

The distortion dimension of \mathbb{Q} –rank 1 lattices

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Abstract

Let $X = G/K$ be a symmetric space of noncompact type and rank $k \geq 2$. We prove that horospheres in X are Lipschitz $(k-2)$ –connected if their centers are not contained in a proper join factor of the spherical building of X at infinity. As a consequence, the distortion dimension of an irreducible \mathbb{Q} –rank-1 lattice Γ in a linear, semisimple Lie group G of \mathbb{R} –rank k is $k-1$. That is, given $m < k-1$, a Lipschitz m –sphere S in (a polyhedral complex quasi-isometric to) Γ , and a $(m+1)$ –ball B in X (or G) filling S , there is a $(m+1)$ –ball B' in Γ filling S such that $\text{vol } B' \sim \text{vol } B$. In particular, such arithmetic lattices satisfy Euclidean isoperimetric inequalities up to dimension $k-1$.

Key words: Lipschitz connectivity, subgroup distortion, horospheres, symmetric spaces, arithmetic groups, Dehn functions

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1 Introduction and main results

Let G be a Lie group equipped with a left invariant metric and $\Gamma \subset G$ a finitely generated discrete subgroup equipped with a word metric. If Γ is cocompact, then Γ is quasi-isometric to G and thus both have the same large-scale geometry. If Γ is not cocompact, the large-scale geometric properties of G and Γ can be very different. For instance, $\Gamma = \text{SL}(2, \mathbb{Z})$ is exponentially distorted in $\text{SL}(2, \mathbb{R})$, see e.g. [11, Ch. 3].

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The higher-dimensional geometry of Γ can also be highly distorted. This was first seen in a result of Epstein, Cannon, Holt, Levy, Paterson, and Thurston, who showed that $\mathrm{SL}(n, \mathbb{Z})$ has an exponentially large $(n - 1)$ -dimensional filling volume function [10, Ch. 10.4]. That is, there is a $c > 0$ such that for any sufficiently large r , there is a $(n - 2)$ -sphere S in $\mathrm{SL}(n, \mathbb{Z})$ of volume r^{n-2} such that any $(n - 1)$ -ball with boundary S has volume at least e^{cr} . On the other hand, Thurston famously conjectured that this distortion does not occur in lower dimensions; that the 2-dimensional filling volume function $\mathrm{FV}_{\mathrm{SL}(k, \mathbb{Z})}^2(r)$ (the Dehn function) should satisfy $\mathrm{FV}_{\mathrm{SL}(k, \mathbb{Z})}^2(r) \sim r^2$ when $k \geq 4$. Gromov generalized this conjecture to arbitrary lattices in semisimple groups; his conjecture posits that if Γ is a nonuniform lattice in G and G has rank k , then the $(k - 1)$ -dimensional filling volume function of Γ is exponential and lower-dimensional filling volume functions are polynomial [11, 5.D].

This conjecture has been studied for many years, and many cases of the conjecture are now known. To state a few:

- C. Druțu showed that the 2-dimensional filling volume function $\mathrm{FV}_{\Gamma}^2(n)$ is at most $n^{2+\epsilon}$ when G has rank at least 3 [7, 8].
- Leuzinger and Pittet showed that if G has rank 2, then there is a $c > 0$ such that $\mathrm{FV}_{\Gamma}^2(n) \gtrsim e^{cr}$ [17].
- Wortman showed that if Γ is an arithmetic subgroup of relative \mathbb{Q} -type A_n, B_n, C_n, D_n, E_6 or E_7 , then there is a $c > 0$ such that $\mathrm{FV}_{\Gamma}^{k-1}(r^{k-2}) \gtrsim e^{cr}$, where k is the rank of G [23].
- Bux, Köhl, and Witzel recently showed that the finiteness length of an S -arithmetic group is equal to its total rank minus 1 [5]. This is a nonquantitative version of Gromov and Thurston's conjecture.

More generally, we may consider a group G of the form $G = \prod_{i=1}^m G_i(k_i)$, where the k_i are locally compact, non-discrete fields and the G_i are connected, absolutely almost simple algebraic groups defined over k_i . In [6], Bux and Wortman formulated a version of Gromov and Thurston's conjecture in terms of the higher-dimensional distortion of such groups. Let X be the product of irreducible symmetric spaces and Euclidean buildings on which G acts. The total rank of G is then equal to the maximal dimension of an isometrically embedded Euclidean space in X , which we call the *geometric rank* of X and denote by $\mathrm{geo-rank}(X)$.

Let Γ be an irreducible lattice in G . For some point $x \in X$ and a real number r define the the following thickening of the orbit $\Gamma \cdot x$ in X

$$X(r) := \{y \in X \mid d(y, \Gamma \cdot x) \leq r\}.$$

Note that by the Milnor–Švarc lemma, the induced inner metric on $X(r)$ is quasi-isometric to (Γ, d_Γ) . Following [6] we define Γ as being *undistorted up to dimension m* if: given any $r \geq 0$, there exist real numbers $r' \geq r, \lambda \geq 1$, and $C \geq 0$ such that for any $k < m$ and any Lipschitz k –sphere $S \subset X(r)$, there is a Lipschitz $(k+1)$ –ball $B_\Gamma \subset X(r')$ with $\partial B_\Gamma = S$ and

$$\text{volume}(B_\Gamma) \leq \lambda \text{ volume}(B_X) + C$$

for all Lipschitz $(k+1)$ –balls $B_X \subset X$ with $\partial B_X = S$. The *distortion dimension of Γ* is then defined as

$$\text{dis-dim}(\Gamma) = \max\{m \mid \Gamma \text{ is undistorted up to dimension } m\}.$$

Bux and Wortman posit that $\text{dis-dim}(\Gamma) = \text{geo-rank}(X) - 1$. This would imply the conjecture of Gromov and Thurston.

In this paper, we will prove the Bux–Wortman conjecture for \mathbb{Q} –rank 1 arithmetic groups in linear, semisimple groups defined over number fields, i.e. finite extensions of \mathbb{Q} . For such lattices the space X above is a symmetric space of noncompact type; that is, there are no building factors. Similar arguments hold for groups defined over general fields, but for simplicity, we restrict ourselves to the case of symmetric spaces.

In our proof, it will be convenient to replace the subset $X(r)$ by the complement of a countable union of horoballs in X (see [15], Thm. 3.6). Like $X(r)$, this is quasi-isometric to (Γ, d_Γ) . A crucial fact is that for \mathbb{Q} –rank 1 lattices these horospheres are *disjoint*.

Theorem A (Distortion dimension). *The distortion dimension of an irreducible \mathbb{Q} –rank 1 lattice in a linear, semisimple Lie group of \mathbb{R} –rank k is $k-1$. If $k \geq 2$, then such an arithmetic lattice satisfies Euclidean isoperimetric inequalities up to dimension $k-1$ and an exponential isoperimetric inequality in dimension k .*

That is, there is a $(k-2)$ –connected complex Y that is equivariantly quasi-isometric to Γ such that for any $m \leq k-2$ and any Lipschitz m –sphere $S \subset Y$, there is a Lipschitz $(m+1)$ –ball $B \subset Y$ such that $\partial B = S$ and

$$\text{vol } B \lesssim (\text{vol } S)^{\frac{m+1}{m}}.$$

Conversely, for $r > 1$, there is a Lipschitz sphere $S : S^{k-1} \rightarrow Y$ such that $\text{vol } S \sim r^{k-1}$ but $\text{vol } B \gtrsim e^r$ for any Lipschitz k –ball $B \subset Y$ such that $\partial B = S$.

Gromov showed that any nonuniform lattice Γ in a semisimple group G of \mathbb{R} –rank 1 is exponentially distorted and thus has distortion dimension 0 (see [11, 3.G]). Link recently

showed this theorem in the case that Γ is an irreducible lattice acting on a product of rank-1 symmetric spaces [18].

In many cases, the lower bound in Theorem A follows from a result of Wortman [23]. If Γ has \mathbb{Q} -rank 1, then its relative \mathbb{Q} -type is either A_1 or BC_1 . In the former case, Wortman showed that Γ has an exponential isoperimetric inequality in dimension k . We will prove the exponential isoperimetric inequality in the general case.

By work of Young [24], non-distortion for subsets of spaces with finite Assouad–Nagata dimension is a consequence of Lipschitz connectivity. Recall that Z is Lipschitz n -connected if for all $d \leq n$ and any Lipschitz map $\alpha: S^d \rightarrow Z$, there is an extension $\beta: D^{d+1} \rightarrow Z$ such that $\text{Lip}(\beta) \lesssim \text{Lip}(\alpha)$. Theorem 1.3 of [24] (see also [25]) states the following:

Proposition 1.1 (Distortion and connectivity, [24, 1.3]). *Let X be a metric space and let $Z \subset X$ be a nonempty closed subset with inner metric induced by the metric of X . Suppose in addition that X is a geodesic metric space such that the Assouad–Nagata dimension $\dim_{AN}(X)$ of X is finite and one of the following is true:*

- *Z is Lipschitz n -connected.*
- *X is Lipschitz n -connected, and if $X_p, p \in P$ are the connected components of $X \setminus Z$, then the sets $H_p = \partial X_p$ are Lipschitz n -connected with uniformly bounded implicit constant.*

Then Z is undistorted up to dimension $n + 1$.

