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The Stochastic Linear Quadratic Control Problem with Singular Estimates

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Abstract

We study an infinite dimensional finite horizon stochastic linear quadratic control problem in an abstract setting. We assume that the dynamics of the problem are generated by a strongly continuous semigroup, while the control operator is unbounded and the multiplicative noise operators for the state and the control are bounded. We prove an optimal feedback synthesis along with well-posedness of the Riccati equation for the finite horizon case. Our results extend the ones proposed in [H] to the case in which disturbance in the control is considered and a final time penalization term is included in the quadratic cost functional.

1 Introduction

We consider the stochastic linear quadratic problem in infinite dimensions with state and control dependent noise for the so-called singular estimate control systems. These systems involve dynamics driven by C_0-semigroups and unbounded control actions, with the control to state kernel satisfying a singular estimate. Such situation is typical in boundary or point control problems where the action of the control operator B is either only densely defined on a control space or its range is outside the state space. In order to quantify the "unboundedness" of control action-singular estimates play a pivotal role. Such estimate describes the amount of blow up of the "transfer function". The latter is necessary for a rigorous analysis of control problems and the associated feedback synthesis -be it deterministic or stochastic.

For deterministic systems, the infinite dimensional LQR problem has been studied extensively in the literature [BK, BDDM, LT2]. The purpose of the theoretical framework is to address optimal control of systems of partial differential equations. For most systems, the controlling mechanism can only be applied from the interface of the system or at finitely many points or curves [BSW] which necessitates developing a framework for studying boundary/point control. Such control actions can be captured mathematically using maps which are not bounded with respect to the state space, but take values in a larger dual space. The most natural class of problems where such description has been used are dynamics driven by analytic semigroups. The analyticity property quantifies naturally the blow up of the "transfer function" when acted upon by an unbounded operator (compatible with fractional powers of the generator). The linear quadratic problem for systems driven by analytic semigroups with these type

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of control actions were studied by [F2, AT, DI, BDDM, LT2]. The situation is much more complicated in the non-analytic case, where there is no natural characterization of singularity other than technical -often brute force- PDE estimates. However, for some classes of control systems which combine hyperbolic and parabolic dynamics, it has been observed that the control-to-state kernel satisfies a singular estimate which generalizes the case of analytic semigroup dynamics [AL, ABL, L1, LT1, LTu1]. Examples of systems which manifest this type of singular estimate arise frequently in thermo-elastic plate models [BLT, BL, LTu2], acoustic-structure interaction equation [AL, BSS, LTu2], and fluid-structure interaction models [LTu3]. In view of the above, a deterministic theory of feedback control has been developed for these classes of problems (singular estimate)- see the references given in [L2]. However, in the stochastic case the only results available in the literature covering unbounded control actions are the ones dealing with *analytic* semigroups [D, GT1, F1]. *The main goal of the present work is to develop a stochastic treatment of unbounded control action problems arising in a general class of dynamical systems which exhibit singular estimates, but are not necessarily analytic.* One of the main challenges is to develop an approximation framework which would provide rigorous justification of stochastic estimates. In the analytic case, such framework is very natural and based on the instant regularizing effect of the dynamics. In the non-anaytic case, a development of regularizing procedures lies at the heart of the problem. This will be accomplished by expanding and building on the results presented in an unpublished reference [H].

The stochastic linear quadratic regulator problem in finite dimensions has been first studied by Kushner (1962) [K] using dynamic programming. The feedback characterization of the optimal control and the derivation of a matrix Riccati equation satisfied by the gain matrix is due to Wonham (1968) [W1, W2]. A complete theory for the stochastic linear quadratic optimal control problem in finite dimensions can be found in [YZ, DMS, FS]. It is notable that the associated Riccati differential equation in the stochastic linear quadratic problem is a deterministic differential equation, and thus the relation between the optimal control and the optimal state which are random variables is purely deterministic. The linear quadratic problem with random coefficients in finite dimension has also been investigated in [CLZ]. In this case, the associated Riccati equation is a backward stochastic equation.

Several early works in the literature have addressed stochastic optimization in infinite dimensions and the application of a semigroup framework to the stochastic setting with bounded inputs [B2, B3, C1]. The infinite dimensional analog for the stochastic linear quadratic problem and the Riccati equation was treated by Ichikawa [I] via a dynamic programming approach, where he considered dynamics driven by C_0 semigroups and bounded control and noise operators. In an another early work, Curtain [C2] provides a semigroup framework for studying the infinite dimensional linear quadratic Gaussian along with several examples and applications. A complete Riccati feedback synthesis of the infinite dimensional problem with disturbance in the state has been addressed by Da Prato [D] for systems with analytic dynamics and a particular unbounded noise operator which captures the first derivative of the state in a parabolic equation. The analysis was extended to boundary controls by Flandoli [F1] and in particular for analytic systems with Neumann type controls. In [GRS], the authors consider a more general cost functional and a semi linear state equation driven by analytic dynamics, and proceed to solve the problem using a Hamilton-Jacobi-Bellman approach. For systems with singular estimates, which is our primary consideration, the stochastic linear quadratic problem has been studied by one of the authors in [H], but with no disturbance in the control ($D = 0$) and without finite time penalization in the cost functional (G=0). In [U], the time varying problem has been also addressed for systems driven by strongly continuous evolutions with bounded control and noise operators. In [D1, D2], the author investigates stochastic linear quadratic differential games involving a stochastic differential equation with fractional Brownian motion with dynamics generated by analytic semigroups. Some recent interesting work has also treated the linear quadratic problem with random coefficients along with the associated backward stochastic Riccati equation [GT1, GT2].

In view of the above the main novel contributions distinguishing this work from other publications are: *(1) this is the first treatment of stochastic unbounded control systems in the non-analytic setting, (2) the framework allows for consideration of terminal penalization as well as control action perturbed by noise.* Indeed, in the present paper, we consider a more general setting including disturbance in the control, and we also consider the case of the Bolza problem which allows for a finite time penalization in the objective functional whose expected value is to be minimized. This latter aspect of Bolza-Meyer problem is particularly challenging in the unbounded control case. As shown [F1], the solution to optimal control problem may not exist, unless a certain closeability hypothesis is introduced. Under such necessary hypothesis, we provide an optimal feedback synthesis and a Riccati equation for the stochastic linear quadratic optimal control in the context of singular estimate control systems with noise dependence in both state and control.

In the deterministic setting, variational analysis is used to obtain explicit formulas for the optimal control before proceeding to derive the associated Riccati equations [LT1, D]. However, such explicit formulas are not available in the stochastic setting -thus preventing applicability of a method of pivotal importance in the deterministic and singular case. Moreover, in our setting, the lack of smoothing does not allow for the application of the stochastic maximum principle or a solution via the Hamilton-Jacobi-Bellman equation unlike the case of analytic dynamics [GRS]. In particular, the state trajectories are mild solutions of the state equations and not necessarily differentiable in the classical sense.

Therefore, in our approach, we derive a differential Riccati equation associated with the optimal stochastic linear quadratic control problem, by first showing the existence of a solution to an expanded system in the integral form of the Riccati equation via a specially crafted fixed point argument. Here we generalize the arguments given in [H]. We then proceed to derive the differential Riccati equation which requires making sense of the weak derivative of the evolution generated by deterministic dynamics with respect to initial time. Here, the obstacle, as in the deterministic case, lies in the fact that the terms of the Riccati equation may not be well defined due to the unboundedness of the control operator. There has been counter examples in the literature where the Riccati equation is not well posed in the case of unbounded control operators [BLT]. Another difficulty is the finite state penalization which gives rise to possible singularities at the final time and require choosing appropriate spaces to make sense of the quadratic term in the differential Riccati equation [LTu1]. Finally, we then use a dynamic programming argument to show that the minimum of the quadratic functional is realized when the control is expressed in feedback form via the solution to the differential Riccati equation. Here, we proceed with the dynamic programming argument on a regularized version of the problem since the Itô formula only applies to C^2 functions, while the state and control trajectories are not differentiable in the classical sense. For this reason, a forward approach via a maximum principle or a variational method to solve for the optimal control before proceeding to derive the differential Riccati equation is not applicable in this setting.

We first formulate the optimal control problem. Let the abstract stochastic differential equation

$$\begin{aligned} dy(t) &= (Ay + Bu)\,dt + (Cy + Du)\,dW_t \\ y(s) &= x \end{aligned} \tag{1.1}$$

be defined on a Hilbert state space H, where A and C are operators on H while B and D are operators acting from the control Hilbert space U to the state space H. We take C and D to be bounded operators but A and B are typically unbounded.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and W_t a one dimensional real valued stochastic Brownian motion on $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_t$ the sigma algebra generated by $\{W_\tau : \tau \leq t\}$. We assume that all function spaces are adapted to the filteration $\mathcal{F}_t$. We denote by $L^2_w([s,T]; H)$ all stochastic processes $X(t,\omega) : [s,T] \times \Omega \to H$ such that

1. $\int_s^T \|X(t)\|_H^2 \, dt < \infty$ a.e. in Ω.
2. $X(t,\cdot)$ is $\mathcal{F}_t$-measurable $\forall t \in [s,T]$.

We also denote by $M^2_w([s,T]; H)$, the space of all strongly measurable square integrable stochastic processes $X : [s,T] \times \Omega \to H$ such that $\int_s^T \mathbb{E}\left(\|X(t)\|_H^2\right) dt < \infty$, and by $L^2(\Omega; H^1([s,T]; U))$ all strongly measurable square integrable stochastic processes $u : [s,T] \times \Omega \to U$ such that $\int_s^T \mathbb{E}\left(\|u(t)\|_U^2\right) dt + \int_s^T \mathbb{E}\left(\|u_t(t)\|_U^2\right) dt < \infty$. The objective is to minimize the quadratic cost functional

$$J(s,x,u) = \mathbb{E}\left(\int_s^T (\|Ry\|_W^2 + \|u\|_U^2)\, dt + \|Gy(T)\|_Z^2\right) \tag{1.2}$$

over all $u \in M^2_w([s,T]; U)$ where R and G are bounded linear observation operators taking values in Hilbert spaces W and Z respectively. The assumptions we consider are the following

Assumptions 1.1. *1. Operator A is linear and generates a C_0-semigroup e^{At} on H.*

2. The linear operator B acts from $U \to [\mathcal{D}(A^\star)]'$ or equivalently $A^{-1}B$ is bounded from $U \to H$.

3. The noise operator $D : U \to H$ is a bounded linear operator.

4. There exists a number $\gamma \in (0, 1/2)$ such that the control to state map kernel $e^{At}B$ satisfies the singular estimate

$$\|e^{At}Bu\|_H \leq \frac{c}{t^\gamma}\|u\|_U \tag{1.3}$$

for every $u \in U$ and $0 < t < 1$.

5. The operators $R : H \to W$, $G : H \to Z$ and $C : H \to H$ are all bounded linear operators.

Remark 1.2. *Our framework also allows for H-valued Brownian motion W_τ where $(Cy + Du)\, dW_\tau$ is interpreted as a Wick product $(Cy+Du) \diamond dW_\tau$ of generalized random variables on White Gaussian probability spaces. See [LMT2] for chaos expansion treatment of the abstract stochastic differential equation and the linear quadratic control problem in Hilbert spaces.*

Remark 1.3. *The singular estimate (1.3) should be interpreted in the following precise sense:*

$$|\langle e^{At}A^{-1}Bu, A^*\phi\rangle| \leq \frac{c_T}{t^\gamma}||u||_U||\phi||_H, \quad \text{for all } \phi \in H.$$

Remark 1.4. *The results can also be extended to the case when D is unbounded operator satisfying a similar singular estimate condition to that satisfied by B in Assumption 1.1 (4). This condition allows the inclusion of systems with noise in the boundary control into the theoretical framework developed below, as illustrated by the example included in the last section. However, to spare the reader further technical details, we will just assume D is bounded throughout the paper.*

Remark 1.5. *In the case when there is no final state penalization i.e. (G=0), the value of γ in (1.3) could be pushed up to* 1 *-as in the deterministic case [LTu1]. However, the majority of "non analytic" examples exhibit singularity of the type assumed in (1.3). For this reason, we focus on this class only.*

In sections 2 and 3, we state our main results and provide some preliminary results on mild solutions to the stochastic abstract differential equation (1.1). In section 4, we prove the existence of local-in-time solution to the integral Riccati equation via a fixed point argument and we investigate the regularity properties of the Riccati operator. In section 5, we derive the differential Riccati equation from the integral form. In section 6, we show the relation between the solution to the Riccati equation and the optimal control or minimizer of the cost functional (1.2) via dynamic programming, and then extend the result globally in time and show uniqueness of solution to the Riccati equation in sections 7 and 8 respectively. We then return to complete the proof of the main results Theorems 2.1 and 2.2 in section 9. We conclude the paper in section 10 with two examples to illustrate the theory: 1) a hinged thermoelastic plate model with noise and control through Neumann boundary condition and 2) a linearized fluid-structure interaction model with boundary control which we briefly discuss in the next section.

