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One-level density for holomorphic cusp forms of arbitrary level

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Abstract

In 2000 Iwaniec, Luo, and Sarnak proved for certain families of L-functions associated to holomorphic newforms of square-free level that, under the Generalized Riemann Hypothesis, as the conductors tend to infinity the one-level density of their zeros matches the one-level density of eigenvalues of large random matrices from certain classical compact groups in the appropriate scaling limit. We remove the square-free restriction by obtaining a trace formula for arbitrary level by using a basis developed by Blomer and Milićević, which is of use for other problems as well.

Keywords: Low lying zeroes, One level density, Cuspidal newforms, Petersson formula

Mathematics Subject Classification: 11M26 (primary), 11M41, 15A52 (secondary)

1 Introduction

Montgomery [1] conjectured that the pair correlation of critical zeros up to height T of the Riemann zeta function $\zeta(s)$ coincides with the pair correlation of eigenvalues of random unitary matrices of dimension N in the appropriate limit as $T, N \to \infty$. This remarkable connection initiated a new branch of number theory concerned with relating the statistics of zeros of $\zeta(s)$, and of L-functions more generally, to those of eigenvalues of random matrices. While additional support for this agreement was obtained by the work of Hejhal [2] on the triple correlation of $\zeta(s)$, Rudnick and Sarnak [3] on the n-level correlation for cuspidal automorphic forms, and Odlyzko [4,5] on the spacings between adjacent zeros of $\zeta(s)$, the story cannot end here as these statistics are insensitive to the behavior of any finite set of zeros. As the zeros at and near the central point play an important role in a variety of problems, this led Katz and Sarnak [6,7] to develop a new statistic which captures this behavior.

Definition 1.1 Let $L(s,f)$ be an L-function with zeros in the critical strip $\rho_f = 1/2 + i\gamma_f$ (note $\gamma_f \in \mathbb{R}$ if and only if the Generalized Riemann Hypothesis holds for f), and let ϕ be an even Schwartz function whose Fourier transform has compact support. The **one-level density** is

$$D_1(f;\phi) := \sum_{\rho_f} \phi\left(\frac{\gamma_f}{2\pi}\log c_f\right), \tag{1.1}$$

where c_f is the analytic conductor.

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Their Density Conjecture [6,7] states that the scaling limits of eigenvalues of classical compact groups near 1 correctly model the behavior of these zeros a family $\mathcal{F}$ of L-functions as the conductors tend to infinity. Specifically, let $\mathcal{F}_N$ be a sub-family of $\mathcal{F}$ with suitably restricted conductors; often one takes all forms of conductor N, or conductor at most N, or conductor in the range $[N, 2N]$. If the symmetry group is $\mathcal{G}$, then we expect

$$\begin{aligned}\mathcal{D}_1(\mathcal{F};\phi) &:= \lim_{N\to\infty}\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_1(f;\phi) = \int_{-\infty}^{\infty}\phi(x)W_1(\mathcal{G})(x)dx \\ &= \int_{-\infty}^{\infty}\widehat{\phi}(t)\widehat{W_1}(\mathcal{G})(t)dt, \end{aligned} \tag{1.2}$$

where $K(y) = \frac{\sin \pi y}{\pi y}$, $K_\epsilon(x,y) = K(x-y) + \epsilon K(x+y)$ for $\epsilon = 0, \pm 1$, and

$$\begin{aligned} W_1(\mathrm{SO(even)})(x) &= K_1(x,x), \\ W_1(\mathrm{SO(odd)})(x) &= K_{-1}(x,x) + \delta_0(x), \\ W_1(\mathrm{O})(x) &= \frac{1}{2}W_1(\mathrm{SO(even)})(x) + \frac{1}{2}W_1(\mathrm{SO(odd)})(x), \\ W_1(\mathrm{U})(x) &= K_0(x,x), \\ W_1(\mathrm{Sp})(x) &= K_{-1}(x,x). \end{aligned} \tag{1.3}$$

While the Fourier transforms of the densities of the orthogonal groups all equal $\delta_0(y) + 1/2$ in $(-1, 1)$, they are mutually distinguishable for larger support (and are distinguishable from the unitary and symplectic cases for any support). There is now an enormous body of work showing the 1-level densities of many families (such as Dirichlet L-functions, elliptic curves, cuspidal newforms, Maass forms, number field L-functions, and symmetric powers of GL_2 automorphic representations) agree with the scaling limits of a random matrix ensemble; see [6–29] for some examples, and [10,27,30] for discussions on how to determine the underlying symmetry. For additional readings on connections between random matrix theory, nuclear physics and number theory see [31–39].

We concentrate on extending the results of Iwaniec, Luo, and Sarnak in [18]. One of their key results is a formula for unweighted sums of Fourier coefficients of holomorphic newforms of a given weight and level. This formula writes the unweighted sums in terms of weighted sums to which one can apply the Petersson trace formula; it is instrumental in performing any averaging over holomorphic newforms, since one can interchange summation and replace the average of Fourier coefficients with Kloosterman sums and Bessel functions, which are amenable to analysis.

A drawback of their formula is that it may only be applied to averages of newforms of square-free level. One reason is that the development of such a formula depends essentially on the construction of an explicit orthonormal basis for the space of cusp forms of a given weight and level, which they only computed in the case of square-free level. In 2011, Rouymi [40] complemented the square-free calculations of Iwaniec, Luo, and Sarnak, finding an orthonormal basis for the space of cusp forms of prime power level, and applying this explicit basis towards the development of a similar sum of Fourier coefficients over all newforms with level equal to a fixed prime power.

In 2015, Blomer and Milićević [41] extended the results of Iwaniec, Luo, and Sarnak and Rouymi by writing down an explicit orthonormal basis for the space of cusp forms (holomorphic or Maass) of a fixed weight and, novelly, arbitrary level.

The purpose of this article is, first, to leverage the basis of Blomer and Milićević to prove an exact formula for sums of Fourier coefficients of holomorphic newforms over all newforms of a given weight and level, where now the level is permitted to be arbitrary (see below, as well as Proposition 5.2 for a detailed expansion). The basis of Blomer and Milićević requires one to split over the square-free and square-full parts of the level; this splitting combined with the loss of several simplifying assumptions for Hecke eigenvalues and arithmetic functions makes the case where the level is not square-free much more complex. As an application, we use this formula to show the 1-level density agrees only with orthogonal symmetry.

1.1 Harmonic averaging

Throughout we assume that $k, N \geq 1$ with k even. By $H_k^\star(N)$ we always mean a basis of arithmetically normalized Hecke eigenforms in the space orthogonal to oldforms. Explicitly, it is a basis of holomorphic cusp forms of weight k and level N which are new of level N in the sense of Atkin and Lehner [42] and whose elements are eigenvalues of the Hecke operators T_n with $(n, N) = 1$ and normalized so that the first Fourier coefficient is 1. We let $\lambda_f(n)$ denote the nth Fourier coefficient of an $f \in H_k^\star(N)$ (see the next section for more details).

For any holomorphic cuspidal newform f, we introduce the renormalized Fourier coefficients

$$\Psi_f(n) := \left(\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\right)^{1/2} ||f||_N^{-1} \lambda_f(n), \tag{1.4}$$

where $\|f\|_N^2 = \langle f, f\rangle_N$ and $\langle \cdot, \cdot, \rangle_N$ denotes the Petersson inner product (see, for instance, [43, (14.11)]). We then define

$$\Delta_{k,N}(m, n) := \sum_{g \in \mathscr{B}_k(N)} \overline{\Psi_g(m)} \Psi_g(n), \tag{1.5}$$

where $\mathscr{B}_k(N)$ is an orthonormal basis for the space of cusp forms of weight k and level N. The importance of $\Delta_{k,N}(m, n)$ is clarified by the introduction of the Petersson formula in the next section.

Using the orthonormal basis $\mathscr{B}_k(N)$ of Milićević and Blomer, we then prove the following (unconditional) formula.

