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Double bubbles in hyperbolic surfaces

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We seek the least-perimeter way to enclose and separate two prescribed areas in certain hyperbolic surfaces.

1. Introduction

The isoperimetric problem of enclosing a given area in a least-perimeter way has been investigated in various surfaces. The classical isoperimetric theorem in the plane asserts that the circle is the shortest curve to enclose a given area in the plane. While this result is widely known, the solution of the isoperimetric problem has proved to be elusive in surfaces aside from the plane. By 1999, the problem had been solved for a handful of Riemannian surfaces, namely, the Euclidean plane, a round sphere, a round projective plane, the hyperbolic plane, a circular cone, a circular cylinder, a flat torus or Klein bottle, and a general surface of revolution [Howards et al. 1999]. Adams and Morgan [1999] obtained further results in hyperbolic surfaces. The related problem of discovering the least perimeter needed to enclose and separate two given volumes has invited exploration as well.

Particular interest has been garnered by the double bubble conjecture. The double bubble conjecture states that three spherical caps meeting at $\frac{2\pi}{3}$ angles (the "standard double bubble") is the least-perimeter way to enclose and separate two given volumes. This has been believed to be true since the nineteenth century, but it was first articulated as a conjecture by Joel Foisy [1991], an undergraduate student at Williams College, in his senior thesis, and it was proved in the planar case in [Foisy et al. 1993]. Joel Hass, Michael Hutchings, and Roger Schlafly [Hass et al. 1995] attacked the conjecture in the $\mathbb{R}^3$ case using heavily computational methods, successfully resolving the problem for the case where the two volumes are equal. Finally, Michael Hutchings, Frank Morgan, Manuel Ritoré, and Antonio Ros [Hutchings et al. 2002] proved the double bubble conjecture for any ratio of two volumes in $\mathbb{R}^3$. Moreover, Andrew Cotton and David Freeman [2002] have

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shown the conjecture to hold for the hyperbolic plane as well as the case of equal volumes in hyperbolic 3-space. In certain hyperbolic surfaces however, the standard double bubble is not perimeter-minimizing. We study this problem, following the work on single bubbles by Adams and Morgan [1999].

Section 2 discusses the existence and regularity of perimeter-minimizing double bubbles. Section 3 considers n-punctured spheres. Proposition 3.6 identifies small perimeter-minimizing double bubbles as horocycles around cusps. Section 4 focuses on double bubbles on the thrice-punctured sphere. Conjecture 4.1 describes perimeter-minimizing double bubbles as horocycles for small areas and θ-curves for large areas. Proposition 4.2 shows that, for equal areas, θ-curves are shorter than horocycles for a specific range of areas through direct computations. Propositions 4.7–4.9 show necessary conditions on the topology of perimeter-minimizing double bubbles using inequalities obtained in Lemmas 4.3–4.5. Section 5 considers the once-punctured torus. Proposition 5.1 proves that for relatively small areas two horocycles around a cusp are shorter than a horocycle with a lens.

2. Existence and regularity

Definition 2.1. A *double bubble* on a surface consists of two disjoint open regions with piecewise smooth boundaries. The *perimeter* refers to the union of the boundaries or its length. We do not assume that each region, or that the perimeter, or that the entire bubble (the union of the regions and the perimeter) is connected. We call the bubble perimeter-minimizing or sometimes just minimizing if it minimizes perimeter for fixed area of each region.

Morgan [1994] examined existence and regularity for soap bubble clusters in $\mathbb{R}^2$ and on compact Riemannian surfaces, and his results and proofs apply to geometrically finite hyperbolic surfaces.

