutative algebras introduced by Fomin and Zelevinsky [2]. They originated in Lie theory but turned out to be related to several areas of mathematics.

Fock and Goncharov [3] recognized that the function $\mathrm{Li}_2(x)$ is *built into* cluster algebra theory *as a Hamiltonian*. To see this, consider the Poisson bracket introduced by [4],

$$\{y_i, y_i\} = b_{ij} y_i y_j, \tag{1.2}$$

where $y_1,\ldots,y_n$ are commutative variables and $B = (b_{ij})_{i,j=1}^n$ is a skew-symmetric matrix. Then, using formula (3.4), we have

$$\{\mathrm{Li}_2(-y_k), y_i\} = -b_{ki}\log(1+y_k)\cdot y_i. \tag{1.3}$$

This is an infinitesimal form (of the automorphism part) of the mutation of y-variables (x-variables in [3]) in cluster algebras. Therefore, one may regard the function $\mathrm{Li}_2(-y_k)$ as a Hamiltonian with continuous time variable, and the ordinary mutation is obtained as the time one flow of this Hamiltonian. This viewpoint naturally guided the authors of [3, 5, 6] to quantize the cluster algebras using the *quantum dilogarithm*.

Meanwhile, there is another story which developed independently, connecting the dilogarithm and cluster algebras, where the *Rogers dilogarithm*

$$L(x) = -\frac{1}{2}\int_0^x \left\{\frac{\log(1-y)}{y} + \frac{\log y}{1-y}\right\} dy = \mathrm{Li}_2(x) + \frac{1}{2}\log x\log(1-x) \tag{1.4}$$

plays the central role. The function $L(x)$ is a variant of $\mathrm{Li}_2(x)$ and it is known that many dilogarithm identities are simplified in terms of $L(x)$. In the 90's, several conjectures on dilogarithm identities for $L(x)$ were given through the study of so called *Y-systems* in integrable models of Yang–Baxter type. Later, the connection between Y-systems and cluster algebras was recognized [7] and these dilogarithm conjectures were solved using the cluster algebraic method [8–11] with the help of the *constancy condition* from [12]. Then, these results were further generalized to the following theorem [13]: *for any period in a cluster algebra, there is an associated dilogarithm identity of $L(x)$.*

It is interesting to understand how these seemingly independent appearances of the Euler and Rogers dilogarithms are intrinsically related. In [14], it was clarified that the dilogarithm identities in [13] are recovered through the semiclassical analysis of the corresponding *quantum dilogarithm identities*, where the Rogers dilogarithm emerges through the relation

$$L\left(\frac{x}{1+x}\right) = -\mathrm{Li}_2(-x) - \frac{1}{2}\log x\log(1+x). \tag{1.5}$$

Furthermore, the result therein yields the following curious observation: *The relation* (1.5) *can be regarded as the Legendre transformation in classical mechanics, where the Euler dilogarithm is the Hamiltonian, while the Rogers dilogarithm is the Lagrangian.* However, to formulate and establish the claim precisely, we need *canonical variables* for the Poisson bracket (1.2).

Motivated by the above question, in this article we introduce canonical variables of the Poisson bracket (1.2), and study the Hamiltonian and Lagrangian formalisms of mutations in cluster algebras. As a consequence, the above observation is justified; furthermore, we can successfully explain how the dilogarithm identities in [13] naturally arise in the Hamiltonian/Lagrangian picture.

1.2 *Outline and main results*

Let us briefly describe the outline and the main results of the article.

In Section 2, we recall basic definitions and properties of mutations in cluster algebras and of the dilogarithm functions which we are going to use.

In Section 3, we introduce the Hamiltonian formalism of mutations in cluster algebras with canonical variables. The x- and y-variables in a cluster algebra are constructed as exponentials of linear combinations of the canonical variables, while the Hamiltonian is given by the Euler dilogarithm. The time one flow of the Hamiltonian, together with the tropical transformation, yields the mutation of the x- and y-variables (Theorem 3.12). This naturally extends the Hamiltonian formalism in [3]. An interesting feature here is that the y-variables mutate properly on the total phase space $M \simeq \mathbb{R}^{2n}$, while the x-variables do so only on a certain subspace M_0 of the phase space. This does not have an effect on the equations of motion; but does affect the *quantization* of the Poisson bracket in the following way:

- The canonical quantization of the Poisson bracket leads to the quantization of the y-variables by [3, 5].
- On the other hand, the Poisson bracket on the small phase space M_0 is redefined via the *Dirac bracket* due to [15]. Then, the 'canonical quantization' of the Dirac bracket leads to the quantization of the x-variables by [16].

Therefore, the formulation here provides a common platform for the quantization of both x- and y-variables.

In Section 4, having the canonical variables at hand, we study the Lagrangian formalism of mutations. A specific feature here is that the Hamiltonian in Section 3 is *singular*. This implies that we do not have a Lagrangian whose Euler–Lagrange equations are fully equivalent to the equations of motion of the Hamiltonian. Despite this deficiency, one can still define the Lagrangian through the Legendre transformation. Then, the following fact holds:

Fact 1. (Proposition 4.4) The Lagrangian coincides with the Rogers dilogarithm on the above small phase space M_0.

This justifies the observation stated in the previous subsection.

In Section 5, we make a little detour to establish the periodicity property of the canonical variables. In the Hamiltonian formalism, we naturally introduce a sign $\varepsilon = \pm$ to decompose each seed mutation into the tropical and non-tropical parts. The mutation of x- and y-variables is independent of the choice of the sign ε, while the mutation of the canonical variables depends on it. Thus, we call these mutations *signed mutations*. We have the following result, which is crucial for obtaining the final result in the next section.

Fact 2. (Proposition 5.11) A sequence of signed mutations enjoys the same periodicity as a sequence of seed mutations, if we choose the sign sequence therein as the *tropical sign sequence*.

In Section 6, we show how the dilogarithm identities in [13] arise from the Hamiltonian/Lagrangian point of view. This is done by considering the *action integral* of the singular Lagrangian from Section 4 along a Hamiltonian flow. There are two key facts:

Fact 3. (Proposition 6.2) The Lagrangian is piecewise constant along the flow.

Fact 4. (Theorem 6.3) The action integral does not depend on the flow, if all flows are periodic for the time span of the integral. (This is regarded as the converse of a finite time analogue of Noether's theorem, see Remark 6.6.)

The dilogarithm identities in [13] are obtained as an immediate consequence of the above Facts 1–4. This is the main result of the article (Theorems 6.8 and 6.9).

Remark 1.1 Theorem 6.8 can be straightforwardly extended to generalized cluster algebras and dilogarithm functions of higher degree studied in [17]. Meanwhile, Theorem 6.9 can be also extended to them if we admit the version of Theorem 6.11 for generalized cluster algebras, which is not yet available.

2. Preliminaries

2.1 *Mutations in cluster algebras*

Let us recall two main notions in cluster algebras, namely, a *seed* and its *mutation*. See [2, 18] for more information on cluster algebras.

Let us fix a positive integer n throughout the article. We say that an $n \times n$ integer matrix $B = (b_{ij})_{i,j=1}^n$ is *skew-symmetrizable* if there is a diagonal matrix $D = \mathrm{diag}(d_1, \dots, d_n)$ with positive integer diagonal entries $d_1, \dots, d_n$ such that DB is skew-symmetric, i.e., $d_i b_{ij} = -d_j b_{ji}$. We call such D a *skew-symmetrizer* of B.

Let us fix a semifield $\mathbb{P}$, that is, an abelian multiplicative group with a binary operation $\oplus$ called the *addition*, which is commutative, associative, and distributive, i.e., $a(b \oplus c) = ab \oplus ac$. Let $\mathbb{ZP}$ be the group ring of $\mathbb{P}$. Since $\mathbb{ZP}$ is a domain [2], the field of fractions of $\mathbb{ZP}$ is well defined and we denoted it by $\mathbb{QP}$. Let $\mathcal{F} = \mathcal{F}_{\mathbb{P}}$ be a purely transcendental field extension of $\mathbb{QP}$ of degree n, that is $\mathcal{F}$ is isomorphic to a rational function field of n variables with coefficients in $\mathbb{QP}$. We call the semifield $\mathbb{P}$ and the field $\mathcal{F}$ the *coefficient semifield* and the *ambient field* (of a cluster algebra under consideration), respectively.

A *seed with coefficients in* $\mathbb{P}$ is a triplet (B, x, y) consisting of an $n \times n$ skew-symmetrizable matrix B, an n-tuple $(x_i)_{i=1}^n$ of algebraically independent elements in $\mathcal{F}$, and an n-tuple $(y_i)_{i=1}^n$ of elements in $\mathbb{P}$. For each $k = 1, \dots, n$, the *mutation of a seed* (B, x, y) *at* k is another seed $(B', x', y') = \mu_k(B, x, y)$, which is obtained from (B, x, y) by the following formulas:

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + [-\varepsilon b_{ik}]_+ b_{kj} + b_{ik}[\varepsilon b_{kj}]_+ & i, j \neq k, \end{cases} \tag{2.1}$$

$$x'_i = \begin{cases} x_k^{-1}\left(\prod_{j=1}^{n} x_j^{[-\varepsilon b_{jk}]_+}\right)\dfrac{1 + \hat{y}_k^{\varepsilon}}{1 \oplus y_k^{\varepsilon}} & i = k \\ x_i & i \neq k, \end{cases} \tag{2.2}$$

$$y'_i = \begin{cases} y_k^{-1} & i = k \\ y_i y_k^{[\varepsilon b_{ki}]_+}(1 \oplus y_k^{\varepsilon})^{-b_{ki}} & i \neq k, \end{cases} \tag{2.3}$$

where

$$\hat{y}_i := y_i \prod_{j=1}^{n} x_j^{b_{ji}}, \tag{2.4}$$

and ε is a sign, $+$ or $-$, which is naturally identified with 1 or -1, respectively. Then we have the following properties:

(1). The right-hand sides of (2.1)–(2.3) are independent of the choice of sign ε.

(2). If D is a skew-symmetrizer of B, then it is also a skew-symmetrizer of B'.

(3). The mutation μ_k is involutive, namely,

$$\mu_k \circ \mu_k = \mathrm{id}. \tag{2.5}$$

(4). The $\hat{y}$-variables (2.4) also mutate in $\mathcal{F}$ as the y-variables; namely,

$$\hat{y}'_i = \begin{cases} \hat{y}_k^{-1} & i = k \\ \hat{y}_i \hat{y}_k^{[\varepsilon b_{ki}]_+} (1 + \hat{y}_k^{\varepsilon})^{-b_{ki}} & i \neq k. \end{cases} \tag{2.6}$$

Let $\mathcal{F}_0^n$ be the set of all n-tuples of algebraically independent elements in $\mathcal{F}$, and, as usual, let $\mathbb{P}^n$ be the set of all n-tuples of elements in $\mathbb{P}$. Let us extract the 'variable part' of the mutation μ_k in (2.2) and (2.3) as

$$\begin{array}{rccc} \mu_k^B : & \mathcal{F}_0^n \times \mathbb{P}^n & \rightarrow & \mathcal{F}_0^n \times \mathbb{P}^n \\ & (x, y) & \mapsto & (x', y'), \end{array} \tag{2.7}$$

and call it the *mutation at k by B*. The involution property (2.5) is equivalent to the inversion relation,

$$\mu_k^{B_k} \circ \mu_k^B = \mathrm{id}, \tag{2.8}$$

where $B_k = B'$ is the one in (2.1).

