Stable Motion and Distributed Topology Control for Multi-Agent Systems with Directed Interactions

Pratik Mukherjee, Andrea Gasparri and Ryan K. Williams

***Abstract*— In this paper, we study stable coordination in multi-agent systems with directed interactions, and apply the results for distributed topology control. Our main contribution is to extend the well-known potential-based control framework originally introduced for undirected networks to the case of networks modeled by a directed graph. Regardless of the particular objective to be achieved, potential-based control for undirected graphs is *intrinsically* stable. Briefly, this can be explained by the positive semidefiniteness of the graph Laplacian induced by the symmetric nature of the interactions. Unfortunately, this energy finiteness guarantee no longer holds when a multi-agent system lacks symmetry in pairwise interactions. In this context, our contribution is twofold: i) we formalize stable coordination of multi-agent systems on directed graphs, demonstrating the graph structures that induce stability for a broad class of coordination objectives; and ii) we design a topology control mechanism based on a distributed eigenvalue estimation algorithm to enforce Lyapunov energy finiteness over the derived class of stable graphs. Simulation results demonstrate a multi-agent system on a directed graph performing topology control and collision avoidance, corroborating the theoretical findings.**

I. INTRODUCTION

Multi-agent coordination has been widely investigated by the control community over the last two decades. A very popular framework in the context of networked multi-agent systems, used for achieving a large variety of collaborative objectives, is the well-known potential-based control methodology. Representative examples of coordination problems are for instance consensus agreement, self-localization, and formation control [1]–[8], with applications ranging from environmental monitoring to collaborative transportation. A typical assumption that has been made over the years for the design of these distributed algorithms is that pairwise interactions among agents are *symmetric*, i.e., interactions occur over an undirected graph. Unfortunately, this symmetry assumption does not always reflect the actual capabilities of networked systems, especially when the pairwise interactions involve relative sensing among agents.

In this work, our goal is to relax the symmetry assumption by allowing communication to remain isotropic, but with sensing that may be anisotropic. This implies that while interactions occurring through communication may still be undirected, interactions occurring through relative sensing are directed in nature. We point out that this assumption better reflects typical hardware features of robotic units, where it is reasonable to assume that over the range of visibility of a sensing device, the communication radio may have an isotropic radiation lobe.

In particular, we focus on extending the potential-field control framework originally introduced for undirected networks to the case of networks modeled by a directed graph. Indeed, while this control design methodology has proven very successful for a wide array of applications in the context of undirected networks, it can no longer be applied to the case of directed networks as is. Intuitively, this can be explained by the fact that potential-based control for undirected graphs is *intrinsically* stable, i.e., the Lyapunov energy remains finite over time, independent of the particular graph topology. This fact is a consequence of the symmetric nature of the interactions, and formally, is due to the positive semidefiniteness of the graph Laplacian that describes the system energy. Clearly, the lack of symmetry suddenly breaks this negative semi-definiteness of the Lyapunov derivative, and the system can no longer be claimed to be stable for all topologies.

Some closely related work to this paper can be summarized as follows. In [9], the authors introduce an edge variant of the well-known consensus protocol for undirected graphs, referred to as the edge agreement problem, for which the dynamics evolve according to the edge Laplacian. In [10], the authors propose a quadratic Lyapunov function that utilizes the structure of the underlying communication topology and the spectral properties of the corresponding edge Laplacian matrix to prove consensus with quantized relative state measurements. Very recently in [11], the authors address the edge agreement problem of second-order non-linear multi-agent systems under quantized measurements for directed graphs. In particular, convergence results are given for quasi-strongly connected graphs, that is graphs for which a directed spanning tree exists. Various prior works such as [12] have demonstrated multi-agent systems over directed networks, often with the *assumption* that the graph is strongly connected. Topology control for directed graphs in comparison is very sparse; the most relevant examples are [13], [14], which each require certain assumptions regarding agents' regions of interaction.

In this context, our contribution is twofold: i) we study stable coordination for directed graphs in the potential-based control framework, demonstrating the graph structures that induce stability for a broad class of coordination objectives; and ii) we design a topology control mechanism and a distributed eigenvalue estimation algorithm to enforce Lyapunov energy finiteness based on the class of stable graphs. Simulation results are provided to corroborate the theoretical findings.

Pratik Mukherjee and Ryan K. Williams are with the Department of Electrical and Computer Engineering , Virginia Polytechnic Institute and State University, Blacksburg, VA USA. E-mail: {*mukhe027, rywilli1*}@*vt.edu.* This work was supported by the NSF CISE Research Initiation Initiative Program under Award #1657235.

A. Gasparri is with the Department of Engineering, University of "Roma Tre", Via della Vasca Navale 79, 00146 Rome, Italy. E-mail: *gasparri@dia.uniroma3.it*

II. PRELIMINARIES

A. Agent and Network Modeling

Consider a multi-agent system composed of n agents, each having motion that evolves according to the following dynamics

$$\dot{x}_i(t) = u_i(t) \tag{1}$$

with $x_i(t) \in \mathbb{R}^d$ the agent state (position), $u_i(t) \in \mathbb{R}^d$ the control input, and time $t \in \mathbb{R}_{\geq 0}$. Stacking agent states and inputs yields the overall system

$$\dot{\mathbf{x}}(t) = \mathbf{u}(t) \tag{2}$$

with $\mathbf{x}(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^{nd}$ and $\mathbf{u}(t) = [u_1(t), \ldots, u_n(t)]^T \in \mathbb{R}^{nd}$ the stacked vector of states and control inputs, respectively[1]. The distance between agents i and j is denoted by $\|x_{ij}(t)\| \triangleq \|x_i(t) - x_j(t)\|$, with the standard Euclidean norm.