Thus, to prove the upper bounds in Theorem A, it suffices to show that horospheres in X are highly Lipschitz connected:

Theorem B (Horospheres are highly Lipschitz connected). *Let $X = G/K$ be a symmetric space of noncompact type and rank k . Then any horosphere in X whose center is not contained in a proper join factor of the boundary of X at infinity is Lipschitz $(k - 2)$ -connected and thus undistorted up to dimension $k - 1$.*

The key step in Theorem B is constructing flats E that face “away from” a chamber α in the boundary at infinity of X (Sec. 3.3). These flats are arranged so that if Z is a horosphere whose center is in α , then $E \cap Z$ is roughly spherical. (In fact, spheres of the form $E \cap Z$ will provide the lower bound in Theorem A.) Similar constructions appear in [8, Sec. 3.3] and [24, Sec 4.3] in the case that X is a building. In those cases, E is constructed using a sequence of ramifications; here, we will show that flats like E are abundant and use their abundance to prove the Lipschitz connectivity of Z .

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2 Sketch of proof

The core of this paper is the proof of Theorem B. This proof is structured similarly to the proof of Theorem 4.1 in [24], which established Lipschitz connectivity for certain horospheres in Euclidean buildings. The proof of Theorem 4.1 in [24] used a version of Morse theory based on the *downward link at infinity* of a vertex in a Euclidean building. Let \mathcal{B} be such a building and let \mathcal{B}_∞ be its boundary at infinity. Then, if a horoball is centered at a point $\tau \in \mathcal{B}_\infty$, the downward link at infinity of a point $x \in \mathcal{B}$ is the subset of \mathcal{B}_∞ consisting of the limit points of geodesic rays from x that point away from τ .

We start by constructing an analogue of the downward link for symmetric spaces. When X is a symmetric space, we let X_∞ be its (geodesic) boundary at infinity and equip X_∞ with the Tits metric associated to the angular metric \angle . Then X_∞ has the additional structure of a spherical building, see [1], Appendix 5 and [3], Ch. II.10. We use the building structure to replace downward links by *shadows*. If H is a horoball and $x \in H$, then each chamber in X_∞ is the limit set of a Weyl chamber based at x . The shadow of x consists of the limit sets of Weyl chambers that “strongly point out of” H (Sec. 3.2). We denote the shadow of x by \mathcal{S}_x ; note that this is a set of chambers in X_∞ viewed as a spherical building.

For $x \in H$ we denote the union of the chambers in \mathcal{S}_x by Σ_x . This is a subset of the geodesic boundary X_∞ . It is a collection of directions (or limits) of geodesic rays starting at x that head toward $Z = \partial H$ quickly. In particular, there is a Lipschitz map $i_x: \Sigma_x \rightarrow Z$ that takes each direction to the point where the corresponding ray intersects Z .

Next, we show that shadows are $(k-2)$ –connected (Sections 3.3 and 3.4) by constructing many flats inside each shadow. For each shadow \mathcal{S}_x , we will find a chamber $\mathfrak{d} \subset X_\infty$ such that for each chamber $e \in \mathcal{S}_x$, there is a flat $E_{\mathfrak{d},e} \subset X$ that is spanned by \mathfrak{d} and e and whose boundary at infinity is contained in a larger shadow. The boundary at infinity of the union of these flats is a union of apartments in the spherical building. This union has the homotopy type of a wedge of $(k-1)$ –spheres, so Σ_x is $(k-2)$ –connected inside a larger shadow.

Finally, we use shadows to prove that Z is Lipschitz $(k - 2)$ -connected (Section 3.5). Let Δ_Z be the infinite-dimensional simplex with vertex set Z . It suffices to construct a map $\Omega: \Delta_Z \rightarrow Z$ with certain metric properties. Since shadows are $(k - 2)$ -connected, we construct a map $\Omega_\infty: \Delta_Z \rightarrow X_\infty$ such that the image of each face of Δ_Z lies in a shadow. We can then use the map i_x above to send these shadows to Z .

3 Proof of Theorem B

3.1 Preliminaries and standing assumptions

Let $X = G/K$ be a symmetric space of noncompact type and rank $k \geq 2$. If $g \in G$, let $[g] = gK \in X$ be the corresponding coset of K . Let X_∞ be the (geodesic) boundary of X at infinity and equip X_∞ with the Tits metric associated to the angular metric \angle . Note that if X is a Riemannian product of irreducible factors, $X = X_1 \times \dots \times X_m$, then its boundary X_∞ is the spherical join of the boundaries of the factors, $X_\infty = (X_1)_\infty * \dots * (X_m)_\infty$ (see [3], II.8.11.).

Let $\tau \in X_\infty$ be a point that is not contained in a proper join factor of X_∞ , i.e., τ is the limit point of a geodesic ray that is not constant on any proper factor. Let $H \subset X$ be a horoball centered at τ and let $Z := \partial H$ be the boundary horosphere. Let $h: X \rightarrow \mathbb{R}$ be the Busemann function centered at τ , oriented so that $H = h^{-1}([0, \infty))$ and $Z = h^{-1}(\{0\})$. We define H_∞ to be the open ball $B_{\pi/2}(\tau)$ in X_∞ , so that $(h \circ r)'(t) > 0$ for any geodesic ray r that is asymptotic to a point in H_∞ . Since τ is not contained in a proper join factor, there is a chamber $\mathfrak{c} \subset X_\infty$ such that $\tau \in \mathfrak{c}$ and $\mathfrak{c} \subset H_\infty$ (see [13], Section 3). This is the main reason for the assumption that τ is not contained in a proper join factor; the fact that $\mathfrak{c} \subset H_\infty$ is crucial to Lemma 3.6. We note that if X is irreducible, a Weyl chamber has diameter less than $\pi/2$. Thus any $\tau \in X_\infty$ and the associated Busemann function also have the above properties.

The stabilizer of \mathfrak{c} in G is a minimal parabolic subgroup P , and the set of (maximal) chambers in X_∞ can be identified with the homogeneous space G/P (see e.g. [22, Ch. 1.2], or [20, Lemma 4.1]). We let $P = NAM$ be its Levi decomposition. Thus N is nilpotent, A abelian, and M is the centralizer of A in K and in particular compact. Note that, by the Iwasawa decomposition $G = NAK$, P acts transitively on X .

For $x \in X$ there is a unique flat E_x containing x and \mathfrak{c} . Let $E_0 = E_{eK} = [A]$ and let \mathfrak{c}^* be the chamber opposite to \mathfrak{c} in E_0 . More generally, let \mathfrak{c}_x^* be the chamber opposite to \mathfrak{c} in E_x . If $x = [p]$ and $p = nam$ for some $n \in N$, $a \in A$, $m \in M$, then $E_x = nE_0 = pE_0$ and $\mathfrak{c}_x^* = p\mathfrak{c}^* = n\mathfrak{c}^*$.

Let $\mathfrak{X}_\infty^0(\mathfrak{c})$ be the set of chambers opposite to \mathfrak{c} , so that $\mathfrak{X}_\infty^0(\mathfrak{c}) = N \cdot \mathfrak{c}^*$. Let $X_\infty^0(\mathfrak{c})$ be the corresponding subcomplex. If $\mathfrak{d} \in \mathfrak{X}_\infty^0(\mathfrak{c})$, we define $E_{\mathfrak{d}}$ to be the flat asymptotic to both \mathfrak{c} and \mathfrak{d} .

The notation $f \lesssim g$ means that there is some constant c such that $f \leq cg$. We write $f \sim g$ if there is some $c > 0$ such that $c^{-1}g \leq f \leq cg$. When c depends on x and y , we write $f \lesssim_{x,y} g$ or $f \sim_{x,y} g$. All of our constants will depend implicitly on X and τ , so we will omit X and τ from these subscripts.

3.2 Shadows

In this section, we define the shadow $\mathcal{S}_x \subset \mathfrak{X}_\infty^0(\mathfrak{c})$ of a point $x \in X$ and prove some of its properties. The shadow of a point in a symmetric space will play a similar role to the downward link at infinity of a point in a Euclidean building in [24].

Definition 3.1. *Let \mathfrak{g} and \mathfrak{n} be the Lie algebras of G and N . The metric on X is induced by an $Ad(K)$ -invariant norm $\|\cdot\|$ on \mathfrak{g} (see [9], 2.7.1). For $n \in N$, define $d_N(n) = \|\log n\|$.*

The r -shadow of $x = [p]$ with respect to \mathfrak{c} is the set of chambers

$$\mathcal{S}_x(r) := \{\mathfrak{c}_{[pn]}^* \in \mathfrak{X}_\infty^0(\mathfrak{c}) \mid n \in N, d_N(n) < r\}.$$

We set $\mathcal{S}_x = \mathcal{S}_x(1)$. Note that $\mathcal{S}_x(r)$ is well-defined, since by $Ad(K)$ -invariance $d_N(mnm^{-1}) = d_N(n)$ for all $m \in M \subset K$ and $n \in N$; moreover $\mathcal{S}_x(r) = p \cdot \mathcal{S}_{[e]}(r)$.