Following the idea of [5], we decompose the mutation μ_k^B into two parts. For each sign $\varepsilon = \pm$, we introduce a map

$$\begin{array}{rccc} \rho_{k,\varepsilon}^B : & \mathcal{F}_0^n \times \mathbb{P}^n & \rightarrow & \mathcal{F}_0^n \times \mathbb{P}^n \\ & (x, y) & \mapsto & (\tilde{x}, \tilde{y}), \end{array} \tag{2.9}$$

$$\tilde{x}_i = x_i \left(\frac{1 + \hat{y}_k^{\varepsilon}}{1 \oplus y_k^{\varepsilon}} \right)^{-\delta_{ki}}, \tag{2.10}$$

$$\tilde{y}_i = y_i (1 \oplus y_k^{\varepsilon})^{-b_{ki}}, \tag{2.11}$$

and also a map

$$\begin{array}{rccc} \tau_{k,\varepsilon}^B : & \mathcal{F}_0^n \times \mathbb{P}^n & \rightarrow & \mathcal{F}_0^n \times \mathbb{P}^n \\ & (x, y) & \mapsto & (x', y'), \end{array} \tag{2.12}$$

$$x'_i = \begin{cases} x_k^{-1} \left(\displaystyle\prod_{j=1}^n x_j^{[-\varepsilon b_{jk}]_+} \right) & i = k \\ x_i & i \neq k, \end{cases} \tag{2.13}$$

$$y'_i = \begin{cases} y_k^{-1} & i = k \\ y_i y_k^{[\varepsilon b_{ki}]_+} & i \neq k. \end{cases} \tag{2.14}$$

Then, for each sign $\varepsilon = \pm$, the mutation μ_k^B is decomposed as

$$\mu_k^B = \tau_{k,\varepsilon}^B \circ \rho_{k,\varepsilon}^B. \tag{2.15}$$

In [5], the transformations $\rho_{k,\varepsilon}^B$ and $\tau_{k,\varepsilon}^B$ for $\varepsilon = +$ were considered, and they were called the *automorphism part* and the *monomial part* of the mutation μ_k^B, respectively. Here, we call $\rho_{k,\varepsilon}^B$ and $\tau_{k,\varepsilon}^B$ the *non-tropical part* and the *tropical part* of the mutation μ_k^B, respectively. See [19] for the background of the terminology.

When the coefficient semifield $\mathbb{P}$ is taken to be the *trivial semifield* $\mathbf{1} = \{1\}$, where $1 \oplus 1 = 1$, we say that the x-variables are *without coefficients*. In that case, the transformations (2.2) and (2.10) are simplified as

$$x_i' = \begin{cases} x_k^{-1}\left(\prod_{j=1}^{n} x_j^{[-\varepsilon b_{jk}]_+}\right)(1+\hat{y}_k^{\varepsilon}) & i = k \\ x_i & i \neq k, \end{cases} \tag{2.16}$$

$$\tilde{x}_i = x_i(1+\hat{y}_k^{\varepsilon})^{-\delta_{ki}}, \tag{2.17}$$

respectively, while (2.4) also reduces to

$$\hat{y}_i = \prod_{j=1}^{n} x_j^{b_{ji}}. \tag{2.18}$$

2.2 *Euler and Rogers dilogarithm functions*

Let us recall the definition of the Euler and Rogers dilogarithms. See [1, 20] for more information.

The *Euler dilogarithm* $\mathrm{Li}_2(x)$ is originally defined as the following convergent series with radius of convergence 1,

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}. \tag{2.19}$$

It has the integral expression

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-y)}{y}dy, \quad (x \leq 1), \tag{2.20}$$

where throughout the text we concentrate on the real region $x \leq 1$ so that there is no ambiguity due to multivaluedness of the integral. Note that (2.20) is also written as

$$\mathrm{Li}_2(-x) = -\int_0^x \frac{\log(1+y)}{y}dy, \quad (-1 \leq x). \tag{2.21}$$

On the other hand, the *Rogers dilogarithm* $L(x)$ is defined by the integral expression

$$L(x) = -\frac{1}{2}\int_0^x \left\{\frac{\log(1-y)}{y} + \frac{\log y}{1-y}\right\}dy, \quad (0 \leq x \leq 1). \tag{2.22}$$

Again, since we concentrate on the real region $0 \leq x \leq 1$, there is no ambiguity due to multivaluedness of the integral.

These two dilogarithms are related by

$$L(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log x \log(1-x), \quad (0 \le x \le 1), \tag{2.23}$$

which can be used as an alternative definition of the Rogers dilogarithm. They are also related by the following less well-known formula:

$$L\left(\frac{x}{1+x}\right) = -\mathrm{Li}_2(-x) - \frac{1}{2}\log x \log(1+x), \quad (0 \le x) \tag{2.24}$$

$$= \frac{1}{2}\int_0^x \left\{\frac{\log(1+y)}{y} - \frac{\log y}{1+y}\right\} dy, \quad (0 \le x). \tag{2.25}$$

Formulas (2.23)–(2.25) can be most easily verified by taking the derivative.

In view of the formulas (2.24) and (2.25), it is convenient to introduce a function

$$\tilde{L}(x) = L\left(\frac{x}{1+x}\right) = \frac{1}{2}\int_0^x \left\{\frac{\log(1+y)}{y} - \frac{\log y}{1+y}\right\} dy, \quad (0 \le x), \tag{2.26}$$

so that it satisfies the equality

$$\tilde{L}(x) = -\mathrm{Li}_2(-x) - \frac{1}{2}\log x \log(1+x), \quad (0 \le x). \tag{2.27}$$

For simplicity, we still call the function $\tilde{L}(x)$ the Rogers dilogarithm.

3. Hamiltonian formalism of mutations

3.1 *Canonical and log-canonical variables*

Let M be a symplectic manifold with a global Darboux chart $\varphi : M \xrightarrow{\sim} \mathbb{R}^{2n}$. Let (u,p), $u = (u_1, \dots, u_n)$, $p = (p_1, \dots, p_n)$, be the *canonical coordinates* of the chart. Then, in the coordinates (u,p), the *Poisson bracket* is given by

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial u_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial u_i}\right) \tag{3.1}$$

for any (smooth) functions f and g on M. We call M the *phase space*.

We recall some basic properties of the Poisson bracket which we use below.

(1) We have

$$\{p_i, u_j\} = \delta_{ij}, \quad \{u_i, u_j\} = \{p_i, p_j\} = 0. \tag{3.2}$$

(2) For any function f on M,

$$\{f, p_i\} = -\frac{\partial f}{\partial u_i}, \quad \{f, u_i\} = \frac{\partial f}{\partial p_i}. \tag{3.3}$$

(3) For any functions f and g on M, and any smooth functions $F(\zeta)$ and $G(\zeta)$ of a single variable ζ,

$$\{F(f), G(g)\} = \{f, g\}F'(f)G'(g). \tag{3.4}$$

In particular, the following formula holds:

$$\{e^f, e^g\} = \{f, g\}e^f e^g. \tag{3.5}$$

Let us fix any $n \times n$ skew-symmetrizable (integer) matrix $B = (b_{ij})_{i,j=1}^n$ with a skew-symmetrizer D. We introduce variables (i.e., functions on M) w_i, x_i, y_i $(i = 1, \dots, n)$ as follows:

$$w_i = \sum_{j=1}^n b_{ji}u_j, \tag{3.6}$$

$$x_i = e^{2u_i}, \tag{3.7}$$

$$y_i = e^{d_i p_i + w_i}. \tag{3.8}$$

LEMMA 3.1 We have the following formulas:

$$\{d_i p_i + w_i, d_j p_j + w_j\} = 2d_i b_{ij}, \quad \{d_i p_i + w_i, u_j\} = d_i \delta_{ij}. \tag{3.9}$$

DEFINITION 3.2 Following [4], we say that a family of variables $z_1, \dots, z_m$ is *log-canonical* if their pairwise Poisson brackets are of the form

$$\{z_i, z_j\} = c_{ij} z_i z_j, \tag{3.10}$$

where each c_{ij} is a constant.

PROPOSITION 3.3 The variables $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are log-canonical with the following Poisson brackets:

$$\{x_i, x_j\} = 0, \quad \{y_i, y_j\} = 2d_i b_{ij} y_i y_j, \quad \{y_i, x_j\} = 2d_i \delta_{ij} y_i x_j. \tag{3.11}$$

Proof. Follows from (3.2), (3.5) and Lemma 3.1. □

3.2 *Hamiltonian for infinitesimal non-tropical mutation*

For any $k \in \{1, \dots, n\}$ and a sign $\varepsilon = \pm$, we introduce the *Hamiltonian function* $H^B_{k,\varepsilon}$ on M by

$$H^B_{k,\varepsilon} = \frac{\varepsilon}{2d_k}\mathrm{Li}_2(-y_k^\varepsilon) = -\frac{\varepsilon}{2d_k}\int_0^{y_k^\varepsilon} \frac{\log(1+z)}{z} dz, \tag{3.12}$$

where $\mathrm{Li}_2(x)$ is the Euler dilogarithm (2.20), and we used the expression (2.21).

Let t be the time variable. We consider the Hamiltonian flow on the phase space M by the Hamiltonian $H^B_{k,\varepsilon}$. Accordingly, we have functions of t, $u_i(t)$, $p_i(t)$, $w_i(t)$, etc., which obey the following *equations of motion*, where we use the standard notation $\dot{f} = df/dt$.

PROPOSITION 3.4

(1) The equations of motion are given as follows:

$$\dot{u}_i(t) = \left\{H^B_{k,\varepsilon}, u_i(t)\right\} = -\frac{1}{2}\delta_{ki}\log\left(1 + y_k(t)^{\varepsilon}\right), \tag{3.13}$$

$$\dot{p}_i(t) = \left\{H^B_{k,\varepsilon}, p_i(t)\right\} = -\frac{1}{2d_i}b_{ki}\log\left(1 + y_k(t)^{\varepsilon}\right), \tag{3.14}$$

$$\dot{w}_i(t) = \left\{H^B_{k,\varepsilon}, w_i(t)\right\} = -\frac{1}{2}b_{ki}\log\left(1 + y_k(t)^{\varepsilon}\right), \tag{3.15}$$

$$d_i\dot{p}_i(t) = \left\{H^B_{k,\varepsilon}, d_ip_i(t)\right\} = -\frac{1}{2}b_{ki}\log\left(1 + y_k(t)^{\varepsilon}\right), \tag{3.16}$$

$$\dot{x}_i(t) = \left\{H^B_{k,\varepsilon}, x_i(t)\right\} = -\delta_{ki}\log\left(1 + y_k(t)^{\varepsilon}\right)\cdot x_i(t), \tag{3.17}$$

$$\dot{y}_i(t) = \left\{H^B_{k,\varepsilon}, y_i(t)\right\} = -b_{ki}\log\left(1 + y_k(t)^{\varepsilon}\right)\cdot y_i(t). \tag{3.18}$$

(2) In particular, $\dot{y}_k(t) = 0$, so that $y_k(t)$ in the right-hand sides of (3.13)–(3.18) does not depend on t.

Proof. For example,

$$\begin{aligned}\left\{H^B_{k,\varepsilon}, u_i\right\} &= \frac{\partial H^B_{k,\varepsilon}}{\partial p_i} = \frac{dH^B_{k,\varepsilon}}{dy_k^{\varepsilon}}\frac{dy_k^{\varepsilon}}{dy_k}\frac{\partial y_k}{\partial p_i} \\ &= \left(-\frac{\varepsilon}{2d_k}\frac{\log(1+y_k^{\varepsilon})}{y_k^{\varepsilon}}\right)(\varepsilon y_k^{\varepsilon-1})(\delta_{ki}d_ky_k) = -\frac{1}{2}\delta_{ki}\log(1+y_k^{\varepsilon}),\end{aligned} \tag{3.19}$$

$$\begin{aligned}\left\{H^B_{k,\varepsilon}, p_i\right\} &= -\frac{\partial H^B_{k,\varepsilon}}{\partial u_i} = -\frac{dH^B_{k,\varepsilon}}{dy_k^{\varepsilon}}\frac{dy_k^{\varepsilon}}{dy_k}\frac{\partial y_k}{\partial u_i} \\ &= -\left(-\frac{\varepsilon}{2d_k}\frac{\log(1+y_k^{\varepsilon})}{y_k^{\varepsilon}}\right)(\varepsilon y_k^{\varepsilon-1})(b_{ik}y_k) = -\frac{1}{2d_i}b_{ki}\log(1+y_k^{\varepsilon}),\end{aligned} \tag{3.20}$$

where $d_ib_{ik} = -d_kb_{ki}$ is used for the last equality. □

From Proposition 3.4, one can also observe the following. (In fact, it is a direct consequence of the fact that $H^B_{k,\varepsilon}$ only depends on the variable $y_k = \exp(d_kp_k + w_k)$.)