Theorem 2.2 (existence and regularity). *In a complete hyperbolic surface, there exists a least-perimeter double bubble, enclosing and separating two regions of prescribed areas. Its perimeter consists of curves of constant curvature meeting in threes at angles of* $\frac{2\pi}{3}$*; all curves separating a specific pair of regions have the same curvature.*

Proof. We explain the extension of Morgan [1994] to the noncompact case. If in a minimizing sequence a region goes out a cusp, its area goes to 0 and it may be discarded. If it goes out a flared end, it can be translated back inside a compact region. □

We are assuming that the sum of the two areas is less than the area of the surface; the complement is a third region. It remains conjectural in general that each of the three regions is connected.

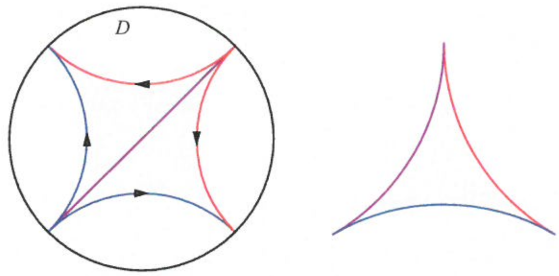

Figure 1. The thrice-punctured sphere can be obtained from the Poincaré disc (D) model of the hyperbolic plan by identifying the two ideal triangles as indicated: the purple side is already identified, blue is glued to blue, and red to red, according to the orientation given.

3. n-punctured spheres

The hyperbolic surfaces we will primarily focus on throughout this paper are n-punctured spheres, mainly because they are at once both simple (having cusps but no handles) and interesting. Proposition 3.5 gives the total area of an n-punctured sphere. Proposition 3.6 shows that for a certain range of areas, perimeter-minimizing double bubbles on an n-punctured sphere have disconnected boundary, a deviation from the topological properties of the standard double bubble.

Definition 3.1. An *n-punctured sphere* is constructed by doubling an ideal n-gon in hyperbolic 2-space and identifying the boundary.

The n-punctured sphere admits a hyperbolic metric for $n \geq 3$, so we assume henceforth that $n \geq 3$. Figure 1 gives an example of this construction in the case of the thrice-punctured sphere.

We have the following helpful proposition on single bubbles on the n-punctured sphere.

Proposition 3.2 [Adams and Morgan 1999, Theorem 2.2]. *For single bubbles on a punctured surface, least-perimeter P is less than or equal to area A with equality precisely for horocycles about cusps. Moreover, if $A < \pi$, then a minimizer consists of horocycles about an arbitrary collection of cusps.*

Remark 3.3. Adams and Morgan [1999] further show that in the case of the thrice-punctured sphere, the hypothesis of this proposition can be extended to $A \leq \pi$.

In the proofs of our results we will make use of the following well-known facts in this area.

Remark 3.4. A horocycle about a cusp has constant curvature 1 and its length is equal to the area of the cusp neighborhood.

Proposition 3.5. *The total area of the n-punctured sphere is $2(n-2)\pi$.*

Proof. The area of an ideal triangle in hyperbolic 2-space is π. Since an ideal n-gon can be triangulated into $n-2$ ideal triangles, the area of the ideal n-gon is $(n-2)\pi$. The n-punctured sphere is composed of two ideal n-gons glued together and thus has area $2(n-2)\pi$. □

Proposition 3.6. *Given $0 < A_1 \le A_2 < \pi - A_1$, the least-perimeter way to enclose and separate areas A_1, A_2 on the n-punctured sphere is horocycles around cusps.*

Proof. Assume to the contrary the perimeter is less than or equal to $A_1 + A_2$ and the regions have common boundary. Then the shared boundary can be eliminated with the remaining boundary enclosing the single area $A_1 + A_2$. By our assumption the length of the remaining boundary is strictly less than $A_1 + A_2$. Since $A_1 + A_2 < \pi$, this is a contradiction of Proposition 3.2. □

4. The thrice-punctured sphere

The thrice-punctured sphere is equipped with unique hyperbolic structure with area 2π and constant Gaussian curvature -1. These features make the thrice-punctured sphere an ideal surface on which to explore the properties of double bubbles. Conjecture 4.1 says that horocycles are perimeter-minimizing for small areas and that a θ-curve is perimeter-minimizing for large areas, with the transition point for equal areas given by Proposition 4.2. Proposition 4.8 shows that for double bubbles with connected perimeter, all three regions must contain a cusp. Proposition 4.9 further restricts the topology.