1.1 Motivating Example-Fluid Structure Interaction

In order to draw the attention of the reader to the significance of the assumptions imposed above on the control problem we provide an example of a fluid-structure interaction control problem with noise which became a motivation for our abstract framework [LTu3]. In the domain Ω, we consider a partition into an interior region Ω_s and an exterior region Ω_f where Ω_f is occupied by a fluid while Ω_s is occupied by a solid body. The interaction between the solid and the fluid takes place on the boundary Γ_s which separates both regions. The dynamics of the fluid are captured by a linear Stokes equation with multiplicative noise satisfied by fluid velocity u and fluid pressure p

$$du - \Delta u\, dt + \nabla p\, dt = c_1 u\, dW_t \qquad \text{in} \qquad \Omega_f \times [0, T] \tag{1.4}$$

$$\operatorname{div} u = 0 \qquad \text{in} \qquad \Omega_f \times [0, T]. \tag{1.5}$$

The dynamics of the solid are modeled by a linear second order equation with multiplicative noise

$$dw_t - \operatorname{div} \sigma(w)\, dt = c_2 w\, dW_t \qquad \text{in} \qquad \Omega_s \times [0, T] \tag{1.6}$$

in the solid displacement variable w, where σ is the stress tensor defined by

$$\sigma_{ij}(w) = \lambda \delta_{ij} \operatorname{div} w + 2\mu \epsilon_{ij}(w)$$

for $i, j = 1, 2, 3$ and constants $\lambda, \mu > 0$, and where ϵ is the strain tensor defined by

$$\epsilon_{ij}(w) = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right).$$

Here, W_t is a real Brownian motion on a complete probability space $(\Sigma, \mathcal{F}, P)$.

The interaction between the two bodies at the common interface Γ_s is captured by the following transmission boundary conditions matching velocities and stresses

$$u = w_t \qquad \text{on} \qquad \Gamma_s \times [0, T] \tag{1.7}$$

$$\epsilon(u)\nu - p\nu = \sigma(w)\nu + g + g\,\dot{W}(t) \qquad \text{on} \qquad \Gamma_s \times [0, T] \tag{1.8}$$

where ν is the outward unit normal and g is a control function acting as a force. On the outer part of the boundary Γ_f, we prescribe the no slip boundary condition

$$u = 0 \qquad \text{on} \qquad \Gamma_f \times [0, T]. \tag{1.9}$$

Given initial conditions in the finite energy space $u_0 \in H \equiv \{L^2(\Omega) : \text{div } u = 0, u \cdot \nu|_{\Gamma_f} = 0\}$ and $(w_0, w_1) \in H^1(\Omega_s) \times L^2(\Omega_s)$, the problem is to find a control $g \in L^2(\Sigma; L^2([0,T]; L^2(\Gamma_s)))$ to minimize the energy functional

$$J(u, w, w_t, g) = \mathbb{E}\left(\int_0^T \left(\|u(t) - u_T(t)\|^2_{L_2(\Omega_f)} + \|g(t)\|^2_{L^2(\Gamma_s)}\right) dt + \|u(T) - u_D\|^2_{L_2(\Omega_f)} + \|w(T) - w_D\|^2_{L_2(\Omega_s)}\right). \tag{1.10}$$

where $u_D \in L_2(\Omega_f), w_D \in L_2(\Omega_s), u_T \in L_2(\Omega_f \times [0,T])$ are given tracking targets.

2 Main Results

We first state the result pertaining to existence, regularity and uniqueness of solution to the optimal control problem.

Theorem 2.1. *Under the assumptions 1.1, there exists a positive self-adjoint operator $P(t) \in C([0,T]; \mathcal{L}(H))$ satisfying the Riccati equation*

$$\begin{aligned} \langle \dot{P}x, y\rangle + \langle PAx, y\rangle + \langle A^\star Px, y\rangle + \langle C^\star PCx, y\rangle + \langle R^\star Rx, y\rangle \\ -\langle (B^\star P + D^\star PC)^\star (I + D^\star PD)^{-1}(B^\star P + D^\star PC)x, y\rangle = 0, \qquad (2.1) \\ I + D^\star P(t) D > 0, \qquad (2.2) \\ P(T)x = G^\star Gx. \qquad (2.3) \end{aligned}$$

for every $x, y \in \mathcal{D}(A)$. Moreover, the following holds
(i) The minimum of the functional (1.2) *is given by*

$$\inf_{u \in M_w([s,T];U)} J(s, x, u) = \langle P(s)x, x\rangle$$

(ii) The solution $P(t)$ is unique in the class of positive self adjoint operators in $C([0,T]; \mathcal{L}(H))$.
(iii) The solution $P(t)$ satisfies the estimate

$$\|P(t)y\|_H \le c\|y\|_H \qquad \forall\ t \in [0,T),\ y \in H. \tag{2.4}$$

(iv) The operator $B^\star P(t)$ satisfies the estimate

$$\|B^\star P y\|_H \le \frac{c}{(T-t)^\gamma}\|y\|_H \qquad \forall\ t \in [0,T),\ y \in H. \tag{2.5}$$

We next state the result on the feedback form of the optimal control and the associated differential Riccati equation satisfied by the gain operator.

Theorem 2.2. *Under assumptions 1.1, the optimal control problem of minimizing* (1.2) *subject to the differential equation* (1.1) *with initial condition $x \in H$ has a unique solution $u^0(s, \cdot; x) \in L^2(\Omega; C([s,T); U))$*

and a corresponding optimal state $y^0(s,\cdot;x) \in L^2(\Omega; C([s,T];H))$. Moreover,
(i) The optimal control u^0 satisfies the estimate

$$\mathbb{E}(\|u^0(s,t;x)\|_U^2) \leq \frac{c}{(T-t)^{2\gamma}}\|x\|_H^2 \qquad \forall\ t \in [s,T). \tag{2.6}$$

(ii) The optimal control y^0 satisfies the estimate

$$\mathbb{E}(\|y^0(s,t;x)\|_H^2) \leq c\|x\|_H^2 \qquad \forall\ t \in [s,T]. \tag{2.7}$$

(iii) The optimal control u^0 has the feedback characterization in terms of the optimal state

$$u^0(t,s;x) = -(I + D^\star P(\tau)D)^{-1}(B^\star P(t) + D^\star P(t)C)y^0(t)$$

where $P(t)$ is the unique solution to the DRE (2.1)-(2.3).

Specific examples motivating the theory presented above include coupled PDE systems with boundary or point control where hyperbolic and parabolic dynamics are interwined. These, in particular include thermoelasticity, fluid structure interactions and models arising in structural acoustics [L2, AL].

Remark 2.3. *The analysis and result above easily extends to the case $1/2 \leq \gamma < 1$ when $G = 0$. However, for nonzero G, this case $1/2 \leq \gamma < 1$ is more challenging since operator*

$$GL_T \equiv G\int_0^T e^{A(T-\tau)}B\,d\tau$$

is no longer bounded $C(L^2(\Omega); L^2([s,T];U)) \to Z$. In fact, the existence of an optimal control in this case requires closability of GL_T [LT1]. Such condition is trivially satisfied when G is bounded invertible $H \to Z$.

3 Preliminaries

Following [DZ1], we say $y(t,s;x)$ is a mild solution of the stochastic differential equation (1.1) if

1. $y(t,s;x) = e^{A(t-s)}x + \int_s^t e^{A(t-\tau)}Bu(\tau)\,d\tau + \int_s^t e^{A(t-\tau)}Cy(\tau)\,dW_\tau + \int_s^t e^{A(t-\tau)}Du(\tau)\,dW_\tau$,
2. $y(t,s;x)$ takes values in $D(C)$,
3. $$P\left(\int_s^T \|y(\tau)\|_H\,d\tau < \infty\right) = 1$$
   and
   $$P\left(\int_s^T \|Cy(\tau)\|_H^2\,d\tau < \infty\right) = 1$$
4. Bu and Du are $\mathcal{F}_t$ measurable Bochner integrable H valued functions.

Results on the existence of mild solutions to (1.1) for a general forcing can be found in [DZ1, HO]. By strong continuity of the semigroup, we know there exists numbers $\alpha, M > 0$ such that $\|e^{At}z\|_H \leq Me^{\alpha t}\|z\|_H$, for all $z \in H$ and $t \in [s,T]$. We start with the existence of milld solution to (1.1), for which the proof is a standard argument, [H].

Theorem 3.1. *Let $\gamma < 1$. Given a function $u \in M_w^2([s,T];U)$ and an initial condition $y(s) = x \in H$, there exists a unique mild solution $y \in M_w^2([s,T];H)$ to the abstract differential equation* (1.1). *Moreover, if $\gamma < 1/2$ then $y \in L^2(\Omega; C([s,T];H))$.*

4 Integral Riccati Equation

In this section, we establish the existence of a solution to an integral form of the Riccati equation. The Riccati equation is, by itself, deterministic. However, its form is generated by the underlying stochastic process. This results in several additional terms (with respect to deterministic processes) which require subtle treatment. In fact, the relevant integral form of the differential Riccati equation is

$$\begin{aligned} P(t) = &\int_t^T e^{A^\star(\tau-t)} R^\star R \Phi(\tau,t)\, d\tau + \int_t^T e^{A^\star(\tau-t)} C^\star P(\tau) C \Phi(\tau,t)\, d\tau \\ &- \int_t^T e^{A^\star(\tau-t)} C^\star P^\star(\tau) D (I + D^\star P(\tau) D)^{-1} (B^\star P(\tau) + D^\star P(\tau) C) \Phi(\tau,t)\, d\tau \\ &+ e^{A^\star(T-t)} G^\star G \Phi(T,t) \end{aligned} \tag{4.1}$$

subject to the condition

$$\langle (I + D^\star P(t) D)x, x\rangle > 0 \quad \forall x \neq 0 \text{ and } x \in U$$

where $\Phi(t,s)$ is the solution to the equation

$$\Phi(t,s)x = e^{A(t-s)}x - \int_s^t e^{A(t-\tau)} B (I + D^\star P^\star(\tau) D)^{-1} (B^\star P(\tau) + D^\star P(\tau) C) \Phi(\tau,s) x\, d\tau. \tag{4.2}$$

Our main result in this section is the existence of local-in-time solutions to the above integral equations.

Theorem 4.1. *The integral equations* (4.1) *and* (4.2) *have unique local-in-time solutions $P(t) \in C([s,T];H)$ and $\Phi(\cdot,s) \in C([s,T];H)$ for $s = T_{max} < T$ chosen such that $T - T_{max}$ is sufficiently small. Moreover, the solution $P(t)$ is a positive self-adjoint operator on the space H and satisfies the estimate*

$$\|B^\star P(t) x\|_H \le \frac{c}{(T-t)^\gamma} \|x\|_H, \quad \forall x \in H, t \in [s,T). \tag{4.3}$$

The solutions will be extended to a global solution on the whole interval $[s,T]$ in section 7. One notices that the integral equation (4.1) depends on composition operators B^*P and PB which a priori are not defined at all. It is not even clear that B^*P can be densely defined (due to the unboundedness of B). However, the validity of the singular estimate will enable a rigorous analysis of this equation. We also notice that in the deterministic case one will only have the first and the last term in (4.1). Instead, in the present stochastic case the appearance of the third term provides quadratic dependence on the composition PB and P. Classical deterministic methods (either variational or direct) are no longer applicable. In order to tackle the problem of existence, we shall formulate a rather special iteration scheme which enables us to "unscramble" the convoluted dependence on the troublesome operator B^*P which a priori has no reason to be even densely defined. After few preliminaries in section 4.1, the proof will proceed in steps.

Step 1: In section 4.2, we first prove existence of a solution $(\widetilde{P}, \hat{\Phi})$ to the linear integral equation

$$\begin{aligned} \widetilde{P}(t) = &\int_t^T e^{A^\star(\tau-t)} R^\star R \hat{\Phi}(\tau,t)\, d\tau + \int_t^T e^{A^\star(\tau-t)} Q^\star(\tau) Q(\tau) \hat{\Phi}(\tau,t)\, d\tau \\ &+ \int_t^T e^{A^\star(\tau-t)} \hat{C}^\star(\tau) \widetilde{P}(\tau) \hat{C}(\tau) \hat{\Phi}(\tau,t)\, d\tau \\ &- \int_t^T e^{A^\star(\tau-t)} \hat{\psi}^\star(\tau) B^\star \widetilde{P}(\tau) \hat{\Phi}(\tau,t)\, d\tau + e^{A^\star(T-t)} G^\star G \hat{\Phi}(T,t), \end{aligned}$$

$$\hat{\Phi}(t,s)x = e^{A(t-s)}x - \int_s^t e^{A(t-z)} B \hat{\psi}(z) \hat{\Phi}(z,s) x\, dz,$$

where $Q(t)$, $\hat{C}(t)$ and $\hat{\psi}(t)$ are given bounded operators satisfying the singular estimate (4.6).

Remark 4.2. *Note these integral equations formally correspond to the system of linear equations*

$$\begin{aligned}
\frac{d}{dt}\widetilde{P}(t) &= -R^\star R - Q^\star(t)Q(t) - A^\star\widetilde{P}(t) - \widetilde{P}(t)A - \hat{C}^\star(t)\widetilde{P}(t)\hat{C}(t) + \hat{\psi}^\star(t)B^\star\widetilde{P}(t),\\
\frac{d}{dt}\hat{\Phi}(t,s) &= (A - B\hat{\psi}(t))\hat{\Phi}(t,s)\\
\widetilde{P}(T) &= G^\star G, \quad \hat{\Phi}(s,s) = I.
\end{aligned}$$

Step 2: In section 4.3, we next show that the solution $\widetilde{P}$ is a positive self-adjoint operator in $C([s,T];\mathcal{L}(H))$ and $\hat{\Phi}(t,s)$ is an evolution while $B^\star\widetilde{P}(t)$ satisfies the estimate (4.3).

Step 3: We now define the initial variables

$$\begin{aligned}
P_0(t) &\equiv e^{A^\star(T-t)}G^\star G e^{A(T-t)},\\
Q_0(\tau) &\equiv (I + D^\star P_0(\tau)D)^{-1}(B^\star P_0(\tau) + D^\star P_0(\tau)C),\\
\hat{C}_0(\tau) &\equiv C - D(I + D^\star P_0(\tau)D)^{-1}(B^\star P_0(\tau) + D^\star P_0(\tau)C),\\
\hat{\psi}_0 &\equiv (I + D^\star P_0(\tau)D)^{-1}(B^\star P_0(\tau) + D^\star P_0(\tau)C).
\end{aligned}$$

This choice of the positive operator P_0 guarantees that $B^\star P_0(t)$ is bounded $H \to U$ for $t \in [s,T)$ and satisfies (4.3), and that $(I + D^\star P_0 D)^{-1}$ is well defined and bounded on U.