If $\mathfrak{d} \in \mathfrak{X}_\infty^0(\mathfrak{c})$ and $x = [p] \in X$, then there is a $q_x(\mathfrak{d}) \in N$ such that $[pq_x(\mathfrak{d})] \in E_{\mathfrak{d}}$. In fact, if $n_{\mathfrak{d}}, n \in N$ and $a \in A$ are such that $E_{\mathfrak{d}} = [n_{\mathfrak{d}}A]$ and $x = [na]$, then we can write $q_x(\mathfrak{d}) = a^{-1}n^{-1}n_{\mathfrak{d}}a$. This is unique up to conjugation by some $m \in M$. We set

$$\rho_x(\mathfrak{d}) = d_N(q_x(\mathfrak{d})),$$

this is well-defined (i.e., independent of the choice of $q_x(\mathfrak{d})$). Further we have: $\mathfrak{d} \in \mathcal{S}_x(r)$ if and only if $\rho_x(\mathfrak{d}) < r$. Roughly, the function ρ_x measures the angle at which the Weyl chamber based at x and asymptotic to \mathfrak{d} meets the apartment E_x . When $\rho_x(\mathfrak{d})$ is small, then \mathfrak{d} is close to E_x , and when it is large, \mathfrak{d} deviates more sharply. We think of \mathcal{S}_x as the “shadow” on $\mathfrak{X}_\infty^0(\mathfrak{c})$ cast by a light at \mathfrak{c} shining on a ball around x with radius roughly 1.

Lemma 3.2. *If $x \in X$ and $\mathfrak{d} \in \mathcal{S}_x(r)$, then $d(x, E_{\mathfrak{d}}) < r$. Conversely, there is a $c > 0$ depending on X such that if $d(x, E_{\mathfrak{d}}) < r$, then $\mathfrak{d} \in \mathcal{S}_x(e^{cr})$.*

Proof. First, if $x = [p] \in X$ and $\mathfrak{d} \in \mathcal{S}_x(r)$, then, as we have seen above, there is an $n = q_x(\mathfrak{d}) \in N$ such that $[pn] \in E_{\mathfrak{d}}$ and $d_N(n) < r$. It follows that $d(x, E_{\mathfrak{d}}) \leq d([p], [pn]) \leq d_N(n) < r$.

Conversely, suppose that $d(x, y) < r$. Without loss of generality, we may take $x = [e]$. Let $n \in N$ and $a \in A$ be such that $y = [na]$. Then $\mathfrak{c}_y^* = \mathfrak{c}_{[n]}^*$, so it suffices to show that $d_N(n)$ is exponentially bounded in r .

The map $[na] \mapsto a$ is a distance-decreasing map from X to A , so

$$d([e], [a]) \leq d([e], [na]) = d(x, y) < r.$$

It follows that

$$d([e], [n]) \leq d([e], [na]) + d([na], [n]) \leq 2r.$$

By [19], n satisfies the inequality

$$\log d_N(n) \lesssim d([e], [n]) \lesssim 1 + \log d_N(n),$$

so $d_N(n) \leq e^{cr}$ as desired. \square

The shadow of x grows exponentially as x moves toward \mathfrak{c} .

Lemma 3.3. *Let $x \in X$ and let $\gamma: [0, \infty) \rightarrow X$ be a unit-speed geodesic ray starting at x and pointing at a point $\sigma \in \text{int } \mathfrak{c}$ in the interior of \mathfrak{c} . There is a constant $\kappa > 0$ depending on σ such that for all $t \geq 0$ and all $\mathfrak{d} \in \mathfrak{X}_\infty^0(\mathfrak{c})$,*

$$\rho_{\gamma(t)}(\mathfrak{d}) \lesssim e^{-\kappa t} \rho_x(\mathfrak{d}).$$

Proof. Without loss of generality, we may assume that $x = [e]$ and that $x' = [\exp tV]$ for a regular unit vector V in the open Weyl chamber in $T_x X$ corresponding to the chamber $\mathfrak{c} \in X_\infty$ (see [1], appendix 5). Let Σ_+ be the corresponding set of positive roots. Let $a(t) = \exp tV$ and let

$$\kappa := \min_{\alpha \in \Sigma_+} \alpha(V) > 0.$$

Let $n = q_x(\mathfrak{d})$ so that $\rho_x(\mathfrak{d}) = d_N(n)$ and $q_{\gamma(t)}(\mathfrak{d}) = a(-t)na(t)$. The Lie algebra \mathfrak{n} of N can be written as the sum of (positive) root spaces $\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$. Thus $\log n = \sum_{\alpha \in \Sigma^+} X_\alpha$ and

$$q_{\gamma(t)}(\mathfrak{d}) = \exp[\text{Ad}(a(-t)) \log n] = \exp \sum_{\alpha \in \Sigma^+} e^{-t\alpha(V)} X_\alpha.$$

Then

$$\rho_{\gamma(t)}(\mathfrak{d}) = d_N(q_{\gamma(t)}(\mathfrak{d})) \lesssim e^{-\kappa t} d_N(n).$$

as desired. \square

Let $A_+ \subset A$ denote the Weyl chamber based at the identity that is asymptotic to \mathfrak{c} . For $p \in P$ and $x = [p] \in X$, let $C_x := [pA_+]$ be the Weyl chamber based at x and asymptotic to \mathfrak{c} . Because shadows grow exponentially with height, we can expand \mathcal{S}_x greatly by replacing x by a point in C_x . Let

$$D_x := \{y \in X \mid d(y, C_x) < 1\}.$$

The set D_x is roughly the set of points whose shadows contain \mathcal{S}_x .

Lemma 3.4. *There is a $\rho > 0$ such that for all $x \in X$ and all $y \in D_x$, $\mathcal{S}_x \subset \mathcal{S}_y(\rho)$.*

Proof. Without loss of generality, suppose that $x = [e]$, so that $C_x = [A_+]$, where A_+ is the Weyl chamber in A corresponding to \mathfrak{c} . If we write $y = [an]$ with $a \in A$, $n \in N$, then $d(a, A^+) \leq 1$ and $d_N(n) \lesssim 1$. We write $a = a_+b$, where $a_+ \in A^+$ and $\|\log b\| < 1$.

Suppose that $n' \in N$ and $d_N(n') < 1$, so that $\mathfrak{c}_{[n']}^* \in \mathcal{S}_x$. Then

$$E_{[n']} = [n'A] = [ann^{-1}(a^{-1}n'a)A]$$

and $\mathfrak{c}_{[n']}^* \in \mathcal{S}_y(\rho)$ if and only if $d_N(n^{-1}(a^{-1}n'a)) < \rho$. But $d_N(n) \lesssim 1$, and

$$d_N(a^{-1}n'a) = \|\text{Ad}(a_+b) \log n'\| \lesssim 1$$

because the eigenvalues of $\text{Ad}(a_+)$ are all at most 1 and $\text{Ad}(b)$ is bounded. It follows that there is a ρ depending on X so that $\mathfrak{c}_{[n']}^* \in \mathcal{S}_y(\rho)$ and $\mathcal{S}_x \subset \mathcal{S}_y(\rho)$. \square

The shadows of a collection of points can all be contained in a larger shadow.

Lemma 3.5. *Let $x \in X$ and let $\gamma: [0, \infty) \rightarrow X$ be a unit-speed geodesic ray starting at x and pointing at a point $\sigma \in \text{int } \mathfrak{c}$ in the interior of \mathfrak{c} . Let $r > 0$. There is a point $x' = \gamma(t)$ with $t \lesssim_\sigma r + 1$ such that*

$$\bigcup_{y \in B_r(x)} \mathcal{S}_y \subset \mathcal{S}_{x'}$$

and $x' \in \bigcap_{y \in B_r(x)} D_y$.

Proof. Without loss of generality, we take $x = [e]$, $a(t) = \exp tV$, and $\gamma(t) = [\exp tV]$ as in the proof of Lemma 3.3.

Suppose that $y \in B_r(x)$. We claim that there is a $c > 0$ such that $\mathcal{S}_y \subset \mathcal{S}_{\gamma(t)}$ and $\gamma(t) \in D_y$ for all $t \geq cr + c$. First, we claim that $\mathcal{S}_y \subset \mathcal{S}_{\gamma(t)}$ when t is large. If $\mathfrak{d} \in \mathcal{S}_y$, Lemma 3.2 implies that $d(y, E_{\mathfrak{d}}) < 1$, so $d(x, E_{\mathfrak{d}}) < r + 1$. If c_0 is as in Lemma 3.2, then $\mathfrak{d} \in \mathcal{S}_x(e^{c_0(r+1)})$, and by Lemma 3.3, there is a c_1 such that $\mathfrak{d} \in \mathcal{S}_{\gamma(t)}$ for all $t > c_1(r + 1)$.

Next, we claim that $\gamma(t) \in D_y$ when t is large. Let $n \in N$, $a \in A$ be such that $y = [na]$ and $E_y = [nA]$. Let $\tilde{\gamma}(t) = n\gamma(t)$ so that $\tilde{\gamma}$ is a geodesic ray toward σ that lies in E_y . Then $d(y, \tilde{\gamma}(0)) = d([a], [e]) < r$, and since $\tilde{\gamma}$ points toward the interior of \mathfrak{c} , there is a c_2 such that $\tilde{\gamma}(t) \in C_y$ for all $t \geq c_2 r$.

If $t > \max\{c_2 r, c_1(r + 1)\}$, then

$$d(\gamma(t), C_y) \leq d(\gamma(t), \tilde{\gamma}(t)) \leq d_N(a(-t)na(t)).$$

and, since $\mathfrak{c}_y^* \in \mathcal{S}_{\gamma(t)}$, we have

$$d_N(a(-t)na(t)) = \rho_{\gamma(t)}(\mathfrak{c}_y^*) < 1.$$

So $\gamma(t) \in D_y$ as desired. \square

Finally, we can use shadows to define a map that projects directions in X_∞ to points in Z .