Conjecture 4.1. *Given two areas $0 < A_1 \le A_2 \le 2\pi - A_1 - A_2$, a perimeter-minimizing double bubble on the thrice-punctured sphere consists of*

(1) *horocycles around cusps if A_1 is relatively small,*

(2) *a θ-curve with each region containing one cusp (unique up to the three-fold symmetry) if A_1 is relatively large (see Figure 2).*

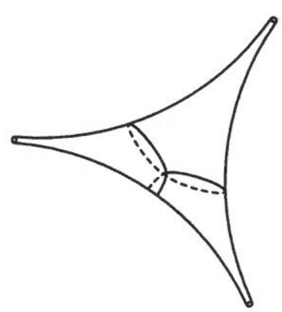

Figure 2. θ-curves as pictured are conjectured to minimize perimeter for relatively large pairs of areas.

Proposition 4.2. *There exists a constant* $A_0 \approx 1.7038$ *such that given* $0 < A_1 = A_2 \leq \frac{2\pi}{3}$, *the symmetric* θ*-curve enclosing areas* A_1, A_2 *is shorter than horocycles (of length* $A_1 + A_2$*) if and only if* $A_1 = A_2 > A_0$.

Proof. Let $H = \{x + yi \in \mathbb{C} \mid y > 0\}$ with the metric $ds = \sqrt{dx^2 + dy^2}/y$; this is the upper half-plane model of hyperbolic space. The length of a parametrized curve $\sigma : [a, b] \to H$ is given by

$$\text{length} = \int_a^b \frac{|\sigma'(t)|}{y(t)}\, dt.$$

The area of a region R is given by

$$\text{area} = \iint_R \frac{1}{y}\, dx\, dy.$$

We consider the following construction in H. The thrice-punctured sphere can be considered as the quotient of two ideal triangles H (the edges of these triangles are shown in blue in Figure 3 with the edges e and f being identified with e' and f' as shown). For computational ease we choose the radii of the semicircles f and f' to be 1.

Given $A_1 = A_2 = \frac{2\pi}{3}$, consider the pink θ-curve ϕ of Figure 3, composed of three geodesics which each contain a cusp and meet at angles of $\frac{2\pi}{3}$. In the upper half-plane this curve consists of four circular arcs of radius 2 and angle $\frac{\pi}{6}$ and two vertical segments. Each of the arcs is centered at a vertex of the ideal triangle and runs from a vertical edge toward the center, while the two vertical segments run from the intersections of the arcs to the edges f and f'.

By symmetry this curve divides the thrice-punctured sphere into three equal parts each having area $\frac{2\pi}{3}$. Due to symmetry the length of ϕ is $6l$, where l is the length of just one of the vertical segments. Computing the length of the segment from $(1, \sqrt{3})$ to $(1, 1)$ using the formula given we obtain $l = \ln\sqrt{3} - \ln 1 = \frac{1}{2}\ln 3$. Thus the length of ϕ is $3\ln 3$.

For $A_1 = A_2 = \frac{2\pi}{3}$, the θ-curve has length $3\ln 3 < \frac{4\pi}{3} = A_1 + A_2$, while for $A_1 = A_2 < \pi$, the horocycles of length $A_1 + A_2$ are minimizing by Proposition 3.6. Moreover, as $A_1 = A_2$ decreases, the symmetric θ-curve gets longer and the horocycles get shorter. Therefore there is a constant $\pi < A_0 < \frac{2\pi}{3}$ such that the θ-curve is shorter if and only if $A_1 = A_2 > A_0$.