Theorem 1.2 *Suppose that* $(n, N) = 1$. *Then*

$$\sum_{f \in H_k^\star(N)} \lambda_f(n) = \frac{k-1}{12} \sum_{LM=N} \mu(L) M \prod_{p^2 | M} \left(\frac{p^2}{p^2-1}\right)^{-1} \sum_{(m,M)=1} m^{-1} \Delta_{k,M}(m^2, n). \tag{1.6}$$

A key part of the proof is a result on weighted sums of products of the Fourier coefficients, which we extract in Lemma 3.1. Note that in many cases, the right-hand side of (1.6) is preferable to the left-hand side, as it is amenable to application of spectral summation formulas such as the Petersson formula (Proposition 2.1) and can be studied via Kloosterman sums, see Proposition 5.2. More generally, this sort of formula has a variety of applications involving the Fourier coefficients of holomorphic cusp forms and L-functions. Rouymi uses his basis and formula to study the non-vanishing at the central point of L-functions attached to primitive cusp forms; we elect to apply our formula to generalize [18, Theorem 1.1] on the one-level density of families of holomorphic new-

form L-functions by removing the condition that N must pass to infinity through the square-free integers.

1.2 The Density Conjecture

Before stating our results, we introduce the L-function $L(s,f)$ associated to a $f \in H_k^\star(N)$ as the Dirichlet series

$$L(s,f) = \sum_1^\infty \lambda_f(n) n^{-s}. \tag{1.7}$$

See Section 3 of [18] for the Euler product, analytic continuation, and functional equation of $L(s,f)$; $L(s,f)$ may be analytically continued to an entire function on $\mathbf{C}$ with a functional equation relating s to $1-s$. We also need similar results for its symmetric square (see [43,44]):

$$L(s, \mathrm{sym}^2 f) = L(s, f \otimes f,)L(s, \chi)^{-1}, \tag{1.8}$$

where χ is the nebentypus of f (we will only consider the case of trivial nebentypus below).

We now assume the Generalized Riemann Hypothesis for $L(s,f)$, and, for technical reasons, $L(s, \mathrm{sym}^2 f)$ as well as for all Dirichlet L-functions (see Remark 1.5). Then we may write all nontrivial zeros of $L(s,f)$ as

$$\varrho_f = \frac{1}{2} + i\gamma_f. \tag{1.9}$$

For any $f \in H_k^\star(N)$, we denote by c_f its analytic conductor; for our family

$$c_f = k^2 N. \tag{1.10}$$

Towards the definition of the one-level density for our families, we start with (1.1), the one-level density for a fixed form f; the ordinates γ_f are counted with their corresponding multiplicities, and $\phi(x)$ is an even function of Schwartz class such that its Fourier transform

$$\widehat{\phi}(y) = \int_{-\infty}^\infty \phi(x) e^{-2\pi i x y} dx \tag{1.11}$$

has compact support so that $\phi(x)$ extends to an entire function.

Our family $\mathscr{F}(N)$ is $H_k^\star(N)$, where the level N is our asymptotic parameter (and $\mathscr{F} = \cup_{N\geq 1}\mathscr{F}(N)$). It is worth mentioning that $\lim_{N\to\infty} |H_k^\star(N)| = \infty$. The one-level density is the expectation of $D_1(f;\phi)$ averaged over our family:

$$D_1(H_k^\star(N);\phi) := \frac{1}{|H_k^\star(N)|} \sum_{f\in H_k^\star(N)} D_1(f;\phi). \tag{1.12}$$

Iwaniec, Luo, and Sarnak [18] prove the Density Conjecture with the support of $\widehat{\phi}$ in $(-2,2)$ and as N runs over square-free numbers. We prove the following theorem with no conditions on how N tends to infinity; new features emerge from the presence of square factors dividing the level.

Theorem 1.3 *Fix any $\phi \in \mathscr{S}(\mathbf{R})$ with $\operatorname{supp}\widehat{\phi} \subset (-2,2)$. Then, assuming the Generalized Riemann Hypothesis for $L(s,f)$ and $L(s,\mathrm{sym}^2 f)$ for $f \in H_k^\star(N)$ and for all Dirichlet L-functions,*

$$\lim_{N\to\infty} \frac{1}{|H_k^\star(N)|} \sum_{f\in H_k^\star(N)} D_1(f;\phi) = \int_{-\infty}^\infty \phi(x) W_1(\mathrm{O})(x)\, dx, \tag{1.13}$$

where $W_1(\mathrm{O})(x) = 1 + \frac{1}{2}\delta_0(x)$*; thus the 1-level density for the family* $H_k^\star(N)$ *agrees only with orthogonal symmetry.*

More generally, under the same assumptions the Density Conjecture holds for the family $H_k^\star(N)$ *for any test function* $\phi(x)$ *whose Fourier transform is supported inside* $(-u, u)$ *with* $u < 2\log(kN)/\log(k^2N)$.

Remark 1.4 While [18] are also able to split the family by the sign of the functional equation, we are unable to do so. The reason is that for square-free level N the sign of the functional equation, ϵ_f, is given by

$$\epsilon_f = i^k \mu(N)\lambda_f(N)N^{1/2} \tag{1.14}$$

(see equation (3.5) of [18]). By multiplying by $\frac{1}{2}(1 \pm \epsilon_f)$ we can restrict to just the even ($\epsilon_f = 1$) or odd ($\epsilon_f = -1$) forms, at the cost of having an additional $\lambda_f(N)$ factor in the Petersson formula. This leads to involved calculations of Bessel-Kloosterman terms, but these sums can be evaluated well enough to obtain support in $(-2, 2)$. Unfortunately there is no analogue of their equation (3.5) for general level.

Remark 1.5 We briefly comment on the use of the various Generalized Riemann Hypotheses. First, assuming GRH for $L(s, f)$ yields a nice spectral interpretation of the 1-level density, as the zeros now lie on a line and it makes sense to order them; note, however, that this statistic is well-defined even if GRH fails. Second, GRH for $L(s, \mathrm{sym}^2 f)$ is used to bound certain sums which arise as lower order terms; in [18] (page 80 and especially page 88) the authors remark how this may be replaced by additional applications of the Petersson formula (assuming GRH allows us to trivially estimate contributions from each form, but a bound on average suffices). Finally, GRH for Dirichlet L-functions is needed when we follow [18] and expand the Kloosterman sums in the Petersson formula with Dirichlet characters; if we do not assume GRH here we are still able to prove the 1-level density agrees with orthogonal symmetry, but in a more restricted range.

The structure of the paper is as follows. Our main goal is to prove the formula for sums of Hecke eigenvalues and then use this to compute the one-level density. We begin in §2 with a short introduction of the theory of primitive holomorphic cusp forms, as well as the Petersson trace formula and the basis of Blomer and Milićević. In §3 we find a formula for $\Delta_{k,N}(m, n)$, which we leverage in §4 to find a formula for the arithmetically weighted sums, $\Delta_{k,N}^\star(n)$ (see [18, (2.53)]); this is Theorem 1.2. Using our formula, we find bounds for $\Delta_{k,N}^\star(n)$ in §5, culminating in the computation of the one-level density in §6 (Theorem 1.3).

2 Preliminaries

In this section we introduce some notation and results to be used throughout, much of which can be found in [43].

2.1 Hecke eigenvalues and the Petersson inner product

Our setup is classical. Throughout k, N are positive integers, with k even. Let $S_k(N)$ be the linear space spanned by cusp forms of weight k and trivial nebentypus which are Hecke eigenforms for the congruence group $\Gamma_0(N)$. Each $f \in S_k(N)$ admits a Fourier

development

$$f(z) = \sum_{n\geq 1} a_f(n)e(nz), \tag{2.1}$$

where $e(z) := e^{2\pi iz}$ and the $a_f(n)$ are in general complex numbers, though as we only consider forms with trivial nebentypus, our Fourier coefficients are real.

It is well known that $S_k(N)$ is a finite-dimensional Hilbert space with respect to the Petersson inner product

$$\langle f, g\rangle_N = \int_{\Gamma_0(N)\backslash\mathfrak{H}} f(z)\overline{g(z)}y^{k-2}dx\,dy, \tag{2.2}$$

where $\mathfrak{H}$ denotes the upper-half plane $\mathfrak{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$. Given a form on $\Gamma_0(M)$, it is possible to induce a form on $\Gamma_0(N)$ for $M \mid N$. We call such forms for which $M < N$ "old forms"; the basis of ones orthogonal to the space spanned by the forms with $M < N$ which are eigenvalues of the Hecke operators are called the "new forms" or "primitive forms". We write the inner product with a subscript N to indicate we are considering f and g as forms on $\Gamma_0(N)$, when perhaps $\langle f, g\rangle_M$ might make sense as well.