Step 4: We next set up the following iteration scheme on the equation from step 1

$$\begin{aligned}
P_{i+1}(t) = &\int_t^T e^{A^\star(\tau-t)}R^\star R\hat{\Phi}_i(\tau,t)\,d\tau + \int_t^T e^{A^\star(\tau-t)}Q_i^\star(\tau)Q_i(\tau)\hat{\Phi}_i(\tau,t)\,d\tau\\
&+ \int_t^T e^{A^\star(\tau-t)}\hat{C_i}^\star(\tau)P_{i+1}(\tau)\hat{C}_i(\tau)\hat{\Phi}_i(\tau,t)\,d\tau\\
&- \int_t^T e^{A^\star(\tau-t)}\hat{\psi_i}^\star(\tau)B^\star P_{i+1}(\tau)\hat{\Phi}_i(\tau,t)\,d\tau + e^{A^\star(T-t)}G^\star G\hat{\Phi}_i(T,t)
\end{aligned}$$

where $\hat{\Phi}_i(t,s)x = e^{A(t-s)}x - \int_s^t e^{A(t-z)}B\hat{\psi}_i(z)\hat{\Phi}_i(z,s)x\,dz$ and

$$\begin{aligned}
Q_i(\tau) &\equiv (I + D^\star P_i(\tau)D)^{-1}(B^\star P_i(\tau) + D^\star P_i(\tau)C),\\
\hat{C}_i(\tau) &\equiv C - D(I + D^\star P_i(\tau)D)^{-1}(B^\star P_i(\tau) + D^\star P_i(\tau)C),\\
\hat{\psi}_i &= (I + D^\star P_i(\tau)D)^{-1}(B^\star P_i(\tau) + D^\star P_i(\tau)C).
\end{aligned}$$

Step 1 guarantees the existence of solution (P_{i+1}, Φ_i) at each step of the iteration, and that P_{i+1} is a positive self adjoint operator, such that $B^\star P_{i+1}$ is bounded for $t \in [s,T)$ and satisfies (4.3). This in turn gives sense to the operator $(I + D^\star P_{i+1}(\tau)D)^{-1}$ in $\mathcal{L}(U)$ which is needed in the next step of the iteration.

Step 5: Passing through the limit, we finally show that the sequence P_i converges to the solution P of the original integral equation (4.1) in $C([s,T];\mathcal{L}(H))$.

4.1 Preliminaries

We first introduce the space $C([s,T];\mathcal{L}(H))$ of the continuous family $P(.)$ of bounded operators on the space H, where

$$\|P\|_{C([s,T],\mathcal{L}(H))} = \sup_{s\le t\le T}\|P(t)\|_{\mathcal{L}(H)}.$$

Following [H], we also introduce the space $C(\mathcal{T}_s;\mathcal{L}(H))$ where

$$\mathcal{T}_s \equiv \{(t,\tau)\in\mathbb{R}^2 : s\le\tau\le t\le T\}.$$

This space $C(\mathcal{T}_s;\mathcal{L}(H))$ is a Banach space equipped with the norm

$$\|f\|_{C(\mathcal{T}_s;\mathcal{L}(H))} = \sup_{(t,\tau)\in\mathcal{T}_s} \|f(t,\tau)\|_{\mathcal{L}(H)}.$$

We also introduce the Banach space $C_\gamma([s,T];Y)$ (following [BDDM]) of continuous functions on $[s,T)$ into a Banach space Y, which is equipped with norm

$$\|f\|_{C_\gamma([s,T];Y)} = \sup_{t\in[s,T]} (T-t)^\gamma \|f(t)\|_Y < \infty.$$

The space accounts for possible singularities at time T of order γ. We start with the following useful lemmas [L1, L2, LTu1].

Lemma 4.3. *(i) The map $L_s \equiv \int_s^t e^{A(t-\tau)}B\,d\tau$ is continuous from $C_\gamma([s,T];U)$ to $C([s,T];H)$ for $\gamma<1/2$.*

(ii) The adjoint map $L_s^\star \equiv \int_t^T B^\star e^{A^\star(\tau-t)}\,d\tau$ is continuous from $C_\gamma([s,T];H)$ to $C([s,T];U)$ for $\gamma<1/2$.

4.2 Linear Integral Equation

We first consider the linear integral equations

$$\begin{aligned}\widetilde{P}(t) = &\int_t^T e^{A^\star(\tau-t)}R^\star R\hat{\Phi}(\tau,t)\,d\tau + \int_t^T e^{A^\star(\tau-t)}Q^\star(\tau)Q(\tau)\hat{\Phi}(\tau,t)\,d\tau \\ &+ \int_t^T e^{A^\star(\tau-t)}\hat{C}^\star(\tau)\widetilde{P}(\tau)\hat{C}(\tau)\hat{\Phi}(\tau,t)\,d\tau \\ &- \int_t^T e^{A^\star(\tau-t)}\hat{\psi}^\star(\tau)B^\star\widetilde{P}(\tau)\hat{\Phi}(\tau,t)\,d\tau + e^{A^\star(T-t)}G^\star G\hat{\Phi}(T,t)\end{aligned} \tag{4.4}$$

and

$$\hat{\Phi}(t,s)x = e^{A(t-s)}x - \int_s^t e^{A(t-z)}B\hat{\psi}(z)\hat{\Phi}(z,s)x\,dz. \tag{4.5}$$

In the next Lemma, we prove existence of solutions $\widetilde{P}$ and $\hat{\Phi}(t,s)$ to integral equations (4.4) and (4.5).

Lemma 4.4. *Assume $Q(t)$, $\hat{C}(t)$, $\hat{\psi}(t)$ are given bounded operators for every $t\in[s,T)$ satisfying the conditions*

$$\|Q(t)x\|_H, \|\hat{C}(t)x\|_H, \|\hat{\psi}(t)x\|_H \le \frac{r\|x\|_H}{(T-t)^\gamma}, \quad \forall x\in H, t\in[s,T) \tag{4.6}$$

for some suitably chosen $r>0$. Then, there exists a unique local-in-time solution $\widetilde{P}\in C([T_0,T];\mathcal{L}(H))$ and $\hat{\Phi}(\cdot,\cdot)\in C(\mathcal{T}_{T_0};\mathcal{L}(H))$ to the set of integral equations (4.4) and (4.5) such that

$$\|B^\star\widetilde{P}(t)x\|_H \le \frac{c}{(T-t)^\gamma}\|x\|_H. \tag{4.7}$$

To prove existence of a solution $\widetilde{P}$ and $\hat{\Phi}$, we use a fixed point argument on the map Λ defined by

$$\Lambda\begin{pmatrix} f\\ g\\ h\end{pmatrix}(t) = \begin{pmatrix} \Lambda_{11}(g)(t)+\Lambda_{12}(g)(t)+\Lambda_{13}(f,g)(t)+\Lambda_{14}(g,h)(t)+\Lambda_{15}(g)(t) \\ \Lambda_2(g)(t) \\ \Lambda_{31}(g)(t)+\Lambda_{32}(g)(t)+\Lambda_{33}(f,g)(t)+\Lambda_{34}(g,h)(t)+\Lambda_{35}(g)(t)\end{pmatrix}$$

for $t \in [s,T]$ on the space $X \equiv C([s,T];\mathcal{L}(H)) \times C(\mathcal{T}_s;\mathcal{L}(H)) \times C_\gamma([s,T];\mathcal{L}(H,U))$ where

$$\begin{aligned}
\Lambda_{11}(g)(t) &\equiv \int_t^T e^{A^\star(\tau-t)} R^\star R g(\tau,t)\, d\tau \\
\Lambda_{12}(g)(t) &\equiv \int_t^T e^{A^\star(\tau-t)} Q^\star(\tau) Q(\tau) g(\tau,t)\, d\tau \\
\Lambda_{13}(f,g)(t) &\equiv \int_t^T e^{A^\star(\tau-t)} \hat{C}^\star(\tau) f(\tau) \hat{C}(\tau) g(\tau,t)\, d\tau \\
\Lambda_{14}(g,h)(t) &\equiv -\int_t^T e^{A^\star(\tau-t)} \hat{\psi}^\star(\tau) h^\star(\tau) g(\tau,t)\, d\tau \\
\Lambda_{15}(g)(t) &\equiv e^{A^\star(T-t)} G^\star G e^{A(T-t)} - e^{A^\star(T-t)} G^\star G \int_t^T e^{A(T-\tau)} B \hat{\psi}(\tau) g(\tau,t)\, d\tau
\end{aligned}$$

and

$$\Lambda_2(f,g,h) = e^{A(t-s)} - L_s B \hat{\psi}(\cdot) g(\cdot,\cdot)(t)$$

while

$$\begin{aligned}
\Lambda_{31}(g)(t) &\equiv \int_t^T B^\star e^{A^\star(\tau-t)} R^\star R g(\tau,t)\, d\tau \\
\Lambda_{32}(g)(t) &\equiv \int_t^T B^\star e^{A^\star(\tau-t)} Q^\star(\tau) Q(\tau) g(\tau,t)\, d\tau \\
\Lambda_{33}(f,g)(t) &\equiv \int_t^T B^\star e^{A^\star(\tau-t)} \hat{C}^\star(\tau) f(\tau) \hat{C}(\tau) g(\tau,t)\, d\tau \\
\Lambda_{34}(g,h)(t) &\equiv -\int_t^T B^\star e^{A^\star(\tau-t)} \hat{\psi}^\star(\tau) h^\star(\tau) g(\tau,t)\, d\tau \\
\Lambda_{35}(g)(t) &\equiv B^\star e^{A^\star(T-t)} G^\star G e^{A(T-t)} - B^\star e^{A^\star(T-t)} G^\star G \int_t^T e^{A(T-\tau)} B \hat{\psi}(\tau) g(\tau,t)\, d\tau
\end{aligned}$$

In order to deal with unboundedness of control operator B we look at a fixed point of the system of three equations defined by three variables (operators) which are, $f = P$, $g = \Phi$ and $h = B^* P$. All these three quantities will be defined on the space X. Clearly we will have $h = B^* f$-which then will lead to "hidden" regularity results obtained for the gain operator $B^* P$. The fixed point f, g, h here represent the operators $P(t)$, $\Phi(t,s)$ and $B^\star P$ respectively.

Lemma 4.5. *The map Λ maps the ball $B_r(0) \subset X$ into itself continuously, and is a contraction on $B_r(0)$ for suitably chosen $r > 0$ and $s = T_0$ such that $T - T_0$ is sufficiently small.*

Proof. Let $[f,g,h]$ be an element in the ball $B_r(0)$. We estimate the norm of $\Lambda[f,g,h]$ in X, by considering every component. We spare the reader the technical details of the estimates. Defining c_s by

$$c_s = \max\left\{ c(T-s), c\frac{(T-s)^{1-\gamma}}{1-\gamma}, c\frac{(T-s)^{1-2\gamma}}{1-2\gamma} \right\},$$

and based on these estimates we impose the condition $6cM^2 e^{2\alpha(T-s)} + 6c_s M e^{\alpha(T-s)}(r^4 + r^3 + r^2 + r) < r$, or equivalently

$$cM^2 e^{2\alpha(T-s)} + c_s M e^{\alpha(T-s)}(r^4 + r^3 + r^2 + r) - r/6 < 0 \tag{4.8}$$

Let $r = 12cM^2e^{2\alpha T}$ and choose s such that $(T-s)$ is sufficiently small and so that

$$c_s < \frac{cMe^{\alpha T}}{r^4 + r^3 + r^2 + r}$$

This gauarantees that Λ acts from $B_r(0)$ into $B_r(0)$ in X for our choice of s and r. The contraction property can be shown by estimating the norm of the difference of $\Lambda[f_1, g_1, h_1]^T$ and $\Lambda[f_2, g_2, h_2]^T$. Choosing $s = T_0$ so that $T - T_0$ is sufficiently small we have that Λ is a contraction on $B_r(X)$ and hence has a unique fixed point $(f, g, h) \in X$.

□

From the above lemma, the fixed point (f, g, h) represent solutions $(\widetilde{P}(t), \hat{\Phi}(t,s), B^\star \widetilde{P}(t)) \in X$ to (4.4) and (4.5). Estimate (4.7) follows from the membership of $B^\star \widetilde{P}$ in $C_\gamma([s,T]; U)$. This proves Lemma 4.4.

4.3 Positivity and Self-Adjointness of $\widetilde{P}$

Let $s = T_0$. In the following lemma, we prove that the solution $\widetilde{P}$ to (4.4) is positive, self-adjoint in addition to the evolution property of $\hat{\Phi}(t,s)$on the space $C(\mathcal{T}_s; \mathcal{L}(H))$.

Lemma 4.6. *(i) The operator $\hat{\Phi}(t,s)$, defined by* (4.5)*, is an evolution operator on $C([s,T]; \mathcal{L}(H))$.*
(ii) The operator $\widetilde{P}$ solving the integral equation (4.4) *is self-adjoint.*
(iii) The operator $\widetilde{P}$ solving the integral equation (4.4) *is positive.*

Proof. (i) This follows from a standard argument using the evolution property of the semigroup.