Using Mathematica we were able to find an approximate value of A_0. For $A_1 = A_2 < \frac{2\pi}{3}$, we consider the same construction as for ϕ, but shift it downwards a euclidean distance of p to the red curve in Figure 3. This is the only possible θ-curve enclosing A_1 and A_2 which satisfies the regularity and constant curvature conditions of a perimeter-minimizing double bubble. By symmetry, the length is given by adding four times the length of one arc (we take the one centered at $(0, p)$) to two

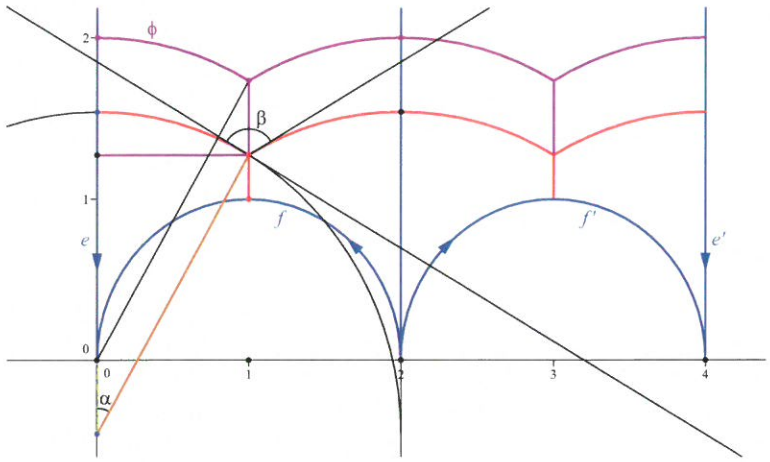

Figure 3. A θ-curve on a thrice-punctured sphere in the upper half-plane allows us to parametrize the area and perimeter of the θ-curve for equal areas.

times the length of one vertical segment (we take the one starting at $(1, 1)$). Using the standard parametrizations for these curves and the length formula given we obtain

$$\text{Perimeter}(p) = 4\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin x - \frac{p}{2}}\,dx - 2\ln(\sqrt{3} - p).$$

The area enclosed by the red curve is given by taking four times area of the region between the red arc and the first half of f. Applying the given formula for computing area we have

$$\text{Area}(p) = 4\int_0^1 \int_{\sqrt{1-(x-1)^2}}^{\sqrt{4-x^2}-p} \frac{1}{y}\,dy\,dx = 4\int_0^1 -\frac{1}{\sqrt{4-x^2}-p} + \frac{1}{1-(x-1)^2}\,dx.$$

Given these parametrizations of area and perimeter, we can plot the perimeter against the area as in Figure 4. Further computations via Mathematica show that a θ-curve is more efficient than horocycles for areas greater than about $3.4076/2$. □

Lemma 4.3. *In the hyperbolic plane, for a disc of area A and perimeter P the following statements hold*:

(1) *If* $A \le \pi$, *then* $P \ge 2.2A$.

(2) *If* $A \le \frac{\pi}{2}$, *then* $P \ge 3A$.

(3) *If* $A \le \frac{4\pi}{9}$, *then* $P \ge \sqrt{10}$.

(4) *If* $A \le \frac{4\pi}{15}$, *then* $P \ge 4A$.

(5) *If* $A < 8\pi/(9 + 3\sqrt{13})$, *then* $P > \frac{4\pi}{3}$.

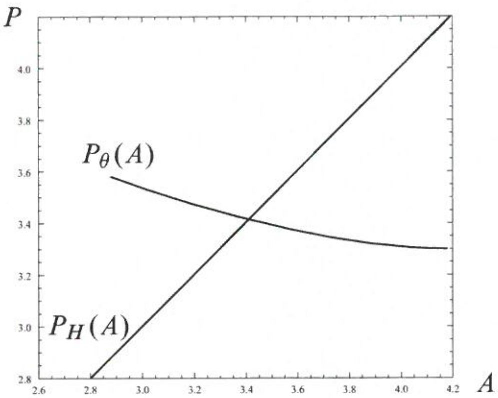

Figure 4. The θ-curve is shorter than horocycles for equal areas greater than about 1.7.