We define two radii $\rho_s(i), \rho_c(i) \in \mathbb{R}_+, \forall\, i \in \{1, \ldots, n\}$, within which sensing and communication can occur for each agent, respectively. A dynamic directed graph is denoted by $\mathcal{G}(t) \triangleq (\mathcal{V}, \mathcal{E}(t))$ with node set $\mathcal{V} \triangleq [v_1, ..., v_n]$ and edge set $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$. We can define the sensing graph as $\mathcal{G}_s = (\mathcal{V}, \mathcal{E}_s)$ with edges $\mathcal{E}_s = \{(i,j) \mid \|x_{ij}\| \leq \rho_s(i), i, j \in \mathcal{V}\}$ and the communication graph $\mathcal{G}_c = (\mathcal{V}, \mathcal{E}_c)$ with edges $\mathcal{E}_c = \{(i,j) \mid \|x_{ij}\| \leq \rho_c(i), i, j \in \mathcal{V}\}$.

Denote by $\mathcal{N}_i^+(t) := \{j \in \mathcal{V} : (i,j) \in \mathcal{E}(t)\}$ the set of *out-neighbors* of agent i and $\mathcal{N}_i^-(t) := \{j \in \mathcal{V} : (j,i) \in \mathcal{E}(t)\}$ the set of *in-neighbors* with the assumption that $(i,i) \notin \mathcal{E}$. The *incidence matrix* is defined as $B(\mathcal{G}(t)) \in \mathbb{R}^{n \times |\mathcal{E}|}$ of a graph $\mathcal{G}$.The *outgoing incidence matrix* $\mathcal{B}_+$ contains only the outgoing parts of the incidence matrix $\mathcal{B}$, with incoming parts set to zero. The *undirected Laplacian matrix* $\mathcal{L} \in \mathbb{R}^{n \times n}$ is obtained as $\mathcal{L} = \mathcal{B}\mathcal{B}^T$, whereas the *directed Laplacian matrix* $\mathcal{L}_d \in \mathbb{R}^{n \times n}$ is computed as $\mathcal{L}_d = \mathcal{B}\mathcal{B}_+^T$. We will also make use of the *undirected edge Laplacian* $\mathcal{L}_\mathcal{E} \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}|}$ defined as $\mathcal{L}_\mathcal{E} = \mathcal{B}^T\mathcal{B}$ and the *directed edge Laplacian* $\mathcal{L}_\mathcal{E}^d \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}|}$ given by $\mathcal{L}_\mathcal{E}^d = \mathcal{B}^T\mathcal{B}_+$. For properties of the edge Laplacian see for example [9] and [11]. We will also apply the *Kronecker product*, which we will denote in the standard manner with $\otimes$. In this paper, we make the following assumption

Assumption 1: The communication radius ρ_c is sufficiently large compared to the sensing radius ρ_s, such that *weak connectivity* of $\mathcal{G}_s$ implies *strong connectivity* of $\mathcal{G}_c$.

Here, we make this assumption to guarantee communication can always support a distributed *consensus* algorithm for use in topology control.

B. Potential-Based Control Framework Overview

Potential-based control design is a commonly used framework for controlling multi-agent systems [1]–[5], [15]. The basic idea is to encode the energy of a system as a potential function $V(\mathbf{x}(t)) \in \mathbb{R}_{\geq 0}$ such that the desired configurations of the multi-agent system correspond to critical points. Thus, a control law can be designed to achieve these configurations by driving the system along the anti-gradient $\mathbf{u} = -\nabla_\mathbf{x} V$.

Control objectives that are pairwise, and thus distributed across a multi-agent system, can be designed by associating a (continuously differentiable[2]) potential function $V_{ij}(\mathbf{x}) \triangleq V_{ij}(x_i, x_j) \in \mathbb{R}_{\geq 0}$ with agents i and j, for which the following properties hold

$$\begin{aligned} \nabla_{x_i} V_{ij} &= \nabla_{x_i} V_{ji} \\ \nabla_{x_i} V_{ij} &= -\nabla_{x_j} V_{ij} \end{aligned} \tag{3}$$

where we note that the first property above is a *symmetry* condition. Then, the global potential function for the multi-agent system can be considered

$$V = \sum_{i=1}^{n} \sum_{j \neq i} V_{ij} \tag{4}$$

where the anti-gradient control term

$$u_i = -\sum_{j \in \mathcal{N}_i(t)} \nabla_{x_i} V_{ij} \tag{5}$$

with undirected neighborhood $\mathcal{N}_i$ can be shown to drive the system to a (local) minimum of the potential function V asymptotically over an undirected graph $\mathcal{G}$. Most importantly, the symmetry properties in (3) and symmetry in agent interactions are critical in the derivation of the convergence result. In this work, we will drop assumptions such as (3) and allow agent interactions *and* potential-based controllers to exhibit asymmetry.