Lemma 3.6. *Let H , h , Z , τ , and \mathfrak{c} be as in the standing assumptions.*

For $u \in H$ such that $h(u) \geq 1$ and $\rho > 0$, let

$$\Sigma_u(\rho) = \bigcup_{\mathfrak{d} \in \mathcal{S}_u(\rho)} \mathfrak{d} \subset X_\infty$$

be the point set in X_∞ determined by the chambers in the shadow $\mathcal{S}_u(\rho)$. We define $i_u: X_\infty^0(\mathfrak{c}) \rightarrow Z$ so that $i_u(\sigma)$ is the point where the geodesic ray from u toward σ intersects Z . The distance traveled before reaching Z is bounded in terms of $h(u)$ and ρ :

$$d(i_u(\sigma), u) \lesssim h(u) + \rho.$$

The map i_u is locally Lipschitz with

$$\text{Lip}(i_u|_{\Sigma_u(\rho)}) \lesssim (\rho + 1)^2 d(u, Z).$$

Furthermore, if $u_1, u_2 \in H$ are such that $h(u_i) \geq 1$ and if $\sigma_i \in \Sigma_{u_i}(\rho)$, then

$$(1) \quad d(i_{u_1}(\sigma_1), i_{u_2}(\sigma_2)) \lesssim_\rho d(u_1, u_2) + \min\{h(u_1), h(u_2)\} \cdot \angle(\sigma_1, \sigma_2).$$

Proof. We proceed similarly to the arguments in Section 4.5 of [24]. Let

$$CX_\infty = (X_\infty \times [0, \infty)) / (X_\infty \times 0)$$

be the infinite cone over X_∞ . We equip CX_∞ with the Euclidean cone metric

$$d((\sigma_1, t_1), (\sigma_2, t_2))^2 = t_1^2 + t_2^2 - 2t_1 t_2 \cos \angle(\sigma_1, \sigma_2)$$

so that the cone over an apartment in X_∞ is isometric to Euclidean space. For $\sigma \in X_\infty$, let $r_{x,\sigma}: [0, \infty) \rightarrow X$ be the unit-speed geodesic ray based at x that is asymptotic to σ , and let $e_x: CX_\infty \rightarrow X$ be the “exponential map”

$$e_x(\sigma, t) = r_{x,\sigma}(t).$$

Because X is a CAT(0) space, this is a distance-decreasing map and if $x, x' \in X$, then

$$(2) \quad d(e_x(\sigma, t), e_{x'}(\sigma, t)) \leq d(x, x').$$

Since τ is not in a proper join factor (by the standing assumptions), there is an $\epsilon > 0$ such that $\mathfrak{c} \subset B_{\pi/2-\epsilon}(\tau)$. Each point $\sigma \in X_\infty^0(\mathfrak{c})$ is opposite to a point in \mathfrak{c} , so $\angle(\sigma, \tau) > \frac{\pi}{2} + \epsilon$. By the concavity of h , it follows that for each such σ , the ray $r_{u,\sigma}$ intersects Z exactly once.

Our first task is to show that $d(i_u(\sigma), u) \lesssim h(u) + \rho$ for all $\sigma \in \Sigma_u(\rho)$. Let $T_u(\sigma) = d(u, i_u(\sigma))$, so that

$$i_u(\sigma) = r_{u,\sigma}(T_u(\sigma)).$$

Without loss of generality, we can take u to be the basepoint $u = [e]$. By Lemma 3.2, we have $d(u, E_\mathfrak{d}) \leq \rho$ for all $\mathfrak{d} \in \mathcal{S}_u(\rho)$.

Suppose that $\sigma \in \Sigma_u(\rho)$ and that $\mathfrak{d} \in \mathcal{S}_u(\rho)$ is a chamber containing σ . Let $n \in N$ be such that $E_\mathfrak{d} = [nA]$; then $h([n]) = h(u) \geq 1$. The geodesic $r_{[n],\sigma}$ is contained in the flat $E_\mathfrak{d}$, so

$$h(r_{[n],\sigma}(t)) = h(u) + t \cos \angle(\sigma, \tau) \leq h(u) - t \sin \epsilon.$$

Since $r_{[n],\sigma}$ and $r_{u,\sigma}$ are asymptotic to the same point, the distance $d(r_{u,\sigma}(t), r_{[n],\sigma}(t))$ is a non-increasing function of t , and

$$\begin{aligned} h(r_{u,\sigma}(t)) &\leq h(r_{[n],\sigma}(t)) + d(r_{u,\sigma}(t), r_{[n],\sigma}(t)) \\ &\leq h(u) - t \sin \epsilon + \rho. \end{aligned}$$

If $t > (h(u) + \rho)(\sin \epsilon)^{-1}$, then $h(r_{u,\sigma}(t)) < 0$, so

$$T_u(\sigma) \leq \frac{h(u) + \rho}{\sin \epsilon} \lesssim h(u) + \rho.$$

as desired. Let $b := \frac{1+\rho}{\sin \epsilon}$; then $b \lesssim \rho + 1$ and $T_u(\sigma) \leq b h(u)$ for all u such that $h(u) \geq 1$.

Next, we bound the Lipschitz constant of T_u . If $\sigma_1, \sigma_2 \in \Sigma_u(\rho)$ and $\angle(\sigma_1, \sigma_2) \geq \frac{1}{2b}$, then

$$\frac{|T_u(\sigma_1) - T_u(\sigma_2)|}{\angle(\sigma_1, \sigma_2)} \leq \frac{bh(u)}{(2b)^{-1}} \lesssim b^2 h(u).$$

Otherwise, consider the case that $\angle(\sigma_1, \sigma_2) < \frac{1}{2b}$. Let $r_i = r_{u, \sigma_i}$ and $T_i = T_u(\sigma_i)$. Without loss of generality, suppose $T_1 \leq T_2$, so that $h(r_1(T_1)) = 0$ and $h(r_2(T_1)) \geq 0$. We will show that $h(r_2(T_1))$ is small and that $(h \circ r_2)(t)$ is decreasing quickly at $t = T_1$.

Since X is CAT(0), we have

$$d(r_1(T_1), r_2(T_1)) \leq bh(u)\angle(\sigma_1, \sigma_2) \leq \frac{h(u)}{2}.$$

It follows that $(h \circ r_2)(T_1) \leq bh(u)\angle(\sigma_1, \sigma_2)$. Furthermore, since $(h \circ r_2)(t)$ is concave down, we have

$$-(h \circ r_2)'(t) > \frac{h(u) - (h \circ r_2)(T_1)}{T_1} \geq \frac{h(u)}{2bh(u)} \gtrsim \frac{1}{b}$$

for all $t \geq T_1$. Consequently,

$$\begin{aligned} T_2 - T_1 &\leq \frac{(h \circ r_2)(T_1)}{-(h \circ r_2)'(T_1)} \\ &\lesssim \frac{bh(u)\angle(\sigma_1, \sigma_2)}{b^{-1}} \\ &\lesssim b^2 h(u)\angle(\sigma_1, \sigma_2), \end{aligned}$$

so $\text{Lip } T_u|_{\Sigma_u(\rho)} \lesssim b^2 h(u)$.

Thus, for all $\sigma_1, \sigma_2 \in \Sigma_u(\rho)$, if r_i and T_i are as above, we have

$$\begin{aligned} d(i_u(\sigma_1), i_u(\sigma_2)) &= d(r_1(T_1), r_2(T_2)) \\ &\leq d(r_1(T_1), r_2(T_1)) + |T_2 - T_1| \\ &\lesssim bh(u)\angle(\sigma_1, \sigma_2) + b^2 h(u)\angle(\sigma_1, \sigma_2) \\ &\lesssim (\rho + 1)^2 h(u)\angle(\sigma_1, \sigma_2), \end{aligned}$$

so

$$(3) \quad \text{Lip}(i_u|_{\Sigma_u(\rho)}) \lesssim (\rho + 1)^2 h(u) \lesssim_\rho h(u).$$

Finally, we prove (1). First, we will show that if $u_1, u_2 \in H$ are such that $h(u_i) \geq 1$ and $\sigma \in \Sigma_{u_1}(\rho) \cap \Sigma_{u_2}(\rho)$, then

$$(4) \quad d(i_{u_1}(\sigma), i_{u_2}(\sigma)) \lesssim bd(u_1, u_2) \lesssim_\rho d(u_1, u_2).$$

Let $r_i = r_{u_i, \sigma}$ and $T_i = T_{u_i}(\sigma)$ and suppose that $T_1 \leq T_2$. By the convexity of $h \circ r_1$, we know that $(h \circ r_1)'(t) < 0$ for all $t \geq T_1$. In fact,

$$\begin{aligned}(h \circ r_1)'(t) &\leq \frac{(h \circ r_1)(T_1) - (h \circ r_1)(0)}{T_1} \\ &\leq \frac{-h(u_1)}{bh(u_1)} = -b^{-1}\end{aligned}$$

for all $t \geq T_1$.

Since X is a CAT(0) space, we have $d(r_1(t), r_2(t)) \leq d(u_1, u_2)$ for all $t \geq 0$. Then

$$\begin{aligned}h(r_2(T_2)) &\leq h(r_1(T_2)) + d(u_1, u_2) \\ &\leq -(T_2 - T_1)b^{-1} + d(u_1, u_2).\end{aligned}$$

But $h(r_2(T_2)) = 0$, so $|T_2 - T_1| \leq bd(u_1, u_2)$. Therefore,

$$d(i_{u_1}(\sigma), i_{u_2}(\sigma)) \leq d(u_1, u_2) + |T_1 - T_2| \lesssim bd(u_1, u_2).$$

Now suppose that $\sigma_i \in \Sigma_{u_i}(\rho)$ for $i = 1, 2$. Without loss of generality, suppose that $\rho \geq 2$ and $h(u_1) \leq h(u_2)$. We consider the cases $d(u_1, u_2) \leq \rho$ and $d(u_1, u_2) \geq \rho$.