Proof. Set $c = P/A$. If we parametrize area and perimeter of such a disc using the hyperbolic radius, s, then

$$c = \frac{2\pi \sinh s}{4\pi \sinh^2 \frac{s}{2}} = \coth \tfrac{s}{2}.$$

Notice that $\coth \frac{s}{2}$ is decreasing with s, whereas $A = 4\pi \sinh^2 \frac{s}{2}$ is increasing with s. Therefore $\coth \frac{s}{2}$ is bounded below by its value at the hyperbolic radius corresponding to the largest A. Suppose $A \leq \pi$. We solve $A \leq 4\pi \sinh^2 \frac{s}{2}$ to find $s \leq \cosh^{-1} \frac{3}{2}$. Hence $c \leq \coth\bigl(\frac{1}{2} \cosh^{-1}\bigl(\frac{3}{2}\bigr)\bigr) \approx 2.22$. Thus $P = cA \geq 2.2A$. Therefore, the first statement holds. Statements (2)–(4) are shown by the same method.

To show (5), we suppose that $A > 8\pi/(9+3\sqrt{13})$. Since $P(A) = \sqrt{A^2 + 4\pi A}$ is strictly increasing for all positive A, we have that for $A > 8\pi/(9+3\sqrt{13})$,

$$P > \sqrt{\left(\frac{8\pi}{9+3\sqrt{13}}\right)^2 + 4\pi \frac{8\pi}{9+3\sqrt{13}}} = \frac{4\pi}{3}. \qquad \square$$

Remark 4.4. In Lemma 4.3(1)–(4) both inequalities of each statement may be made strict and the statements will still hold. The method of proof is the same.

Lemma 4.5. *For two regions on the thrice-punctured sphere with areas A_1 and A_2 such that $A_1, A_2 \leq A_3 = 2\pi - A_1 - A_2$, we have that $A_1, A_2 \leq \pi$.*

Proof. If this was not true, the total area $A_1 + A_2 + A_3$ would exceed 2π, which is the area of the thrice-punctured sphere (Proposition 3.5). $\square$

Lemma 4.6. *Given a double bubble with regions of areas $0 < A_1, A_2 \leq 2\pi - A_1 - A_2$ and perimeters P_i, the total perimeter P satisfies $P \geq A_1 + \frac{1}{2}P_2$.*

Proof. Denote the area and perimeter of the complementary region by A_3 and P_3. By Lemma 4.5, $A_1 \leq \pi$. Thus Proposition 3.2 implies that $P_1 \geq A_1$. If $A_3 < \pi$, then

$$P_3 \geq A_3 = 2\pi - A_1 - A_2 \geq (2A_1 + A_2) - A_1 - A_2 = A_1.$$

If $A_3 > \pi$, then

$$P_3 \geq 2\pi - A_3 = 2\pi - (2\pi - A_1 - A_2) = A_1 + A_2 \geq A_1.$$

Therefore the total perimeter satisfies

$$P = \tfrac{1}{2}(P_1 + P_2 + P_3) \geq \tfrac{1}{2}(2A_1 + P_2) = A_1 + \tfrac{1}{2}P_2. \qquad \square$$

Proposition 4.7. *On a thrice-punctured sphere, a curve enclosing and separating regions R_i of perimeters P_i and areas $A_1, A_2 \leq 2\pi - A_1 - A_2$ has total perimeter $P > A_1 + A_2$ if R_1 or R_2 is a union of topological discs. In particular, it is not perimeter-minimizing.*