III. DIRECTED CONTROL FRAMEWORK

A. Motivating Example

Continuing along the lines of Section II-B, for the undirected case it is known that the system energy derivative is

$$\dot{V} \sim -\sum_{i=1}^{n} \left\| \sum_{j \in \mathcal{N}_i} \nabla_{x_i} V_{ij} \right\|^2 \tag{6}$$

which by inspection means the system is *inherently* stable because $\dot{V} \leq 0$ for *all* graph topologies, which implies energy finiteness and thus stability. In systems with asymmetric interactions, this luxury is no longer present as there is often a subset of graphs $\mathcal{G}$ for which the system lacks even the stability property. A basic illustration of instability in an asymmetric system is given in Figure 1. If the task for agent 2 is to maintain interactions with agents 1 and 3, but agent 1 and 3 progressively move away from agent 2, the system is in some sense degenerate and an instability may occur. Such degeneracies must be avoided at all costs as they may threaten the predictability and safety of a system. It follows that if we can identify the class of controllers and asymmetric agent interactions that preserve our ability to encode desired configurations with stable system energy, then we can for example guarantee desirable topological properties (as in Section III-E).

[1]Note that dependence on time, state, and/or a graph will only be shown when introducing new concepts or symbols. Subsequent usage will drop these dependencies for clarity of presentation.

[2]A generalization to handle non-smooth potential functions can be found in [15]. Here, for the sake of simplicity, smooth pairwise potentials are assumed.

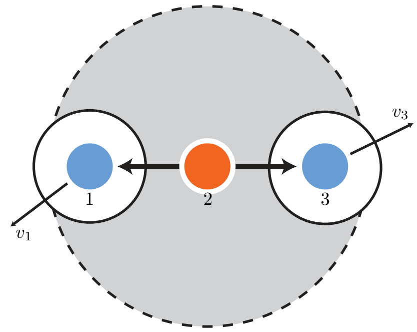


Fig. 1. Three agents with asymmetric disks of interaction.

B. Encoding Topology in Directed Lyapunov Analysis

On a directed graph $\mathcal{G}$ the control input for the ith agent is defined as

$$u_i = -\sum_{j\in\mathcal{N}_i^+} \nabla_{x_i} V_{ij}(\|x_{ij}\|) \tag{7}$$

where the potentials V_{ij} are smooth functions of inter-agent distance $\|x_{ij}\|$, and $\nabla_{x_i} V_{ij} = 0$ for $(i,j) \notin \mathcal{E}$. Furthermore, we assume potential smoothness continues to hold as the graph $\mathcal{G}$ changes over time.

Now, let the Lyapunov function $V : \mathbb{R}^{nd} \rightarrow \mathbb{R}_{\geq 0}$ be a continuously differentiable function, defined in the standard manner:

$$V(\mathbf{x}(t)) = \sum_{i=1}^{n} \sum_{j\neq i} V_{ij}(\|x_{ij}\|) \tag{8}$$

with time derivative

$$\dot{V} = (\nabla_{\mathbf{x}} V)^T \, \dot{\mathbf{x}} \tag{9}$$

from simple application of the chain rule. For the directed case, we will derive an *edge-based* form of $\nabla_{\mathbf{x}} V$ and $\dot{\mathbf{x}}$ that will eventually reveal the graph topology. First, consider the general form for the gradient of the Lyapunov function, given by

$$\nabla_{\mathbf{x}} V = \sum_{i=1}^{n} \sum_{j\in\mathcal{N}_i^+} \nabla_{\mathbf{x}} V_{ij} \tag{10}$$

and assuming a *local* property of potentials, $\nabla_{x_i} V_{ji} = -\nabla_{x_j} V_{ji}$ we have

$$\nabla_{\mathbf{x}} V = \Bigg[\sum_{j\in\mathcal{N}_1^+} \nabla_{x_1} V_{1j} - \sum_{j\in\mathcal{N}_1^-} \nabla_{x_j} V_{j1}, \; \ldots \; \sum_{j\in\mathcal{N}_n^+} \nabla_{x_n} V_{nj} - \sum_{j\in\mathcal{N}_n^-} \nabla_{x_j} V_{jn} \Bigg]^T \tag{11}$$

This motivates us to define the stacked vector of potential field gradients, $\xi \in \mathbb{R}^{d|\mathcal{E}|}$, given by

$$\xi = \left[\nabla_{e_1(1)} V_{e_1}, \nabla_{e_2(1)} V_{e_2}, \ldots, \nabla_{e_{|\mathcal{E}|}(1)} V_{e_{|\mathcal{E}|}} \right]^T \tag{12}$$

where $e_k(1)$ denotes the starting vertex v_i of the kth edge (i,j), and thus $\nabla_{e_k(1)} V_{e_k} \in \mathbb{R}^d$ denotes the gradient with respect to x_i of potential function V_{ij}. Now, from (12) and (11), $\nabla_{\mathbf{x}} V$ can be written as:

$$\nabla_{\mathbf{x}} V = (\mathcal{B} \otimes I_d)\, \xi \tag{13}$$

with $\mathcal{B}$ the incidence matrix associated with the directed sensing graph $\mathcal{G}_s$ as defined in Section II-A and I_d the $d \times d$ identity matrix. Similarly, we can proceed in computing $\dot{\mathbf{x}}$ as

$$\dot{\mathbf{x}} = -\Bigg[\sum_{j\in\mathcal{N}_1^+} \nabla_{\mathbf{x_1}} V_{\mathbf{1j}}, \; \ldots \; , \sum_{j\in\mathcal{N}_n^+} \nabla_{\mathbf{x_n}} V_{\mathbf{nj}} \Bigg]^T \tag{14}$$

Unlike before, (14) has contributions only from the starting vertex of an edge and thus we can conclude that

$$\dot{\mathbf{x}} = -(\mathcal{B}_+ \otimes I_d)\, \xi \tag{15}$$

with $\mathcal{B}_+$ the outgoing portion of the incidence matrix associated with the directed sensing graph $\mathcal{G}_s$ as defined in Section II-A.