(ii) Taking the inner product of (4.4) with $y \in H$ and substituting the expression

$$e^{A(\tau-t)}y = \hat{\Phi}(\tau,t)y + \int_t^\tau e^{A(\tau-z)}B\hat{\psi}(z)\hat{\Phi}(z,t)y\,dz$$

from (4.5) into the equation, we have

$$\begin{aligned}
\langle \widetilde{P}(t)x, y\rangle = &\int_t^T \langle R\hat{\Phi}(\tau,t)x, R\hat{\Phi}(\tau,t)y\rangle\,d\tau \\
&+ \int_t^T \langle R^\star R\hat{\Phi}(\tau,t)x, \int_t^\tau e^{A(\tau-z)}B\hat{\psi}(z)\hat{\Phi}(z,t)y\,dz\rangle\,d\tau \\
&+ \int_t^T \langle Q(\tau)\hat{\Phi}(\tau,t)x, Q(\tau)\hat{\Phi}(\tau,t)y\rangle\,d\tau \\
&+ \int_t^T \langle Q^\star(\tau)Q(\tau)\hat{\Phi}(\tau,t)x, \int_t^\tau e^{A(\tau-z)}B\hat{\psi}(z)\hat{\Phi}(z,t)y\,dz\rangle\,d\tau \\
&+ \int_t^T \langle \hat{C}^\star(\tau)\widetilde{P}(\tau)\hat{C}(\tau)\hat{\Phi}(\tau,t)x, \hat{\Phi}(\tau,t)x\rangle\,d\tau \\
&+ \int_t^T \langle \hat{C}^\star(\tau)\widetilde{P}(\tau)\hat{C}(\tau)\hat{\Phi}(\tau,t)x, \int_t^\tau e^{A(\tau-z)}B\hat{\psi}(z)\hat{\Phi}(z,t)y\,dz\rangle\,d\tau \\
&- \int_t^T \langle \hat{\psi}^\star(\tau)B^\star\widetilde{P}(\tau)\hat{\Phi}(\tau,t)x, \hat{\Phi}(\tau,t)y\rangle\,d\tau \\
&- \int_t^T \langle \hat{\psi}^\star(\tau)B^\star\widetilde{P}(\tau)\hat{\Phi}(\tau,t)x, \int_t^\tau e^{A(\tau-z)}B\hat{\psi}(z)\hat{\Phi}(z,t)y\,dz\rangle\,d\tau \\
&+ \langle G\hat{\Phi}(T,t)x, G\hat{\Phi}(T,t)y\rangle + \langle G^\star G\hat{\Phi}(T,t)x, \int_t^T e^{A(T-z)}B\hat{\psi}(z)\hat{\Phi}(z,t)y\,dz\rangle.
\end{aligned}$$

Changing the order of integration, the second, fourth, sixth and eighth term combine into

$$\begin{aligned}
&\int_t^T \int_z^T \langle B^\star e^{A^\star(\tau-z)} R^\star R\hat{\Phi}(\tau,t)x, \hat{\psi}(z)\hat{\Phi}(z,t)y \,\rangle \, d\tau \, dz \\
&\quad + \int_t^T \int_z^T \langle B^\star e^{A^\star(\tau-z)} Q^\star(\tau)Q(\tau)\hat{\Phi}(\tau,t)x, \hat{\psi}(z)\hat{\Phi}(z,t)y \,\rangle \, d\tau \, dz \\
&\quad + \int_t^T \int_z^T \langle B^\star e^{A^\star(\tau-z)} \hat{C}^\star(\tau)\widetilde{P}(\tau)\hat{C}(\tau)\hat{\Phi}(\tau,t)x, \hat{\psi}(z)\hat{\Phi}(z,t)y\rangle \, d\tau \, dz \\
&\quad - \int_t^T \int_z^T \langle B^\star e^{A^\star(\tau-z)} \hat{\psi}^\star(\tau)B^\star\widetilde{P}(\tau)\hat{\Phi}(\tau,t)x, \hat{\psi}(z)\hat{\Phi}(z,t)y\rangle \, d\tau \, dz \\
&\quad + \int_t^T \langle B^\star e^{A^\star(T-z)} G^\star G\hat{\Phi}(T,t)x, \hat{\psi}(z)\hat{\Phi}(z,t)y \,\rangle dz \\
&\quad = \int_t^T \langle B^\star \widetilde{P}(z)\hat{\Phi}(z,t)x, \hat{\psi}(z)\hat{\Phi}(z,t)y \,\rangle dz
\end{aligned}$$

which cancels with the fifth term. Therefore we have

$$\begin{aligned}
\langle \widetilde{P}(t)x, y\rangle = &\int_t^T \langle R\hat{\Phi}(\tau,t)x, R\hat{\Phi}(\tau,t)y\rangle \, d\tau + \int_t^T \langle Q(\tau)\hat{\Phi}(\tau,t)x, Q(\tau)\hat{\Phi}(\tau,t)y\rangle \, d\tau \\
&+ \int_t^T \langle \hat{C}^\star(\tau)\widetilde{P}(\tau)\hat{C}(\tau)\hat{\Phi}(\tau,t)x, \hat{\Phi}(\tau,t)y\rangle \, d\tau + \langle G\hat{\Phi}(T,t)x, G\hat{\Phi}(T,t)y\rangle
\end{aligned} \tag{4.9}$$

On the other hand, we have

$$\begin{aligned}
\langle \widetilde{P}^\star(t)x, y\rangle = &\int_t^T \langle R\hat{\Phi}(\tau,t)x, R\hat{\Phi}(\tau,t)y\rangle \, d\tau + \int_t^T \langle Q(\tau)\hat{\Phi}(\tau,t)x, Q(\tau)\hat{\Phi}(\tau,t)y\rangle \, d\tau \\
&+ \int_t^T \langle \hat{C}^\star(\tau)\widetilde{P}^\star(\tau)\hat{C}(\tau)\hat{\Phi}(\tau,t)x, \hat{\Phi}(\tau,t)y\rangle \, d\tau + \langle G\hat{\Phi}(T,t)x, G\hat{\Phi}(T,t)y\rangle.
\end{aligned}$$

Taking the difference of the two last equations, we get

$$\langle [\widetilde{P} - \widetilde{P}^\star](t)x, y\rangle = \int_t^T \langle \hat{C}^\star(\tau)[\widetilde{P} - \widetilde{P}^\star](\tau)\hat{C}(\tau)\hat{\Phi}(\tau,t)x, \hat{\Phi}(\tau,t)y\rangle \, d\tau$$

Estimating the left side, and taking the supremum over all x of unit norm and all y in H, we obtain

$$\|\widetilde{P}(t) - \widetilde{P}^\star(t)\|_{\mathcal{L}(H)} \leq cr^4 \int_t^T \|\widetilde{P}(\tau) - \widetilde{P}^\star(\tau)\|_{\mathcal{L}(H)} \, d\tau.$$

Using Gronwall's inequality we conclude that the left hand side is zero and hence $P(t) = P^\star(t)$ for all $t \in [s,T]$.

(ii) To prove positivity, we appeal to (4.9). The operator $\widetilde{P}$ is then the unique fixed point of the map S on $C([s,T];\mathcal{L}(H))$ defined by

$$\begin{aligned}
\langle S(P)(t)x, y\rangle = &\int_t^T \langle R\hat{\Phi}(\tau,t)x, R\hat{\Phi}(\tau,t)y\rangle \, d\tau + \int_t^T \langle Q(\tau)\hat{\Phi}(\tau,t)x, Q(\tau)\hat{\Phi}(\tau,t)y\rangle \, d\tau \\
&+ \int_t^T \langle \hat{C}^\star(\tau)P(\tau)\hat{C}(\tau)\hat{\Phi}(\tau,t)x, \hat{\Phi}(\tau,t)y\rangle \, d\tau + \langle G\hat{\Phi}(T,t)x, G\hat{\Phi}(T,t)y\rangle.
\end{aligned}$$

The map S clearly maps positive operators to positive operators. The set of positive operators denoted by Σ_+ in $\mathcal{L}(H)$ is a convex set, and the existence of a unique fixed point for S on $C([T_0,T];\Sigma_+)$ follows by the contraction mapping theorem, for T_0 chosen so that $T - T_0$ is sufficiently small. □

4.4 Step 4: Proof of Theorem 4.1

Proof. To derive the integral equation (4.1), we use the following iteration scheme

$$
\begin{aligned}
P_{i+1}(t) = &\int_t^T e^{A^\star(\tau-t)} R^\star R \hat{\Phi}_i(\tau,t)\,d\tau + \int_t^T e^{A^\star(\tau-t)} Q_i^\star(\tau) Q_i^\star(\tau) \hat{\Phi}_i(\tau,t)\,d\tau \\
&+ \int_t^T e^{A^\star(\tau-t)} \hat{C}_i^{\,\star}(\tau) P_{i+1}(\tau) \hat{C}_i(\tau) \hat{\Phi}_i(\tau,t)\,d\tau \\
&- \int_t^T e^{A^\star(\tau-t)} \hat{\psi}_i^{\,\star}(\tau) B^\star P_{i+1}(\tau) \hat{\Phi}_i(\tau,t)\,d\tau + e^{A^\star(T-t)} G^\star G \hat{\Phi}_i(T,t), \qquad (4.10)
\end{aligned}
$$

where

$$
\begin{aligned}
Q_i(\tau) &\equiv (I + D^\star P_i(\tau) D)^{-1} (B^\star P_i(\tau) + D^\star P_i(\tau) C), \\
\hat{C}_i(\tau) &\equiv C - D(I + D^\star P_i(\tau) D)^{-1} (B^\star P_i(\tau) + D^\star P_i(\tau) C), \\
\hat{\psi}_i &= (I + D^\star P_i(\tau) D)^{-1} (B^\star P_i(\tau) + D^\star P_i(\tau) C), \\
P_0(t) &= e^{A^\star(T-t)} G^\star G e^{A(T-t)},
\end{aligned}
$$

and $\hat{\Phi}_i$ solves

$$
\hat{\Phi}_i(t,s)x = e^{A(t-s)}x - \int_s^t e^{A(t-z)} B \hat{\psi}_i(z) \hat{\Phi}_i(z,s) x\,dz. \qquad (4.11)
$$

Using the results of Lemma 4.1 and Lemma 4.6 from previous sections, each iteration P_i is well defined, positive self adjoint and bounded with

$$
\begin{aligned}
\|P_i\|_{C([s,T];\mathcal{L}(H))} &\le r \\
\|B^\star P_i(t) x\|_H &\le \frac{r}{(T-t)^\gamma} \|x\|_H
\end{aligned}
$$

$\forall x \in H$ and $\forall i \in \mathbb{N}$, while $\Phi_i \in C(\mathcal{T}_s; \mathcal{L}(H))$ such that

$$
\|\Phi_i\|_{C(\mathcal{T}_s;\mathcal{L}(H))} \le r
$$

and this guarantees that the inverse $(I + D^\star P_i(t) D)^{-1}$ is well defined and bounded on H at each step. Using standard estimates, it is not difficult to show that the sequence $\{P_i, \Phi_i, B^\star P_i\}$ is Cauchy in X for $s = T_{max} \ge T_0$ chosen such that $T - T_{max}$ is sufficiently small, and thus converging to some $(P(t), \Phi, h(t)) \in X$ with $h(t) = B^\star P(t)$. Passing through the limit in (4.10) and (4.11), we obtain (4.1) and (4.2).

□

5 The Differential Riccati Equation

In this section, we derive the differential Riccati equation from the integral Riccati equation (4.1). Our main result is then

Theorem 5.1. *The Riccati operator $P(t)$ solving the integral Riccati equation* (4.1) *is a solution to the differential Riccati equation*

$$
\begin{aligned}
\langle \dot{P}(t)x, y\rangle = &-\langle Rx, Ry\rangle - \langle Ax, P(t)y\rangle - \langle A^\star P(t)x, y\rangle - \langle C^\star P(t) Cx, y\rangle) \\
&+ \langle (I + D^\star P(t) D)^{-1} (B^\star P(t) + D^\star P(t) C)x, (B^\star P(t) + D^\star P(t) C)y\rangle \qquad (5.1)
\end{aligned}
$$

for all $x, y \in \mathcal{D}(A)$.

A critical step in this process is to establish a "singular estimate" on the transfer function corresponding to the controlled dynamics. This amounts to the estimate of singularity on the composition operator $\Phi(t,s)B$. To accomplish this we need several preliminary results. To carry out the derivation, we shall need to make sense of the derivative of the evolution $\Phi(t,s)$ with respect to the initial time s (in the weak sense).

5.1 Preliminaries

We first define the operator $\mathcal{M}$.

Definition 5.1. *Denote by* $\mathcal{M} \equiv \int_s^t e^{A(t-\tau)}B(I + D^\star P(\tau)D)^{-1}(B^\star P(\tau) + D^\star P(\tau)C)\,d\tau$.

We also define the space ${}_\gamma C([s,T];H)$ following [BDDM].

Definition 5.2. ${}_\gamma C([s,T];H) \equiv \{f \in C((s,T];H) : \sup_{t\in[s,T]} (t-s)^\gamma \|f(t)\|_H < \infty\}$

which is indeed a Banach space with the norm

$$\|f\|_{{}_\gamma C} = \sup_{t\in[s,T]} (t-s)^\gamma \|f(t)\|_H,$$

for $\gamma < 1/2$. In the following Lemma, we establish some of the properties of the operator $\mathcal{M}$.

Lemma 5.2. *(i) The operator* $e^{A(\cdot-s)}Bx \in {}_\gamma C([s,T];H) \quad \forall x \in U$ *and satisfies the estimate*

$$\|e^{A(t-s)}Bx\|_{{}_\gamma C([s,T];H)} \leq c\|x\|_U. \tag{5.2}$$

(ii) The operator $\mathcal{M}$ *is bounded on* ${}_\gamma C([s,T];H)$ *and satisfies the estimate*

$$\|\mathcal{M}g\|_{{}_\gamma C([s,T];H)} \leq c(T-s)^{1-\gamma}\|g\|_{{}_\gamma C([s,T];H)} \tag{5.3}$$

for every $g \in {}_\gamma C([s,T];H)$.