Proof. Suppose R_2 is the union of topological discs. Let P_i denote the perimeter of R_i. Since the disc is isoperimetric in the hyperbolic plane, P_2 is greater than or equal to the perimeter of a hyperbolic disc of the same area. By Lemma 4.5, $A_2 \leq \pi$. Thus, by Lemma 4.3(1), $P_2 \geq 2.2A_2$. By Lemma 4.6, the total perimeter P satisfies

$$P \geq A_1 + \tfrac{1}{2}P_2 > A_1 + \tfrac{1}{2}(2.2)A_2 > A_1 + A_2.$$

Therefore it cannot be perimeter-minimizing, because horocycles on two separate cusps have perimeter $A_1 + A_2$. $\square$

Proposition 4.8. *In a perimeter-minimizing double bubble with connected perimeter containing regions R_i of perimeters P_i and areas $A_1, A_2 \leq A_3 = 2\pi - A_1 - A_2$, all three regions contain a cusp.*

Proof. Both regions must have a component which is not a topological disc; otherwise horocycles enclosing the same area would be shorter than the perimeter of our double bubble by Proposition 4.7, contradicting the fact that our bubble is perimeter-minimizing. These components of regions which aren't topological discs must contain cusps (they can't be annular regions since the perimeter is connected).

Suppose that R_3 is the union of topological discs. Then P_3 is greater than or equal to the perimeter of the hyperbolic disc of area A_3. Since dP/dA of the hyperbolic disc is always positive and $A_3 \geq \frac{2\pi}{3}$, P_3 is greater than or equal to the perimeter of the hyperbolic disc of area $\frac{2\pi}{3}$, which is $\frac{2\sqrt{7}\pi}{3}$. Therefore we have $P > P_3 > \frac{2\sqrt{7}\pi}{3} > \frac{4\pi}{3} \geq A_1 + A_2$, a contradiction. $\square$

Proposition 4.9. *Consider a double bubble enclosing areas $0 < A_1, A_2 \leq 2\pi - A_1 - A_2$, consisting of four region components C_i of areas $A_1 - a_1$, a_1, a_2, $A_2 - a_2$, where $a_1 \leq a_2$, and each C_i is adjacent only to C_{i-1} and C_{i+1} for $1 < i < 4$. Suppose*

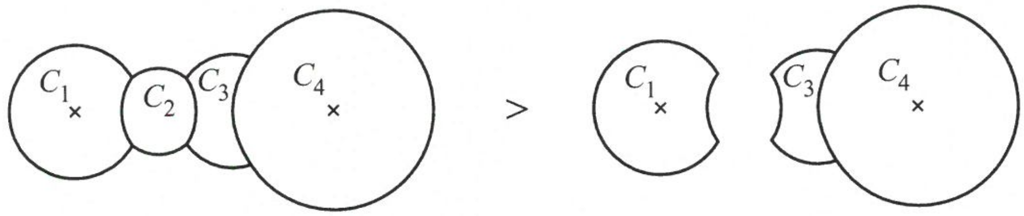

Figure 5. Lower bound on perimeter of four components of regions with two cusps.

that C_1 and C_4 have no common boundary and are each connected and contain cusps, and that the union of C_2 and C_3 (not necessarily connected) is the union of topological discs. Then the total perimeter satisfies $P > A_1 + A_2$, and the double bubble is not perimeter-minimizing.

Proof. Suppose that $\left(\frac{\sqrt{10}}{2} - 1\right)a_2 \leq a_1$ and $\frac{4\pi}{15} \leq a_2$. Then

$$a_1 + a_2 \geq \frac{\sqrt{10}}{2} a_2 \geq \frac{\sqrt{10}}{2}\frac{4\pi}{15} > \frac{8\pi}{3(3+\sqrt{13})}.$$

Therefore $a_1 + a_2 > 8\pi/(3(3+\sqrt{13}))$. We conclude that at least one of the following conditions must be satisfied:

(1) $8\pi/(3(3+\sqrt{13})) < a_1 + a_2$.

(2) $a_1 \leq \left(\frac{\sqrt{10}}{2} - 1\right)a_2$ and $a_2 < \frac{4\pi}{9}$.