Suppose that $d(u_1, u_2) \leq \rho$. Let $\mathfrak{d} \in \mathcal{S}_{u_2}$ be a chamber such that $\sigma_2 \in \mathfrak{d}$. By Lemma 3.2, we have $d(u_1, E_{\mathfrak{d}}) \leq \rho + d(u_1, u_2) \leq 2\rho$, so there is a $c > 0$ such that $\sigma_2 \in \mathfrak{d} \subset \Sigma_{u_1}(e^{2c\rho})$. Let $\rho' = e^{2c\rho}$. By (3) and (4),

$$\begin{aligned}d(i_{u_1}(\sigma_1), i_{u_2}(\sigma_2)) &\leq d(i_{u_1}(\sigma_1), i_{u_1}(\sigma_2)) + d(i_{u_1}(\sigma_2), i_{u_2}(\sigma_2)) \\ &\lesssim_{\rho'} h(u_1) \angle(\sigma_1, \sigma_2) + d(u_1, u_2).\end{aligned}$$

Since ρ' depends on ρ , this implies (1).

If $d(u_1, u_2) \geq \rho$, then by Lemmas 3.3 and 3.5, there is a $u \in H$ such that $\mathcal{S}_{u_i}(\rho) \subset \mathcal{S}_{u'}$ for $i = 1, 2$ and

$$d(u_i, u) \lesssim 1 + \log \rho + d(u_1, u_2) \lesssim d(u_1, u_2).$$

Then $\sigma_1, \sigma_2 \in \mathcal{S}_u$, so by (3), (4), and the fact that $\angle(\sigma_1, \sigma_2) \leq \pi$,

$$\begin{aligned}d(i_{u_1}(\sigma_1), i_{u_2}(\sigma_2)) &\leq d(i_{u_1}(\sigma_1), i_u(\sigma_1)) + d(i_u(\sigma_1), i_u(\sigma_2)) + d(i_u(\sigma_2), i_{u_2}(\sigma_2)) \\ &\lesssim_{\rho} d(u_1, u) + h(u) \angle(\sigma_1, \sigma_2) + d(u, u_2) \\ &\lesssim d(u_1, u_2) + (h(u_1) + d(u_1, u_2)) \angle(\sigma_1, \sigma_2) \\ &\lesssim d(u_1, u_2) + h(u_1) \angle(\sigma_1, \sigma_2).\end{aligned}$$

□

3.3 Finding opposite flats

We will need to show that the shadows of points are highly connected. To that end we will show in this section that shadows of points contain the boundaries of many flats. First, we claim that X_∞^0 , the complex of chambers opposite to \mathfrak{c} , contains many flats.

Definition 3.7. *If X is a symmetric space of rank k and $\mathfrak{c} \subset X_\infty$ is a chamber, and $E \subset X$ is a k -flat, then we say that E is opposite to \mathfrak{c} if every chamber in the boundary of E at infinity, $E_\infty \subset X_\infty$, is opposite to \mathfrak{c} .*

Lemma 3.8. *If $\mathfrak{c} \subset X_\infty$ is a chamber, then there is some k -flat E such that E is opposite to \mathfrak{c} .*

Proof. Let the stabilizer of \mathfrak{c} in G be the minimal parabolic subgroup $P = NAM$ and identify the set of (maximal) chambers in X_∞ with the homogeneous space G/P . Recall that $\mathfrak{X}_\infty^0(\mathfrak{c}) \subset X_\infty$ denotes the set of chambers opposite to \mathfrak{c} and if \mathfrak{c}^* is one such chamber, then $\mathfrak{X}_\infty^0(\mathfrak{c}) = N \cdot \mathfrak{c}^*$. Under the identification of the set of chambers in X_∞ with G/P , we find $\mathfrak{X}_\infty^0(\mathfrak{c}) = Nw^*P$, where w^* is the longest element in the Weyl group of X . This orbit is the big cell in the Bruhat decomposition of G and its complement has measure zero. In fact, its complement has codimension at least 1 (see [22], Prop. 1.2.4.9 or [12], Ch. IX, Cor. 1.8), so we can view $\mathfrak{X}_\infty^0(\mathfrak{c})$ as an open submanifold of G/P whose complement has codimension at least 1.

Let F be a maximal flat in X . Its boundary at infinity $F_\infty \subset X_\infty$ consists of finitely many chambers, say $\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_m$. The set $\mathfrak{X}_\infty^0(\mathfrak{c}_i)$ of chambers that are opposite to \mathfrak{c}_i is an open submanifold whose complement has codimension at least 1 for all $i = 1, \dots, m$. Thus the set of all chambers in X_∞ simultaneously opposite to all chambers of F is the intersection $\cap_{i=1}^m \mathfrak{X}_\infty^0(\mathfrak{c}_i)$. This is an open submanifold of G/P whose complement has codimension at least 1. In particular, there is a chamber \mathfrak{c}' opposite to F . If we write the given chamber \mathfrak{c} as $\mathfrak{c} = g\mathfrak{c}'$ for some $g \in G$, then $E = gF$ satisfies the claim of the Lemma. \square

Remark 3.9. The above proof shows that if a chamber \mathfrak{c} is opposite to a fixed flat E , then so are all chambers in an open neighborhood of \mathfrak{c} . Similarly, all flats of the form hE for h in an open neighborhood of $e \in G$ are opposite to a fixed \mathfrak{c} .

We use Lemma 3.8 to prove lower bounds on the filling invariants of horospheres.

Proposition 3.10. *For some $c > 0$ and for all sufficiently large r , there is a Lipschitz map $\alpha: S^{k-1} \rightarrow Z$ with $\text{Lip}(\alpha) \sim r$ such that any Lipschitz extension $\beta: D^k \rightarrow Z$ satisfies*

$$\text{vol } \beta \geq e^{cr}.$$

Proof. Our bound is based on an estimate of the k th divergence of X due to Leuzinger [14]. Leuzinger showed that there are $c_0 > 0$ and $R > 1$ such that if F is a flat in X , $x \in F$ and $r > R$, then the $(k-1)$ -sphere centered at x with radius r in F has exponentially large $\frac{1}{2}$ -avoidant filling volume. That is, if $\alpha_0: S^{k-1} \rightarrow F$ is the sphere of radius r centered at x , then any extension $\beta_0: D^k \rightarrow X$ whose image avoids the ball $B_{r/2}(x)$ has exponentially large volume, i.e., $\text{vol}(\beta_0) \geq e^{c_0 r}$.

To use this result, we find a flat that intersects Z in a sphere. Let F_0 be a flat opposite to \mathfrak{c} and let x_0 be a point in F_0 . (We can take x_0 to be the point on F_0 where h is maximized, but this is not necessary.) The boundary of F_0 at infinity, $(F_0)_\infty$, consists of finitely many chambers, so there is some $\rho > 0$ such that $(F_0)_\infty \subset \Sigma_{x_0}(\rho)$. Choose r so that $r > R$ and let $x \in X$ be such that $h(x) = r$. Let $p \in P$ be a group element such that $px_0 = x$ and let $F = pF_0$.

We identify F_∞ with S^{k-1} and define $\alpha: S^{k-1} \rightarrow Z$ by letting $\alpha(\sigma) = i_x(\sigma)$ for all $\sigma \in F_\infty$. By Lemma 3.6, this is a Lipschitz map with $\text{Lip}(\alpha) \sim r$. The image of α is the intersection $F \cap Z$, and we claim that α has exponentially large filling area in Z .

Let $\alpha_0 \subset F$ be the sphere centered at x with radius r , as in Leuzinger's bound. The spheres α_0 and α both lie in the flat F , and since $d(x, Z) = r$, α_0 is on the inside of α . If $\beta: D^k \rightarrow Z$ is an extension of α , then we can attach an annulus A of volume $\text{vol}A \lesssim r^k$ to β to construct an extension β_0 of α_0 . This extension lies outside $B_{r/2}(x)$, so by Leuzinger's bound, $\text{vol}\beta_0 \gtrsim e^{c_0 r}$. Then $\text{vol}\beta = \text{vol}\beta_0 - \text{vol}A$, and if r is sufficiently large, we have $\text{vol}\beta \geq e^{c_0 r/2}$, as desired. \square

In order to prove upper bounds, we will need a few more flats. As in Section 3.2, for $x \in X$, let $C_x \subset X$ be the Weyl chamber in X based at x and asymptotic to \mathfrak{c} .

Lemma 3.11. *If $x \in X$ and \mathfrak{c} is a chamber of X_∞ , then there is some $x' \in C_x$ such that $d(x, x') \lesssim 1$ and some chamber $\mathfrak{d} \subset \mathcal{S}_{x'}$ such that for all $\mathfrak{e} \in \mathcal{S}_{x'}$:*

1. *The chambers \mathfrak{e} and \mathfrak{d} are opposite.*
2. *The flat $E_{\mathfrak{e}, \mathfrak{d}}$ is opposite to \mathfrak{c} .*
3. *The boundary at infinity of $E_{\mathfrak{e}, \mathfrak{d}}$ is contained in $\mathcal{S}_{x'}$.*

Proof. Without loss of generality we can assume that $x = [e]$ and let $E = E_x = [A]$, and let $\mathfrak{c}^* = \mathfrak{c}_x^*$.