PROPOSITION 3.5

(1) The variables $u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_n$ and p_k are integrals of motion in involution. Therefore, the Hamiltonian $H^B_{k,\varepsilon}$ is completely integrable.
(2) The flows of the variables u_k and $p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_n$ are linear in t.

Let us consider a *time one flow*

$$\begin{array}{rccc}\rho^B_{k,\varepsilon}: & \mathbb{R}^{2n} & \to & \mathbb{R}^{2n} \\ & (u,p) & \mapsto & (\tilde{u},\tilde{p}),\end{array} \tag{3.21}$$

which is defined by the Hamiltonian flow from time $t = 0$ to $t = 1$. Let $\tilde{w}_i, \tilde{x}_i, \tilde{y}_i$ be the corresponding flows of w_i, x_i, y_i, respectively. Let $\mathbb{R}_+$ be the semifield of all positive real numbers, where the multiplication and the addition are given by the ordinary ones for real numbers.

PROPOSITION 3.6 We have the following formulas:

$$\tilde{u}_i = u_i - \frac{1}{2}\delta_{ki}\log(1 + y_k^{\varepsilon}), \tag{3.22}$$

$$\tilde{p}_i = p_i - \frac{1}{2d_i}b_{ki}\log(1 + y_k^{\varepsilon}), \tag{3.23}$$

$$\tilde{w}_i = w_i - \frac{1}{2}b_{ki}\log(1 + y_k^{\varepsilon}), \tag{3.24}$$

$$d_i\tilde{p}_i = d_ip_i - \frac{1}{2}b_{ki}\log(1 + y_k^{\varepsilon}), \tag{3.25}$$

$$\tilde{x}_i = x_i(1 + y_k^{\varepsilon})^{-\delta_{ki}}, \tag{3.26}$$

$$\tilde{y}_i = y_i(1 + y_k^{\varepsilon})^{-b_{ki}}. \tag{3.27}$$

In particular, the transformation (3.27) coincides with the non-tropical part of the mutation of the y-variables in (2.11) with $\mathbb{P} = \mathbb{R}_+$.

Proof. This follows from Proposition 3.4. □

Therefore, the Hamiltonian $H^B_{k,\varepsilon}$ provides the infinitesimal generator of the non-tropical mutation of y-variables of seeds. This Hamiltonian viewpoint of mutations (without employing the canonical variables u_i and p_i) was first stated in [3, Section 1.3] for $\varepsilon = 1$.

3.3 *Small phase space and x-variables*

The transformation (3.26) of x-variables is comparable to the non-tropical part of the mutation of x-variables *without coefficients* from (2.17). However, in contrast to the y-variable case, they do *not* exactly match due to the discrepancy between y_k and $\hat{y}_k$ therein. To remedy this situation, we introduce a subspace of the phase space M,

$$M_0 = \{\varphi^{-1}(u,p) \in M \mid d_ip_i - w_i = 0,\ (i = 1,\dots,n)\}, \tag{3.28}$$

and we call it the *small phase space*.

PROPOSITION 3.7 The small phase space M_0 is preserved under the Hamiltonian flow by $H^B_{k,\varepsilon}$.

Proof. This follows from (3.15) and (3.16). □

Let us consider the variables $\hat{y}_i$ $(i = 1,\dots,n)$ in (2.4) *without coefficients*, namely,

$$\hat{y}_i := e^{2w_i} = e^{2\sum_{j=1}^n b_{ji}u_j} = \prod_{j=1}^n x_j^{b_{ji}}. \tag{3.29}$$

Since $d_i p_i = w_i$ on M_0, we have

$$y_i = \hat{y}_i \quad \text{on } M_0. \tag{3.30}$$

Thus, the transformation (3.26) assumes the desired form on M_0:

PROPOSITION 3.8

$$\tilde{x}_i = x_i(1 + \hat{y}_k^{\varepsilon})^{-\delta_{ki}} \quad \text{on } M_0. \tag{3.31}$$

In particular, the transformation (3.31) restricted to M_0 coincides with the non-tropical part of the mutation of the x-variables without coefficients in (2.17) under the specialization of x-variables in $\mathbb{R}_+$.

3.4 *Tropical transformation*

To complete the picture, we also give a realization of the tropical transformations (2.13) and (2.14) through a *change of coordinates* of the phase space M. For any $k \in \{1, \dots, n\}$ and a sign $\varepsilon = \pm$, we consider the following transformation:

$$\begin{array}{cccc} \tau^B_{k,\varepsilon}: & \mathbb{R}^{2n} & \to & \mathbb{R}^{2n} \\ & (u, p) & \mapsto & (u', p') \end{array} \tag{3.32}$$

$$u'_i = \begin{cases} -u_k + \sum_{j=1}^{n} [-\varepsilon b_{jk}]_+ u_j & i = k \\ u_i & i \neq k, \end{cases} \tag{3.33}$$

$$p'_i = \begin{cases} -p_k & i = k \\ p_i + [-\varepsilon b_{ik}]_+ p_k & i \neq k. \end{cases} \tag{3.34}$$

We call the transformation $\tau^B_{k,\varepsilon}$ a *tropical transformation*. Note that it is an ordinary linear transformation, not a *piecewise linear* transformation.

PROPOSITION 3.9 Let $\tau^B_{k,\varepsilon}(u, p) = (u', p')$.

(1) We have

$$\sum_{i=1}^{n} u'_i p'_i = \sum_{i=1}^{n} u_i p_i. \tag{3.35}$$

(2) The transformation $\tau^B_{k,\varepsilon}$ is canonical; namely, we have

$$\{p'_i, u'_j\} = \delta_{ij}, \quad \{u'_i, u'_j\} = \{p'_i, p'_j\} = 0. \tag{3.36}$$

Proof. We write the linear transformations (3.33) and (3.34) in the matrix form as $u' = Mu$ and $p' = Np$. Then, $N^T M = I$ holds. Both properties (1) and (2) follow from it. □

By Proposition 3.9 (2), one can introduce a new global Darboux chart $\varphi' : M \xrightarrow{\sim} \mathbb{R}^{2n}$ with canonical coordinates (u', p') by the following commutative diagram:

$$\begin{tikzcd} & M \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\ \mathbb{R}^{2n} \arrow[rr, "\tau^B_{k,\varepsilon}"] & & \mathbb{R}^{2n} \end{tikzcd} \tag{3.37}$$

Let $B' = B_k$. We employ a common skew-symmetrizer D for B and B', and we define primed variables w'_i, x'_i, y'_i for $(u', p') = \tau^B_{k,\varepsilon}(u, p)$,

$$w'_i = \prod_{j=1}^{n} b'_{ji} u'_j, \tag{3.38}$$

$$x'_i = e^{2u'_i}, \tag{3.39}$$

$$y'_i = e^{d_i p'_i + w'_i}. \tag{3.40}$$

PROPOSITION 3.10 We have the following formulas:

$$w'_i = \begin{cases} -w_k & i = k \\ w_i + [\varepsilon b_{ki}]_+ w_k & i \neq k, \end{cases} \tag{3.41}$$

$$d_i p'_i = \begin{cases} -d_k p_k & i = k \\ d_i p_i + [\varepsilon b_{ki}]_+ d_k p_k & i \neq k, \end{cases} \tag{3.42}$$

$$x'_i = \begin{cases} x_k^{-1} \left(\prod_{j=1}^{n} x_j^{[-\varepsilon b_{jk}]_+} \right) & i = k \\ x_i & i \neq k, \end{cases} \tag{3.43}$$

$$y'_i = \begin{cases} y_k^{-1} & i = k \\ y_i y_k^{[\varepsilon b_{ki}]_+} & i \neq k. \end{cases} \tag{3.44}$$

In particular, the transformations (3.43) and (3.44) coincide with the tropical part of the mutation of the x- and y-variables in (2.13) and (2.14), respectively.

Proof. To prove (3.41), we use (2.1) together with (3.33) and (3.34). □

PROPOSITION 3.11 In the new global Darboux chart $\varphi' : M \xrightarrow{\sim} \mathbb{R}^{2n}$, the small phase space M_0 is given by

$$M_0 = \{\varphi'^{-1}(u', p') \in M \mid d_i p'_i - w'_i = 0,\ (i = 1, \ldots, n)\}. \tag{3.45}$$

Proof. This follows from (3.41) and (3.42). □

3.5 *Signed mutations*

Let us introduce a composition of maps

$$\mu^B_{k,\varepsilon} = \tau^B_{k,\varepsilon} \circ \rho^B_{k,\varepsilon} : \begin{array}{ccc} \mathbb{R}^{2n} & \to & \mathbb{R}^{2n} \\ (u,p) & \mapsto & (u',p'). \end{array} \tag{3.46}$$

Summarizing Propositions 3.6, 3.8 and 3.10, we have the following conclusion.

THEOREM 3.12 Let $\mu^B_{k,\varepsilon}(u,p) = (u',p')$. Then, we have the following formulas:

$$u'_i = \begin{cases} -u_k + \displaystyle\sum_{j=1}^{n} [-\varepsilon b_{jk}]_+ u_j + \frac{1}{2}\log(1+y_k^{\varepsilon}) & i = k \\ u_i & i \neq k, \end{cases} \tag{3.47}$$

$$p'_i = \begin{cases} -p_k & i = k \\ p_i + [-\varepsilon b_{ik}]_+ p_k - \dfrac{1}{2d_i} b_{ki} \log(1+y_k^{\varepsilon}) & i \neq k, \end{cases} \tag{3.48}$$

$$w'_i = \begin{cases} -w_k & i = k \\ w_i + [\varepsilon b_{ki}]_+ w_k - \dfrac{1}{2} b_{ki} \log(1+y_k^{\varepsilon}), & i \neq k, \end{cases} \tag{3.49}$$

$$d_i p'_i = \begin{cases} -d_k p_k & i = k \\ d_i p_i + [\varepsilon b_{ki}]_+ d_k p_k - \dfrac{1}{2} b_{ki} \log(1+y_k^{\varepsilon}) & i \neq k, \end{cases} \tag{3.50}$$

$$x'_i = \begin{cases} x_k^{-1} \left(\displaystyle\prod_{j=1}^{n} x_j^{[-\varepsilon b_{jk}]_+} \right) (1+y_k^{\varepsilon}) & i = k \\ x_i & i \neq k, \end{cases} \tag{3.51}$$

$$y'_i = \begin{cases} y_k^{-1} & i = k \\ y_i y_k^{[\varepsilon b_{ki}]_+} (1+y_k^{\varepsilon})^{-b_{ki}} & i \neq k. \end{cases} \tag{3.52}$$

In particular, the transformation (3.52) coincides with the mutation of the y-variables in (2.3) with $\mathbb{P} = \mathbb{R}_+$, while the transformation (3.51) restricted to M_0 coincides with the mutation of the x-variables without coefficients in (2.16) under the specialization of x-variables in $\mathbb{R}_+$.

As a corollary of Theorem 3.12, x' on M_0 and y' therein do not depend on the choice of the sign ε. However, u' and p' *do* depend on ε. Thus, we call the map $\mu^B_{k,\varepsilon}$ in (3.46) the *signed mutation at k by B with sign ε*.