Atkin and Lehner [42] showed that the space $S_k(N)$ has the following canonical orthogonal decomposition in terms of the newforms described in §1.1: let $H_k^\star(M)$ be a basis of arithmetically normalized Hecke eigenformsforms for the space of newforms of weight k and level M (typically we choose M to be a divisor of N). Then

$$S_k(N) = \bigoplus_{LM=N} \bigoplus_{f\in H_k^\star(M)} S_k(L;f), \tag{2.3}$$

where $S_k(L;f)$ is the linear space spanned by the forms

$$f_{|\ell}(z) = \ell^{\frac{k}{2}} f(\ell z) \quad \text{with } \ell \mid L. \tag{2.4}$$

Though the forms $f_{|\ell}(z)$ are linearly independent, they are not orthogonal.

If $f \in H_k^\star(M)$ then f is an eigenfunction of all Hecke operators $T_M(n)$, where

$$(T_M(n)f)(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n\\(a,M)=1}} \left(\frac{a}{d}\right)^{k/2} \sum_{b \pmod d} f\left(\frac{az+b}{d}\right). \tag{2.5}$$

For a fixed $f \in H_k^\star(M)$, let $\lambda_f(n)$ denote the eigenvalue of $T_M(n)$; i.e.,

$$T_M(n)f = \lambda_f(n)f \tag{2.6}$$

for all $n \geq 1$. The Hecke eigenvalues are multiplicative; more precisely, they satisfy the following identity for any $m, n \geq 1$:

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d\mid(m,n)\\(d,M)=1}} \lambda_f(mn/d^2). \tag{2.7}$$

We normalize so that

$$a_f(1) = 1. \tag{2.8}$$

Then $a_f(n)$ and $\lambda_f(n)$ are related by

$$a_f(n) = \lambda_f(n)n^{(k-1)/2}. \tag{2.9}$$

Deligne showed that the Weil conjectures imply the Ramanujan-Petersson conjecture for holomorphic cusp forms, and then proved them. As a consequence, for $f \in H_k^\star(N)$ we have the bound

$$\left|\lambda_f(n)\right| \leq \tau(n), \tag{2.10}$$

where $\tau(n)$ is the divisor function, and if $f \in H_k^\star(M)$ and $p \mid M$, then

$$\lambda_f(p)^2 = \begin{cases} \frac{1}{p} & \text{if } p \mid\mid M \\ 0 & \text{if } p^2 \mid M, \end{cases} \tag{2.11}$$

see, for instance, [40, (2.8)].

We recall the definition (1.4) of the normalized Fourier coefficients $\Psi_f(n)$ attached to any cusp form f:

$$\Psi_f(n) = \left(\frac{\Gamma(k-1)}{(4\pi n)^{k-1}}\right)^{1/2} ||f||_N^{-1} a_f(n) \ll_f \tau(n). \tag{2.12}$$

Let $\mathscr{B}_k(N)$ be an orthogonal basis of $S_k(N)$. Then

$$\left|\mathscr{B}_k(N)\right| = \dim S_k(N) \asymp \nu(N)k, \tag{2.13}$$

where

$$\nu(N) := [\Gamma_0(1) : \Gamma_0(N)] = N \prod_{p \mid N} (1 + \tfrac{1}{p}). \tag{2.14}$$

From the Atkin-Lehner decomposition, we also deduce

$$\dim S_k(N) = \sum_{LM=N} \tau(L) \left|H_k^\star(M)\right|. \tag{2.15}$$

Recall Definition (1.5) of $\Delta_{k,N}(m,n)$:

$$\Delta_{k,N}(m,n) := \sum_{g \in \mathscr{B}_k(N)} \overline{\Psi_g(m)} \Psi_g(n). \tag{2.16}$$

The importance of $\Delta_{k,N}(m,n)$ is established by the Petersson trace formula.

Proposition 2.1 (Petersson [45]) *For any $m, n \geq 1$ we have*

$$\Delta_{k,N}(m,n) = \delta(m,n) + 2\pi i^k \sum_{c \equiv 0 \pmod N} c^{-1} S(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right). \tag{2.17}$$

Though the quantity $\Delta_{k,N}(m,n)$ is independent of the choice of an orthonormal basis, we would like to compute with the Petersson trace formula using an explicit basis $\mathscr{B}_k(N)$ to average over newforms. However, as remarked, the spaces $S_k(L;f)$ do not have a distinguished orthogonal basis. Therefore, to produce a basis $\mathscr{B}_k(N)$, we need a basis for the spaces $S_k(L;f)$. Iwaniec, Luo, and Sarnak [18] write down an explicit basis when N is square-free. As we will see in the next section, Blomer and Milićević [41] have recently obtained a basis for arbitrary level N. Our first key idea, a kind of trace formula for sums of Hecke eigenvalues over newforms in the case N is arbitrary, is an explicit computation with this new basis. Our second key idea on the one-level density of the L-functions $L(s,f)$ for $f \in H_k^\star(N)$ uses our first key idea in an essential way to reduce the problem to the one already treated by Iwaniec, Luo, and Sarnak.

To $H_k^\star(M)$ we often associate $\chi_{0;M}$, the trivial character mod M:

$$\chi_{0;M}(n) = \begin{cases} 1 & \text{if } (n,M) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

2.2 An orthonormal basis for $S_k(N)$

For $f \in H_k^\star(M)$ consider the following arithmetic functions, which coincide with the ones defined in [41] up to a few corrections [46]:

$$r_f(c) := \sum_{b|c} \frac{\mu(b)\lambda_f(b)^2}{b\sigma_{-1}^{\text{twisted}}(b)^2}, \quad \alpha(c) := \sum_{b|c} \frac{\chi_{0;M}(b)\mu(b)}{b^2}, \quad \beta(c) := \sum_{b|c} \frac{\chi_{0;M}(b)\mu^2(b)}{b}, \tag{2.19}$$

where $\mu_f(c)$ is the multiplicative function given implicitly by

$$L(f,s)^{-1} = \sum_c \frac{\mu_f(c)}{c^s}, \tag{2.20}$$

or explicitly on prime powers by

$$\mu_f(p^j) = \begin{cases} -\lambda_f(p) & j = 1, \\ \chi_{0;M}(p) & j = 2, \\ 0 & j > 2, \end{cases} \tag{2.21}$$

and

$$\sigma_{-1}^{\text{twisted}}(b) = \sum_{r|b} \frac{\chi_{0;M}(r)}{r}. \tag{2.22}$$

For $\ell \mid d$ define

$$\xi_d'(\ell) := \frac{\mu(d/\ell)\lambda_f(d/\ell)}{r_f(d)^{1/2}(d/\ell)^{1/2}\beta(d/\ell)}, \quad \xi_d''(\ell) := \frac{\mu_f(d/\ell)}{(d/\ell)^{1/2}(r_f(d)\alpha(d))^{1/2}}. \tag{2.23}$$

Write $d = d_1 d_2$ where d_1 is square-free, d_2 is square-full, and $(d_1, d_2) = 1$. Thus $p \,||\, d$ implies $p \mid d_1$ and $p^2 \mid d$ implies $p^2 \mid d_2$. Then for $\ell \mid d$ define

$$\xi_d(\ell) := \xi_{d_1}'((d_1, \ell))\xi_{d_2}''((d_2, \ell)). \tag{2.24}$$

We record the following identities; while these are not needed for the arguments in this paper, they were of use in an earlier draft in investigating related problems, and thus may be of use to others. For prime powers, $\xi_d(\ell)$ simplifies as

$$\begin{aligned} &\xi_1(1) = 1, && \xi_{p^\nu}(p^\nu) = \left(r_f(p)\left(1 - \chi_{0;M}(p)/p^2\right)\right)^{-1/2}, \\ &\xi_p(p) = r_f(p)^{-1/2}, && \xi_{p^\nu}(p^{\nu-1}) = \tfrac{-\lambda_f(p)}{\sqrt{p}}\xi_{p^\nu}(p^\nu), \\ &\xi_p(1) = \tfrac{-\lambda_f(p)}{\sqrt{p}(1+\chi_{0;M}(p)/p)}\xi_p(p), && \xi_{p^\nu}(p^{\nu-2}) = \tfrac{\chi_{0;M}(p)}{p}\xi_{p^\nu}(p^\nu), \quad \text{for } \nu \geq 2. \end{aligned} \tag{2.25}$$

Blomer and Milićević prove the following.