The chambers in \mathcal{S}_x are all close to \mathfrak{c}^* , so we first choose \mathfrak{d}_0 so that the flat $E_{\mathfrak{d}_0, \mathfrak{c}^*}$ spanned by \mathfrak{d}_0 and \mathfrak{c}^* is opposite to \mathfrak{c} . To that end let E_0 be a flat opposite to \mathfrak{c} , and let $n \in N$ be

such that $\mathfrak{c}^* \subset (nE_0)_\infty$. If \mathfrak{d}_0 is the chamber opposite to \mathfrak{c}^* in nE_0 , then $nE_0 = E_{\mathfrak{d}_0, \mathfrak{c}^*}$ and it is opposite to $n\mathfrak{c} = \mathfrak{c}$.

By the above remark, there is an open neighborhood U of \mathfrak{c}^* so that for any $\mathfrak{c}' \in \text{closure}(U)$, the flat $E_{\mathfrak{d}_0, \mathfrak{c}'}$ is also opposite to \mathfrak{c} . We will use Lemma 3.3 to find an element $a \in A$ such that a sends \mathcal{S}_x into U .

Let $0 < r < 1$ be such that $\mathcal{S}_x(r) \subset U$. Let \mathfrak{a} be the Lie algebra of A and let $V \in \mathfrak{a}$ be a unit vector in the open Weyl chamber corresponding to \mathfrak{c} . By Lemma 3.3, there is a $t > 0$ such that $t \lesssim -\log r$ and

$$\mathcal{S}_{\exp(-tV)} = \exp(-tV)\mathcal{S}_x \subset \mathcal{S}_x(r).$$

Let $a := \exp(-tV)$ and let $\mathfrak{d} := a^{-1}\mathfrak{d}_0$. Then for any $\mathfrak{c}' \in \mathcal{S}_x$, we have $a\mathfrak{c}' \in U$ and thus $E_{a\mathfrak{c}', \mathfrak{d}_0}$ is opposite to \mathfrak{c} . It follows that $a^{-1}E_{a\mathfrak{c}', \mathfrak{d}_0} = E_{\mathfrak{c}', \mathfrak{d}}$ is opposite to \mathfrak{c} .

Finally, we choose x' . The union

$$V = \bigcup_{\mathfrak{c}' \in \mathcal{S}_x} (E_{\mathfrak{c}', \mathfrak{d}})_\infty$$

is contained in a compact set and consists of chambers opposite to \mathfrak{c} , so $V \subset \mathcal{S}_x(r')$ for some r' . By Lemma 3.3, there is some $x' \in C_x$ such that $d(x, x') \lesssim \log(r' + 1)$ and $V \subset \mathcal{S}_{x'}$. \square

3.4 $(k - 2)$ -connectivity at infinity

Next, we show that $X_\infty^0(\mathfrak{c})$ is highly connected. First, we consider spheres that lie in a single shadow. Let $\Sigma_x = \bigcup_{\mathfrak{d} \in \mathcal{S}_x} \mathfrak{d} \subset X_\infty$ be the subset of X_∞ covered by the chambers in \mathcal{S}_x as in Section 3.2. The following lemma is an analogue of Lemma 4.17 of [24]. Recall that C_x is the Weyl chamber based at x and asymptotic to \mathfrak{c} .

Lemma 3.12. *If $x \in X$, then there is some $x' \in C_x$ such that $d(x, x') \lesssim 1$, $\mathcal{S}_x \subset \mathcal{S}_{x'}$, and Σ_x is $(k - 2)$ -connected inside $\Sigma_{x'}$.*

That is, if $\alpha: S^m \rightarrow \Sigma_x$ is Lipschitz and if $m \leq k - 2$, then there is an extension $\beta: D^{m+1} \rightarrow \Sigma_{x'}$ such that $\text{Lip } \beta \lesssim \text{Lip } \alpha + 1$.

Proof. Let $\Sigma_x^{(m)}$ be the m -skeleton of Σ_x given by the Tits building structure on X_∞ . Let $\alpha: S^m \rightarrow \Sigma_x$ be a Lipschitz map. There is a simplicial approximation $\alpha': S^m \rightarrow \Sigma_x^{(m)}$ and a homotopy $h_0: S^m \times [0, 1] \rightarrow \Sigma_x$ from α to α' such that $\text{Lip } \alpha' \lesssim \text{Lip } \alpha$ and $\text{Lip } h_0 \lesssim \text{Lip } \alpha + 1$. (One can construct α' by triangulating S^m into simplices of diameter $\epsilon_k \text{Lip}(\alpha)^{-1}$, then letting $\alpha'(v)$ be the closest vertex to $\alpha(v)$ for every vertex $v \in S^m$. If ϵ_k is sufficiently small, then α' extends to a simplicial map and there is a straight-line homotopy h_0 from α to α' .)

We first contract α' . Let x' and $\mathfrak{d} \in \mathcal{S}_{x'}$ be as in Lemma 3.11 and let u be the barycenter of \mathfrak{d} . For any $v \in \Sigma_x$, there is a flat E that contains u and v and whose boundary is contained in $\mathcal{S}_{x'}$, so any Tits geodesic from u to v lies in $\Sigma_{x'}$. Furthermore, if $v \in \Sigma_x^{(m)}$, then u and v are not opposite to one another, so this geodesic is unique.

Let $h_1: S^m \times [0, 1] \rightarrow \Sigma_{x'}$ be the map which sends $v \times [0, 1]$ to the geodesic between $\alpha'(v)$ and u . This is a null-homotopy of α' , and

$$\text{Lip } h_1 \lesssim 1 + \text{Lip } \alpha'.$$

The constant in the inequality depends on the distance between u and the m -skeleton of X_∞ . By concatenating h_0 and h_1 we obtain a disc $\beta: D^{m+1} \rightarrow \Sigma_{x'}$ with boundary α , and $\text{Lip } \beta \lesssim \text{Lip } \alpha + 1$ as desired. \square

Let Δ_Z be the infinite-dimensional simplex with vertex set labeled by Z and let $\langle z_0, \dots, z_k \rangle$ denote the k -simplex with vertices z_0, \dots, z_k . We will use Lemma 3.12 to construct a map $\Omega_\infty: \Delta_Z \rightarrow X_\infty$ that sends each vertex $\langle v \rangle$ to a direction in the shadow of v and sends each simplex δ to a simplex in the shadow of some point x_δ . If δ is a simplex of Δ_Z , we denote its set of vertex labels by $\mathcal{V}(\delta) \subset Z$, so that $\mathcal{V}(\langle z_0, \dots, z_k \rangle) = \{z_0, \dots, z_k\}$.

Lemma 3.13 (see [24, 4.16]). *There is a cellular map*

$$\Omega_\infty: \Delta_Z^{(k-1)} \rightarrow X_\infty,$$

a constant $c > 0$ depending on X , and a family of points $x_\delta \in X$, where δ ranges over the simplices of $\Delta_Z^{(k-1)}$. This map is c -Lipschitz, and for every δ :

1. $d(x_\delta, \mathcal{V}(\delta)) \lesssim \text{diam } \mathcal{V}(\delta) + 1$ (and consequently, $h(x_\delta) \lesssim \text{diam } \mathcal{V}(\delta) + 1$).
2. $\Omega_\infty(\delta) \subset \Sigma_{x_\delta}$.
3. If $\delta' \subset \delta$, then $h(x_\delta) \geq h(x_{\delta'})$ and $x_\delta \in D_{x_{\delta'}}$, where D_x is a neighborhood of the chamber C_x as in Section 3.2.
4. $h(x_\delta) \geq 1$.

Proof. We will construct Ω_∞ one dimension at a time using Lemma 3.12. Let τ_0 be the barycenter of \mathfrak{c} and let $\theta := \angle(\tau_0, \tau)$. By the standing assumptions, we have $\theta < \pi/2$. For $x \in X$, let $r_x: [0, \infty) \rightarrow X$ be the geodesic ray based at x and asymptotic to τ_0 , so that

$$h(r_x(t)) = h(x) + t \cos \theta$$

for all t .

We start by defining Ω_∞ on the vertices of Δ_Z . For $z \in Z$, let b_z be the barycenter of the chamber c_z^* and let

$$\Omega_\infty(\langle z \rangle) = b_z$$

and let $x_{\langle z \rangle} = r_z(\sec \theta)$. This satisfies all of the desired conditions.

Now, suppose by induction that $0 < m \leq k-1$, that we have defined Ω_∞ on $\Delta_Z^{(m-1)}$, and that we have defined $x_{\delta'}$ for every simplex δ' with $\dim \delta' < m$. Let δ be an m -simplex and let $z \in Z$ be one of its vertices.

By part 1 of the lemma, the points $x_{\delta'}$ are contained in a ball $B_z(R)$ with $R \sim \text{diam } \mathcal{V} + 1$. By Lemma 3.5, there is a $t \lesssim R + 1$ such that if $x_0 = r_z(t)$ and δ' is a face of δ , then $\mathcal{S}_{x_{\delta'}} \subset \mathcal{S}_{x_0}$ and $x_0 \in D_{x_{\delta'}}$.

By part 2 and Lemma 3.12, there is an $x' \in C_{x_0}$ such that $\Omega_\infty|_{\partial\delta}$ is contractible in $\Sigma_{x'}$ and $d(x_0, x') \lesssim 1$. We let $x_\delta = x'$ and define the extension $\Omega_\infty|_\delta: \delta \rightarrow \Sigma_{x'}$ using Lemma 3.12. The desired properties of x_δ and $\Omega_\infty|_\delta$ are easy to check. \square

3.5 Lipschitz connectivity of Z

Finally, we show that Z is Lipschitz $(k-2)$ -connected. Our main tool is a lemma similar to Lemma 3.2 of [24].