The inversion relation (2.8) of the unsigned mutation μ^B_k is replaced with the following one:

PROPOSITION 3.13 Define $\rho^B_{k,\varepsilon}$ and $\rho^{B_k}_{k,-\varepsilon}$ by a common skew-symmetrizer D of B and B_k. Then, the following inversion relation holds:

$$\mu^{B_k}_{k,-\varepsilon} \circ \mu^B_{k,\varepsilon} = \mathrm{id}. \tag{3.53}$$

Proof. One can directly verify it by (3.47) and (3.48). □

3.6 *Canonical quantization*

One can canonically quantize the Poisson brackets in (3.2) by replacing them with the canonical commutation relations,

$$[P_i, U_j] = \frac{\hbar}{\sqrt{-1}}\delta_{ij}, \quad [U_i, U_j] = [P_i, P_j] = 0. \tag{3.54}$$

Then, we have

$$[d_iP_i + W_i, d_jP_j + W_j] = \frac{2\hbar}{\sqrt{-1}}d_ib_{ij} = 2\hbar\sqrt{-1}d_jb_{ji}. \tag{3.55}$$

Let us set

$$q = e^{\hbar\sqrt{-1}}. \tag{3.56}$$

We recall a special case of the Baker–Campbell–Hausdorff formula. For any non-commutative variables A and B such that $[A, B] = C$ and $[C, A] = [C, B] = 0$, we have

$$e^Ae^B = e^{C/2}e^{A+B} \tag{3.57}$$

or

$$e^Ae^B = e^Ce^Be^A. \tag{3.58}$$

Applying it for (3.55), we have the commutation relation for $Y_i = e^{d_iP_i+W_i}$,

$$Y_iY_j = q^{2d_jb_{ji}}Y_jY_i. \tag{3.59}$$

This coincides with the quantization of y-variables due to Fock and Goncharov [3, 5].

REMARK 3.14 The realization of quantum y-variables by the canonical variables presented here appeared in [6, 14, 17]. In fact, the construction of x- and y-variables in (3.7) and (3.8) is deduced from the quantum ones in [14, 17].

3.7 *Quantization of x-variables through Dirac bracket*

Since x-variables are in involution for the Poisson bracket (3.11), the canonical quantization in (3.54) only provides the trivial quantization for them. However, by Proposition 3.8, they should be restricted to the small phase space M_0 to be identified with the x-variables in a seed. Therefore, we should apply Dirac's method [15] to obtain the Poisson structure on M_0.

Recall that the space M_0 is given by a family of constraints $\chi_i = 0$ $(i = 1, \dots, n)$, where

$$\chi_i = d_ip_i - w_i. \tag{3.60}$$

Following [15], let us consider the $n \times n$ matrix $A = (a_{ij})_{i,j=1}^n$ defined by

$$a_{ij} := \{\chi_i, \chi_j\} = -2d_i b_{ij}. \tag{3.61}$$

To proceed, we have to assume that the matrix $A = -2DB$ is invertible, or equivalently, B is invertible. Then, we have $A^{-1} = -(1/2)B^{-1}D^{-1}$.

LEMMA 3.15 The matrix B^{-1} is skew-symmetrizable with D being its skew-symmetrizer.

Proof. By assumption, we have $DBD^{-1} = -B^T$. Taking its inverse, we have $DB^{-1}D^{-1} = -(B^{-1})^T$. □

The *Dirac bracket* is defined by

$$\{f,g\}_D := \{f,g\} - \sum_{i,j=1}^n \{f,\chi_i\}(A^{-1})_{ij}\{\chi_j, g\}, \tag{3.62}$$

where $(A^{-1})_{ij}$ is the (i,j)-component of A^{-1}.

Here are some basic properties of the Dirac bracket.

(1) It defines a new Poisson bracket on M.

(2) For any constraint function χ_i and any function f on M,

$$\{f,\chi_i\}_D = 0 \tag{3.63}$$

holds. Thus, for any function g on M, $\{f, g\chi_i\}_D = \{f,g\}_D\chi_i$ vanishes on M_0. As a consequence, it defines a Poisson bracket on M_0.

(3) For any function f on M,

$$\{H^B_{k,\varepsilon}, f\}_D = \{H^B_{k,\varepsilon}, f\}, \tag{3.64}$$

since $\{H^B_{k,\varepsilon}, \chi_i\} = 0$ as stated in Proposition 3.7. Therefore, the equations of motion do not change.

It is convenient to set $B^{-1} = \Omega = (\omega_{ij})_{i,j=1}^n$.

PROPOSITION 3.16 We have the following formulas:

$$\{p_i, u_j\}_D = \frac{1}{2}\delta_{ij}, \quad \{u_i, u_j\}_D = -\frac{1}{2}d_i\omega_{ij}, \quad \{p_i, p_j\}_D = \frac{1}{2d_j}b_{ij}, \tag{3.65}$$

$$\{x_i, x_j\}_D = -2d_i\omega_{ij}x_ix_j, \quad \{\hat{y}_i, \hat{y}_j\}_D = 2d_ib_{ij}\hat{y}_i\hat{y}_j, \quad \{\hat{y}_i, x_j\}_D = 2d_i\delta_{ij}\hat{y}_ix_j, \tag{3.66}$$

where $\hat{y}_i$ is defined by (3.29).

Proof. The formulas in (3.65) are obtained by explicit calculations from the definition (3.62). Note that $\hat{y}_i = e^{2d_ip_i}$ on M_0. Then, we apply (3.5) to obtain (3.66). □

In particular, the Dirac brackets in (3.66) for x- and $\hat{y}$-variables coincide with the Poisson brackets in [4].

Now we 'canonically quantize' the Dirac brackets in (3.65) by replacing them with the commutation relations,

$$[P_i, U_j] = \frac{\hbar}{\sqrt{-1}}\frac{1}{2}\delta_{ij}, \quad [U_i, U_j] = \frac{\hbar}{\sqrt{-1}}\frac{-1}{2}d_i\omega_{ij}, \quad [P_i, P_j] = \frac{\hbar}{\sqrt{-1}}\frac{1}{2d_j}b_{ij}. \tag{3.67}$$

Then, in the same manner as in the previous subsection, we obtain the following commutation relation for $X_i = e^{2U_i}$,

$$X_iX_j = q^{2d_i\omega_{ij}}X_jX_i. \tag{3.68}$$

This coincides with the quantization of x-variables (without coefficients) due to Berenstein and Zelevinsky [16], where the skew-symmetric matrix Λ therein is related to Ω via $\Lambda^T = D\Omega$, and q therein is identified with q^{-2} here.

4. Lagrangian formalism and Rogers dilogarithm

4.1 *Legendre transformation*

Let us recall some basic facts on the Legendre transformation of a Hamiltonian. See, for example, [21, 22] for more information.

For simplicity, let us consider a Hamiltonian H on the space $\mathbb{R}^{2n}$ with the canonical coordinates (u, p). The space $\mathbb{R}^{2n}$ is naturally identified with the cotangent bundle $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$, where $\pi(u, p) = u$ and $p = (p_i)_{i=1}^n$ represents the 1-form $\sum_{i=1}^n p_i du_i$. Then, the Hamiltonian H induces the following fibre preserving map:

$$\begin{array}{rccc} F_H: & T^*\mathbb{R}^n & \to & T\mathbb{R}^n \\ & (u,p) & \mapsto & (u,\dot{u}). \end{array} \tag{4.1}$$

DEFINITION 4.1 We say that a Hamiltonian $H(u, p)$ is *regular* if the map F_H is a diffeomorphism.

The *Lagrangian* $\mathscr{L}$ for the Hamiltonian H is formally defined by the *Legendre transformation*,

$$\mathscr{L} = \sum_{i=1}^{n} \dot{u}_i p_i - H. \tag{4.2}$$

Here we use the symbol $\mathscr{L}$ so that it is not confused with the Rogers dilogarithm $L(x)$ or $\tilde{L}(x)$. From the definition $\mathscr{L}$ is a function of $(u, p) \in T^*\mathbb{R}^n$. Assume that the Hamiltonian H is regular. Then, by the inverse map of F_H, one can convert it to a function of $(u, \dot{u})$, which is the Lagrangian function $\mathscr{L}(u, \dot{u})$. The equations of motion of the Hamiltonian H are equivalent to the *Euler–Lagrange equations*

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{u}_i}\right) = \frac{\partial \mathscr{L}}{\partial u_i} \tag{4.3}$$

together with the identification of variables p_i,

$$p_i = \frac{\partial \mathscr{L}}{\partial \dot{u}_i}. \tag{4.4}$$

There is a parallel notion of regularity for a Lagrangian.

DEFINITION 4.2 We say that a Lagrangian $\mathscr{L}(u, \dot{u})$ is *regular* if the map

$$\begin{array}{cccc} F_{\mathscr{L}}: & T\mathbb{R}^n & \to & T^*\mathbb{R}^n \\ & (u,\dot{u}) & \mapsto & (u,p) \end{array} \tag{4.5}$$

is a diffeomorphism, where p_i is defined by (4.4).

It is known (e.g., [21, Section 3.6]) that from a regular Hamiltonian one obtains a regular Lagrangian by the Legendre transformation; conversely, from a regular Lagrangian one obtains a regular Hamiltonian by the Legendre transformation. In either case, the two systems are equivalent in the above sense.

4.2 *Lagrangian and Rogers dilogarithm*

Let us consider the Hamiltonian $H = H^B_{k,\varepsilon}$ from (3.12) in the canonical coordinates (u, p). By (3.13), we have

$$\dot{u}_i = -\frac{1}{2}\delta_{ki} \log(1 + y_k^{\varepsilon}). \tag{4.6}$$

Thus, the map F_H in (4.1) is far from surjective. Therefore, the Hamiltonian H is singular (i.e., not regular), unfortunately.

Nevertheless, let us write the Lagrangian in (4.2) explicitly, for now, as a function of (u, p),

$$\mathscr{L}^B_{k,\varepsilon}(u,p) = -\frac{1}{2}\log(1 + y_k^{\varepsilon})p_k - \frac{\varepsilon}{2d_k}\mathrm{Li}_2(-y_k^{\varepsilon}). \tag{4.7}$$

Inverting the relation (4.6) for $i = k$, we regard y_k as a function of $\dot{u}_k$. Then, we also regard p_k as a function of $\dot{u}_k$ and $u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_n$ by the relation

$$p_k = d_k^{-1}(\log y_k - w_k). \tag{4.8}$$

Thus, the function $\mathscr{L}^B_{k,\varepsilon}(u, p)$ is converted to a function of $(u, \dot{u})$ by

$$\mathscr{L}^B_{k,\varepsilon}(u,\dot{u}) = -\frac{1}{2d_k}\log(1 + y_k^{\varepsilon})(\log y_k - w_k) - \frac{\varepsilon}{2d_k}\mathrm{Li}_2(-y_k^{\varepsilon}), \tag{4.9}$$

despite the fact that the Hamiltonian is singular.

Of course, we have to pay some price. The Lagrangian $\mathscr{L}^B_{k,\varepsilon}(u, \dot{u})$ is *singular*, since it is independent of the variables $\dot{u}_i$ for $i \neq k$. Moreover, it is *not* equivalent to the original Hamiltonian system.

PROPOSITION 4.3

(1) For $i = k$, the equation (4.3), together with (4.4), yields

$$y_k(t) = e^{d_k p_k(t) + w_k(t)}, \quad \dot{p}_k(t) = 0. \tag{4.10}$$

(2) For $i \neq k$, the equation (4.3), together with (4.4), yields

$$p_i(t) = 0, \quad b_{ki} \log(1 + y_k(t)^\varepsilon) = 0. \tag{4.11}$$

Proof.

(1) For $i = k$,

$$\frac{\partial \mathscr{L}}{\partial \dot{u}_k} = \frac{1}{d_k}(\log y_k - w_k), \tag{4.12}$$

$$\frac{\partial \mathscr{L}}{\partial u_k} = 0. \tag{4.13}$$

(2) For $i \neq k$,

$$\frac{\partial \mathscr{L}}{\partial \dot{u}_i} = 0, \tag{4.14}$$

$$\frac{\partial \mathscr{L}}{\partial u_i} = \frac{b_{ik}}{2d_k} \log(1 + y_k^\varepsilon) = -\frac{b_{ki}}{2d_i} \log(1 + y_k^\varepsilon). \tag{4.15}$$

□

Therefore, the Euler–Lagrange equation for $i = k$ is a part of the equations of motion of the original Hamiltonian system in Proposition 3.4. On the other hand, the ones for $i \neq k$ set an unwanted restriction (4.11), and we have to avoid using them.