Proposition 2.2 (Blomer and Milićević [41, Lemma 9]) *Let*

$$f_d(z) := \sum_{\ell|d} \xi_d(\ell) f|_\ell(z), \tag{2.26}$$

where $N = LM$ and $f \in H_k^\star(M)$ is Petersson-normalized with respect to the Petersson norm on $S_k(N)$. Then $\{f_d : d \mid L\}$ is an orthonormal basis of $S_k(L; f)$.

Note that in our application we are not using the Petersson normalization but instead have normalized our forms to have first coefficient 1; thus for us below we have an orthogonal basis which becomes orthonormal upon dividing the forms by their norm.

In addition, we will also use of the following lemma. Originally stated in the context of square-free level, the same proof holds in general.

Lemma 2.3 (Iwaniec, Luo, Sarnak [18, Lemma 2.5]) *Let $M|N$ and $f \in H_k^\star(M)$. Then*

$$\langle f,f\rangle_N \;=\; (4\pi)^{1-k}\Gamma(k)\frac{\nu(N)\varphi(M)}{12M}Z(1,f), \tag{2.27}$$

where φ is the Euler totient function, and

$$Z(s,f) \;:=\; \sum_{n=1}^{\infty}\lambda_f(n^2)n^{-s} \;\text{ with }\; \mathrm{Res}_{s=1}L(s,f\otimes f) \;=\; Z(1,f)M/\varphi(M). \tag{2.28}$$

It is often convenient to work with the local zeta function

$$Z_N(s,f) \;:=\; \sum_{\ell|N^\infty}\lambda_f(\ell^2)\ell^{-s}. \tag{2.29}$$

If $f \in H_k^\star(N)$, then we may write $Z(1,f) = \prod_p Z(1,f)$ and one deduces from (2.11) and (2.7) that the local Euler factors $Z_p(1,f)$ are given by

$$Z_p(1,f) \;=\; \begin{cases} \left(1+\frac{1}{p}\right)^{-1}\rho_f(p)^{-1} & \text{if } p\nmid N,\\ \left(1+\frac{1}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^{-1} & \text{if } p\,||\,N,\\ 1 & \text{if } p^2\mid N, \end{cases} \tag{2.30}$$

where $\rho_f(c)$ is the multiplicative function

$$\rho_f(c) \;=\; \sum_{b|c}\mu(b)b\left(\frac{\lambda_f(b)}{\nu(b)}\right)^2 \;=\; \prod_{p|c}\left(1-p\left(\frac{\lambda_f(p)}{p+1}\right)^2\right). \tag{2.31}$$

Assume now that $N = LM$ and $f \in H_k^\star(M)$ is normalized so that $\lambda_f(n) = 1$, and, writing

$$\mathfrak{p}(L,M) \;:=\; \prod_{\substack{p^\beta||L\\ p|M}} p^\beta, \tag{2.32}$$

note that

$$Z_{LM/\mathfrak{p}(L,M)}(s,f) \;=\; Z_N(s,f). \tag{2.33}$$

Thus, without loss of generality, we assume for the next calculation that $(L,M) = 1$; in particular, no prime divides both L and M. Specializing to $s = 1$, we find

$$\begin{aligned} &\frac{MN}{\phi(M)\nu(N)}\prod_{\substack{p|L\\ p\nmid M}}\rho_f(p)^{-1}\prod_{p^2|M}\left(\frac{p^2-1}{p^2}\right)\\ &\quad=\prod_{p^2|M}\left(1-\frac{1}{p}\right)^{-1}\left(1+\frac{1}{p}\right)^{-1}\prod_{p||M}\left(1-\frac{1}{p}\right)^{-1}\left(1+\frac{1}{p}\right)^{-1}\\ &\qquad\times\prod_{\substack{p|N\\ p\nmid M}}\left(1+\frac{1}{p}\right)^{-1}\prod_{\substack{p|L\\ p\nmid M}}\rho_f(p)^{-1}\prod_{p^2|M}\left(\frac{p^2-1}{p^2}\right)\\ &\quad=\prod_{p||M}\left(1-\frac{1}{p}\right)^{-1}\left(1+\frac{1}{p}\right)^{-1}\prod_{\substack{p\nmid M\\ p|N}}\left(1+\frac{1}{p}\right)^{-1}\rho_f(p)^{-1}\\ &\quad=\prod_{p|N}Z_p(1,f), \end{aligned} \tag{2.34}$$

since $f \in H_k^\star(M)$. We obtain

$$\frac{MN}{\varphi(M)\nu(N)} \prod_{\substack{p|L \\ p\nmid M}} \rho_f(p)^{-1} = Z_N(1,f) \prod_{p^2|M} \left(\frac{p^2}{p^2-1}\right). \tag{2.35}$$

We also note that if $p \nmid M$, then $r_f(p) = \rho_f(p)$, and if $p \mid M$, then $r_f(p) = 1 - \lambda_f(p)^2/p$.

3 A formula for $\Delta_{k,N}(m,n)$

In this section we provide the following explicit formula for $\Delta_{k,N}(m,n)$ in terms of Hecke eigenvalues. We start with a generalization of Lemma 2.7 of [18] to general N.

Lemma 3.1 *Suppose* $(m,N) = 1$ *and* $(n,N) = 1$. *Then*

$$\Delta_{k,N}(m,n) = \frac{12}{(k-1)N} \prod_{p^2|N} \left(\frac{p^2}{p^2-1}\right) \sum_{LM=N} \sum_{f \in H_k^\star(M)} \frac{Z_N(1,f)}{Z(1,f)} \lambda_f(m)\lambda_f(n). \tag{3.1}$$

Before we can prove the above lemma, we begin with a result about the coefficients inherited from the orthonormal basis defined in Proposition 2.2. Note that if $f(z) \in H_k^\star(M)$ has Fourier expansion

$$f(z) = \sum_{n\geq 1} a_f(n) e(nz), \tag{3.2}$$

then

$$f_d(z) := \sum_{\ell|d} \xi_d(\ell) f\,|_\ell\,(z) = \sum_{\ell|d} \xi_d(\ell)\ell^{k/2} f(\ell z), \tag{3.3}$$

so the coefficients of the Fourier expansion of $f_d(z)$ are given by

$$a_{f_d}(n) = \sum_{\ell|(d,n)} \xi_d(\ell)\ell^{k/2} a_f\left(\tfrac{n}{\ell}\right). \tag{3.4}$$

Let $N = LM$ and let f be a newform of weight k and level M. Let $f' = f/||f||_N$ so that f' is Petersson-normalized with respect to level N (i.e., $||f'||_N = 1$) and note that

$$a_{f'}(n) = \frac{\lambda_f(n) n^{(k-1)/2}}{||f||_N}. \tag{3.5}$$

Then by Proposition 2.2, the set $\{f_d' : d \mid L\}$ is an orthonormal basis of $S_k(L;f)$. Let

$$\mathscr{B}_k(N) = \bigcup_{LM=N} \bigcup_{f \in H_k^\star(M)} \bigcup_{d|L} \{f_d'\} \tag{3.6}$$

be our orthonormal basis for $S_k(N)$. We have

$$\begin{aligned}
\Delta_{k,N}(m,n) &= \sum_{g \in \mathscr{B}_k(N)} \overline{\left(\frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\right)^{1/2} ||g||_N^{-1} a_g(m)} \left(\frac{\Gamma(k-1)}{(4\pi n)^{k-1}}\right)^{1/2} ||g||_N^{-1} a_g(n) \\
&= (4\pi)^{1-k}(mn)^{\frac{1-k}{2}} \Gamma(k-1) \sum_{LM=N} \sum_{f \in H_k^\star(M)} \sum_{f_d' : d|L} ||f_d'||_N^{-2} \overline{a_{f_d'}(m)} a_{f_d'}(n) \\
&= (4\pi)^{1-k}(mn)^{\frac{1-k}{2}} \Gamma(k-1) \\
&\quad \times \sum_{LM=N} \sum_{f \in H_k^\star(M)} \sum_{d|L} \frac{1}{||f||_N^2} \overline{\left(\sum_{\ell|(d,m)} \xi_d(\ell)\ell^{k/2} \lambda_f\left(\tfrac{m}{\ell}\right)(m/\ell)^{(k-1)/2}\right)}
\end{aligned}$$

$$\times \left(\sum_{\ell|(d,n)} \xi_d(\ell)\ell^{k/2} a_f(\tfrac{n}{\ell})(n/\ell)^{(k-1)/2} \right)$$
$$= \frac{12}{(k-1)\nu(N)} \sum_{LM=N} \frac{M}{\varphi(M)} \sum_{f \in H_k^\star(M)} \frac{1}{Z(1,f)}$$
$$\times \sum_{d|L} \left(\sum_{\ell|(d,m)} \xi_d(\ell)\ell^{1/2}\lambda_f(\tfrac{m}{\ell}) \right) \left(\sum_{\ell|(d,n)} \xi_d(\ell)\ell^{1/2}\lambda_f(\tfrac{n}{\ell}) \right), \tag{3.7}$$

where the last equality follows from (2.27).