Lemma 3.14. *Suppose that $Z \subset X$, that X is Lipschitz $(k-2)$ -connected and that for any r , there is a Lipschitz retraction $R_r: N_r(Z) \rightarrow Z$, where $N_r(Z) = \{x \in X \mid d(x, Z) < r\}$. Then, if there is a map $\Omega: \Delta_Z^{(k-1)} \rightarrow Z$ such that*

1. *for all $z \in Z$, $d(\Omega(\langle z \rangle), z) \lesssim 1$, and*
2. *for any simplex $\delta \subset \Delta_Z$, we have*

$$\text{Lip } \Omega|_\delta \lesssim \text{diam } \mathcal{V}(\delta) + 1$$

then Z is Lipschitz $(k-2)$ -connected.

Proof. The proof is very similar to the proof of Lemma 3.2 of [24], which constructs a Lipschitz extension using a Whitney decomposition. The main difference is that we assume that $\text{Lip } \Omega|_\delta \lesssim \text{diam } \mathcal{V}(\delta) + 1$ rather than $\text{Lip } \Omega|_\delta \lesssim \text{diam } \mathcal{V}(\delta)$. The weaker inequality means that the Lipschitz constant of Ω on small simplices can be unbounded, so we use the Lipschitz connectivity of X to extend the map near the boundary.

Let $L > 0$, let $D^{k-1}(L) \subset \mathbb{R}^{k-1}$ be the closed ball of radius L , and let $S^{k-2}(L) := \partial D^{k-1}(L)$. It suffices to show that for any $L > 0$ and any 1-Lipschitz map $\alpha: S^{k-2}(L) \rightarrow Z$, there is an extension $\beta: D^{k-1}(L) \rightarrow Z$ such that $\text{Lip}(\beta) \lesssim 1$.

Let α be such a map. By the Whitney covering lemma, we can decompose the interior of $D := D^{k-1}(L)$ into a countable union of dyadic cubes with disjoint interiors such that $\text{diam } C \sim d(C, \partial D)$ for each cube C . The barycentric subdivision of this cover is a triangulation T of the interior of D such that each simplex is bilipschitz equivalent to a scaling of the standard simplex. Let $c > 1$ be such that for each simplex σ with $\dim \sigma > 0$, we have

$$c^{-1}d(\sigma, \partial D) \leq \text{diam } \sigma \leq cd(\sigma, \partial D).$$

Define $h: T^{(0)} \rightarrow \partial D$ so that for all $v \in T^{(0)}$, we have $d(v, h(v)) = d(v, \partial D)$. For each edge $e = (v, w)$, we have

$$d(h(v), h(w)) \leq d(v, \partial D) + d(v, w) + d(w, \partial D).$$

Since $d(e, \partial D) \leq c \text{diam } e$, this implies that $d(h(v), h(w)) \lesssim d(v, w)$ and thus $\text{Lip}(h) \lesssim 1$.

For each $v \in T^{(0)}$, let

$$\beta(v) := \begin{cases} \Omega(\langle \alpha(h(v)) \rangle) & \text{if } d(v, \partial D) \geq c^{-1} \\ \alpha(h(v)) & \text{otherwise.} \end{cases}$$

If $e = (v, w)$ is an edge of T such that $\ell(e) < c^{-2}/2$, then $d(v, \partial D) < c^{-1}$ and $d(w, \partial D) < c^{-1}$. Then

$$d(\beta(v), \beta(w)) = d(\alpha(h(v)), \alpha(h(w))) \leq \text{Lip}(\alpha) \text{Lip}(h) d(v, w) \lesssim d(v, w).$$

If $e = (v, w)$ is an edge of T such that $\ell(e) \geq c^{-2}/2$, then Ω may introduce bounded error:

$$d(\beta(v), \beta(w)) \lesssim d(\alpha(h(v)), \alpha(h(w))) + 1 \lesssim d(v, w).$$

It follows that $\text{Lip}(\beta|_{T^{(0)}}) \sim 1$.

If $\sigma = \langle v_0, \dots, v_k \rangle$ is a simplex of T with diameter at least 1, then $d(v_i, \partial D) \geq c^{-1}$, so we may extend β to σ so that it sends σ to $\Omega(\langle g_0(v_0), \dots, g_0(v_k) \rangle)$. This extension is Lipschitz with bounded Lipschitz constant, and it remains to extend β to simplices with diameter less than 1.

We work one dimension at a time. We have already defined β on the 0-skeleton of T in a Lipschitz fashion. Suppose that $d \leq k - 2$, that β has been defined on $T^{(d-1)}$, and that there is

a $c_{d-1} > 0$, independent of α , such that $\text{Lip } \beta|_{T^{(d-1)}} \leq c_{d-1}$. We claim that there is an extension of β to $T^{(d)}$ and a c_d independent of α such that $\text{Lip } \beta|_{T^{(d)}} \leq c_d$.

Suppose that σ is a d -simplex of T such that $\text{diam } \sigma < 1$. Then $\beta|_{\partial\sigma}: \partial\sigma \rightarrow Z$ is a map with $\text{Lip } \beta|_{\partial\sigma} \leq c_{d-1}$. Since σ is bilipschitz equivalent to a scaling of the standard simplex, we can identify $\partial\sigma$ with a scaled version of S^{d-1} in a bilipschitz way and use the Lipschitz connectivity of X to produce an extension $f: \sigma \rightarrow X$ with Lipschitz constant $\text{Lip } f \lesssim c_{d-1}$. Then there is a r_d such that $f(\sigma) \subset N_{r_d}(Z)$, and if $\beta|_\sigma = R_{r_d} \circ f$, then

$$\text{Lip } \beta|_\sigma \lesssim c_{d-1} \text{Lip } R_{r_d},$$

as desired. Repeating this process for each small simplex in $T^{(d)}$, we obtain the desired extension. \square

We use Ω_∞ and the map i_u constructed in Lemma 3.6 to construct an Ω that satisfies Lemma 3.14.

Lemma 3.15. *There is a map $\Omega: \Delta_Z^{(k-1)} \rightarrow Z$ satisfying the conditions of Lemma 3.14. Consequently, Z is Lipschitz $(k-2)$ -connected.*

Proof. For $\rho > 0$, let

$$Y(\rho) := \{(\sigma, x) \in X_\infty^0(\mathbb{C}) \times X \mid h(x) \geq 1, \sigma \in \Sigma_x(\rho)\}.$$

Let $d_Y: Y(\rho) \times Y(\rho) \rightarrow \mathbb{R}$ be the function

$$d_Y((\sigma_1, x_1), (\sigma_2, x_2)) := d(x_1, x_2) + \min\{h(x_1), h(x_2)\} \cdot \angle(\sigma_1, \sigma_2).$$

This does not satisfy the triangle inequality, but in Lemma 3.6, we showed that the map $I: Y(\rho) \rightarrow X$ given by $I(v, x) = i_x(v)$ is ‘‘Lipschitz’’ in the sense that

$$d(I(v, x), I(v', x')) \leq L_I d_Y((v, x), (v', x'))$$

for some L_I depending on ρ . The map Ω will be a composition $\Omega = I \circ W$, where $W: \Delta_Z^{(k-1)} \rightarrow Y(\rho)$ is a map based on the map Ω_∞ and the points x_δ constructed in Lemma 3.13.

To construct W , we use the exploded simplices used in [24]. If Δ is a simplex, the exploded simplex $E(\Delta)$ is a cellular subdivision of a simplex Δ with the following properties (see Figure 1):

1. $E(\Delta)$ contains a copy Δ' of Δ at its center.

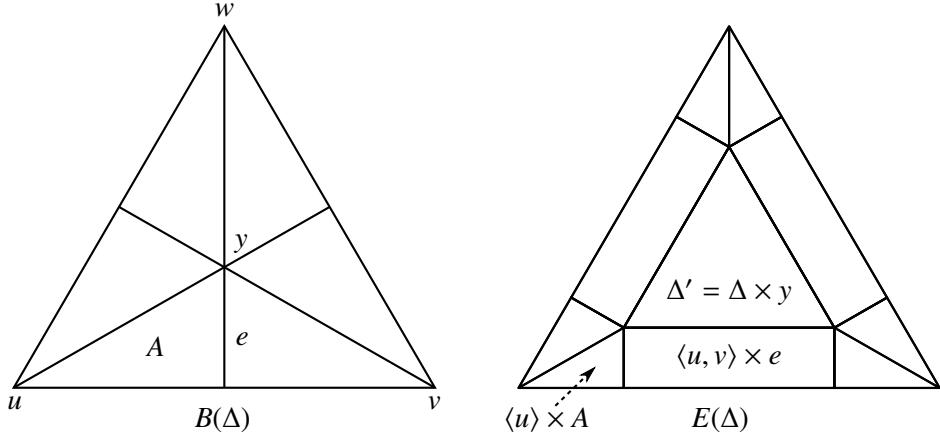


Figure 1: Each cell of the “exploded simplex” $E(\delta)$ is a product of a cell of δ and a cell of $B(\delta)$.

2. $E(\Delta)$ subdivides each face of Δ into an exploded simplex of lower dimension.
3. Each cell of $E(\Delta)$ is of the form $\Delta_1 \times \Delta_2$, where Δ_1 is a face of Δ and Δ_2 is a face of the barycentric subdivision $B(\Delta)$.