Putting this defect aside, let us evaluate $\mathscr{L}^B_{k,\varepsilon}$ on the small phase space M_0. Recall that we have $y_k = e^{2d_k p_k}$ on M_0. Thus, we have

$$p_k = \frac{1}{2d_k} \log y_k = \frac{\varepsilon}{2d_k} \log y_k^\varepsilon \quad \text{on } M_0. \tag{4.16}$$

Thus, $\mathscr{L}^B_{k,\varepsilon}$ depends only on y_k, or equivalently, on y_k^ε. So, let us write it is as a function of y_k^ε as $\mathscr{L}^B_{k,\varepsilon}(y_k^\varepsilon)$ for our convenience. Then, putting it in (4.7), we obtain

$$\mathscr{L}^B_{k,\varepsilon}(y_k^\varepsilon) = -\frac{\varepsilon}{4d_k} \log y_k^\varepsilon \log(1 + y_k^\varepsilon) - \frac{\varepsilon}{2d_k} \mathrm{Li}_2(-y_k^\varepsilon). \tag{4.17}$$

Now the Rogers dilogarithm emerges in our picture.

PROPOSITION 4.4 The function $\mathscr{L}^B_{k,\varepsilon}(y^\varepsilon_k)$ is given by the Rogers dilogarithm $\tilde{L}(x)$ in (2.26) as

$$\mathscr{L}^B_{k,\varepsilon}(y^\varepsilon_k) = \frac{\varepsilon}{2d_k}\tilde{L}(y^\varepsilon_k). \tag{4.18}$$

Proof. This follows from (2.27) and (4.17). □

Let us boldly phrase the above result as 'the Rogers dilogarithm is a Legendre transformation of the Euler dilogarithm.' This justifies the observation stated in Section 1.1.

5. Periodicity of canonical variables

5.1 *Universal and tropical semifields*

Let us recall two important classes of semifields, following [18].

DEFINITION 5.1 Let $y = (y_1, \dots, y_n)$ be an n-tuple of formal commutative variables.

(1) Define a semifield

$$\mathbb{Q}_+(y) = \left\{ \frac{p(y)}{q(y)} \in \mathbb{Q}(y) \mid p(y) \text{ and } q(y) \text{ are non-zero polynomials of } y \right.$$
$$\left. \text{with non-negative integer coefficients} \right\}, \tag{5.1}$$

where the multiplication and the addition $\oplus$ are given by the ordinary ones for the rational function field $\mathbb{Q}(y)$. We call it the *universal semifield of* y.

(2) Define a semifield

$$\mathrm{Trop}(y) = \left\{ \prod_{i=1}^n y_i^{a_i} \mid a_i \in \mathbb{Z} \right\}, \tag{5.2}$$

where the multiplication is given by the ordinary one for monomials of y, while the addition $\oplus$ is given by the *tropical sum*

$$\left(\prod_{i=1}^n y_i^{a_i}\right) \oplus \left(\prod_{i=1}^n y_i^{b_i}\right) = \prod_{i=1}^n y_i^{\min(a_i,b_i)}. \tag{5.3}$$

We call it the *tropical semifield of* y.

There is a semifield homomorphism

$$\begin{array}{rccc} \pi_{\mathrm{trop}} & : \mathbb{Q}_+(y) & \to & \mathrm{Trop}(y) \\ & y_i & \mapsto & y_i, \end{array} \tag{5.4}$$

which we call the *tropicalization map*.

5.2 *Periodicity of seeds and tropical periodicity*

Let $y = (y_1, \dots, y_n)$ be an n-tuple of formal commutative variables. We consider a sequence of seed mutations *with coefficients in the universal semifield* $\mathbb{Q}_+(y)$,

$$\begin{aligned}(B, x, y) = (B[0], x[0], y[0]) &\overset{\mu_{k_0}}{\mapsto} (B[1], x[1], y[1]) \overset{\mu_{k_1}}{\mapsto} \cdots \\ \cdots &\overset{\mu_{k_{T-1}}}{\mapsto} (B[T], x[T], y[T]).\end{aligned} \tag{5.5}$$

Note that the initial y-variables y are set to be the generators y of $\mathbb{Q}_+(y)$.

We may view the sequence (5.5) as a discrete dynamical system with a discrete time $s = 0, 1, \dots, T$. Let us introduce the notion of periodicity for this system.

Let $\mathcal{F}_0^n$ be the one defined in Section 2.1 with $\mathbb{P} = \mathbb{Q}_+(y)$. We define a (left) action of a permutation σ of $\{1, \dots, n\}$ on $\mathcal{F}_0^n \times \mathbb{Q}_+(y)^n$ by

$$\begin{array}{cccc}\sigma: & \mathcal{F}_0^n \times \mathbb{Q}_+(y)^n & \to & \mathcal{F}_0^n \times \mathbb{Q}_+(y)^n \\ & (x, z) & \mapsto & (x', z'),\end{array} \tag{5.6}$$

$$x'_i = x_{\sigma^{-1}(i)}, \tag{5.7}$$

$$z'_i = z_{\sigma^{-1}(i)}. \tag{5.8}$$

DEFINITION 5.2 Let σ be a permutation of $\{1, \dots, n\}$. We say that a sequence of mutations (5.5) is *σ-periodic* if the following conditions hold for any $1 \le i, j \le n$:

$$b_{\sigma^{-1}(i)\sigma^{-1}(j)}[T] = b_{ij}[0], \tag{5.9}$$

$$x_{\sigma^{-1}(i)}[T] = x_i[0], \tag{5.10}$$

$$y_{\sigma^{-1}(i)}[T] = y_i[0]. \tag{5.11}$$

Note that the conditions (5.10) and (5.11) are also expressed as the following equality on $\mathcal{F}_0^n \times \mathbb{Q}_+(y)^n$:

$$\sigma \circ \mu_{k_{T-1}}^{B[T-1]} \cdots \circ \mu_{k_1}^{B[1]} \circ \mu_{k_0}^{B[0]} = \mathrm{id}. \tag{5.12}$$

The following fact is known.

PROPOSITION 5.3 ([17, Proposition 4.3]) Let $D = \mathrm{diag}(d_1, \dots, d_n)$ be any common skew-symmetrizer of $B[s]$ $(s = 0, \dots, T-1)$. Suppose that the condition (5.9) holds. Then, the following equality holds:

$$d_{\sigma(i)} = d_i. \tag{5.13}$$

Let us consider the ‘tropicalization’ of the sequence (5.5). By applying the tropicalization map π_{trop} in (5.4) to each y-variable $y_i[s]$, $(s = 0, \dots, T)$ in the sequence (5.5), we obtain a monomial of initial y-variables y,

$$\pi_{\mathrm{trop}}(y_i[s]) = \prod_{j=1}^{n} y_j^{c_{ji}[s]}. \tag{5.14}$$

The integer vector $c_i[s] = (c_{ji}[s])_{j=1}^n$ is called the *c-vector* of $y_i[s]$.

The following fact is of fundamental importance in the theory of cluster algebras.

THEOREM 5.4 (Sign-coherence of c-vectors, [23, Theorem 1.7] with [18, Proposition 5.6] and [24, Corollary 5.5]) Each c-vector is a non-zero vector and its components are either all non-negative or all non-positive.

Based on this theorem, we define the following notion.

DEFINITION 5.5 The *tropical sign* $\varepsilon = \varepsilon(y_i[s])$ *of* $y_i[s]$ is given by $+$ (respectively, $-$) if the components of the c-vector of $y_i[s]$ are all non-negative (respectively, non-positive).

We introduce a sequence of signs, $\varepsilon_0, \dots, \varepsilon_{T-1}$, where

$$\varepsilon_s = \varepsilon(y_{k_s}[s]), \quad (s = 0, \dots, T-1), \tag{5.15}$$

and k_s is the one in (5.5). We call it the *tropical sign sequence* of (5.5). Accordingly, we have the following sequence of transformations associated with the sequence (5.5):

$$\mathcal{F}_0^n \times \mathbb{Q}_+(y)^n \overset{\tau^{B[0]}_{k_0,\varepsilon_0}}{\rightarrow} \mathcal{F}_0^n \times \mathbb{Q}_+(y)^n \overset{\tau^{B[1]}_{k_1,\varepsilon_1}}{\rightarrow} \cdots \overset{\tau^{B[T-1]}_{k_{T-1},\varepsilon_{T-1}}}{\rightarrow} \mathcal{F}_0^n \times \mathbb{Q}_+(y)^n, \tag{5.16}$$

where $\tau^B_{k,\varepsilon}$ is the one in (2.12).

DEFINITION 5.6 Let σ be a permutation of $\{1, \dots, n\}$. We say that a sequence of transformations (5.16) is σ*-periodic* if the following equality holds on $\mathcal{F}_0^n \times \mathbb{Q}_+(y)^n$:

$$\sigma \circ \tau^{B[T-1]}_{k_{T-1},\varepsilon_{T-1}} \cdots \circ \tau^{B[1]}_{k_1,\varepsilon_1} \circ \tau^{B[0]}_{k_0,\varepsilon_0} = \mathrm{id}. \tag{5.17}$$

Note that we do not assume the condition (5.9) here.

Each mutation in (5.5) is a rational transformation, while each transformation in (5.16) is (the exponential form of) a linear transformation, which is much simpler. Surprisingly, the two periodicities in Definitions 5.2 and 5.6 are equivalent. The if-part of the following statement is very non-trivial, and our proof is based on the recent result by [25].

PROPOSITION 5.7 The sequence of mutations (5.5) is σ-periodic if and only if the sequence of transformations (5.16) is σ-periodic.

Proof. First, we note that, by [26, Proposition 1.3] and Theorem 5.4, the sequence of transformations (5.16) is the exponential form of the transformations of the corresponding c-vectors $c_i[s] = (c_{ji}[s])_{j=1}^n$ and g-vectors $g_i[s] = (g_{ji}[s])_{j=1}^n$ along the sequence (5.5), where g-vectors are defined in [18, Section 6]. Thus, the σ-periodicity of the sequence (5.16) is equivalent to the σ-periodicity of c- and g-vectors, i.e.,

$$c_{ij}[T] = c_{i\sigma(j)}[0] = \delta_{i\sigma(j)}, \quad g_{ij}[T] = g_{i\sigma(j)}[0] = \delta_{i\sigma(j)}. \tag{5.18}$$

(Only-if-part.) Assume that the sequence (5.5) is σ-periodic. The σ-periodicity of c-vectors directly follows from the σ-periodicity of y-variables (5.11) by applying the tropicalization map in (5.4). Then, the σ-periodicity of g-vectors follows from the duality of c- and g-vectors in [26, Equation (3.11)].

(If-part.) Assume that the sequence (5.16) is σ-periodic. The σ-periodicity of c-vectors implies the σ-periodicity of exchange matrices $B[s]$ thanks to [26, Equation (2.9)]. Furthermore, let $F_i[s]$ be the F-polynomials along the sequence (5.5), which are defined in [18, Section 3]. Then, by [25, Theorem 2.5], the σ-periodicity of c-vectors implies the σ-periodicity of F-polynomials, i.e.,

$$F_{\sigma^{-1}(i)}[T] = F_i[0]. \tag{5.19}$$

Then, the σ-periodicity of x-variables (respectively, y-variables) follows form the formula in [18, Corollary 6.3] (respectively, [18, Proposition 3.13]). □

REMARK 5.8 Note that the above proof of the only-if-part uses only the assumption (5.11).

5.3 *Periodicity of canonical variables and tropical periodicity*

Let us consider the counterparts of the two periodicities in Definitions 5.2 and 5.6 for canonical variables.

Let us define a (left) action of a permutation σ of $\{1,\dots,n\}$ on $\mathbb{R}^{2n}$,

$$\begin{array}{cccc} \sigma: & \mathbb{R}^{2n} & \rightarrow & \mathbb{R}^{2n} \\ & (u,p) & \mapsto & (u',p'), \end{array} \tag{5.20}$$

$$u'_i = u_{\sigma^{-1}(i)}, \tag{5.21}$$

$$p'_i = p_{\sigma^{-1}(i)}. \tag{5.22}$$

Since the map σ is a canonical transformation, we may regard it as a change of canonical coordinates on the phase space M.