At this point, by using (13) and (15), it follows that (9) can be rewritten as:

$$\begin{aligned} \dot{V} &= -\Big[(\mathcal{B} \otimes I_d)\, \xi \Big]^T \Big[(\mathcal{B}_+ \otimes I_d)\, \xi \Big] \\ &= -\xi^T \left[\left(\mathcal{B}^T \mathcal{B}_+ \right) \otimes I_d \right] \xi \\ &= -\xi^T \left[\mathcal{L}_{\mathcal{E}}^d \otimes I_d \right] \xi \end{aligned} \tag{16}$$

with $\mathcal{L}_{\mathcal{E}}^d$ the directed edge Laplacian. As the directed edge Laplacian is asymmetric and indefinite in general, studying the Lyapunov stability of system (2) with controllers (7) is not straightforward. This implies that topology control is needed for the *stability* of directed systems, unlike undirected systems where topology control only serves secondary objectives like consensus, while motion is inherently stable[3].

C. Stable Directed Topologies

Our goal now is to answer a fundamental question: *assuming general potential field controls, for which directed topologies is a multi-agent system guaranteed to be stable*? By *general potential field controls* we mean having the form

$$u_i(t) = -\sum_{j\in\mathbb{N}_i^+} \nabla_{x_i} V_{ij} = -\sum_{j\in\mathbb{N}_i^+} a_{ij}(\|x_{ij}(t)\|) x_{ij} \tag{17}$$

where $a_{ij}(\|x_{ij}(t)\|) \in \mathbb{R}$ is a smooth, time-varying scalar weight function that can take arbitrary values for edges $(i,j) \in \mathcal{E}$, and where a_{ij} does not necessarily equal a_{ji}. Applying the form (17) to the definition of ξ as in (12), we now have a weighted version of ξ defined as

$$\xi = W(t)(\mathcal{B}^T \otimes I_d)\, \mathbf{x} \tag{18}$$

with $W(t) = \mathrm{diag}\left(\left[a_{e_1}, \ldots, a_{e_{|\mathcal{E}|}} \right] \right)$ where it can be noticed that further topological structure is given to the variable ξ. With this redefinition, the time derivative of the Lyapunov function is now

$$\dot{V} = -\mathbf{x}^T \left[\left(\mathcal{B} W \mathcal{L}_{\mathcal{E}}^d W \mathcal{B}^T \right) \otimes I_d \right] \mathbf{x} \tag{19}$$

[3]Recall that stability does not imply convergence to a desired equilibrium, although our methods certainly do not preclude such convergence.

Notice that with the addition of the time-varying weight matrix W, proving stability means it is necessary to prove that for a *every* matrix W encoding a certain potential-based control at any time t, the asymmetric matrix $\mathcal{B}W\mathcal{L}^d_{\mathcal{E}}W\mathcal{B}^T \in \mathbb{R}^{n\times n}$ is positive semidefinite. Instead, we want to know for *any* weight matrix W if the system is stable. This question is captured by the following result.

Lemma 3.1: For any $W = \operatorname{diag}\left(\left[a_{e_1}, \ldots, a_{e_{|\mathcal{E}|}}\right]\right) \in \mathbb{R}^{|\mathcal{E}|\times|\mathcal{E}|}$ associated with $\mathcal{G}_s$, there exists $\widehat{W}$ such that:

$$W(\mathcal{B}^T \otimes I_d) = (\mathcal{B}^T \otimes I_d)\,\widehat{W} \tag{20}$$

whenever $\mathcal{B}^T\mathcal{B} \triangleq \mathcal{L}_{\mathcal{E}}$ is invertible.

Proof: The above relation is valid when there exists a right inverse, $\mathcal{B}^{-1}_{right} = (\mathcal{B})(\mathcal{B}^T\mathcal{B})^{-1}$ and $(\mathcal{B}^T\mathcal{B})$ is invertible. Then $\widehat{W} = (\mathcal{B})(\mathcal{B}^T\mathcal{B})^{-1}W(\mathcal{B}^T \otimes I_d)$ such that

$$\begin{aligned}(\mathcal{B}^T \otimes I_d)\,\widehat{W} &= (\mathcal{B}^T \otimes I_d)(\mathcal{B})(\mathcal{B}^T\mathcal{B})^{-1}W(\mathcal{B}^T \otimes I_d)\\ &= (\mathcal{B}^T(\mathcal{B})(\mathcal{B}^T\mathcal{B})^{-1} \otimes I_d)W(\mathcal{B}^T \otimes I_d)\\ &= W(\mathcal{B}^T \otimes I_d)\end{aligned} \tag{21}$$

which yields the result. ■

Remark 3.1: Notice that the result is a sufficient condition. The generality of the result, however, comes at the price of a restricted class of stable topologies for which $\mathcal{B}^T\mathcal{B}$ is invertible.

Next, by applying the above Lemma to (16), we obtain:

$$\begin{aligned}\dot{V} &= -\xi^T\left[\,(\mathcal{B}^T\mathcal{B}_+) \otimes I_d\right]\xi\\ &= -\left[(\mathcal{B}^T \otimes I_d)\widehat{W}\mathbf{x}\right]^T\left[\,(\mathcal{B}^T\mathcal{B}_+) \otimes I_d\right]\left[(\mathcal{B}^T \otimes I_d)\,\widehat{W}\mathbf{x}\right]\\ &= -\mathbf{z}^T\left[\underbrace{(\mathcal{L}\mathcal{L}_d^T)}_{S} \otimes I_d\right]\mathbf{z}\end{aligned} \tag{22}$$

with $\mathbf{z} = \widehat{W}\mathbf{x} \in \mathbb{R}^{nd}$ and $S \in \mathbb{R}^{n\times n}$ the *structural Lypaunov matrix*, where by structural we refer to the fact that this matrix is by construction compatible with the network sensing graph $\mathcal{G}_s$ and independent of the system state between changes in $\mathcal{G}_s$.