(iii) The operator $(I+\mathcal{M})$ *is invertible on* ${}_\gamma C([s,T];H)$ *and the inverse satisfies the estimate*

$$\|(I+\mathcal{M})^{-1}g\|_{{}_\gamma C([s,T];H)} \leq c(T-s)\|g\|_{{}_\gamma C([s,T];H)}.$$

(iv) The evolution $\Phi(t,s)$ *satisfies*

$$\Phi(\cdot,s)x = (I+\mathcal{M})^{-1}e^{A(\cdot-s)}x \qquad \forall x \in H. \tag{5.4}$$

Proof. The proofs are similar to the deterministic case in which $C = D = 0$, see [LTu1, Tu]. □

5.2 Regularity of the "Transfer Function"

We now make sense of the transfer function $\Phi(t,s)B$ and the derivative of the evolution $\Phi(t,s)$ with respect to initial time in an appropriate singular space, which is crucial in the derivation of the differential Riccati equation.

Proposition 5.3. *(i) For all $x \in U$ and $\gamma < 1/2$, we have $\Phi(t,s)Bx \in {}_\gamma C([s,T];H)$ and*

$$\|\Phi(t,s)Bx\|_H \le \frac{c}{(t-s)^\gamma}\|x\|_U, \quad \forall x \in U. \tag{5.5}$$

(ii) For all $x \in \mathcal{D}(A)$, the derivative of the evolution $\Phi(t,s)x$ with respect to initial time in the weak sense is

$$\frac{\partial}{\partial s}\Phi(\cdot,s)x = -\Phi(\cdot,s)(A - B(I + D^\star P(s)D)^{-1}(B^\star P(s) + D^\star P(s)C))x \in {}_\gamma C([s,T];H)$$

and satisfies the estimate

$$\left\|\frac{\partial}{\partial s}\Phi(t,s)Bx\right\|_H \le c\|x\|_{\mathcal{D}(A)} + \frac{c}{(t-s)^\gamma}\|x\|_U. \tag{5.6}$$

Proof. The proof follows from Lemma 5.2, see [LT1, Tu]. □

5.3 Proof of Theorem 5.1

Proof. Let $x, y \in \mathcal{D}(A)$ and consider the integral Riccati equation satisfied by $P(t)$ in (4.1). Taking the derivative with respect to t, we have

$$\begin{aligned}\langle \dot P(t)x,y\rangle = &-\langle R^\star Rx,y\rangle - \langle C^\star P(t)Cx,y\rangle + \langle C^\star P(t)D(I+D^\star P(t)D)^{-1}(B^\star P(t)+D^\star P(t)C)x,y\rangle\\ &-\langle A^\star P(t)x,y\rangle\\ &+\left\langle \int_t^T e^{A^\star(\tau-t)}R^\star R\frac{\partial}{\partial t}\Phi(\tau,t)x,y\right\rangle + \left\langle \int_t^T e^{A^\star(\tau-t)}C^\star P(\tau)C\frac{\partial}{\partial t}\Phi(\tau,t)x,y\right\rangle\\ &-\left\langle \int_t^T e^{A^\star(\tau-t)}C^\star P(\tau)D(I+D^\star P(\tau)D)^{-1}(B^\star P(\tau)+D^\star P(\tau)C)\frac{\partial}{\partial t}\Phi(\tau,t)x,y\right\rangle\end{aligned}$$

We now appeal to Proposition 5.3 (ii), where the expression for $\frac{\partial}{\partial t}\Phi(\tau,t)$ was derived so that we obtain

$$\begin{aligned}\langle \dot P(t)x,y\rangle = &-\langle R^\star Rx,y\rangle - \langle C^\star P(t)Cx,y\rangle + \langle C^\star P(t)D(I+D^\star P(t)D)^{-1}(B^\star P(t)+D^\star P(t)C)x,y\rangle\\ &-\langle A^\star P(t)x,y\rangle\\ &-\langle P(t)(A - B(I+D^\star P(t)D)^{-1})(B^\star P(t)+D^\star P(t)C)x,y\rangle\end{aligned}$$

where the last term is well defined by boundedness of $P(t)B$ and its adjoint. Rearranging terms, we obtain the differential Riccati equation

$$\begin{aligned}\langle \dot P(t)x,y\rangle = &-\langle R^\star Rx,y\rangle - \langle A^\star P(t)x,y\rangle - \langle P(t)Ax,y\rangle - \langle C^\star P(t)Cx,y\rangle\\ &+\langle (P(t)B + C^\star P(t)D)(I+D^\star P(t)D)^{-1}(B^\star P(t)+D^\star P(t)C)x,y\rangle.\end{aligned}$$

□

Remark 5.4. *The differential form of the Riccati equation holds for any elements $x, y \in D(A)$. This form will be used for elements x, y resulting from a stochastic process. Since stochastic equations do not posses strong solutions, the applicability of DRE in the stochastic context is questionable. To resolve this issue, we shall introduce an approximation procedure which consists of two steps. Step one: Regularity Lemma page 48 [H] allows one to define the derivative of P on a stochastic process which originates in the domain of A, with twice differentiable controls and smooth observations C, D. In the second step we shall regularize the state y by changing variable to v_n. This will allow the application of Itô's formula.*

Here we state a regularity Lemma and justify the form of DRE when acting on a stochastic process, -page 48 in [H].

Lemma 5.5. *If we have the additional assumptions that the operators* $AC, AD \in \mathcal{L}(H)$*, and* $u \in L^2(\Omega; H_0^1([s,T];U))$ *then given* $x \in \mathcal{D}(A)$ *we have*

$$\mathbb{E}(\langle P(t)X(t), AX(t)\rangle_H) < \infty$$

for all $t \in [s,T]$ *where* $X(t)$ *is a solution of the stochastic differential equation*

$$\begin{aligned} dX &= (AX + Bu)\,dt + (CX + Du)\,dW_t \\ X(s) &= x \in \mathcal{D}(A). \end{aligned}$$

Proof. We first write the form of the mild solution to the abstract differential equation as

$$X(t) = e^{A(t-s)}x + \int_s^t e^{A(t-\tau)}Bu(\tau)\,d\tau + \int_s^t e^{A(t-\tau)}CX(\tau)\,dW_\tau + \int_s^t e^{A(t-\tau)}Du(\tau)\,dW_\tau.$$

We apply operator A to each side and then split the term $AX(t)$ into two parts $AX(t) = Y_1 + Y_2$ where

$$Y_1(t) = e^{A(t-s)}Ax + \int_s^t e^{A(t-\tau)}ACX(\tau)\,dW_\tau + \int_s^t e^{A(t-\tau)}ADu(\tau)\,dW_\tau.$$

and $Y_2(t) = \int_s^t e^{A(t-\tau)}ABu(\tau)\,d\tau$.

We then estimate the norm of Y_1 in $L^2(\Omega; C([s,T];H))$ to obtain

$$\begin{aligned} \mathbb{E}(\|Y_1(t)\|_H^2) \le 3M^2e^{2\alpha(T-s)}\|Ax\|_H^2 + 3M^2e^{2\alpha(T-s)}\|AC\|_{\mathcal{L}(H)}^2 \int_s^t \mathbb{E}(\|X(\tau)\|_H^2)\,d\tau \\ + 3M^2e^{2\alpha(T-s)}\|AD\|_{\mathcal{L}(U,H)}^2 \int_s^t \mathbb{E}(\|u(\tau)\|_U^2)\,d\tau \end{aligned}$$

where we used the Itô isometry to estimate the stochastic integrals. Since $X(t)$ is the solution to the abstract differential equation, by Theorem 3.1, its norm in $M_w^2([s,T];H)$ is bounded and satisfies

$$\|X(t)\|_{M_w^2([s,T];H)}^2 \le c\|x\|_H^2 + c\|u\|_{M_w^2([s,T];U)}^2.$$

Hence, we have that $\mathbb{E}(\|Y_1(t)\|_H^2) \le cQ(\|AC\|_{\mathcal{L}(H)}, \|AD\|_{\mathcal{L}(H)}, \|u\|_{L^2(\Omega;H_0^1([s,T];U))}, \|Ax\|_H)$ where Q is a polynomial in the indicated norms. We next express Y_2 as

$$Y_2(t) = -Bu(t) + \int_s^t e^{A(t-\tau)}Bu'(\tau)\,d\tau = -Bu(t) + I(t) \tag{5.7}$$

via integration by parts in time where we used the fact $u(s) = 0$ since $u \in H_0^1([s,T];U))$. The second term can be estimated via the singular estimate condition and Hölder's inequality as

$$\begin{aligned} \mathbb{E}\left(\left\|\int_s^t e^{A(t-\tau)}Bu'(\tau)\,d\tau\right\|_H^2\right) &\le \mathbb{E}\left(\int_s^t \frac{c}{(t-\tau)^\gamma}\|u'(\tau)\|_U\,d\tau\right)^2 \\ &\le c(T-s)^{1-2\gamma}\mathbb{E}(\|u\|_{H_0^1([s,T];U)}^2). \end{aligned}$$

We are now ready to estimate the term $\mathbb{E}(\langle P(t)X(t), AX(t)\rangle_H$ as

$$\begin{aligned} \mathbb{E}(\langle P(t)X(t), AX(t)\rangle_H &\le |\mathbb{E}(\langle P(t)X(t), Y_1(t)\rangle_H)| + |\mathbb{E}(\langle P(t)X(t), I(t)\rangle_H)| + |\mathbb{E}(\langle P(t)X(t), Bu(t)\rangle_H)| \\ &\le \|P(t)\|_{\mathcal{L}(H)}\mathbb{E}(\|X(t)\|_H)\mathbb{E}(\|Y_1(t)\|_H) + \|P(t)\|_{\mathcal{L}(H)}\mathbb{E}(\|X(t)\|_H)\mathbb{E}(\|I(t)\|_H) \\ &\quad + \|B^\star P(t)\|_{\mathcal{L}(H,U)}\mathbb{E}(\|u(t)\|_U)\mathbb{E}(\|X(t)\|_H) \\ &\le cQ(\|AC\|_{\mathcal{L}(H)}, \|AD\|_{\mathcal{L}(H)}, \|P(t)\|_{\mathcal{L}(H)}, \|B^\star P(t)\|_{\mathcal{L}(H,U)}, \|u\|_{L^2(\Omega;H_0^1([s,T];U))}, \|Ax\|_H), \end{aligned}$$

where we used the continuous embedding $H_0^1([s,T];U) \subset C([s,T];U)$ in the last step and where Q is a polynomial in the indicated norms. The right hand side is finite which yields the desired result.

□

6 Dynamic Programming: The Riccati Equation and the Optimal Control

In the following lemma, we relate the optimization problem to the solution of the differential Riccati equation via a dynamic programing argument. This technique is paramount to a completion of squares technique which furnishes an expression for the cost functional in which the minimizer and minimum value of the cost functional can be immediately deduced. However, the use of Itô's formula in this argument requires C^2 trajectories, which means that the argument has to be performed on an approximate regularized version of the abstract SDE, before passing through the limit.

Lemma 6.1. *The quadratic cost functional* (1.2) *has the form*

$$J(t,x,u) = \mathbb{E}\left(\int_t^T \|(I + D^\star P(\tau)D)^{1/2}u(\tau) + (I + D^\star P(\tau)D)^{-1/2}(B^\star P(\tau) + D^\star P(\tau)C)y(\tau)\|_U^2 \, d\tau\right) + \langle P(t)x, x\rangle \tag{6.1}$$

for $s \le t \le T$ *and* $s = T_{max}$*, where* $P(t)$ *is a solution to the differential Riccati equation* (5.1) *and* y *is the solution to* (1.1) *corresponding to* $u \in M_w^2([s,T];U)$.