Let $\varepsilon_0,\dots,\varepsilon_{T-1}$ continue to be the tropical sign sequence of (5.5). Let $(u[0],p[0])$ be an arbitrary point in $\mathbb{R}^{2n}$. In parallel to the sequence of mutations (5.5), we consider a sequence of signed mutations on $\mathbb{R}^{2n}$,

$$(u[0],p[0]) \overset{\mu^{B[0]}_{k_0,\varepsilon_0}}{\mapsto} (u[1],p[1]) \overset{\mu^{B[1]}_{k_1,\varepsilon_1}}{\mapsto} \cdots \overset{\mu^{B[T-1]}_{k_{T-1},\varepsilon_{T-1}}}{\mapsto} (u[T],p[T]), \tag{5.23}$$

where

$$\mu^{B[s]}_{k_s,\varepsilon_s} = \tau^{B[s]}_{k_s,\varepsilon_s} \circ \rho^{B[s]}_{k_s,\varepsilon_s}, \quad (s = 0,\dots,T-1) \tag{5.24}$$

and $\rho^{B[s]}_{k_s,\varepsilon_s}$ are defined by (3.46) under the following assumption:

ASSUMPTION 5.9 We employ a common skew-symmetrizer D of $B[s]$'s to define $\rho^{B[s]}_{k_s,\varepsilon_s}$'s.

DEFINITION 5.10 Let σ be a permutation of $\{1,\dots,n\}$. We say that a sequence of signed mutations (5.23) is *σ-periodic* if the following conditions hold for any initial point $(u[0],p[0]) \in \mathbb{R}^{2n}$ and for any $1 \le i \le n$:

$$b_{\sigma^{-1}(i)\sigma^{-1}(j)}[T] = b_{ij}[0], \tag{5.25}$$

$$u_{\sigma^{-1}(i)}[T] = u_i[0], \tag{5.26}$$

$$p_{\sigma^{-1}(i)}[T] = p_i[0]. \tag{5.27}$$

The conditions (5.26) and (5.27) are also expressed as the following equality on $\mathbb{R}^{2n}$:

$$\sigma \circ \mu^{B[T-1]}_{k_{T-1},\varepsilon_{T-1}} \cdots \circ \mu^{B[1]}_{k_1,\varepsilon_1} \circ \mu^{B[0]}_{k_0,\varepsilon_0} = \mathrm{id}. \tag{5.28}$$

PROPOSITION 5.11 The sequence of mutations (5.5) is σ-periodic if and only if the sequence of signed mutations (5.23) is σ-periodic.

The proof is a little lengthy, and it will be given in Section 5.4.

EXAMPLE 5.12 The inversion relation (2.5) is the simplest example of a σ-periodic sequence of mutations (5.5) with $T = 2$, $k_1 = k_2 = k$, $\sigma = \mathrm{id}$, and the tropical sign sequence is $\varepsilon_1 = +$, $\varepsilon_2 = -$. The corresponding σ-periodic sequence of signed mutations is the inversion relation (3.53) with $\varepsilon = +$.

Next, let us consider the counterpart of the sequence (5.16) for canonical variables. For the sequence (5.23), we introduce the following sequence of transformations,

$$\mathbb{R}^{2n} \overset{\tau^{B[0]}_{k_0,\varepsilon_0}}{\longrightarrow} \mathbb{R}^{2n} \overset{\tau^{B[1]}_{k_1,\varepsilon_1}}{\longrightarrow} \cdots \overset{\tau^{B[T-1]}_{k_{T-1},\varepsilon_{T-1}}}{\longrightarrow} \mathbb{R}^{2n}, \tag{5.29}$$

where $\tau^B_{k,\varepsilon}$ is the one in (3.32).

DEFINITION 5.13 Let σ be a permutation of $\{1,\dots,n\}$. We say that a sequence of transformations (5.29) is *σ-periodic* if the following equality holds on $\mathbb{R}^{2n}$:

$$\sigma \circ \tau^{B[T-1]}_{k_{T-1},\varepsilon_{T-1}} \cdots \circ \tau^{B[1]}_{k_1,\varepsilon_1} \circ \tau^{B[0]}_{k_0,\varepsilon_0} = \mathrm{id}. \tag{5.30}$$

PROPOSITION 5.14 The sequence of transformations (5.16) is σ-periodic if and only if the sequence of transformations (5.29) is σ-periodic.

Proof. (Only-if-part.) Assume that the sequence (5.16) is σ-periodic. Since (3.33) is a log-version of (2.13), the σ-periodicity of u-variables follows from the σ-periodicity of g-vectors in Proposition 5.7. Similarly, since (3.42) is a log-version of (2.14), the σ-periodicity of variables $(d_i p_i)_{i=1}^n$ follows from the σ-periodicity of c-vectors in Proposition 5.7. Then, the σ-periodicity of p-variables follows from this using Proposition 5.3.

(If-part.) One can easily convert the above argument. □

Combining Propositions 5.7, 5.11 and 5.14, we reach the following conclusion.

THEOREM 5.15 The following four conditions are equivalent to each other:

(a). The sequence of mutations (5.5) is σ-periodic.

(b). The sequence of transformations (5.16) is σ-periodic.

(c). The sequence of signed mutations (5.23) is σ-periodic.

(d). The sequence of transformations (5.29) is σ-periodic.

Proof. We have (a) $\Longleftrightarrow$ (b) by Proposition 5.7, (a) $\Longleftrightarrow$ (c) by Proposition 5.11, and (b) $\Longleftrightarrow$ (d) by Proposition 5.14. ☐

5.4 *Proof of Proposition 5.11*

5.4.1 *If-part* Let us prove the if-part of the proposition, which is easier. Suppose that the sequence of signed mutations (5.23) is σ-periodic. By the conditions (5.25)–(5.27) and Proposition 5.3, the y-variables defined by (3.8) satisfy the desired σ-periodicity (5.11) in $\mathbb{R}_+$. Furthermore, the initial y-variables $y_1, \ldots, y_n$ are algebraically independent in $\mathbb{R}_+$ for a generic choice of the initial point $(u[0], p[0])$. Therefore, the σ-periodicity (5.11) holds in $\mathbb{Q}_+(y)$.

To prove the periodicity of x-variables, we make use of Proposition 5.7. As noted in Remark 5.8, from the σ-periodicity (5.11) of the y-variables for the sequence (5.5), the σ-periodicity of the sequence of transformations (5.16) holds. Then, by the if-part of Proposition 5.7, the σ-periodicity of the sequence of mutations (5.5) holds.

In the rest of this subsection, we prove the only-if-part of the proposition.

5.4.2 *Hamiltonian point of view* In our proof, keeping the Hamiltonian point of view in mind is very useful. To make the presentation simple, we consider the case $T = 2$ in (5.5). Although this is a toy example, it fully contains the idea of the proof for the general case. To lighten the notation, let us abbreviate the flow in the left-hand side of the sequence (5.28) as

$$(u,p) \overset{\mu}{\mapsto} (u',p') \overset{\mu'}{\mapsto} (u'',p''), \overset{\sigma}{\mapsto} (u''',p'''). \tag{5.31}$$

Using the decomposition (5.24) with a similar abbreviation, we write it in the following way:

$$\begin{tikzcd}
 & (\tilde{u}',\tilde{p}') \arrow[r, mapsto, "\tau'"] & (u'',p'') \arrow[r, mapsto, "\sigma"] & (u''',p''') \ . \\
(\tilde{u},\tilde{p}) \arrow[r, mapsto, "\tau"] & (u',p') \arrow[u, mapsto, "\rho'"] & & \\
(u,p) \arrow[u, mapsto, "\rho"] & & &
\end{tikzcd} \tag{5.32}$$

From the Hamiltonian point of view, the vertical maps ρ and ρ' are Hamiltonian flows from $t = 0$ to 1 and from $t = 1$ to 2 in the phase space M, respectively, while the horizontal maps τ, τ', σ are changes of canonical coordinates of M so that points do not move in M. Let us gather the piecewise Hamiltonian flow from $t = 0$ to 2 in the initial chart. This can be done by the pullback along the horizontal arrows as follows:

$$\begin{tikzcd}
(\tilde{\tilde{u}}, \tilde{\tilde{p}}) \arrow[r, mapsto, "\tau"] & (\tilde{u}', \tilde{p}') \arrow[r, mapsto, "\tau'"] & (u'', p'') \arrow[r, mapsto, "\sigma"] & (u''', p''') \ . \\
(\tilde{u}, \tilde{p}) \arrow[r, mapsto, "\tau"] \arrow[u, mapsto, dashed] & (u', p') \arrow[u, mapsto, "\rho'"] & & \\
(u, p) \arrow[u, mapsto, "\rho"] & & &
\end{tikzcd} \tag{5.33}$$

The σ-periodicity (5.28), which we are going to show, states that the points (u, p) and (u''', p''') coincide, but this coincidence happens in different charts. On the other hand, the tropical periodicity of (5.30) guaranteed by Proposition 5.14 means that

$$\sigma \circ \tau' \circ \tau = \mathrm{id}. \tag{5.34}$$

Thus, we have

$$(\tilde{\tilde{u}}, \tilde{\tilde{p}}) = (u''', p'''). \tag{5.35}$$

Therefore, the σ-periodicity (5.28) is equivalent to the equality in the initial chart,

$$(\tilde{\tilde{u}}, \tilde{\tilde{p}}) = (u, p). \tag{5.36}$$

In other words, *the flow is periodic in the phase space M*. We will show this separately for u- and p-variables.

5.4.3 *Periodicity of p-variables* We start with p-variables. Consider

$$\begin{aligned}
\Delta p := \tilde{\tilde{p}} - p &= (\tilde{\tilde{p}} - \tilde{p}) + (\tilde{p} - p) \\
&= \tau^{-1}(\tilde{p}' - p') + (\tilde{p} - p).
\end{aligned} \tag{5.37}$$

By (3.23), we have

$$\tilde{p}_i - p_i = -\frac{1}{2d_i} b_{ki} \log(1 + y_k^{\varepsilon}), \tag{5.38}$$

$$\tilde{p}'_i - p'_i = -\frac{1}{2d_i} b'_{k'i} \log(1 + {y'_{k'}}^{\varepsilon'}), \tag{5.39}$$

where we keep the same system of abbreviation. Recall that y-variables here are defined by

$$y_i = e^{d_i p_i + w_i}, \quad y'_i = e^{d_i p'_i + w'_i}, \tag{5.40}$$

and they obey the mutation rule (3.52). In particular, it is uniquely determined by the initial y-variables y_i.

Our goal is to show that $\Delta p = 0$. For this purpose, we compare the above flow of p-variables with the (logarithm of) y-variables in the sequence of mutations (5.5). In the same spirit of (5.33), we write a diagram for the sequence (5.5),

$$\begin{tikzcd}
(\tilde{\tilde{x}}, \tilde{\tilde{y}}) \arrow[r, mapsto, "\tau"] & (\tilde{x}', \tilde{y}') \arrow[r, mapsto, "\tau'"] & (x'', y'') \arrow[r, mapsto, "\sigma"] & (x''', y''') \ . \\
(\tilde{x}, \tilde{y}) \arrow[u, mapsto, dashed] \arrow[r, mapsto, "\tau"] & (x', y') \arrow[u, mapsto, "\rho'"] & & \\
(x, y) \arrow[u, mapsto, "\rho"] & & &
\end{tikzcd} \tag{5.41}$$

Let us set $v_i = \log y_i / d_i$, $v'_i = \log y'_i / d_i$, and so on. Here, $\log y_i$ ($y_i \in \mathbb{Q}_+(y)$) is a formal notation such that the multiplication in $\mathbb{Q}_+(y)$ is written additively. In this notation, the linear aspect of the transformation $\tau^B_{k,\varepsilon}$ in (2.12) is more transparent. Note that we have $v'''_i = v''_{\sigma^{-1}(i)}$ thanks to Proposition 5.3. Then, we have

$$\begin{aligned} \Delta v := \tilde{\tilde{v}} - v &= (\tilde{\tilde{v}} - \tilde{v}) + (\tilde{v} - v) \\ &= \tau^{-1}(\tilde{v}' - v') + (\tilde{v} - v). \end{aligned} \tag{5.42}$$

By (2.11), we have

$$\tilde{v}_i - v_i = -\frac{1}{d_i} b_{ki} \log(1 + y_k^{\varepsilon}), \tag{5.43}$$

$$\tilde{v}'_i - v'_i = -\frac{1}{d_i} b'_{k'i} \log(1 + {y'_{k'}}^{\varepsilon'}). \tag{5.44}$$

Let us compare them with (5.38) and (5.39). Note that the y-variables here are elements in $\mathbb{Q}_+(y)$, and they are different from the ones for (5.38) and (5.39). However, they mutate by the rule (2.3), which is the same rule as for the y-variables in (5.41). Moreover, the initial y-variables y_i in (5.5) are the generators of $\mathbb{Q}_+(y)$, which are formal (algebraically independent) variables. Therefore, one can specialize them arbitrarily in $\mathbb{R}_+$, so that they exactly match the initial y-variables for (5.41). Under this specialization, we have

$$\Delta p = \frac{1}{2} \Delta v, \tag{5.45}$$

where we also used the fact that τ is a linear transformation. On the other hand, by the periodicity assumption we have $\Delta v = 0$. Therefore, $\Delta p = 0$ holds.