We now prove a result regarding the rank of the asymmetric matrix S.

Lemma 3.2: The *structural Lyapunov matrix* S is *rank-compatible* with the graph $\mathcal{G}_s$ in the sense that there exists a direct relation between the rank of the S, and the rank of the incidence matrix $\mathcal{B}$ of graph $\mathcal{G}_s$.

Proof: Using Theorem 8.3.1 in [16] for a directed graph $\mathcal{G}_s$ with n vertices and c connected components, it is shown that the rank of the incidence matrix $\mathcal{B}$ is given by $Rank[\mathcal{B}] = n - c$ where c is the dimension of the null space of $\mathcal{B}$. It is known that the *structural Lyapunov matrix* is of the form shown in equation (22). Therefore, using Theorem 8.3.1 and a simple property of the equality of the rank of a matrix and the rank of its transpose, it can be stated that $Rank[\mathcal{B}] = Rank[\mathcal{B}^T] = Rank[\mathcal{B}_+]$. From Lemma 3.1, it is known that the special class of graphs for which stability can be proved using the *structural Lyapunov matrix* are the ones with the equal number of edges and vertices or graphs with more vertices than edges which means $\mathcal{B}$ is a $n \times |\mathcal{E}|$ matrix where $n \geq |\mathcal{E}|$. Now it can be shown that $Rank[\mathcal{B}\mathcal{B}^T] = Rank[\mathcal{B}]$. Let $Z = \mathcal{B}^T$ and $Z^T = \mathcal{B}$. Suppose $x \in N(\mathcal{Z})$ where $N(\mathcal{Z})$ is the null space of $\mathcal{Z}$. $\mathcal{Z}x = 0 \Rightarrow \mathcal{Z}^T\mathcal{Z}x = 0 \Rightarrow x \in N(\mathcal{Z}^T\mathcal{Z})$. Hence $N(\mathcal{Z}) \subseteq N(\mathcal{Z}^T\mathcal{Z})$. Now $x \in N(\mathcal{Z}^T\mathcal{Z})$. Then $\mathcal{Z}^T\mathcal{Z}x = 0 \Rightarrow x^T\mathcal{Z}^T\mathcal{Z}x = 0 \Rightarrow (\mathcal{Z}x)^T(\mathcal{Z}x) = 0 \Rightarrow (\mathcal{Z}x) = 0 \Rightarrow x \in N(\mathcal{Z})$. This implies $N(\mathcal{Z}^T\mathcal{Z}) \subseteq N(\mathcal{Z}^T) \Rightarrow dim(N(\mathcal{Z}^T\mathcal{Z})) = dim(N(\mathcal{Z}^T))$

Therefore, $N(\mathcal{Z}^T\mathcal{Z}) = N(\mathcal{Z})$

$$\Rightarrow Rank(\mathcal{Z}^T\mathcal{Z}) = Rank(\mathcal{Z}) = Rank(\mathcal{B})) = Rank(\mathcal{B}\mathcal{B}^T) \tag{23}$$

Now, using a known property of matrices, it can be stated that $Rank(\mathcal{B}\mathcal{B}^T\mathcal{B}_+) \leq \mathbf{min}(Rank(\mathcal{B}\mathcal{B}^T), Rank(\mathcal{B}_+))$ which implies $Rank(\mathcal{B}\mathcal{B}^T\mathcal{B}_+) \leq Rank(\mathcal{B}\mathcal{B}^T) = Rank(\mathcal{B}) = Rank(\mathcal{B}_+)$. Further it can be stated that $Rank(\mathcal{B}\mathcal{B}^T\mathcal{B}_+\mathcal{B}^T) = Rank(\mathcal{S}) \leq \mathbf{min}(Rank(\mathcal{B}\mathcal{B}^T\mathcal{B}_+), Rank(\mathcal{B}))$. If $Rank(\mathcal{B}\mathcal{B}^T\mathcal{B}_+) = Rank(\mathcal{B}) \Rightarrow Rank(\mathcal{S}) \leq Rank(\mathcal{B})$. However, if $Rank(\mathcal{B}\mathcal{B}^T\mathcal{B}_+) < Rank(\mathcal{B}) \Rightarrow Rank(\mathcal{S}) \leq Rank(\mathcal{B}\mathcal{B}^T\mathcal{B}_+)$. Therefore, from this statement, it is now known that the *structural Lyapunov matrix* has a direct relation to the rank of the incidence matrix of $\mathcal{G}_s$. ■

D. Stability Analysis

The Lyapunov time derivative in (22) is in a typical quadratic form and the characteristics of this quadratic equation are dependent on the properties of the *structural Lyapunov matrix* S. The Lyapunov stability analysis is carried out on the symmetrized S, $S^+ = \frac{1}{2}(S + S^T)$, as positive semi-definiteness of S^+ implies positive semi-definiteness of S.

Theorem 3.1: Assuming the conditions for Lemma 3.1 hold, and $1/2[(\mathcal{B}^T\mathcal{B}_+) + (\mathcal{B}_+)^T(\mathcal{B})]$ is positive definite, the system (1) with agent controls (17) is stable in the sense that if V is initially finite it remains finite for all time $t > 0$.