Proof. In order to apply Itô's formula, we must use an appropriate approximate problem satisfied by a sufficiently regular random variable, and in particular a strong solution of an SDE. We follow [H] closely and consider the following stochastic differential equation

$$dy_n = (Ay_n + Bu)\, dt + (C_n y_n + D_n u)\, dW_t,$$

where $R(n,A) = (nI - A)^{-1}$ is the resolvent of A, and C_n is defined by $C_n \equiv nR(n,A)C$ while $D_n \equiv nR(n,A)D$. Taking $u \in L^2(\Omega; H_0^1([s,T];U))$, we set

$$v_n = y_n + A^{-1}Bu$$

Now, let $P(t) \in C([s,T];\mathcal{L}(H))$ be a self adjoint positive operator satisfying the differential Riccati equation (5.1) such that $B^\star P(\cdot) \in C_\gamma([s,T];\mathcal{L}(H,U))$. We rewrite $\langle P(t)y_n(t), y_n(t)\rangle$ in terms of v_n as

$$\psi(t, v_n, u) = \langle P(t)v_n(t), v_n(t)\rangle - 2\langle P(t)v_n(t), A^{-1}Bu(t)\rangle + \langle P(t)A^{-1}Bu(t), A^{-1}Bu(t)\rangle. \tag{6.2}$$

We next observe that v_n is a strong solution of the equation

$$dv_n = (Av_n + A^{-1}Bu')\, dt + (C_n y_n + D_n u)\, dW_t, \tag{6.3}$$

where u' denotes $\frac{d}{dt}u$. In particular, taking $y(s) = x \in \mathcal{D}(A)$, and by the variation of parameters formula we get

$$y_n(t) = e^{A(t-s)}x + \int_s^t e^{A(t-\tau)}Bu(\tau)\, d\tau + \int_s^t e^{A(t-\tau)}C_n y_n(\tau)\, dW_\tau + \int_s^t e^{A(t-\tau)}D_n u(\tau)\, dW_\tau.$$

Integrating by parts in time in the first integral, we get

$$y_n(t) = e^{A(t-s)}x - A^{-1}Bu(t) + e^{A(t-s)}A^{-1}Bu(s) + \int_s^t e^{A(t-\tau)}A^{-1}Bu'(\tau)\,d\tau + \int_s^t e^{A(t-\tau)}(C_n y_n + D_n u)\,dW_\tau.$$

Adding $A^{-1}Bu(t)$ to both sides, we have

$$v_n(t) = e^{A(t-s)}(x + A^{-1}Bu(s)) + \int_s^t e^{A(t-\tau)}A^{-1}Bu'(\tau)\,d\tau + \int_s^t e^{A(t-\tau)}C_n y_n\,dW_\tau + \int_s^t e^{A(t-\tau)}D_n u\,dW_\tau,$$

which shows that v_n is a solution to (6.3). Now, we can verify that $v_n(t) \in \mathcal{D}(A)$. Indeed, applying A to the right hand side, we have

$$e^{A(t-s)}(Ax + Bu(s)) + \int_s^t e^{A(t-\tau)}Bu'(\tau)\,d\tau + \int_s^t e^{A(t-\tau)}AC_n y_n\,dW_\tau + \int_s^t e^{A(t-\tau)}AD_n u\,dW_\tau,$$

and $x \in \mathcal{D}(A)$ while

$$\begin{aligned}\mathbb{E}\left(\left\|\int_s^t e^{A(t-\tau)}Bu'(\tau)\,d\tau\right\|_H^2\right) &\le \mathbb{E}\left(\int_s^t \frac{c}{(t-\tau)^\gamma}\|u'(\tau)\|_U\,d\tau\right)^2 \\ &\le \widetilde{c}T^{1-2\gamma}\mathbb{E}(\|u\|^2_{H^1([s,T];U)}),\end{aligned}$$

(note $\gamma < 1/2$) where we used the singular estimate condition and Hölder's inequality in the last step. Moreover, we have by boundedness of AC_n and using Itô's isometry that

$$\mathbb{E}\left(\|\int_s^t e^{A(t-\tau)}AC_n y_n(\tau)\,dW_\tau\|_H^2\right) \le c_1\int_s^t \mathbb{E}(\|y_n(\tau)\|_H^2)\,d\tau < \infty$$

where we used Theorem 3.1 in the last step. Moreover,

$$\mathbb{E}\left(\|\int_s^t e^{A(t-\tau)}AD_n u\,dW_\tau\|_H^2\right) \le c_1\int_s^t \mathbb{E}(\|u(\tau)\|_U^2)\,d\tau \le c_1\|u\|^2_{M^2_\omega([s,T];U)}.$$

Hence, $v_n \in \mathcal{D}(A)$ which means it is a strong solution of equation (6.3).

We now can differentiate the expression for $\psi(t, v_n(t), u(t))$ in (6.2) using Itô's formula [DZ1] to obtain

$$\begin{aligned}d\psi(\tau, v_n(\tau), u(\tau)) = {}&\langle P'(\tau)v_n(\tau), v_n(\tau)\rangle\,d\tau + 2\langle P(\tau)v_n(\tau), Av_n(\tau) + A^{-1}Bu'(\tau)\rangle\,d\tau \\ &+ 2\langle P(\tau)v_n(\tau), C_n y_n(\tau) + D_n u(\tau)\rangle\,dW_\tau + \langle P(\tau)(C_n y_n(\tau) + D_n u(\tau)), C_n y_n(\tau) + D_n u(\tau)\rangle\,d\tau \\ &- 2\langle P'(\tau)v_n(\tau), A^{-1}Bu(\tau)\rangle\,d\tau - 2\langle P(\tau)(Av_n(\tau) + A^{-1}Bu'(\tau)), A^{-1}Bu(\tau)\rangle\,d\tau \\ &- 2\langle P(\tau)(C_n y_n(\tau) + D_n u(\tau)), A^{-1}Bu(\tau)\rangle\,dW_\tau - 2\langle P(\tau)v_n(\tau), A^{-1}Bu'(\tau)\rangle\,d\tau \\ &+ \langle P'(\tau)A^{-1}Bu(\tau), A^{-1}Bu(\tau)\rangle\,d\tau + 2\langle P(\tau)A^{-1}Bu'(\tau), A^{-1}Bu(\tau)\rangle\,d\tau.\end{aligned}$$

Substituting $y_n(\tau)$ back to eliminate $v_n(\tau)$ using self adjointness of $P'(\tau)$, we obtain

$$\begin{aligned}d\langle P(\tau)y_n(\tau), y_n(\tau)\rangle = {}&\langle P'(\tau)y_n(\tau), y_n(\tau)\rangle\,d\tau + 2\langle P(\tau)y_n(\tau), Ay_n(\tau) + Bu(\tau)\rangle\,d\tau \\ &+ 2\langle P(\tau)y_n(\tau), C_n y_n(\tau) + D_n u(\tau)\rangle\,dW_\tau + \langle P(\tau)(C_n y_n(\tau) + D_n u(\tau)), C_n y_n(\tau) + D_n u(\tau)\rangle\,d\tau\end{aligned}$$

We now recall that $P(\tau)$ solves the differential Riccati equation and hence, we have

$$\begin{aligned}d\langle P(\tau)y_n(\tau), y_n(\tau)\rangle = {}&-\langle A^\star P(\tau)y_n(\tau), y_n(\tau)\rangle\,d\tau - \langle P(\tau)Ay_n(\tau), y_n(\tau)\rangle\,d\tau - \langle R^\star R y_n(\tau), y_n(\tau)\rangle\,d\tau \\ &- \langle C^\star P(\tau)Cy_n(\tau), y_n(\tau)\rangle\,d\tau + \langle (B^\star P(\tau) + D^\star P(\tau)C)y_n(\tau), (I + D^\star P(\tau)D)^{-1}(B^\star P(\tau) + D^\star P(\tau)C)y_n(\tau)\rangle\,d\tau \\ &+ 2\langle P(\tau)y_n(\tau), Ay_n(\tau)\rangle\,d\tau + 2\langle P(\tau)y_n(\tau), Bu(\tau)\rangle\,d\tau + 2\langle P(\tau)y_n(\tau), C_n y_n(\tau) + D_n u(\tau)\rangle\,dW_\tau \\ &+ \langle P(\tau)(C_n y_n + D_n u), C_n y_n + D_n u\rangle\,d\tau\end{aligned}$$

which simplifies to

$$
\begin{aligned}
d\langle P(\tau)y_n(\tau), y_n(\tau)\rangle &= -\|Ry_n(\tau)\|_Z^2\, d\tau - \langle (C^\star P(\tau)C - C_n^\star P(\tau)C_n)y_n(\tau), y_n(\tau)\rangle\, d\tau \\
&+ \|(I + D^\star P(\tau)D)^{-1/2}(B^\star P(\tau) + D^\star P(\tau)C)y_n(\tau)\|_U^2\, d\tau \\
&+ 2\langle B^\star P(\tau)y_n(\tau), u(\tau)\rangle\, d\tau + 2\langle D_n^\star P(\tau)C_n y_n(\tau), u(\tau)\rangle\, d\tau \\
&+ \langle D_n^\star P(\tau)D_n u(\tau), u(\tau)\rangle\, d\tau + 2\langle P(\tau)y_n(\tau), C_n y_n(\tau) + D_n u(\tau)\rangle\, dW_\tau,
\end{aligned}
$$

where $(I + D^\star P(\tau)D)^{-1/2}$ is well defined since $I + D^\star P(\tau)D$ is a positive operator.

Adding $\|u(\tau)\|_U^2\, d\tau$ to both sides and adding and subtracting the term

$$
2\langle D^\star P(\tau)Du(\tau), u(\tau)\rangle\, d\tau + 2\langle D^\star P(\tau)Cy_n(\tau), u(\tau)\rangle\, d\tau
$$

to the right hand side, we get

$$
\begin{aligned}
\|u(\tau)\|_U^2\, d\tau + d\langle P(\tau)y_n(\tau), y_n(\tau)\rangle &= -\|Ry_n(\tau)\|_Z^2\, d\tau(-\langle (C^\star P(\tau)C - C_n^\star P(\tau)C_n)y_n(\tau), y_n(\tau)\rangle\, d\tau \\
&+ \|(I + D^\star P(\tau)D)^{-1/2}(B^\star P(\tau) + D^\star P(\tau)C)y_n(\tau)\|_U^2\, d\tau \\
&+ 2\langle (B^\star P(\tau) + D^\star P(\tau)C)y_n(\tau), u(\tau)\rangle\, d\tau + 2\langle (D_n^\star P(\tau)C_n - D^\star P(\tau)C)y_n(\tau), u(\tau)\rangle\, d\tau \\
&+ \langle (I + D^\star P(\tau)D)u(\tau), u(\tau)\rangle\, d\tau + \langle (I + D_n^\star P(\tau)D_n - D^\star P(\tau)D)u(\tau), u(\tau)\rangle\, d\tau \\
&+ 2\langle P(\tau)y_n(\tau), C_n y_n(\tau) + D_n u(\tau)\rangle\, dW_\tau.
\end{aligned}
$$

This simplifies to

$$
\begin{aligned}
\|u(\tau)\|_U^2\, ds + d\langle P(\tau)y_n(\tau), y_n(\tau)\rangle &= -\|Ry_n(\tau)\|_Z^2\, d\tau - \langle (C^\star P(\tau)C - C_n^\star P(\tau)C_n)y_n(\tau), y_n(\tau)\rangle\, d\tau \\
&+ \|(I + D^\star P(\tau)D)^{-1/2}(B^\star P(\tau) + D^\star P(\tau)C)y_n(\tau) - (I + D^\star P(\tau)D)^{1/2}u\|_U^2\, d\tau \\
&+ 2\langle (D_n^\star P(\tau)C_n - D^\star P(\tau)C)y_n(\tau), u(\tau)\rangle\, d\tau \\
&+ \langle (I + D_n^\star P(\tau)D_n - D^\star P(\tau)D)u(\tau), u(\tau)\rangle\, d\tau + 2\langle P(\tau)y_n(\tau), C_n y_n(\tau) + D_n u(\tau)\rangle\, dW_\tau.
\end{aligned}
$$

Integrating from t to T and using the condition $P(T) = G^\star G$ and $y_n(t) = x$, we have

$$
\begin{aligned}
\int_t^T \|u(\tau)\|_U^2\, d\tau + \int_t^T \|Ry_n(\tau)\|_W^2\, d\tau + \|Gy_n(T)\|_Z^2 &= \langle P(t)x, x\rangle - \int_t^T \langle (C^\star P(\tau)C - C_n^\star P(\tau)C_n)y_n(\tau), y_n(\tau)\rangle\, d\tau \\
&+ \int_t^T \|(I + D^\star P(\tau)D)^{-1/2}(B^\star P(\tau) + D^\star P(\tau)C)y_n(\tau) - (I + D^\star P(\tau)D)^{1/2}u\|_U^2\, d\tau \\
&+ 2\int_t^T \langle (D_n^\star P(\tau)C_n - D^\star P(\tau)C)y_n(\tau), u(\tau)\rangle\, d\tau \\
&+ \int_t^T \langle (I + D_n^\star P(\tau)D_n - D^\star P(\tau)D)u(\tau), u(\tau)\rangle\, d\tau + 2\int_t^T \langle P(\tau)y_n(\tau), C_n y_n(\tau) + D_n u(\tau)\rangle\, dW_\tau.
\end{aligned}
$$

Since $\langle P(\tau)y_n(\tau), C_n y_n(\tau) + D_n u(\tau)\rangle$ is not $L^2(\Omega; L^2([0,T], \mathbb{R}))$, we can not simply apply the expected value to the equation above. However, we appeal to Proposition 7.10 in [D2], from which it suffices that all the integrands are $L^1(\Omega; L^1([0,T], \mathbb{R}))$ to conclude that $\langle P(T)y_n(T), y_n(T)\rangle$ or $\|Gy_n(T)\|_Z^2$ is $L^1(\Omega; \mathbb{R})$

which means $\mathbb{E}(\|Gy_n(T)\|_Z^2) < \infty$ and that the expected value is

$$\begin{aligned}
\mathbb{E}(\|Gy_n(T)\|_Z^2) &= \langle P(t)x, x\rangle - \mathbb{E}\left(\int_t^T \|u(\tau)\|_U^2\, d\tau\right) \\
&- \mathbb{E}\left(\int_t^T \|Ry_n(\tau)\|_W^2\, d\tau - \int_t^T \langle (C^\star P(\tau)C - C_n^\star P(\tau)C_n)y_n(\tau), y_n(\tau)\rangle\, d\tau\right) \\
&+ \mathbb{E}\left(\int_t^T \|(I + D^\star P(\tau)D)^{-1/2}(B^\star P(\tau) + D^\star P(\tau)C)y_n(\tau) - (I + D^\star P(\tau)D)^{1/2}u\|_U^2\, d\tau\right) \\
&+ 2\mathbb{E}\left(\int_t^T \langle (D_n^\star P(\tau)C_n - D^\star P(\tau)C)y_n(\tau), u(\tau)\rangle\, d\tau\right) \\
&+ \mathbb{E}\left(\int_t^T \langle (I + D_n^\star P(\tau)D_n - D^\star P(\tau)D)u(\tau), u(\tau)\rangle\, d\tau\right).
\end{aligned}$$

Rearranging, we have

$$\begin{aligned}
J_n \equiv J(t, x, u) &= \langle P(t)x, x\rangle - \mathbb{E}\left(\int_t^T \langle (C^\star P(\tau)C - C_n^\star P(\tau)C_n)y_n(\tau), y_n(\tau)\rangle\, d\tau\right) \\
&+ \mathbb{E}\left(\int_t^T \|(I + D^\star P(\tau)D)^{-1/2}(B^\star P(\tau) + D^\star P(\tau)C)y_n(\tau) - (I + D^\star P(\tau)D)^{1/2}u\|_U^2\, d\tau\right) \\
&+ 2\mathbb{E}\left(\int_t^T \langle (D_n^\star P(\tau)C_n - D^\star P(\tau)C)y_n(\tau), u(\tau)\rangle\, d\tau\right) \\
&+ \mathbb{E}\left(\int_t^T \langle (D_n^\star P(\tau)D_n - D^\star P(\tau)D)u(\tau), u(\tau)\rangle\, d\tau\right). \qquad (6.4)
\end{aligned}$$

We next must show that $y_n \to y \in M_\omega^2([s, T]; H)$ while the second and the last two terms in (6.4) go to zero as $n \to \infty$.