We now specialize to $(n, N) = 1$ and $(m, N) = 1$, as these assumptions simplify the calculations significantly and are sufficient for our application to the one-level density. In particular, as $d|L|N$ and $(m, N) = (n, N) = 1$, $\ell|(d, m)$ implies $\ell = 1$ (and similarly for $\ell|(d, n)$). Thus the previous equation simplifies to

$$\Delta_{k,N}(m, n) = \frac{12}{(k-1)\nu(N)} \sum_{LM=N} \frac{M}{\varphi(M)} \sum_{f \in H_k^\star(M)} \frac{\lambda_f(m)\lambda_f(n)}{Z(1,f)} \sum_{d|L} \xi_d(1)^2. \tag{3.8}$$

The task is now to understand

$$\sum_{d|L} \xi_d(1)^2,$$

which we investigate in the following lemma.

Lemma 3.2 *Let f be as before, with $LM = N$. We have*

$$\sum_{d|L} \xi_d(1)^2 = \prod_{\substack{p|L \\ p\nmid M}} \rho_f(p)^{-1} \prod_{\substack{p^2|N \\ p^2\nmid M}} \frac{p^2}{p^2-1}. \tag{3.9}$$

Proof Using the definition of $\xi_d(\ell)$, writing $d = d_1 d_2$, where d_1 is squarefree and d_2 is squarefull and $(d_1, d_2) = 1$, we have

$$\sum_{d|L} \xi_d(1)^2 = \sum_{d|L} \frac{\lambda_f(d_1)^2 \mu_f(d_2)^2}{r_f(d) d \beta(d_1)^2 \alpha(d_2)}. \tag{3.10}$$

Recall that $\mu_f(1) = 1$, $\mu_f(p^2) = \chi_{0;M}(p)$, and $\mu_f(p^{e_p}) = 0$ for all $e_p > 2$. As all functions in the sum above are multiplicative, we can factor as follows:

$$\sum_{d|L} \xi_d(1)^2 = \prod_{p^{e_p}\|L} \left(1 + \frac{\lambda_f(p)^2}{r_f(p) p \beta(p)^2} + \frac{\chi_{0;M}(p)(1-\mu(p^{e_p})^2)}{r_f(p^2) p^2 \alpha(p^2)} \right). \tag{3.11}$$

We now break into cases when $(p, M) = 1$ and $(p, M) \neq 1$ to remove the $\chi_{0;M}(p)$:

$$\sum_{d|L} \xi_d(1)^2 = \prod_{\substack{p^{e_p}\|L \\ (p,M)=1}} \left(1 + \frac{\lambda_f(p)^2}{r_f(p) p \beta(p)^2} + \frac{1-\mu(p^{e_p})^2}{r_f(p^2) p^2 \alpha(p^2)} \right) \prod_{\substack{p^{e_p}\|L \\ (p,M)\neq 1}}$$
$$\times \left(1 + \frac{\lambda_f(p)^2}{r_f(p) p \beta(p)^2} \right). \tag{3.12}$$

We can now simplify many of the terms as follows. If $(p, M) = 1$, then

$$\begin{aligned}\beta(p)^2 &= (1+1/p)^2,\\ \alpha(p^2) &= (1-1/p^2),\\ r_f(p) &= \rho_f(p).\end{aligned} \tag{3.13}$$

If $(p, M) \neq 1$, we have

$$\begin{aligned}\beta(p) &= \alpha(p^2) = 1,\\ r_f(p) &= 1 - \frac{\lambda_f(p)^2}{p}.\end{aligned} \tag{3.14}$$

In addition, note that $r_f(p) = r_f(p^2)$. Thus we can write the right hand side of (3.12) as

$$\prod_{\substack{p^{e_p}\|L\\(p,M)=1}} \left(1 + \frac{\lambda_f(p)^2}{\rho_f(p)p\left(1+\frac{1}{p}\right)^2} + \frac{1-\mu(p^{e_p})^2}{\rho_f(p)p^2\left(1-\frac{1}{p^2}\right)}\right) \prod_{\substack{p^{e_p}\|L\\(p,M)\neq 1}} \left(1 + \frac{\lambda_f(p)^2}{p\left(1-\frac{\lambda_f(p)^2}{p}\right)}\right). \tag{3.15}$$

Recall that because $f \in H_k^\star(M)$,

$$\lambda_f(p)^2 = \begin{cases} \frac{1}{p} & \text{if } p\|M,\\ 0 & \text{if } p^2 \mid M.\end{cases} \tag{3.16}$$

Therefore we can rewrite the second product in (3.15), and obtain

$$\sum_{d|L} \xi_d(1)^2 = \prod_{\substack{p^{e_p}\|L\\(p,M)=1}} \left(1 + \frac{\lambda_f(p)^2}{\rho_f(p)p\left(1+\frac{1}{p}\right)^2} + \frac{1-\mu(p^{e_p})^2}{\rho_f(p)p^2\left(1-\frac{1}{p^2}\right)}\right) \prod_{\substack{p|L\\p\|M}} \left(\frac{p^2}{p^2-1}\right). \tag{3.17}$$

We now simplify the first product above as

$$\begin{aligned}&\prod_{\substack{p^{e_p}\|L\\(p,M)=1}} \left(\frac{\rho_f(p)p\left(1+\frac{1}{p}\right)^2(p^2-1) + \lambda_f(p)^2(p^2-1) + p\left(1+\frac{1}{p}\right)^2(1-\mu(p^{e_p})^2)}{\rho_f(p)p\left(1+\frac{1}{p}\right)^2(p^2-1)}\right)\\ &\quad\times \prod_{\substack{p|L\\p\|M}} \left(\frac{p^2}{p^2-1}\right)\\ &= \prod_{\substack{p^{e_p}\|L\\(p,M)=1}} \left(\frac{\left(p\left(1+\frac{1}{p}\right)^2 - \lambda_f(p)^2\right)(p^2-1) + \lambda_f(p)^2(p^2-1) + p\left(1+\frac{1}{p}\right)^2(1-\mu(p^{e_p})^2)}{\rho_f(p)p\left(1+\frac{1}{p}\right)^2(p^2-1)}\right)\\ &\quad\times \prod_{\substack{p|L\\p\|M}} \left(\frac{p^2}{p^2-1}\right)\end{aligned}$$

$$= \prod_{\substack{p^{e_p}\|L \\ (p,M)=1}} \left(\frac{p\left(1+\frac{1}{p}\right)^2 (p^2 - \mu(p^{e_p})^2)}{\rho_f(p) p \left(1+\frac{1}{p}\right)^2 (p^2-1)} \right) \prod_{\substack{p|L \\ p\|M}} \left(\frac{p^2}{p^2-1} \right)$$

$$= \prod_{\substack{p|L \\ p\nmid M}} \rho_f(p)^{-1} \prod_{\substack{p^2|N \\ p^2\nmid M}} \left(s \frac{p^2}{p^2-1} \right), \tag{3.18}$$

which completes the proof. □

Combining Lemma 3.2 with equations (2.35) and (3.8) yields Lemma 3.1.