Proof: To begin, consider the following:

$$\begin{aligned}S^+ &= 1/2[(\mathcal{B}\mathcal{B}^T\mathcal{B}_+\mathcal{B}^T) + (\mathcal{B}\mathcal{B}^T\mathcal{B}_+\mathcal{B}^T)^T]\\ \text{Since}\quad &(\mathcal{B}\mathcal{B}^T)^T = \mathcal{B}\mathcal{B}^T \quad \text{from Lemma 1}\\ &= 1/2\mathcal{B}[(\mathcal{B}^T\mathcal{B}_+) + (\mathcal{B}_+)^T(\mathcal{B})]\mathcal{B}^T\\ C &= 1/2[(\mathcal{B}^T\mathcal{B}_+) + (\mathcal{B}_+)^T(\mathcal{B})]\end{aligned} \tag{24}$$

If C is a real positive definite symmetric matrix, then using Cholesky decomposition, $C = \mathcal{Q}\mathcal{Q}^T$ is a product of lower triangular matrix $\mathcal{Q}$. $S^+ = \mathcal{B}[C]\mathcal{B}^T = \mathcal{B}[\mathcal{Q}\mathcal{Q}^T]\mathcal{B}^T = \mathcal{B}\mathcal{Q}(\mathcal{B}\mathcal{Q})^T$. Then take $F = \mathcal{B}\mathcal{Q}$ and $S^+ = FF^T$. Now the eigendecomposition of S^+ is $\lambda = vFF^Tv^T = vF(vF)^T$. Since λ can be written as the inner product of vector vF by itself, $\lambda \geq 0$ and so $\mathcal{S}^+$ has non-negative eigenvalues. Since positive semi-definiteness of S^+ implies positive semi-definiteness of S, $\dot{V}$ as defined in (22) is negative semi-definite and if V is initially finite it will remain finite for all $t > 0$. ■

E. Topology Control

The previous section demonstrated the conditions for stable potential-based controllers on directed sensing graphs. Recalling that stability required certain topological properties of $\mathcal{G}$ to hold, we can demonstrate topology control for the purpose of preserving stability. We proceed by first carefully selecting sensing edges for the purposes of motion control. We also emphasize that the previous work of the authors' on undirected topology control in [5] can be replicated to the directed setting. The intuition of our above stability result is that a network may have many sensing edges available to it, but only a subset are appropriate for coordinated motion control on directed graphs. This is a key factor differentiating directed settings from undirected ones: all sensing graphs are inherently stable in undirected settings, while the same does not hold in directed settings. Thus, we allow the agents to differentiate between the *sensing graph* $\mathcal{G}_s$ and a *motion control graph* $\mathcal{G}_m$, where $\mathcal{G}_m \subseteq \mathcal{G}_s$ so that stable motion can be achieved while exploiting all available sensing information (e.g., for localization). For this section, we assume that all previously described matrices are derived from the motion control graph $\mathcal{G}_m$. Our goal will then be to select $\mathcal{G}_m$ over time such that coordinated motion is stable.

Thus, modifying [5] for a directed setting we have the following control law for maintaining a desired topological property $\mathbb{P}$:

$$\begin{aligned} u_i &= -\nabla_{x_i}\left(\sum_{j\in\mathcal{N}_i^+} V_{ij}^o + \sum_{j\in\mathcal{D}_i^d} V_{ij}^d\right) \\ &= -\nabla_{x_i}\Psi_i(\|x_{ij}\|) \end{aligned} \tag{25}$$

where we consider a *generalized coordination objective* driven by potential field $V_{ij}^o: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$. Choosing $V_{ij}^d \rightarrow \infty$ as $\|x_{ij}\| \rightarrow \rho_2$, where ρ_2 is the outer radius of the agent interactions described in [5] with decision set $\mathcal{D}_i^d$ containing neighbors to keep according to $\mathbb{P}$ and neighbors that maintain stable directed coordination, we can then ensure the overall Lyapunov energy is stable. Specifically, notice from (25) that we can write the actuation for agent i in the form $-\nabla_{x_i}\Psi_i(\|x_{ij}\|)$, and thus choosing V_{ij}^o and V_{ij}^d appropriately will yield a form $a_{ij}(\|x_{ij}\|)x_{ij}$ for u_i, allowing for direct application of our stability result.

F. Distributed Eigenvalue Estimation

The online eigenvalue evaluation of the symmetrized *structural Lyapunov matrix*, $\mathcal{S}^+$ (or equivalently a symmetrized $\mathcal{B}^T\mathcal{B}_+$), using the *Lanczos Biorthogonalization algorithm* is now obtained in a decentralized manner.

The decentralized *Lanczos Biorthogonalization algorithm*, shown in Algorithm 1, enables local evaluation of all the global terms in the original centralized algorithm using the surplus-based average consensus method (ASC) [17], under Assumption 1 which guarantees strong connectivity of $\mathcal{G}_c$ and thus consensus convergence (given certain parameters settings defined in [17]). Since $\mathcal{S}^+$ is a symmetric matrix, Algorithm 1 is related to the decentralized algorithm of [18], however we point out that our proposed algorithm has general applicability also to asymmetric matrices. The decentralized eigenvalue estimation in Algorithm 1 computes the coefficients, α_k^j, β_k^j and δ_k^j at every iteration $j = 1, ..., M$ for the kth agent, which are used to populate a tridiagonal matrix, T_k^M for each agent k, similar to T^M . The eigenvalues of this matrix correspond to