5.4.4 *Periodicity of u-variables* Due to the relation (3.8), the periodicities of y- and p-variables imply the same periodicity of w-variables. Therefore, when the matrix $B[0]$ is *invertible*, the periodicity of u-variables immediately follows. This reasoning, however, is not applicable when the matrix $B[0]$ is not invertible. Therefore, we prove the claim directly by comparing the mutations of u- and x-variables in a similar way to the previous case. The proof is parallel, but it requires some extra argument.

Consider

$$\begin{aligned}\Delta u := \tilde{\tilde{u}} - u &= (\tilde{\tilde{u}} - \tilde{u}) + (\tilde{u} - u)\\ &= \tau^{-1}(\tilde{u}' - u') + (\tilde{u} - u),\end{aligned} \tag{5.46}$$

where, by (3.22),

$$\tilde{u}_i - u_i = -\frac{1}{2}\delta_{ki}\log(1 + y_k^{\varepsilon}), \tag{5.47}$$

$$\tilde{u}'_i - u'_i = -\frac{1}{2}\delta_{k'i}\log(1 + {y'_{k'}}^{\varepsilon'}) \tag{5.48}$$

for the same y_i and y'_i in (5.40).

Let us compare the flow of u-variables with the (logarithm of) x-variables in the sequence of mutations (5.41). Again, let us introduce formal logarithms, $z_i = \log x_i$, $z'_i = \log x'_i$, etc., to write the multiplication in $\mathcal{F}_{\mathbb{Q}_+(y)}$ in the additive way. Then, we have

$$\begin{aligned}\Delta z := \tilde{\tilde{z}} - z &= (\tilde{\tilde{z}} - \tilde{z}) + (\tilde{z} - z)\\ &= \tau^{-1}(\tilde{z}' - z') + (\tilde{z} - z),\end{aligned} \tag{5.49}$$

where, by (2.10),

$$\tilde{z}_i - z_i = -\delta_{ki}\log(1 + \hat{y}_k^{\varepsilon}) + \delta_{ki}\log(1 \oplus y_k^{\varepsilon}), \tag{5.50}$$

$$\tilde{z}'_i - z'_i = -\delta_{k'i}\log(1 + {\hat{y}'_{k'}}^{\varepsilon'}) + \delta_{k'i}\log(1 \oplus {y'_{k'}}^{\varepsilon'}). \tag{5.51}$$

It follows that we have the following expression of $\Delta z_i := \tilde{\tilde{z}}_i - z_i$,

$$\Delta z_i = -\log f_i(\hat{y}) + \log f_i(y), \tag{5.52}$$

where $f_i(y) \in \mathbb{Q}_+(y)$ is a rational function of the initial y-variables y, and $f_i(\hat{y}) \in \mathcal{F}_{\mathbb{Q}_+(y)}$ is the one obtained from $f_i(y)$ by replacing y with the initial $\hat{y}$-variables $\hat{y}$ and also replacing the addition in $\mathbb{Q}_+(y)$ with the one in $\mathcal{F}_{\mathbb{Q}_+(y)}$. In the same way as (5.45), we have

$$\Delta u_i := \tilde{\tilde{u}}_i - u_i = -\frac{1}{2}\log f_i(y) \tag{5.53}$$

under some specialization of y in the right-hand side.

Now we set $\Delta z_i = 0$ for all i by the assumption of the periodicity of x-variables. This is equivalent to the equality $f_i(y) = f_i(\hat{y})$ as elements of $\mathcal{F}_{\mathbb{Q}_+(y)}$. We first claim that $f_i(y)$ is a *Laurent monomial in y*, possibly with some coefficients in $\mathbb{Q}_+$. In fact, if $f_i(y)$ is not a Laurent monomial, then it includes the addition in $\mathbb{Q}_+(y)$. It follows that $f_i(\hat{y})$ includes the addition in $\mathcal{F}_{\mathbb{Q}_+(y)}$. However, the addition in $\mathcal{F}_{\mathbb{Q}_+(y)}$ is an operation outside of $\mathbb{Q}_+(y)$. Thus, $f_i(\hat{y})$ is not an element in $\mathbb{Q}_+(y) \subset \mathcal{F}_{\mathbb{Q}_+(y)}$. In particular, the equality $f_i(y) = f_i(\hat{y})$ could never occur. We next claim that actually we have

$$f_i(y) = 1. \tag{5.54}$$

To see it, we consider the limit $y_i \to 0$ for all i. Then, thanks to the definition of the tropical sign, we have $y_k^{\varepsilon}, {y'_{k'}}^{\varepsilon'} \to 0$. Thus, by (5.49)–(5.52), we have $\log f_i(y) \to 0$. Therefore, $f_i(y)$=1. Thus, we conclude that $\Delta u_i = 0$ by (5.53).

This completes the proof of Proposition 5.11.

6. Dilogarithm identities and action integral

6.1 *Dilogarithm identities*

The following theorem was proved in [13] by a cluster algebraic method with the help of the *constancy condition* from [12]. See also [17].

THEOREM 6.1 (Dilogarithm identity [13, Theorems 6.4 and 6.8]) Suppose that the sequence of mutations (5.5) is σ-periodic. Let $\varepsilon_0, \dots, \varepsilon_{T-1}$ be the tropical sign sequence of (5.5). Let $D = \mathrm{diag}(d_1, \dots, d_n)$ be any skew-symmetrizer of the initial matrix B in (5.5). Then, the following identity of the Rogers dilogarithm $\tilde{L}(x)$ in (2.26) holds:

$$\sum_{s=0}^{T-1} \frac{\varepsilon_s}{d_{k_s}} \tilde{L}(y_{k_s}^{\varepsilon_s}[s]) = 0, \tag{6.1}$$

where $y_{k_s}[s]$ are evaluated by any semifield homomorphism

$$\mathrm{ev}_y : \mathbb{Q}_+(y) \to \mathbb{R}_+. \tag{6.2}$$

Below, we give an alternative proof of the theorem, based on the Hamiltonian/Lagrangian picture presented in this article.

6.2 *Action integral*

We consider the sequence of signed mutations (5.23).

For each time span $[s, s+1]$ ($s = 0, \dots, T-1$), we have the Hamiltonian

$$H[s] = \frac{\varepsilon_s}{2d_k} \mathrm{Li}_2(-y_{k_s}[s]^{\varepsilon_s}) \quad \text{for } [s, s+1] \tag{6.3}$$

in the sth canonical coordinates $(u[s], p[s])$. A Hamiltonian flow from time 0 to T is a piecewise linear movement. A schematic diagram of a Hamiltonian flow is depicted in Fig. 1 for $T = 2$. Let

$$\mathscr{L}[s] = \dot{u}_{k_s}[s] p_{k_s}[s] - H[s] \quad \text{for } [s, s+1] \tag{6.4}$$

be the corresponding singular Lagrangian from (4.7).

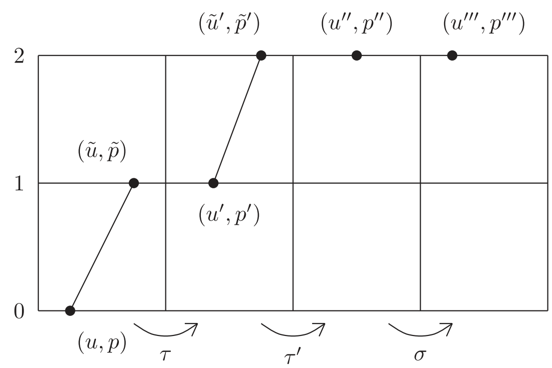

FIG. 1. Schematic diagram of a Hamiltonian flow for $T = 2$, where for simplicity we use the same abbreviation as in (5.32).

Let us consider the action integral S along a Hamiltonian flow,

$$S = \sum_{s=0}^{T-1} S[s], \tag{6.5}$$

$$S[s] = \int_{s}^{s+1} \mathscr{L}[s](u(t), \dot{u}(t))dt, \tag{6.6}$$

where $u(t)$ in (6.6) is (the u-part of) a Hamiltonian flow in the sth coordinates $(u[s], p[s])$.

Here is the first key observation.

PROPOSITION 6.2 The value of the Lagrangian (6.4) along a Hamiltonian flow is constant in t in each time span $[s, s+1]$. Thus, we have

$$S = \sum_{s=0}^{T-1} \mathscr{L}[s]. \tag{6.7}$$

Proof. Since the Hamiltonian is constant along the flow, it is enough to show that the term $\dot{u}_{k_s}[s]p_{k_s}[s]$ is constant. This is true since we have $\ddot{u}_{k_s}[s] = \dot{p}_{k_s}[s] = 0$ by Proposition 3.4. □

6.3 *Invariance of action integral*

Our next key observation is as follows.

THEOREM 6.3 Suppose that the sequence of signed mutations (5.23) is σ-periodic. Then, for any Hamiltonian flow,

$$S = 0. \tag{6.8}$$

The rest of this subsection is devoted to a proof of this theorem.

First we show that the value $S = S(u)$ is independent of a Hamiltonian flow $u(t)$, using the standard variational calculus for a Lagrangian. However, we have to be careful because the Lagrangian here is singular as discussed in Section 4.2.

For a given Hamiltonian flow $u(t)$, we consider an infinitesimal variation $u(t) + \delta u(t)$, *which is also assumed to be a Hamiltonian flow*. Let

$$\delta S[s] := S[s](u + \delta u) - S[s](u). \tag{6.9}$$

be the variation of the action integral $S[s]$ in the time span $[s, s+1]$. We show that it is given by the boundary values of the time span as follows.

LEMMA 6.4

$$\delta S[s] = \sum_{i=1}^{n} \tilde{p}_i[s]\delta\tilde{u}_i[s] - \sum_{i=1}^{n} p_i[s]\delta u_i[s], \tag{6.10}$$

where $(u[s], p[s])$ and $(\tilde{u}[s], \tilde{p}[s])$ are the points of the flow at $t = s$ and $s + 1$, respectively, in the sth coordinates $(u[s], p[s])$.