4 An inversion and a change from weighted to pure sums

We now introduce the arithmetically weighted sums, as defined in [18, (2.53)],

$$\Delta^*_{k,N}(m,n) = \sum_{f \in H^\star_k(N)} \frac{\lambda_f(n)\lambda_f(m) Z_N(1,f)}{Z(1,f)}. \tag{4.1}$$

This allows us to state one of our main results, which generalizes work of Iwaniec, Luo, and Sarnak [18, Proposition 2.8] and Rouymi [40, Proposition 2.3].

Proposition 4.1 *Suppose* $(m,N)=1$ *and* $(n,N)=1$. *Then*

$$\Delta_{k,N}(m,n) = \frac{12}{(k-1)N} \prod_{p^2|N} \left(\frac{p^2}{p^2-1} \right) \sum_{LM=N} \sum_{\substack{\ell | L^\infty \\ (\ell,M)=1}} \ell^{-1} \Delta^\star_{k,M}(m\ell^2, n) \tag{4.2}$$

and

$$\Delta^\star_{k,N}(m,n) = \frac{k-1}{12} \sum_{LM=N} \mu(L) M \prod_{p^2|M} \left(\frac{p^2}{p^2-1} \right)^{-1} \sum_{\substack{\ell | L^\infty \\ (\ell,M)=1}} \ell^{-1} \Delta_{k,M}(m\ell^2, n). \tag{4.3}$$

Proof of Proposition 4.1 We first prove (4.2). Note the following: $(m,N)=1$, $(n,N)=1$ and $\ell \mid L^\infty$ imply $(m,M)=1$, $(n,M)=1$, and $(\ell,m)=1$.

These observations together with Lemma 3.1 imply

$$\begin{aligned}
\Delta_{k,N}(m,n) &= \frac{12}{(k-1)N} \prod_{p^2|N} \left(\frac{p^2}{p^2-1} \right) \\
&\quad \times \sum_{LM=N} \sum_{f \in H^\star_k(M)} \frac{Z_{L/\mathfrak{p}(L,M)}(1,f) Z_M(1,f)}{Z(1,f)} \lambda_f(m)\lambda_f(n) \\
&= \frac{12}{(k-1)N} \prod_{p^2|N} \left(\frac{p^2}{p^2-1} \right) \\
&\quad \times \sum_{LM=N} \sum_{f \in H^\star_k(M)} \Bigg(\sum_{\substack{\ell | L^\infty \\ (\ell,M)=1}} \lambda_f(\ell^2)\ell^{-1} \Bigg) \frac{Z_M(1,f)}{Z(1,f)} \lambda_f(m)\lambda_f(n) \\
&= \frac{12}{(k-1)N} \prod_{p^2|N} \left(\frac{p^2}{p^2-1} \right) \sum_{LM=N} \sum_{\substack{\ell | L^\infty \\ (\ell,M)=1}} \ell^{-1} \Delta^\star_{k,M}(m\ell^2, n).
\end{aligned} \tag{4.4}$$

We are now ready to prove (4.3) using Möbius inversion. We begin with

$$\begin{aligned}
&\frac{k-1}{12}\sum_{LM=N}\mu(L)M\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)^{-1}\sum_{\substack{\ell|L^\infty\\(\ell,M)=1}}\ell^{-1}\Delta_{k,M}(m\ell^2,n)\\
&=\frac{k-1}{12}\sum_{LM=N}\mu(L)M\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)^{-1}\sum_{\substack{\ell|L^\infty\\(\ell,M)=1}}\ell^{-1}\frac{12}{(k-1)M}\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)\\
&\quad\times\sum_{QW=M}\sum_{f\in H_k^\star(W)}\frac{Z_M(1,f)}{Z(1,f)}\lambda_f(m\ell^2)\lambda_f(n)\\
&=\sum_{LM=N}\mu(L)\sum_{QW=M}\sum_{f\in H_k^\star(W)}\Big(\sum_{\substack{\ell|L^\infty\\(\ell,M)=1}}\lambda_f(\ell^2)\ell^{-1}\Big)\frac{Z_M(1,f)}{Z(1,f)}\lambda_f(m)\lambda_f(n)\\
&=\sum_{LM=N}\mu(L)\sum_{QW=M}\sum_{f\in H_k^\star(W)}\frac{Z_N(1,f)}{Z(1,f)}\lambda_f(m)\lambda_f(n).
\end{aligned}\tag{4.5}$$

Let

$$\mathcal{Z}_N(W):=\sum_{f\in H_k^\star(W)}\frac{Z_N(1,f)}{Z(1,f)}\lambda_f(m)\lambda_f(n).\tag{4.6}$$

Interchanging orders of summation yields

$$\sum_{LM=N}\mu(L)\sum_{QW=M}\mathcal{Z}_N(W)=\sum_{W|N}\mathcal{Z}_N(W)\sum_{L|\frac{N}{W}}\mu(L)=\Delta_{k,N}^\star(m,n),\tag{4.7}$$

as the Möbius sum vanishes unless $W=N$ and $\mathcal{Z}_N(N)=\Delta_{k,N}^\star(m,n)$. □

One of our primary applications of Proposition 4.1 is to obtain a formula for pure sums of Hecke eigenvalues. We define the pure sum

$$\Delta_{k,N}^\star(n):=\sum_{f\in H_k^\star(N)}\lambda_f(n)\tag{4.8}$$

(not to be confused with $\Delta_{k,N}^\star(m,n)$, which is weighted) and prove Theorem 1.2 from the introduction, which we restate here for convenience.

Theorem 1.2 *Suppose that* $(n,N)=1$. *Then*

$$\Delta_{k,N}^\star(n)=\frac{k-1}{12}\sum_{LM=N}\mu(L)M\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)^{-1}\sum_{(m,M)=1}m^{-1}\Delta_{k,M}(m^2,n).\tag{4.9}$$

Proof We remove the weights in (4.1) by summing $m^{-1}\Delta_{k,N}^\star(m^2,n)$ over all $(m,N)=1$. We will need to replace $\sum_{\ell|N^\infty}\sum_{(m,N)=1}(\ell m)^{-1}\lambda_f(\ell^2)\lambda_f(m^2)$ with $\sum_{r\geq1}r^{-1}\lambda_f(r^2)$; some care is required as we do not have absolute convergence. This can be handled replacing $(\ell m)^{-1}$ and r^{-1} by $(\ell m)^{-s}$ and r^{-s}, and then taking the limit as s tends to 1 from above. We do not need absolute convergence of the series to justify the limit; it is permissible as the Dirichlet series are continuous in the region of convergence and all sums at $s=1$ exist as the sum of the coefficients grow sub-linearly.

On one side we have

$$\begin{aligned}\sum_{(m,N)=1} m^{-1}\Delta^{\star}_{k,N}(m^2,n) &= \sum_{(m,N)=1} m^{-1} \sum_{f\in H^{\star}_k(N)} \frac{\lambda_f(m^2)\lambda_f(n)Z_N(1,f)}{Z(1,f)} \\ &= \sum_{f\in H^{\star}_k(N)} \frac{\lambda_f(n)}{Z(1,f)} \sum_{(m,N)=1} \sum_{\ell\mid N^{\infty}} (\ell m)^{-1}\lambda_f(\ell^2)\lambda_f(m^2) \\ &= \sum_{f\in H^{\star}_k(N)} \frac{\lambda_f(n)}{Z(1,f)} \sum_{r\geq 1} r^{-1}\lambda_f(r^2) \\ &= \sum_{f\in H^{\star}_k(N)} \lambda_f(n) \\ &= \Delta^{\star}_{k,N}(n). \end{aligned} \tag{4.10}$$