Algorithm 1 Decentralized Lanczos Biorthogonalization

Choose two vectors v_1, w_1 such that $dot(v_k^1, w_k^1) = 1$
Set $\beta_k^1 = \delta_k^1 \equiv 0 \quad w_k^0 = v_k^0 \equiv 0$
for iteration j=1 to M Do:
  for *all nodes k* do
$s_k^{(j)} = ASC_n[\mathcal{S}^+ v^{(j)}]$
$q_k^{(j)} = ASC_n[\mathcal{S}^{+T} w^{(j)}]$
$\alpha_k^{(j)} = ASC_1[s_k^{(j)}, w_k^{(j)}]$
$s_k^{(j+1)} = Local[s_k^{(j)} - \beta_k^{(j)} v_k^{(j-1)} - \alpha_k^{(j)} v_k^{(j)}]$
$q_k^{(j+1)} = Local[q_k^{(j)} - \delta_k^{(j)} w_k^{(j-1)} - \alpha_k^{(j)} w_k^{(j)}]$
$\delta_k^{(j+1)} = \sqrt{K * ASC_1[|(s_k^{(j+1)}, q_k^{(j+1)})|]}.$
If $\delta_k^{(j+1)} = 0$ Stop
$\beta_k^{(j+1)} = K * ASC_1[(s_k^{(j+1)}, q_k^{(j+1)})]/\delta_k^{(j+1)}$
$v_k^{(j+1)} = Local[s_k^{(j+1)}/\delta_k^{(j+1)}]$
$w_k^{(j+1)} = Local[q_k^{(j+1)}/\beta_k^{(j+1)}]$
  END do
END Do

the eigenvalues of the S^+ matrix, and can be computed locally using a simplified QR method in $O(n)$ time.

To obtain the elements α_k^j (principal diagonal), δ_k^j (subdiagonal) and β_k^j (superdiagonal) of the T_k^M matrix distributively, ASC for directed graphs is applied. The (ASC) algorithm is first run n times, where n is equal to the total number of agents in the graph, to obtain $s_k^{(j)}$ and $q_k^{(j)}$ for each agent k. Specifically, computing $\mathcal{S}^+ v^{(j)}$ or $(\mathcal{S}^+)^T w^{(j)}$ distributively by multiplying the row elements of $\mathcal{S}^+$ or $(\mathcal{S}^+)^T$ with $v^{(j)}$ and $w^{(j)}$ will require ASC to be executed n times such that a column vector of size n is generated where each k^{th} element of the newly generated vector represents the k^{th} agent's element, with the portion of rows of S^+ that are local to each agent. Next, the terms α_k^j,δ_k^j and β_k^j are scalar global values in Algorithm 1 because they are derived by taking the dot product of individual elements that are information available unique to each agent. Therefore a single surplus-based average consensus, ASC_1, is run to obtain a common scalar value for each agent in a decentralized manner. Now, once all the *global* information is obtained using ASC, $s_k^{(j+1)}$, $q_k^{(j+1)}$, $v_k^{(j+1)}$, and $w_k^{(j+1)}$ can be computed locally by each agent, yielding a distributed estimate for M eigenvalues of S^+.

IV. Simulations

In this section, the results for the convergence the *decentralized Lanczos Biorthogonalization* algorithm to the smallest eigenvalue, 0, of the symmetrized *Structural Lyapunov matrix*, $\mathcal{S}^+$, over 10,000 random, directed graphs are presented. In addition, a leader-follower objective is demonstrated on a directed graph with both topology control and collision avoidance.

In Fig. 2, the results of smallest eigenvalue estimation for each individual agent as well as the mean squared error (MSE) for the smallest eigenvalue for a Monte Carlo simulation over 10,000 random connected graphs is generated. Fig. 2 clearly shows that the MSE of the smallest eigenvalue is less

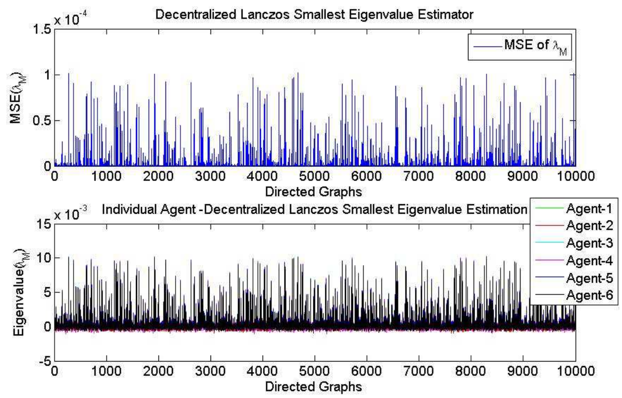
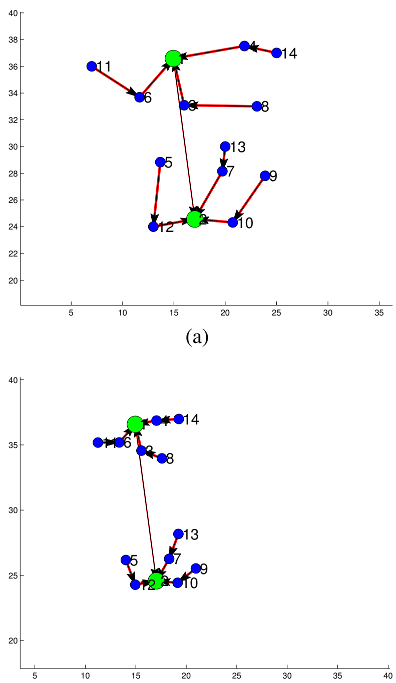


Fig. 2. Mean Squared Error value of smallest eigenvalue for 6 agents and the smallest eigenvalue plot for individual agents.