6.1 Passing through the limit as $n \to \infty$

Estimating the norm of the difference $\mathbb{E}(\|y_n - y\|_H^2)$ we have

$$\begin{aligned}
\mathbb{E}(\|y_n(t) - y(t)\|_H^2) &\le c\mathbb{E}\left(\int_s^t \|e^{A(t-\tau)}(C_n y_n - Cy)\|_H\, dW_\tau\right)^2 + \mathbb{E}\left(\int_s^t \|e^{A(t-\tau)}(D_n u - Du)\|_H\, dW_\tau\right)^2 \\
&\le c\int_s^t \|C_n - C\|_{\mathcal{L}(H)}^2 \mathbb{E}(\|y\|_H^2)\, d\tau + c\int_s^t \|C_n\|_{\mathcal{L}(H)}^2 \mathbb{E}(\|y_n - y\|_H^2)\, d\tau \\
&\quad + c\int_s^t \|D_n - D\|_{\mathcal{L}(U;H)}^2 \mathbb{E}(\|u\|_U^2)\, d\tau.
\end{aligned}$$

Applying Gronwall's inequality, we obtain

$$\mathbb{E}(\|y_n(t) - y(t)\|_H^2) \le c\left(\|C_n - C\|_{\mathcal{L}(H)}^2 \|y\|_{M_w^2([s,T];H)}^2 + \|D_n - D\|_{\mathcal{L}(H)}^2 \|u\|_{M_w^2([s,T];U)}^2\right) \|C_n\|_{\mathcal{L}(H)}^2 t.$$

Integrating in time and noting that the sequence C_n is uniformly bounded by a constant M in norm (since $C_n \to C$), then choosing n sufficiently large we finally get

$$\int_s^T \mathbb{E}(\|y_n(t) - y(t)\|_H^2)\, dt \le \left(c\epsilon\|y\|_{M_w^2([s,T];H)}^2 + \epsilon\|u\|_{M_w^2([s,T];U)}^2\right) M\frac{T^2}{2}.$$

This shows that $y_n \to y$ in $M_w^2([s,T];H)$.

Using standard arguments we can easily show that $\mathbb{E}\left(\int_t^T \langle (C^\star P(\tau)C - C_n^\star P(\tau)C_n)y_n(\tau), y_n(\tau)\rangle\, d\tau\right) \to 0$ as $n \to \infty$. Similarly,

$$2\mathbb{E}\left(\int_t^T \langle (D_n^\star P(\tau)C_n - D^\star P(\tau)C)y_n(\tau), u(\tau)\rangle\, d\tau\right) \to 0$$

and

$$\mathbb{E}\left(\int_t^T \langle (D_n^\star P(\tau)D_n - D^\star P(\tau)D)u(\tau), u(\tau)\rangle\, d\tau\right) \to 0$$

as $n \to \infty$.

As for the second term in (6.4), we have

$$\begin{aligned}&\mathbb{E}\left(\int_t^T \|(I + D^\star P(\tau)D)^{-1/2}(B^\star P(\tau) + D^\star P(\tau)C)y_n(\tau) - (I + D^\star P(\tau)D)^{1/2}u\|_U^2\, d\tau\right)\\ \to\, &\mathbb{E}\left(\int_t^T \|(I + D^\star P(\tau)D)^{-1/2}(B^\star P(\tau) + D^\star P(\tau)C)y(\tau) - (I + D^\star P(\tau)D)^{1/2}u\|_U^2\, d\tau\right).\end{aligned}$$

Therefore, the functional J_n given in (6.4) converges to

$$J(t,x,u) = \langle P(t)x, x\rangle + \mathbb{E}\left(\int_t^T \|(I + D^\star P(\tau)D)^{-1/2}(B^\star P(\tau) + D^\star P(\tau)C)y(\tau) - (I + D^\star P(\tau)D)^{1/2}u\|_U^2\, d\tau\right).$$

6.2 Extending (6.1) for all $u \in M_w^2([s,T];U)$

By density of $L^2(\Omega; H^1([s,T];U)) \subset M_w^2([s,T];U)$, we approximate $u \in M_w^2([s,T];U)$ by a sequence $u_n \in L^2(\Omega; H^1([s,T];U))$, and pass through the limit. It is easy to show show that $y(u_n) \to y(u)$ in $M_w^2([s,T];H)$ (continuous dependence of y on the control u). Hence, passing through the limit in $u_n \to u$, we have $y_n \to y(u)$ and (6.1) is valid for $u \in M_w^2([s,T];U)$. Since the argument in passing through the limit in J is similar, it will not be repeated.

□

7 Extension of Theorem 5.1 to a Global Solution on any time interval $[s,T]$

We now extend the solution of the Riccati equation from $[T_{max}, T]$ to any time interval $[s,T]$. We establish a global bound on $P(t)$ since

$$\begin{aligned}\langle P(t)x, x\rangle \le J(t, x; u = 0) &= \mathbb{E}\left(\int_t^T \|Ry(\tau)\|^2\, d\tau + \|Gy(T)\|_Z^2\right)\\ &\le cM^2Te^{2\alpha T}\|x\|_H^2 + cM^2e^{2\alpha T}\|x\|_H^2 = C_T\|x\|_H^2\end{aligned}$$

for all $t \in [T_{\max}, T]$ and thus $\|P(t)\|_{\mathcal{L}(H)} \le \|P^{1/2}(t)\|_{\mathcal{L}(H)}^2 \le C_T$. This bound can be used to reiterate the proofs of Lemma 4.4 and Theorem 4.1 on new interval $[T_1, T_{max}]$ with $G = P^{1/2}(T_{\max})$. The bound insures that the choice of the constant c (which depends on G) in (4.8) is global and all the estimates are uniform and that r and the time step $T_{max} - T_1$ are the same. Hence, the results can be extended by repeated iteration on equal time steps to any initial time $s \ge 0$.

8 Uniqueness of Solution to the Differential Riccati Equation

Theorem 8.1. *The solution to the differential Riccati equation is unique in the class of self-adjoint operators in $C([0,T];\mathcal{L}(H))$ satisfying $B^\star P \in C_\gamma([s,T];\mathcal{L}(H,U))$.*

Proof. Assume there is another solution $\widetilde{P}(t)$ to the Riccati equation in this class, then the same dynamic programming argument from the previous section leads to

$$\min J(t,x,u) = \langle P(t)x, x\rangle = \langle \widetilde{P}(t)x, x\rangle$$

for all $x \in H$. Hence, we have for any $x, y \in H$ that

$$\begin{aligned} 0 &= \langle (P(t) - \widetilde{P}(t))(x+y), (x+y)\rangle \\ &= \langle (P(t) - \widetilde{P}(t))x, x\rangle + \langle (P(t) - \widetilde{P}(t))x, y\rangle + \langle (P(t) - \widetilde{P}(t))y, x\rangle + \langle (P(t) - \widetilde{P}(t))y, y\rangle \\ &= 2\langle (P(t) - \widetilde{P}(t))x, y\rangle \end{aligned}$$

by self-adjointness of P and $\widetilde{P}$. Thus, $P(t) = \widetilde{P}(t)$. □

9 Proof of Main Theorems 2.1 and 2.2

We finally obtain our main results in this paper stated in Theorems 2.1 and 2.2. We start with Theorem 2.1.

Proof. (i) From equation (6.1) in Lemma 6.1, the functional J satisfies

$$\inf_{u \in M_\omega([s,T];U)} J(s,x;u) = \langle P(s)x, x\rangle$$

where $P(t)$ is the solution to the differential Riccati equation.
(ii) The existence of solution to the Differential Riccati equation in $C([s,T];\mathcal{L}(H)$ follows from Theorem 5.1, and the uniqueness was established in section 8.
(iii), (iv) The regularity properties of $P(t)$ and $B^*P(t)$ were established in Theorem 4.1. □

Finally, we prove Theorem 2.2.

Proof. (i), (iii) To show that the minimum of J is realized in (6.1), we can establish the existence of a unique solution $u^0 \in M_w^2([s,T];U)$ to the equation

$$u^0(t,s;x) = -(I + D^\star P(t)D)^{-1}(B^\star P(t) + D^\star P(t)C)y(t,s,u^0;x)$$

via a fixed point argument on $M_w^2([s,T];U)$. Thus,

$$u^0(s,t;x) = -(I + D^\star P(t)D)^{-1}(B^\star P(t) + D^\star P(t)C)y^0(t,s;x),$$

so that $J(s,x;u^0) = \langle P(s)x, x\rangle$.

(ii) If follows from Theorem 3.1 that the corresponding optimal state $y^0 \in L^2(\Omega; C([s,T];H))$.

(iv) It then follows by regularity properties of $B^\star P$ in (4.3) that

$$\|u^0(t,s;x)\|_{L^2(\Omega;U)} \le \frac{c}{(T-t)^\gamma}\|x\|_H.$$

□

10 Applications to Control of PDEs

This section is devoted to an illustration of the theory presented to concrete PDE systems with unbounded control actions.

10.1 Theremoelastic plates with boundary control

We consider a stochastic model for a Hinged Thermoelastic plate with Neumann thermal boundary control. Let W_t be a one dimensional Wiener process on a complete probability space $(\Sigma, \mathcal{F}, \mathcal{P})$. The system consists of a heat equation and a plate equation

$$\left.\begin{array}{rl} [I-\rho\Delta]dw_t + \Delta^2 w\,dt + \Delta\theta\,dt = (\nabla w + bw_t)\,dW_t & \Omega\times[0,T] \\ d\theta - \Delta\theta\,dt - \Delta w_t\,dt = (C_{31}\Delta w + C_{32}\nabla w_t + C_{33}\theta)\,dW_t, & \Omega\times[0,T] \end{array}\right\} \tag{10.1}$$

where $w(\omega,x,t)$ is the transversal displacement and $\theta(\omega,x,t)$ is the temperature of the plate which occupies the open domain Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$, subject to the hinged boundary conditions

$$w = \Delta w = 0, \quad \partial\Omega\times[0,T] \tag{10.2}$$

and thermal control u on the boundary

$$\frac{\partial\theta}{\partial\nu} + b\theta = u(x,t) + u(x,t)\dot{W}(t), \quad \partial\Omega\times[0,T]. \tag{10.3}$$

The problem is defined for the random variables $y(\omega,x,t) \equiv (w(\omega,x,t), w_t(\omega,x,t), \theta(\omega,x,t))$ which take values in the finite energy space $\mathcal{H}$ defined by $\mathcal{H} \equiv H^2(\Omega)\cap H_0^1(\Omega)\times H_0^1(\Omega)\times L^2(\Omega)$.

We are particularly interested in a Bolza type optimal control of this system with the objective of minimizing an energy functional

$$\begin{aligned} J(u,w,w_t,\theta) = \mathbb{E}\bigg(\int_0^T \|u(\cdot,t)\|^2_{L^2(\partial\Omega)} + \|w(\cdot,t)\|^2_{H^2(\Omega)} + \|w_t(\cdot,t)\|^2_{H^1(\Omega)} + \|\theta(\cdot,t)\|^2_{L^2(\Omega)}\,dt \\ + \|w(\cdot,T)\|^2_{H^2(\Omega)} + \|w_t(\cdot,T)\|^2_{H^1(\Omega)}\bigg), \end{aligned} \tag{10.4}$$

over all boundary controls $u \in M_w^2([0,T];L^2(\partial\Omega))$, given initial data in the $(w_0,w_1,\theta_0)\in\mathcal{H}$ the finite energy space. This problem can be adapted to the abstract setting of the stochastic linear quadratic regulator, since the deterministic uncontrolled system is driven by a C_0-semigroup e^{At} while a control operator B from the boundary to interior satisfies the singular estimate [BL].

Following [LT1, BL], we introduce the self adjoint operator $\mathcal{A}$ on $L^2(\Omega)$ defined by

$$\mathcal{A}h = \Delta^2 h \tag{10.5}$$

with domain

$$\mathcal{D}(\mathcal{A}) = \{h\in H^4(\Omega) : h|_{\partial\Omega} = \frac{\partial}{\partial\nu}h|_{\partial\Omega} = 0\}. \tag{10.6}$$

The fractional power $\mathcal{A}^{1/2}$ of this operator has a domain which can be identified with the space $H^2(\Omega) \times H_0^1(\Omega)$. We also introduce the self-adjoint operator A_N on $L^2(\Omega)$

$$A_N h = -\Delta h \tag{10.7}$$

with domain

$$\mathcal{D}(A_N) = \{h \in H^2(\Omega) : \frac{\partial}{\partial \nu} h + h = 0 \;\text{ on } \partial\Omega\}. \tag{10.8}$$

The operator $-A_N$ is well known to generate an analytic semigroup $e^{-A_N t}$ on the space $L^2(\Omega)$.