On the other hand we have, using (4.3), for $(n,N)=1$,

$$\begin{aligned}&\sum_{(m,N)=1} m^{-1}\Delta^{*}_{k,N}(m^2,n) \\ &= \sum_{(m,N)=1} m^{-1}\frac{k-1}{12} \sum_{LM=N} \mu(L)M \prod_{p^2\mid M}\left(\frac{p^2}{p^2-1}\right)^{-1} \sum_{\substack{\ell\mid L^{\infty}\\(\ell,M)=1}} \frac{1}{\ell}\Delta_{k,M}((m\ell)^2,n) \\ &= \frac{k-1}{12} \sum_{LM=N} \mu(L)M \prod_{p^2\mid M}\left(\frac{p^2}{p^2-1}\right)^{-1} \sum_{(m,N)=1} \sum_{\substack{\ell\mid L^{\infty}\\(\ell,M)=1}} \frac{1}{m\ell}\Delta_{k,M}((m\ell)^2,n) \\ &= \frac{k-1}{12} \sum_{LM=N} \mu(L)M \prod_{p^2\mid M}\left(\frac{p^2}{p^2-1}\right)^{-1} \sum_{(m,M)=1} \frac{1}{m}\Delta_{k,M}(m^2,n). \end{aligned} \tag{4.11}$$

This completes the proof. □

5 Estimating tails of pure sums

One might inquire about the convergence of the innermost sum in Theorem 1.2. It is assured by the holomorphy of $L(s,\operatorname{sym}^2 f)$, but is not absolute (see [18, p. 79] for a full discussion). For this reason, following [18, §2], we begin our work towards the Density Conjecture by splitting

$$\Delta^{\star}_{k,N}(n) = \Delta'_{k,N}(n) + \Delta^{\infty}_{k,N}(n), \tag{5.1}$$

where

$$\Delta'_{k,N}(n) := \frac{k-1}{12} \sum_{\substack{LM=N\\L\leq X}} \mu(L)M \prod_{p^2\mid M}\left(\frac{p^2}{p^2-1}\right)^{-1} \sum_{\substack{(m,M)=1\\m\leq Y}} m^{-1}\Delta_{k,M}(m^2,n), \tag{5.2}$$

and $\Delta^{\infty}_{k,N}(n)$ is the complementary sum. Here $X, Y \geq 1$ are free parameters.

We consider sequences $\mathscr{A} = \{a_q\}$ that satisfy

$$\sum_{(q,nN)=1} \lambda_f(q)a_q \ll (nkN)^{\varepsilon} \tag{5.3}$$

for all $f \in H^{\star}_k(M)$ with $M \mid N$ such that the implied constant depends only on ε. The sequence we need for our application, given by

$$a_q = p^{-1/2}\log p \qquad \text{if } q = p \leq Q, \tag{5.4}$$

and $a_q = 0$ elsewhere, satisfies this property provided $\log Q \ll \log kN$, assuming GRH for $L(s,f)$; see [18, p. 80] for more details. The lemma below is a straightforward modification of Lemma 2.12 of [18] to the case of general N.

Lemma 5.1 *Suppose* $(n,N) = 1$ *and that* $\mathscr{A}$ *satisfies* (5.3). *Then*

$$\sum_{(q,nN)=1} \Delta_{k,N}^{\infty}(nq)a_q \ll kN(X^{-1} + Y^{-1/2})(nkNXY)^{\varepsilon}. \tag{5.5}$$

Proof Suppose $(q,nN) = 1$. We use Theorem 1.2 and the various definitions of the sums of the λ_f's to obtain the expansion for the complementary sum $\Delta_{k,N}^{\infty}(nq)$, and then sum this weighted by a_q over q relatively prime to nN. We simplify some of the resulting sums by grouping them with Lemma 3.1. Thus

$$\begin{aligned}\Delta_{k,N}^{\infty}(nq) = &\sum_{\substack{KLM=N\\ L>X}} \mu(L) \sum_{f\in H_k^{\star}(M)} \lambda_f(nq)\\ &+ \sum_{\substack{KLM=N\\ L\le X}} \mu(L) \sum_{f\in H_k^{\star}(M)} \lambda_f(nq)R_f(KM;Y),\end{aligned} \tag{5.6}$$

where

$$R_f(KM;Y) := \frac{Z_{KM}(1,f)}{Z(1,f)} \sum_{\substack{(m,KM)=1\\ m>Y}} m^{-1}\lambda_f(m^2). \tag{5.7}$$

By the Riemann Hypothesis for $L(s,\mathrm{sym}^2(f))$ we have

$$R_f(KM;Y) \ll Y^{-1/2}(kKMY)^{\varepsilon}. \tag{5.8}$$

Combining this fact with the Deligne bound for $|\lambda_f(n)|$, we have

$$\begin{aligned}\sum_{(q,nN)=1} \Delta_{k,N}^{\infty}(nq)a_q &= \sum_{\substack{KLM=N\\ L>X}} \mu(L) \sum_{f\in H_k^{\star}(M)} \lambda_f(n) \sum_{(q,nN)=1} \lambda_f(q)a_q\\ &\quad + \sum_{\substack{KLM=N\\ L\le X}} \mu(L) \sum_{f\in H_k^{\star}(M)} \lambda_f(n)R_f(KM;Y) \sum_{(q,nN)=1} \lambda_f(q)a_q\\ &\ll \sum_{\substack{KLM=N\\ L>X}} |\mu(L)H_k^{\star}(M)|\tau(n)(nkN)^{\varepsilon}\\ &\quad + \sum_{\substack{KLM=N\\ L\le X}} |\mu(L)H_k^{\star}(M)|\tau(n)Y^{-1/2}(kKMY)^{\varepsilon}(nkN)^{\varepsilon}\\ &\ll \sum_{\substack{KLM=N\\ L>X}} |\mu(L)| \left(\frac{k-1}{12}\right) \varphi(M)\tau(n)(nkN)^{\varepsilon}\\ &\quad + \sum_{\substack{KLM=N\\ L\le X}} |\mu(L)| \left(\frac{k-1}{12}\right) \varphi(M)\tau(n)Y^{-1/2}(kKMY)^{\varepsilon}(nkN)^{\varepsilon}\\ &\ll \sum_{\substack{KLM=N\\ L>X}} k\tfrac{N}{X}(nkN)^{\varepsilon} + \sum_{\substack{KLM=N\\ L\le X}} kNY^{-1/2}(kKMY)^{\varepsilon}(nkN)^{\varepsilon}\\ &\ll kN(X^{-1} + Y^{-1/2})(nkNXY)^{\varepsilon}.\end{aligned} \tag{5.9}$$

This establishes the lemma. □

We now substitute the Petersson formula (Proposition 2.1) for each instance of $\Delta_{k,M}(m^2, n)$ to obtain an exact formula for $\Delta'_{k,N}(n)$ in terms of Kloosterman sums; this is a generalization of Proposition 2.12 of [18].

Proposition 5.2 *Suppose* $(n, N) = 1$. *Then*

$$\begin{aligned}\Delta'_{k,N}(n) &= \delta_Y(m^2, n)\frac{k-1}{12}n^{-1/2}\sum_{\substack{LM=N\\L\le X}}\mu(L)M\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)^{-1}\\ &\quad+\frac{k-1}{12}\sum_{\substack{LM=N\\L\le X}}\mu(L)M\prod_{p^2|M}\left(\frac{p^2}{p^2-1}\right)^{-1}\\ &\quad\times\sum_{\substack{(m,M)=1\\m\le Y}}m^{-1}2\pi i^k\sum_{c\equiv 0 \pmod M}c^{-1}S(m^2, n; c)J_{k-1}\left(\frac{4\pi m\sqrt{n}}{c}\right),\end{aligned}$$

where

$$\delta_Y(m^2, n) = \begin{cases}1 & \text{if } n = m^2 \text{ and } m \le Y,\\ 0 & \text{otherwise.}\end{cases} \tag{5.10}$$

We recover the bounds for $|H^\star_k(N)|$ given by Martin in [47, Theorem 6(c)], and immediately obtain the following result (for completeness the calculation, which is standard, is given in the appendix of [48]).

Proposition 5.3 *We have that as* $kN \to \infty$

$$\begin{aligned}&\frac{k-1}{12}\varphi(N)\prod_p\left(1-\frac{1}{p^2-p}\right)+O\left((kN)^{2/3}\right)\\ &\quad\le\ \left|H^\star_k(N)\right|\ \le\ \frac{k-1}{12}\varphi(N)+O\left((kN)^{2/3}\right).\end{aligned} \tag{5.11}$$

6 The Density Conjecture for $H^\star_k(N)$

Fix some $\phi \in \mathscr{S}(\mathbf{R})$ with $\widehat{\phi}$ supported in $(-u, u)$. We reprise some basic definitions from the introduction.