(a)

(b)

Fig. 3. Starting (a) and final (b) configurations for a multi-agent leader-follower objective with collision avoidance and topology control on a time-varying directed graph. Leaders are green, followers blue, and motion control edges in red (other sensing edges removed for clarity).

than 1×10^{-4}. The accuracy obtained from the estimation of the eigenvalues is sufficient for the purpose of this paper to successfully distinguish between a stable and an unstable graph topology. Fig. 3 demonstrates the working of the estimator in a simulation of leader-following with 2 leaders (green) and 12 followers (blue), where the subset of edges for control in $\mathcal{G}_m$ are depicted in red (other sensing edges in $\mathcal{G}_s$ omitted for clarity). In addition to rendezvous with the leaders, the previously outlined topology control mechanism and collision avoidance are implemented (two opposing objectives), due to the generality of the allowable potential field control. Fig. 3 demonstrates system stability as expected.

V. Conclusions

In this paper, we have derived the class of stable topologies for general potential-based controllers on directed graphs. As multi-agent systems on directed graphs can exhibit inherent instabilities, the class of stable topologies is applied to inform a topology control mechanism that acts to guarantee stability. A distributed Lanczos Biorthogonalization is derived to estimate topologies that are stable, and simulation results for both distributed estimation and multi-agent control were given to corroborate the theoretical findings.

References

[1] M. Ji and M. Egerstedt, “Distributed coordination control of multiagent systems while preserving connectedness,” *IEEE Transactions on Robotics*, vol. 23, no. 4, pp. 693–703, Aug 2007.

[2] D. V. Dimarogonas and K. J. Kyriakopoulos, “Connectedness preserving distributed swarm aggregation for multiple kinematic robots,” *IEEE Transactions on Robotics*, vol. 24, no. 5, pp. 1213–1223, Oct 2008.

[3] M. M. Zavlanos, H. G. Tanner, A. Jadbabaie, and G. J. Pappas, “Hybrid control for connectivity preserving flocking,” *IEEE Transactions on Automatic Control*, vol. 54, no. 12, pp. 2869–2875, Dec 2009.

[4] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, “Flocking in fixed and switching networks,” *IEEE Transactions on Automatic Control*, vol. 52, no. 5, pp. 863–868, May 2007.

[5] R. Williams and G. Sukhatme, “Constrained interaction and coordination in proximity-limited multiagent systems,” *Robotics, IEEE Transactions on*, vol. 29, no. 4, pp. 930–944, Aug 2013.

[6] R. K. Williams, A. Gasparri, G. Ulivi, and G. S. Sukhatme, “Generalized Topology Control for Nonholonomic Teams With Discontinuous Interactions,” *Robotics, IEEE Transactions on*, vol. 33, no. 4, pp. 994–1001, 2017.

[7] R. K. Williams and G. S. Sukhatme, “Locally constrained connectivity control in mobile robot networks,” in *2013 IEEE International Conference on Robotics and Automation (ICRA)*. IEEE, 2013, pp. 901–906.

[8] ——, “Observability in topology-constrained multi-robot target tracking,” in *2015 IEEE International Conference on Robotics and Automation (ICRA)*. IEEE, 2015, pp. 1795–1801.

[9] D. Zelazo, A. Rahmani, and M. Mesbahi, “Agreement via the edge laplacian,” in *2007 46th IEEE Conference on Decision and Control*, Dec 2007, pp. 2309–2314.

[10] M. Guo and D. V. Dimarogonas, “Consensus with quantized relative state measurements,” *Automatica*, vol. 49, no. 8, pp. 2531 – 2537, 2013. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0005109813002719

[11] Z. Zeng, X. Wang, and Z. Zheng, “Edge agreement of multi-agent system with quantised measurements via the directed edge laplacian,” *IET Control Theory Applications*, vol. 10, no. 13, pp. 1583–1589, 2016.

[12] J. Mei, W. Ren, and J. Chen, “Distributed consensus of second-order multi-agent systems with heterogeneous unknown inertias and control gains under a directed graph,” *IEEE Transactions on Automatic Control*, vol. 61, no. 8, pp. 2019–2034, 2016.

[13] M. M. Asadi, A. Ajorlou, and A. G. Aghdam, “Distributed control of a network of single integrators with limited angular fields of view,” *Automatica*, vol. 63, pp. 187–197, 2016.

[14] L. Sabattini, C. Secchi, and N. Chopra, “Decentralized estimation and control for preserving the strong connectivity of directed graphs,” *IEEE Trans Cybern*, vol. 45, no. 10, pp. 2273–2286, Oct. 2015.

[15] R. K. Williams, A. Gasparri, G. S. Sukhatme, and G. Ulivi, “Global connectivity control for spatially interacting multi-robot systems with unicycle kinematics,” in *2015 IEEE International Conference on Robotics and Automation (ICRA)*, May 2015, pp. 1255–1261.

[16] C. Godsil and G. F. Royle, *Algebraic graph theory*. Springer Science & Business Media, 2013, vol. 207.

[17] K. Cai and H. Ishii, “Average consensus on arbitrary strongly connected digraphs with time-varying topologies,” *IEEE Transactions on Automatic Control*, vol. 59, no. 4, pp. 1066–1071, 2014.

[18] F. Penna and S. Stańczak, “Decentralized largest eigenvalue test for multi-sensor signal detection,” in *Global Communications Conference (GLOBECOM), 2012 IEEE*. IEEE, 2012, pp. 3893–3898.