Specifically, recall that the vertex set of $B(\Delta)$ is the set of faces of Δ . If $\delta_0, \dots, \delta_k$ are faces of Δ that form a flag—that is, if $\delta_0 \subset \dots \subset \delta_k$ —then $\langle \delta_0, \dots, \delta_k \rangle$ is a simplex of $B(\Delta)$. Each cell of $E(\Delta)$ is of the form

$$\delta \times \langle \delta_0, \dots, \delta_k \rangle$$

for some flag $\delta_0, \dots, \delta_k$ and some face δ of δ_0 .

4. Since each cell of $E(\Delta)$ is a cell of $\Delta \times B(\Delta)$, we can define maps $p_1: E(\Delta) \rightarrow \Delta$ and $p_2: E(\Delta) \rightarrow B(\Delta)$ (ρ_1 and ρ_2 in [24]) coming from the projections to the first and second factors. These maps are Lipschitz. The map p_1 expands the central simplex to cover Δ and shrinks the collar to the boundary. The map p_2 collapses the central simplex to the barycenter of Δ , sends the central simplices of all the faces to the corresponding barycenters, and sends the collar surjectively to Δ .

If Q is a simplicial complex, we can form a cellular subdivision $E(Q)$ by exploding each simplex. The maps p_1 and p_2 on each simplex agree on overlaps, so we combine them to form maps $p_1: E(Q) \rightarrow Q$ and $p_2: E(Q) \rightarrow B(Q)$ defined on all of $E(Q)$.

By convention, we will write the vertices of a simplex of $B(Q)$ in ascending order, so if $\langle \delta_0, \dots, \delta_d \rangle$ is a simplex of $B(Q)$, then the δ_i 's are simplices of Q and $\delta_0 \subset \dots \subset \delta_d$.

Let $Q = \Delta_Z^{(k-1)}$. For each simplex δ of Q , we define Ω on δ' by

$$(5) \quad \Omega(\delta') = i_{x_\delta}(\Omega_\infty(\delta)),$$

where δ' is the central simplex of $E(\delta)$. We will extend Ω to the collars of the $E(\delta)$'s by using the projections p_1 and p_2 .

Specifically, we will define a map $F: B(Q) \rightarrow X$ and let $\Omega = I \circ W$, where

$$W = (\Omega_\infty \circ p_1, F \circ p_2).$$

The map F will satisfy:

1. The complex $B(Q)$ has a vertex b_δ at the barycenter of each simplex δ of Q . For all δ , $F(b_\delta) = x_\delta$. (This ensures that the extension agrees with the map defined in equation (5).)
2. If $\Delta = \langle \delta_0, \dots, \delta_d \rangle \subset B(Q)$, then $\text{Lip } F|_\Delta \lesssim \text{diam } \mathcal{V}(\delta_d) + 1$.
3. There is a $\rho > 0$ such that if

$$y \in \langle \delta_0, \dots, \delta_d \rangle \subset B(Q),$$

then $\mathcal{S}_{F(y)}(\rho) \supset \mathcal{S}_{x_{\delta_0}}$.

We define F one dimension at a time. For each vertex v of Q , we define $F(v) = x_v$. If δ is a simplex of Q and we have already defined F on $\partial\delta$, we extend F on the rest of δ by coning it off to x_δ . That is, every point in δ is on a line segment between b_δ and a point $y \in \partial\delta$. We send b_δ to the point x_δ and we send each such segment to a geodesic segment from x_δ to $F(y)$.

This satisfies condition 1 by construction. Condition 2 follows from the fact that X is CAT(0) and Lemma 3.13.1. It only remains to check condition 3. Suppose that $\Delta = \langle \delta_0, \dots, \delta_d \rangle \subset B(Q)$ and that $y \in \Delta$. For $i = 0, \dots, d$, let $x_i = x_{\delta_i}$. Let $D_0 = D_{x_0}$ be a neighborhood of C_{x_0} as in Section 3.2. By Lemma 3.13.3, for all $i = 0, \dots, d$, we have $x_i \in D_0$. Since C_{x_0} is convex, D_0 is convex. Since $F(\Delta)$ is contained in the convex hull of the x_i , we have $F(y) \in D_0$. By Lemma 3.4, there is a ρ depending on X such that $\mathcal{S}_x \subset \mathcal{S}_{F(y)}(\rho)$, as desired.

It follows that the image of W lies in $Y(\rho)$. Suppose that $q \in Q$ and let $\delta \times \langle \delta_0, \dots, \delta_k \rangle$ be a cell of $E(Q)$ containing q . Here, δ is a face of Q and $\delta \subset \delta_0 \subset \dots \subset \delta_k$. Then $p_1(q) \in \delta$ and $p_2(q) \in \langle \delta_0, \dots, \delta_k \rangle$. By Lemma 3.13.2, we have

$$\Omega_\infty(p_1(q)) \in \Sigma_{x_\delta} \subset \Sigma_{x_{\delta_0}}.$$

By property 3 of F , we have $\Sigma_{x_{\delta_0}} \subset \Sigma_{F(p_2(q))}(\rho)$, so

$$W(q) = (\Omega_\infty(p_1(q)), F(p_2(q))) \in Y(\rho).$$

We may thus define $\Omega = I \circ W$.

Finally, we claim that Ω satisfies the conditions in Lemma 3.14. First, if $z \in Z$, let $v = \langle z \rangle$ be the corresponding vertex of Δ_Z . Then

$$\Omega(v) = I(\Omega_\infty(v), F(v)) = i_{x_z}(\sigma),$$

where $\sigma = \Omega_\infty(v) \in \Sigma_{x_z}(\rho)$, and

$$\begin{aligned} d(z, \Omega(v)) &\leq d(z, x_z) + d(x_z, i_{x_z}(\sigma)) \\ &\lesssim 1 + h(x_z) + \rho \lesssim 1 \end{aligned}$$

by Lemmas 3.6 and 3.13. This proves property 1 of Lemma 3.14.

If $\delta \subset \Delta_Z$ and if $q_1, q_2 \in \delta$, properties 1 and 2 of F imply that

$$h(F(p_2(q_i))) \lesssim \text{diam } \mathcal{V}(\delta) + 1.$$

It follows that

$$\begin{aligned} d(\Omega(q_1), \Omega(q_2)) &\leq L_I d_Y(W(q_1), W(q_2)) \\ &\lesssim d(F(p_2(q_1)), F(p_2(q_2))) + (\text{diam } \mathcal{V}(\delta) + 1) \cdot \angle(\Omega_\infty(p_1(q_1)), \Omega_\infty(p_1(q_2))) \\ &\lesssim (\text{diam } \mathcal{V}(\delta) + 1) d(q_1, q_2), \end{aligned}$$

implying property 2. □

Lemma 3.14 then implies that Z is Lipschitz $(k-2)$ -connected. This concludes the proof.

4 Proof of Theorem A

We use the following result proved in [15], Thm. 3.6 (see also [4], §13).

Proposition 4.1. *Let Γ be an arithmetic lattice of \mathbb{Q} –rank 1 in a linear, semisimple Lie group G and let $X = G/K$ be the associated symmetric space. Then any orbit of Γ in X is quasi-isometric to a set $Y := X \setminus \bigcup_i B_i$, where the B_i comprise a countable set of disjoint horoballs.*

We can write X as a Riemannian product of irreducible symmetric spaces, $X = \prod_{i=1}^m X_i$ (corresponding to the decomposition $G = \prod_{i=1}^m G_i$ of G into simple factors). The boundary of X at infinity is the spherical join of the boundaries of the factors.

Assume now (as in Theorem A) that the lattice Γ is *irreducible*. We claim that none of the centers of the horoballs in the above proposition are contained in a proper join factor of X_∞ . By way of contradiction, assume that $m \geq 2$ and that one of the horoballs, say B , is centered in a join factor associated to, say, $J = X_1 \times \cdots \times X_n$, for $n < m$. Then B is a sublevel set of the Busemann function associated to a geodesic in X of the form $(c(t), p_{n+1}, \dots, p_m)$ and B has the form $B = H \times \prod_{i=n+1}^m X_i$, where H is a horoball in J . The projection of a Γ –orbit to J then avoids H , so the projection of Γ to the factor $G_1 \times \cdots \times G_n$ cannot be dense. This contradicts irreducibility, see [21], Cor. 5.21 (5).

By Theorem B, the boundary of each horoball B_i is Lipschitz $(k-2)$ –connected, where $k = \text{rank } X$. By part 2 of Proposition 1.1, the subspace $Y \subset X$ is undistorted up to dimension $k-1$. The claim on its isoperimetric inequalities is a consequence of [16], which asserts that a symmetric space X satisfies Euclidean isoperimetric inequalities below the rank.

Finally, the lower bound in Theorem A follows from Proposition 3.10. By the proposition, for each i and for all sufficiently large r , there is a Lipschitz sphere $\alpha: S^{k-1} \rightarrow \partial B_i$ such that $\text{Lip}(\alpha) \sim r$ and such that any Lipschitz extension $\beta: D^k \rightarrow \partial B_i$ satisfies $\text{vol } \beta \geq e^{cr}$.

Let $p: Y \rightarrow \partial B_i$ be the nearest-point projection; since B_i is convex, this is a distance-decreasing map. If $\beta': D^k \rightarrow Y$ is an extension of α , then $p \circ \beta': D^k \rightarrow \partial B_i$ is also an extension, and

$$\text{vol } \beta' \geq \text{vol}(p \circ \beta') \geq e^{cr}.$$

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