We also follow [BL] in introducing the operator $\mathcal{M}$ on $L^2(\Omega)$ given by

$$\mathcal{M} = (I + \rho A_N) \tag{10.9}$$

with the well defined bounded inverse $\mathcal{M}^{-1}$. Additionally, we also introduce the Neumann map $N : L^2(\partial\Omega) \to L^2(\Omega)$ defined by

$$Ng = h \qquad \Longleftrightarrow \qquad \begin{aligned} &\Delta h = 0 \text{ in } \;\Omega \\ &\frac{\partial h}{\partial \nu} + h = g \text{ on } \;\partial\Omega \end{aligned}$$

It is well known that $A^{3/4-\epsilon}N$ is bounded $L^2(\partial\Omega) \to L^2(\Omega)$. The system can then be expressed in abstract form as

$$dy(t) = (Ay + Bu)\,dt + (Cy + Du)\,dW_t,$$

where

$$y(t) = \begin{pmatrix} w \\ w_t \\ \theta \end{pmatrix} \tag{10.10}$$

and

$$A = \begin{pmatrix} 0 & I & 0 \\ -\mathcal{M}^{-1}\mathcal{A} & 0 & \mathcal{M}^{-1}A_N \\ 0 & -A_N & -A_N \end{pmatrix} \tag{10.11}$$

and with domain

$$\mathcal{D}(A) = \mathcal{D}(\mathcal{A}^{3/4}) \times \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(A_N). \tag{10.12}$$

Moreover, the control operators B, D are

$$B = D = \begin{pmatrix} 0 \\ 0 \\ A_N N \end{pmatrix} \tag{10.13}$$

and the noise operator C is

$$C = \begin{pmatrix} 0 & 0 & 0 \\ \nabla & b & 0 \\ C_{31}\Delta & C_{32}\nabla & C_{33} \end{pmatrix},$$

for real parameters $C_{31}, C_{32}, C_{33}, b$. Note here that the adjoint $B^\star : \mathcal{D}(A^\star) \to L^2(\partial\Omega)$ is defined by

$$B^\star[x_1, x_2, x_3] = N^\star A_N x_3 = x_3|_{\partial\Omega}$$

which is the restriction to the boundary $\partial\Omega$. As for the observation operators in (10.4), we take $R = I$ and $G = [I, I, 0]$ on the state space $\mathcal{H}$.

It was shown in [BL] that the set of Assumptions 1.1 are indeed satisfied. In particular, the critical singular estimate does hold with any $\gamma > 1/4$

$$\|e^{At}Bu\|_{\mathcal{H}} \le \frac{C}{t^{1/4+\epsilon}}\|u\|_{L^2(\partial\Omega)}$$

for every $u \in L^2(\partial\Omega)$, and $A^{-1}B$ is bounded from $L^2(\partial\Omega)$ to $\mathcal{H}$. Thus, we are in a position to apply the conclusions of Theorems 2.1 and 2.2. Thus we have the following theorem:

Theorem 10.1. *Given initial data $(\theta_0, w_0, w_1) \in \mathcal{H}$, there exists a unique optimal control $u^0 \in M_w^2([s,T]; L^2(\partial\Omega))$ to the stochastic Thermoelastic plate system* (10.1) *with Hinged boundary conditions* (10.2) *and Neumann thermal boundary control* (10.3)*, which minimizes the cost functional* (10.4)*. Moreover,*

1. *The optimal control $u^0 \in C([s,T); L^2(\Sigma, \partial\Omega))$ and*

$$\mathbb{E}(\|u^0(t)\|^2_{L^2(\partial\Omega)}) \le \Big(\frac{c}{t^{1/4+\epsilon}}(\|w_0\|_{H^2(\Omega)} + \|w_1\|_{H^1(\Omega)} + \|\theta_0\|_{L^2(\Omega)})\Big)^2.$$

2. *The corresponding optimal state $(\theta^0(t), w^0(t), w^0_t(t)) \in C([s,T]; L^2(\Sigma, \mathcal{H}))$ and*

$$\mathbb{E}(\|w^0(t)\|^2_{H^2(\Omega)}) + \mathbb{E}(\|w^0_t(t)\|^2_{H^1(\Omega)}) + \mathbb{E}(\|\theta^0(t)\|^2_{L^2(\Omega)}) \le c(\|w_0\|^2_{H^2(\Omega)} + \|w_1\|^2_{H^1(\Omega)} + \|\theta_0\|^2_{L^2(\Omega)}).$$

3. *The optimal control is given in feedback form*

$$u^0(t) = -(I + D^\star P(t)D)^{-1}(D^\star P(t)C + B^\star P(t))[w^0(t), w^0_t(t), \theta^0(t)]^T$$

for B, D and C defined above and where $P(t)$ is a self-adjoint positive operator on $\mathcal{H}$ satisfying the differential Riccati equation

$$\begin{aligned}
&\langle \mathcal{A}^{1/2}p_{1t}, \mathcal{A}^{1/2}y_1\rangle + \langle \mathcal{M}^{1/2}p_{2t}, \mathcal{M}^{1/2}y_2\rangle + \langle p_{3t}, y_3\rangle = -\langle \mathcal{A}^{1/2}p_1, \mathcal{A}^{1/2}y_2\rangle + \langle p_2, \mathcal{A}y_1\rangle - \langle p_2, A_N y_3\rangle \\
&+ \langle A_N p_3, y_2\rangle + \langle A_N p_3, y_3\rangle - \langle \mathcal{A}^{1/2}x_2, \mathcal{A}^{1/2}\hat{p}_1\rangle + \langle \mathcal{A}x_1, \hat{p}_2\rangle - \langle A_N x_3, \hat{p}_2\rangle + \langle A_N x_2, \hat{p}_3\rangle + \langle A_N x_3, \hat{p}_3\rangle \\
&- \langle P(t)Cx, Cy\rangle + \langle (I + D^\star P(t)D)^{-1}(B^\star P(t) + D^\star PC)x, (B^\star P(t) + D^\star PC)y\rangle_{\partial\Omega}, \\
&[p_1(T), p_2(T), p_3(T)] = [x_1, x_2, 0],
\end{aligned}$$

for all $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathcal{D}(A)$, where we denote $P(t)x = [p_1(t), p_2(t), p_3(t)]$ and $P(t)y = [\hat{p}_1(t), \hat{p}_2(t), \hat{p}_3(t)]$, and by $\langle \cdot, \cdot \rangle$ the L^2 inner product on Ω.

10.2 Fluid Structure Interaction

Here we shall revisit the motivating example introduced in the section 1.1. In particular, the system (1.4)-(1.6) with boundary conditions (1.7)–(1.9) can be expressed in the abstract form

$$dY = \mathcal{A}_{FS}Y\,dt + Bg\,dt + CY\,dW_t + Dg\,dW_t$$

with

$$\mathcal{A}_{FS} = \begin{pmatrix} A_N & A_N N\sigma & 0 \\ 0 & 0 & I \\ 0 & \mathrm{div}(\sigma) & 0 \end{pmatrix} \tag{10.14}$$

where $A_N : V \to V'$ is defined by $\langle A_N\phi, v'\rangle = -\langle \epsilon(\phi), \epsilon(v)\rangle$ and V is the space

$$V \equiv \{v \in H^1(\Omega_f) : \mathrm{div}\, v = 0, v|_{\Gamma_f} = 0\}$$

while $N : H^{-1/2}(\Gamma_s) \to V$ is the map defined by

$$Ng = h \iff \langle A_N h, v\rangle = \langle g, v\rangle_{\Gamma_s}$$
$$h|_{\Gamma_f} = 0$$

for every $v \in V$ which is well defined by the Lax-Milgram Theorem [LTu3]. Denoting the finite energy space $H \times H^1(\Omega_s) \times L^2(\Omega_s)$ by $\mathcal{H}$, the operator $\mathcal{A}_{SF}$ generates a C_0-semigroup on the space $\mathcal{H}$. The control operators B and D are defined by

$$B = D = \begin{pmatrix} A_N N \\ 0 \\ 0 \end{pmatrix}. \tag{10.15}$$

and $B : L^2(\Gamma_s) \to [\mathcal{D}(\mathcal{A}^\star_{SF})]'$ is the control operator [LTu3] which satisfies an incrementally weaker form of the singular estimate [LTu3]

$$\|e^{\mathcal{A}_{SF}t}Bf\|_{\mathcal{H}^{-\alpha}} \le \frac{c}{t^{1/4+\epsilon}}\|f\|_{L^2(\Gamma_s)}$$

for $\alpha > 0$, where $\mathcal{H}^\alpha$ is the lower topology space $\mathcal{H}^\alpha = H \times H^{1-\alpha}(\Omega_s) \times H^{-\alpha}(\Omega_s)$.

However, this estimate is sufficient in order to address the control functional (1.10) with $\alpha = 1$, cf. [LTu3]. In particular, we take our operator $R = [I, 0, 0]$ and $G = [I, I, 0]$ and take the observation space $W \equiv \mathcal{H}$ and $Z \equiv \mathcal{H}^{-1}$. Moreover, we determine the noise operator C as

$$C = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_2 & 0 \end{pmatrix}. \tag{10.16}$$

which is a bounded operator on the state space $\mathcal{H}$. Note here that the adjoint $B^\star : \mathcal{D}(A^\star) \to L^2(\Gamma_s)$ is defined by

$$B^\star[x_1, x_2, x_3] = N^\star A_N x_1 = x_1|_{\Gamma_s}$$

Now that the assumptions of 1.1 are all satisfied by the system, we can specialize theorems 2.1 and 2.2 to this system to obtain the following optimal control result:

Theorem 10.2. *Given initial data* $(u_0, w_0, w_1) \in \mathcal{H}$*, there exists a unique optimal control* $g^0 \in M^2_w([s,T]; L^2(\Gamma_s))$ *to the stochastic Fluid-Structure Interaction system* (1.4)-(1.6) *with boundary conditions* (1.7)-(1.9)*, which minimizes the cost functional* (1.10)*. Moreover,*

1. *The optimal control* $g^0 \in C([s,T); L^2(\Sigma, \Gamma_s))$ *and*

$$\mathbb{E}(\|g^0(t)\|^2_{L^2(\Gamma_s)}) \le \Big(\frac{c}{t^{1/4+\epsilon}}(\|u_0\|_{L^2(\Omega_f)} + \|w_0\|_{L^2(\Omega_s)} + \|w_1\|_{L^2(\Omega_s)})\Big)^2.$$

2. *The corresponding optimal state* $(\theta^0(t), w^0(t), w^0_t(t)) \in L^2([s,T]; L^2(\Sigma, \mathcal{H})) \cap C([s,T]; L^2(\Sigma, \mathcal{H}_{-1}))$ *and*

$$\mathbb{E}(\|u^0(t)\|^2_{L^2(\Omega_f)}) + \mathbb{E}(\|w^0(t)\|^2_{L^2(\Omega_s)}) + \mathbb{E}(\|w^0_t(t)\|^2_{H^{-1}(\Omega_s)}) \le c(\|u_0\|^2_{L^2(\Omega_f)} + \|w_0\|^2_{H^1(\Omega_s)} + \|w_1\|^2_{L^2(\Omega_s)}).$$

3. *The optimal control is given in feedback form*

$$g^0(t) = -(I + D^\star P(t)D)^{-1}(D^\star P(t)C + B^\star P(t))[u^0(t), w^0(t), w^0_t(t)]^T$$

for B, D and C defined above and where $P(t)$ is a self-adjoint positive operator on $\mathcal{H}$ satisfying the differential Riccati equation

$$\begin{aligned}&\langle p_{1t}, y_1\rangle_f + \langle \nabla p_{2t}, \nabla y_2\rangle_s + \langle p_{3t}, y_3\rangle_s = -\langle A_N x_1, \hat{p}_1\rangle_f - \langle A_N N\sigma(x_2), \hat{p}_1\rangle_f - \langle \nabla x_3, \nabla \hat{p}_2\rangle_s \\ &- \langle \operatorname{div}\sigma(x_2), \hat{p}_3\rangle_s - \langle p_1, A_N y_1\rangle_f - \langle p_1, A_N N\sigma(y_2)\rangle_f - \langle \nabla p_2, \nabla y_3\rangle_s - \langle p_3, \operatorname{div}\sigma(y_2)\rangle_s \\ &- \langle c_1 p_1, c_1 y_1\rangle_f - \langle c_2 p_3, c_2 y_2\rangle_s - \langle x_1, y_1\rangle_f + \langle (I + D^\star P(t) D)^{-1}(1+c_1)p_1|_{\Gamma_s}, (1+c_1)\hat{p}_1|_{\Gamma_s}\rangle_{\Gamma_s}, \\ &[p_1(T), p_2(T), p_3(T)] = [x_1, x_2, 0].\end{aligned}$$

for every $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in \mathcal{D}(\mathcal{A}_{FS})$, where $P(t)x = [p_1(t), p_2(t), p_3(t)]$ and $P(t)y = [\hat{p}_1(t), \hat{p}_2(t), \hat{p}_3(t)]$, while $\langle\cdot,\cdot\rangle_f$ and $\langle\cdot,\cdot\rangle_s$ denote the L^2 inner product on Ω_f and Ω_s respectively.

Remark 10.3. *The results of this theorem require extending the results to a generalized singular estimate condition on the observation spaces $\|Re^{At}Bf\|_W \le \frac{c}{t^\gamma}\|f\|_U$ and $\|Ge^{At}Bf\|_Z \le \frac{c}{t^\gamma}\|f\|_U$, $\quad \forall f \in U$ for some $\gamma \in (0, 1/2)$, cf. [Tu]. This leads to the continuity in time property to be satisfied by the observed optimal state space Ry^0 only on the observation space W as stated in (2).*

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