(3) $a_2 < \frac{4\pi}{15}$.

Therefore it suffices to show $P > A_1 + A_2$ for the three cases where at least one of these conditions is satisfied.

Case 1: Since the union of C_2 and C_3 is the union of topological discs with boundary and the disc is isoperimetric in the hyperbolic plane, the length of the boundary of their union is greater than the perimeter of a hyperbolic disc of area $a_1 + a_2$. Therefore, by Lemma 4.3(5), $P > \frac{4\pi}{3} > A_1 + A_2$.

To show the remaining cases, we remove the unshared perimeter of C_2 (see Figure 5) and consider the sum of P_1 and the total perimeter of C_3 and C_4. Since $A_1 - a_1 \leq A_1 \leq \pi$ (Lemma 4.5), by Proposition 3.2, $P_1 \geq A_1 - a_1$. Since C_3 is the union of topological discs, the total perimeter of C_3 and C_4 is bounded below by $A_2 - a_2 + \frac{1}{2}P_3$, by Lemma 4.6. Thus $P \geq A_1 - a_1 + A_2 - a_2 + \frac{1}{2}P_3$.

Case 2: Since $a_2 < \frac{4\pi}{9}$, by Lemma 4.3(2) we have $P_3 > \sqrt{10}a_2$; thus

$$P \geq A_1 - a_1 + A_2 - a_2 + \tfrac{1}{2}P_3 > A_2 - a_2 + A_1 - a_1 + \tfrac{\sqrt{10}}{2}a_2.$$

Since $a_1 \leq \left(\frac{\sqrt{10}}{2} - 1\right)a_2$, we have

$$\begin{aligned} P &> A_2 - a_2 + A_1 - a_1 + \left(\tfrac{\sqrt{10}}{2} - 1\right)a_2 + a_2 \\ &\geq A_2 - a_2 + A_1 - a_1 + a_1 + a_2 = A_1 + A_2. \end{aligned}$$

Case 3: By Lemma 4.3(3), we have $P_3 > 4a_2$; thus

$$\begin{aligned} P &\geq A_2 - a_2 + A_1 - a_1 + \tfrac{1}{2}P_3 > A_2 - a_2 + A_1 - a_1 + 2a_2 \\ &\geq A_2 - a_2 + A_1 - a_1 + a_2 + a_1 = A_1 + A_2. \end{aligned}$$

We conclude that $P > A_1 + A_2$. Hence it is not perimeter-minimizing, as horocycles on separate cusps have perimeter $A_1 + A_2$. □

5. Once-punctured surfaces

Some of the methods employed in Section 4, can be applied to other hyperbolic surfaces of constant Gaussian curvature -1 that share some features of the thrice-punctured sphere, such as having area of 2π and at least one cusp, but lack its fixed hyperbolic structure. For example, a once punctured torus has many hyperbolic structures, yet all have area 2π. Proposition 5.1 shows that for relatively small areas on such a surface, two horocycles have less perimeter than one horocycle with a lens.

Proposition 5.1. *Given two areas $0 < A_1, A_2 \leq \frac{4\pi}{15}$ on a punctured surface of area 2π, the union of two horocycles about the cusp enclosing and separating A_1 and A_2 is shorter than a horocycle with a lens.*

Proof. Without loss of generality suppose that A_1 is not on the cusp. Since $A_1 \leq \pi$ (Lemma 4.5), by Proposition 3.2, our surface has the same isoperimetric profile for single bubbles as the thrice-punctured sphere. Thus Lemma 4.6 holds, and the total perimeter, P, of our enclosure satisfies the inequality $P \geq A_2 + \frac{1}{2}P_1$. By Lemma 4.3(4), $P_1 \geq 4A_1$ for $A_1 \geq \frac{4\pi}{15}$. Thus $P \geq A_2 + 2A_1$. □

Acknowledgements

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