To a holomorphic newform f, we associate the L-function

$$L(s, f)\ =\ \sum_1^\infty \lambda_f(n)n^{-s}. \tag{6.1}$$

Assuming the Riemann Hypothesis for $L(s, f)$, we can write its non-trivial zeros as

$$\varrho_f\ =\ \frac{1}{2}+i\gamma_f, \tag{6.2}$$

where $\gamma_f \in \mathbb{R}$. We are interested in the one-level densities of low-lying zeroes. We recall the definition of $D_1(f; \phi)$ in (1.1):

$$D_1(f;\phi)\ =\ \sum_{\gamma_f}\phi\left(\frac{\gamma_f}{2\pi}\log c_f\right), \tag{6.3}$$

where c_f is the analytic conductor of f which in our case is k^2N. We also introduce a scaling parameter R which we take to satisfy $1 < R \asymp k^2N$.

Iwaniec, Luo, and Sarnak [18, §4] establish that for $f \in H_k^\star(N)$,

$$D_1(f;\phi) \;=\; E(\phi) - P(f;\phi) + O\left(\frac{\log\log kN}{\log R}\right), \tag{6.4}$$

where

$$E(\phi) \;=\; \widehat{\phi}(0) + \frac{1}{2}\phi(0) \tag{6.5}$$

and

$$P(f;\phi) \;=\; \sum_{p\nmid N} \lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}. \tag{6.6}$$

Note that their argument does not depend on N being square-free. The Density Conjecture concerns the average over $H_k^\star(N)$, so we consider the sum

$$\mathscr{B}_k^\star(\phi) \;=\; \sum_{f\in H_k^\star(N)} D_1(f;\phi). \tag{6.7}$$

Although our main goal is to investigate the behavior as $N\to\infty$, we keep the notation $\mathscr{B}_k^\star(\phi)$ to remain consistent with [18]. Substituting (6.4) into the above we find that

$$\mathscr{B}_k^\star(\phi) \;=\; \left|H_k^\star(N)\right| E(\phi) - \mathscr{P}_k^\star(\phi) + O\left(\left|H_k^\star(N)\right|\frac{\log\log kN}{\log R}\right), \tag{6.8}$$

where

$$\mathscr{P}_k^\star(\phi) \;=\; \sum_{p\nmid N} \Delta_{k,N}^\star(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}. \tag{6.9}$$

In order to establish that as $kN\to\infty$ that the main term of $\mathscr{B}_k^\star(\phi)/\left|H_k^\star(N)\right|$ is $E(\phi)$, we need to establish that $\mathscr{P}_k^\star(\phi) = o(k\varphi(N))$. This is sufficient because $\left|H_k^\star(N)\right| \asymp k\varphi(N)$, as we showed in Proposition 5.3.

Using the decomposition (5.1) we can now write

$$\mathscr{P}_k^\star(\phi) \;=\; \sum_{p\nmid N} \left(\Delta'_{k,N}(p) + \Delta^\infty_{k,N}(p)\right)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}. \tag{6.10}$$

We first bound

$$\sum_{p\nmid N} \Delta^\infty_{k,N}(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}. \tag{6.11}$$

Let a_q be as in (5.4) for $q \le R^u$ and 0 for $q > R^u$ (the latter is due to the appearance of $\widehat{\phi}$, which is zero for $P > R^u$). We see that this sequence satisfies the condition on Q in the definition (5.4), and since ϕ is of Schwartz class, we may apply Lemma 5.1 with $X = Y = (kN)^\delta$ for small positive δ to find

$$\sum_{p\nmid N} \Delta^\infty_{k,N}(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R} \;\ll\; kN(X^{-1} + Y^{-1/2})(kNXY)^\varepsilon \;=\; o(k\varphi(N)). \tag{6.12}$$

Next we must estimate the other term from (6.10),

$$\mathscr{M}_k^\star(\phi) \;:=\; \sum_{p\nmid N} \Delta'_{k,N}(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}. \tag{6.13}$$

To begin, define

$$Q^\star_{k;N}(m;c) \;=\; 2\pi i^k \sum_{p\nmid N} S(m^2,p;c) J_{k-1}\left(\frac{4\pi m}{c}\sqrt{p}\right)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}. \tag{6.14}$$

Then we apply Lemma 5.2 to each instance of $\Delta'_{k,N}$. Note that the first term in Lemma 5.2 disappears because p is never a square. Then, moving the initial summation over $p \nmid N$ into the expression, we can rewrite in terms of $Q^\star_{k;N}(m;c)$:

$$\mathscr{M}^\star_k(\phi) \;=\; \frac{k-1}{12} \sum_{\substack{LM=N\\ L\le X}} \mu(L) M \prod_{p^2\mid M} \left(\frac{p^2}{p^2-1}\right)^{-1} \sum_{\substack{(m,M)=1\\ m\le Y}} m^{-1} \sum_{c\equiv 0(M)} c^{-1} Q^\star_{k;N}(m;c). \tag{6.15}$$

Iwaniec, Luo, and Sarnak [18, §6] prove the bound (which still holds for N not square-free)

$$Q^\star_{k;N}(m;c) \;\ll\; \widetilde{\gamma}(z) m P^{1/2} (kN)^{\varepsilon} (\log 2c)^{-2}, \tag{6.16}$$

where $z = 4\pi m\sqrt{P}/c$, $P = R^{u'}$ with some $u' < u$, and $\widetilde{\gamma}(z) = 2^{-k}$ if $3z \le k$; this bound appears after their equation (6.17), and uses GRH for Dirichlet L-functions (they expand the Kloosterman sums with Dirichlet characters). In order to apply this bound we need to secure $12\pi m P^{1/2} \le kc$, $(3z \le k)$ so as to satisfy a condition on an estimate for the Bessel function, given in their equation (2.11‴). Noting that $m \le Y$ and $c \ge M \ge N/X$, it suffices to have $12\pi XYP^{1/2} \le kN$. Taking logarithms, this becomes a condition on u, namely

$$u \;\le\; \frac{2(1-2\delta)\log(kN)}{\log(k^2 N)}. \tag{6.17}$$

For u in this range we can apply the estimate (6.16) to find

$$\begin{aligned}
\mathscr{M}^\star_k(\phi) \;&\ll\; \frac{k-1}{12} \sum_{\substack{LM=N\\ L\le X}} |\mu(L)| M \\
&\quad\times \prod_{p^2\mid M} \left(\frac{p^2}{p^2-1}\right)^{-1} \sum_{\substack{(m,M)=1\\ m\le Y}} m^{-1} \sum_{c\equiv 0(M)} c^{-1} 2^{-k} m P^{1/2} (kN)^{\varepsilon} (\log 2c)^{-2} \\
&\ll\; \frac{k-1}{12} 2^{-k} P^{1/2} (kN)^{\varepsilon} \sum_{\substack{LM=N\\ L\le X}} |\mu(L)| M \prod_{p^2\mid M} \left(\frac{p^2}{p^2-1}\right)^{-1} \\
&\quad\times \sum_{\substack{(m,M)=1\\ m\le Y}} \sum_{c\equiv 0(M)} c^{-1} (\log 2c)^{-2}.
\end{aligned} \tag{6.18}$$

Trivial estimation plus the bound

$$\sum_{c\equiv 0(M)} \frac{1}{c(\log c)^2} \;\ll\; \frac{1}{M} \tag{6.19}$$

yields

$$\sum_{p\nmid N} \Delta'_{k,N}(p) \widehat{\phi}\left(\frac{\log p}{\log R}\right) \frac{2\log p}{\sqrt{p}\log R} \;\ll\; \frac{k-1}{12} 2^{-k} P^{1/2} (kN)^{\varepsilon} XY, \tag{6.20}$$

which is $o(k\varphi(N))$ for $\varepsilon + 2\delta < 1/2$.

Thus by taking δ sufficiently small (to satisfy (6.17)) and applying the combined estimates for the completed sums, (6.12) and (6.20), we have established that $\mathscr{P}^\star_k(\phi) = o(k\varphi(N))$ where the (closed) support of $\widehat{\phi}$ is contained in $(-u, u)$ and

$$u < \frac{2\log kN}{\log k^2 N}, \tag{6.21}$$

which implies Theorem 1.3. □

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