Proof. We fix the discrete time s. For simplicity, we suppress the index s everywhere, in particular we write k instead of k_s. Recall that, by Proposition 3.4, for $i \neq k$,

$$\dot{u}_i = 0. \tag{6.11}$$

Thus, we have, for $i \neq k$,

$$\delta\dot{u}_i = 0. \tag{6.12}$$

Therefore, it is natural to separate the variation $\delta S = \delta S[s]$ into two parts:

$$\delta S = \delta S_1 + \delta S_2, \tag{6.13}$$

$$\delta S_1 = \int_s^{s+1} \left(\frac{\partial \mathscr{L}}{\partial u_k}\delta u_k + \frac{\partial \mathscr{L}}{\partial \dot{u}_k}\delta\dot{u}_k \right) dt, \tag{6.14}$$

$$\delta S_2 = \sum_{\substack{i=1\\ i\neq k}}^{n} \int_s^{s+1} \left(\frac{\partial \mathscr{L}}{\partial u_i}\delta u_i \right) dt. \tag{6.15}$$

Recall that, by Proposition 4.3 (1), the Euler–Lagrange equation for $i = k$ follows from the equations of motion. By using it and also (4.4), we have

$$\delta S_1 = \int_s^{s+1} \left(\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{u}_k} \right)\delta u_k + \frac{\partial \mathscr{L}}{\partial \dot{u}_k}\delta\dot{u}_k \right) dt = \left[\frac{\partial \mathscr{L}}{\partial \dot{u}_k}\delta u_k \right]_s^{s+1} = \left[p_k\delta u_k \right]_s^{s+1}. \tag{6.16}$$

On the other hand, for $i \neq k$, by (4.15) we have an explicit expression for $\partial \mathscr{L}/\partial u_i$ and (6.11) shows δu_i is independent of t. Combining with (3.25) gives

$$\delta S_2 = \sum_{\substack{i=1\\ i\neq k}}^{n} (-1)\frac{b_{ki}}{2d_i}\log(1+y_k^{\varepsilon})\delta u_i = \sum_{\substack{i=1\\ i\neq k}}^{n} (\tilde{p}_i - p_i)\delta u_i. \tag{6.17}$$

(Note that in (6.17) we did not use the Euler–Lagrange equations for $i \neq k$, which are not valid here as noted after Proposition 4.3.) Recall from (3.21) that $(\tilde{u}[s], \tilde{p}[s])$ is obtained from $(u[s], p[s])$ by the time one flow of the Hamiltonian $H[s]$. Thus gathering (6.16) with (6.17) and noting that $\delta u_i = \delta \tilde{u}_i$ for $i \neq k$, we obtain (6.10). □

Let σ be the one in (5.20), and let us introduce

$$(u[T+1], p[T+1]) := \sigma(u[T], p[T]). \tag{6.18}$$

Lemma 6.5 We have the equalities

$$\sum_{i=1}^{n} p_i[s+1]\delta u_i[s+1] = \sum_{i=1}^{n} \tilde{p}_i[s]\delta \tilde{u}_i[s], \quad (s = 0, \dots, T-1), \tag{6.19}$$

$$\sum_{i=1}^{n} p_i[T+1]\delta u_i[T+1] = \sum_{i=1}^{n} p_i[T]\delta u_i[T]. \tag{6.20}$$

Proof. The first equality is due to Proposition 3.9 (1). The second equality is clear from the definition of σ. □

Consider the total variation

$$\delta S = \sum_{s=0}^{T-1} \delta S[s]. \tag{6.21}$$

Combining Lemmas 6.4 and 6.5, we see that it is given by the boundary values,

$$\delta S = \sum_{i=1}^{n} p_i[T+1]\delta u_i[T+1] - \sum_{i=1}^{n} p_i[0]\delta u_i[0]. \tag{6.22}$$

On the other hand, by the assumption of the σ-periodicity of the sequence of signed mutations (5.23), we have

$$p_i[T+1] = p_i[0], \quad \delta u_i[T+1] = \delta u_i[0]. \tag{6.23}$$

Therefore, we conclude that

$$\delta S = 0. \tag{6.24}$$

Since this holds for any infinitesimal variation of any flow $u(t)$, S is constant.

It remains to determine the constant value of S. We evaluate it in the limit $p_i[0] \to -\infty$ for all i. Then, all initial y-variables $y_i[0] = \exp(d_i p_i[0] + w_i[0])$ go to 0. Accordingly, all $y_{k_s}[s]^{\varepsilon_s}$ $(s = 0, \dots, T-1)$ also go to 0 due to the definition of the tropical sign ε_s. So, by (4.9), the Lagrangians $\mathscr{L}[s]$ go to 0 as well. Thus, we have $S \to 0$. Therefore, $S = 0$.

This completes the proof of Theorem 6.3.

REMARK 6.6 The meaning of Theorem 6.3 and its proof becomes more transparent if we compare them with *Noether's theorem* (e.g., [22, Theorem 1.3]).

For simplicity, let us consider the variation of a general regular Lagrangian $\mathscr{L}$ under an infinitesimal transformation $u_i \mapsto u_i + \epsilon a_i$ for some i, where ϵ is an infinitesimal and a_i is a function of u. Then, by the Euler–Lagrange equation for i, we have

$$\delta \mathscr{L} = \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{u}_i}\right)\epsilon a_i + \frac{\partial \mathscr{L}}{\partial \dot{u}_i}\epsilon \dot{a}_i = \epsilon \frac{d}{dt}\left(p_i a_i\right). \tag{6.25}$$

Thus, $\delta\mathscr{L} = 0$, i.e., the invariance of the Lagrangian in the order of ϵ, implies that the generator $X = p_i a_i$ is an integral of motion; that is Noether's theorem. Moreover, we see in (6.25) that the converse is also true, namely, if $X = p_i a_i$ is an integral of motion, then $\delta\mathscr{L} = 0$.

Next we consider a finite time analogue of the above. Namely, we consider a variation of the action integral S under an infinitesimal transformation of Hamiltonian flows $u_i(t) \mapsto u_i(t) + \epsilon a_i(t)$ for some i, where ϵ is an infinitesimal and $a_i(t)$ depends on $u(t)$. Then, again by the Euler–Lagrange equation, we have

$$\delta S = \int_{t_0}^{t_1} \left(\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{u}_i}\right)\epsilon a_i + \frac{\partial \mathscr{L}}{\partial \dot{u}_i}\epsilon \dot{a}_i\right) dt = \epsilon \left[p_i a_i\right]_{t_0}^{t_1}. \tag{6.26}$$

Thus, $\delta S = 0$, i.e., the invariance of the action integral in the order of ϵ, implies that the generator $X = p_i a_i$ is periodic at t_0 and t_1, and *vice versa*.

6.4 *Main results*

By combining Propositions 4.4 and 6.2 with Theorem 6.3, we have the following theorem.

THEOREM 6.7 Suppose that the sequence of signed mutations (5.23) is σ-periodic. Then, for the Hamiltonian flow of the Hamiltonian in (6.3) with any initial point in the phase space M, we have

$$\sum_{s=0}^{T-1} \mathscr{L}[s] = 0. \tag{6.27}$$

In particular, if the initial point $(u[0], p[0])$ in (5.23) satisfies the condition

$$d_i p_i[0] = w_i[0], \quad (i = 1, \dots, n), \tag{6.28}$$

we have the following identity of the Rogers dilogarithm $\tilde{L}(x)$ in (2.26):

$$\sum_{s=0}^{T-1} \frac{\varepsilon_s}{d_{k_s}} \tilde{L}(\hat{y}_{k_s}^{\varepsilon_s}[s]) = 0, \tag{6.29}$$

where

$$\hat{y}_i[s] = e^{2w_i[s]}. \tag{6.30}$$

By combining Proposition 5.11 and Theorem 6.7, we obtain a slightly different version of Theorem 6.1.

THEOREM 6.8 Suppose that the sequence of mutations (5.5) is σ-periodic. Let $\varepsilon_0, \dots, \varepsilon_{T-1}$ be the tropical sign sequence of (5.5). Let $D = \mathrm{diag}(d_1, \dots, d_n)$ be any skew-symmetrizer of the initial matrix B in (5.5). We set the y-variables in (5.5) to be trivial by the specialization $\mathbb{Q}_+(y) \to \mathbf{1}$. Let $\mathbb{Q}_+(x) \subset \mathcal{F}$ be a semifield generated by the initial x-variables in (5.5). Then, the following identity for the Rogers dilogarithm $\tilde{L}(x)$ in (2.26) holds:

$$\sum_{s=0}^{T-1} \frac{\varepsilon_s}{d_{k_s}} \tilde{L}(\hat{y}_{k_s}^{\varepsilon_s}[s]) = 0, \tag{6.31}$$

where

$$\hat{y}_i[s] = \prod_{j=1}^{n} x_j[s]^{b_{ji}[s]} \tag{6.32}$$

are evaluated by any semifield homomorphism

$$\mathrm{ev}_x : \mathbb{Q}_+(x) \to \mathbb{R}_+. \tag{6.33}$$

The only difference between Theorems 6.1 and 6.8 is the ranges of the initial y- and $\hat{y}$-variables therein under the evaluations (6.2) and (6.33). Namely, each of the initial y-variables in Theorem 6.1 independently takes *any* value in $\mathbb{R}_+$, since they are independent variables. Meanwhile, each of the initial $\hat{y}$-variables in Theorem 6.8, defined by (6.32) with $s = 0$, does so if and only if the initial matrix $B = B[0]$ is *invertible*. Therefore, Theorem 6.8 is apparently weaker than Theorem 6.1.

Nevertheless, we can show the following fact, which completes our derivation of Theorem 6.1.

THEOREM 6.9 Theorem 6.8 implies Theorem 6.1. Therefore, Theorems 6.1 and 6.8 are equivalent.

To show Theorem 6.9, we use the following notion.

DEFINITION 6.10 Let B and $\tilde{B}$ be skew-symmetrizable (integer) matrices of size n and m ($n < m$), respectively. We call $\tilde{B}$ an *extension of* B if B is a principal submatrix of $\tilde{B}$. If an extension of B is invertible, then we call it an *invertible extension of B*

For any skew-symmetrizable matrix B, there is an invertible extension $\tilde{B}$ of B. For example, if D is a skew-symmetrizer of B, then we have the following invertible extension of B:

$$\tilde{B} = \left(\begin{array}{c|c} B & -I \\ \hline D & 0 \end{array} \right). \tag{6.34}$$

The following general fact on cluster algebras is key to proving Theorem 6.9.

THEOREM 6.11 (Extension Theorem (cf. [13, Theorem 4.3])) Let B and $\tilde{B}$ be any skew-symmetrizable matrices of size n and m, respectively, such that $\tilde{B}$ is an extension of B. Assume, for simplicity, that B is the principal submatrix of $\tilde{B}$ for the first n indices $1, \dots, n$ of $\tilde{B}$. Then, if the sequence (5.5) with the initial matrix $B = B[0]$ is σ-periodic, the sequence (5.5) with the initial matrix being replaced with $\tilde{B}$ is also σ-periodic. Here, a permutation σ of $\{1, \dots, n\}$ is naturally identified with a permutation of $\{1, \dots, m\}$ such that $\sigma(i) = i$ for $n + 1 \le i \le m$.

REMARK 6.12 The above theorem is shown in [13, Theorem 4.3] when B is skew-symmetric and $\sigma = \mathrm{id}$.

Proof. Since the proof is parallel to the one in [13, Theorem 4.3]), we only give a sketch of a proof. We first show the σ-periodicity of c-vectors. Let $c_i[s] = (c_{ji}[s])_{j=1}^{m}$ be the c-vectors for the sth seed in the sequence (5.5) with the initial matrix $\tilde{B}$. Then, repeating the argument in the proof of [13, Theorem 4.3]), one can show the σ-periodicity,

$$c_{ij}[T] = c_{i\sigma(j)}[0] = \delta_{i\sigma(j)}, \tag{6.35}$$

where we use the sign-coherence property in Theorem 5.4, the duality of c- and g-vectors in [26, Equation (3.11)], and Proposition 5.3. Then, by the proof of the if-part of the proof of Proposition 5.7, the σ-periodicity of seeds is recovered from the σ-periodicity of c-vectors. □

Proof of Theorem 6.9. As already mentioned, when the initial matrix $B = B[0]$ is invertible, Theorem 6.8 implies Theorem 6.1. Suppose that B is not invertible. Then, replace the initial matrix B with any invertible extension $\tilde{B}$ of B. Thanks to Theorem 6.11, the sequence (5.5) enjoys the same σ-periodicity. A crucial observation is that the functional identity (6.31) remains the same even if the initial matrix B is replaced with $\tilde{B}$. This is because, for B and its extension $\tilde{B}$, the matrix mutation in (2.1), and also the exchange relation of y-variables in (2.3), exactly coincide if all indices therein are restricted to the ones for B. On the other hand, since $\tilde{B}$ is invertible, each of the initial y-variables in the sequence (6.31) now independently takes any value in $\mathbb{R}_+$ under the specialization (6.33). This is the desired result. □

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