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Borel–de Siebenthal pairs, global Weyl modules and Stanley–Reisner rings

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Abstract We develop the theory of integrable representations for an arbitrary standard maximal parabolic subalgebra of an affine Lie algebra. We see that such subalgebras can be thought of as arising in a natural way from a Borel–de Siebenthal pair of semisimple Lie algebras. We see that although there are similarities with the representation theory of the standard maximal parabolic subalgebra there are also very interesting and non-trivial differences; including the fact that there are examples of non-trivial global Weyl modules which are irreducible and finite-dimensional. We also give a presentation of the endomorphism ring of the global Weyl module; although these are no longer polynomial algebras we see that for certain parabolics these algebras are Stanley–Reisner rings which are both Koszul and Cohen–Macaulay.

1 Introduction

The category of integrable representations of the current algebra $\mathfrak{g}[t]$ (or equivalently the standard maximal parabolic subalgebra in an untwisted affine Lie algebra) has been intensively studied in recent years. One reason for the interest in the subject is its connections with quantum affine algebras [8], Demazure modules [6,12,22], the theory of crystal bases [25], the theory of Macdonald polynomials [4,9,20], q-Whittaker functions [2,10] and more

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recently with the hypergeometric functions [1]. These connections exploit the fact that the current algebra can also be realized as a particular maximal parabolic subalgebra of the affine Lie algebra. It is natural, in this context to ask if other maximal parbolic subalgebras of an affine Lie algebra have an interesting representation theory with interesting applications.

Recall that for a simple finite-dimensional Lie algebra all Borel subalgebras are conjugate. In the case of affine Lie algebras, or more genrally, Kac–Moody algebras, there is more than one conjugacy class of Borel subalgebras. However the most natural one is the standard Borel subalgebra which comes from the definition of the Kac–Moody algebra via generators and relations. In this paper we shall be interested in parabolic subalgebras in the affine Lie algebra which contain the standard Borel subalgebra. The maximal parabolc subalgebras are indexed by the nodes of the affine Dynkin diagram. The current algebra is obtained by dropping the node zero, or more generally any node whose label in the affine Dynkin diagram is one. In this paper, our focus is on the other nodes of the Dynkin diagram.

We show that such subalgebras can be realized as the set of fixed points of a finite group action on the current algebra. In other words they are examples of equivariant map algebras as defined in [23]. Given a finite group Γ acting on a Lie algebra $\mathfrak{a}$ by Lie algebra automorphisms and on a commutative associative algebra by algebra automorphisms the equivariant map algebra is the set of fixed points of the group action on the Lie algebra $\mathfrak{a} \otimes A$.

The representation theory of equivariant map algebras has been developed in [11,13,23]. However much of the theory depends on the group acting freely on the maximal spectrum of A; in which case it is proved that the representation theory is essentially the same as that of $\mathfrak{a} \otimes A$. We prove that this is not the case for the non-standard parabolics and there are many interesting and non-trivial differences with the representation theory of the current algebra $\mathfrak{g} \otimes \mathbb{C}[t]$.

Recall that two important families of integrable representations of the current algebras are the global and local Weyl modules. The global Weyl modules are indexed by dominant integral weights $\lambda \in P^+$ and are universal objects in the category. Moreover the ring of endomorphisms $\mathbf{A}_\lambda$ in this category is commutative. It is known through the work of [8] that $\mathbf{A}_\lambda$ is a polynomial algebra in a finite number of variables depending on the weight λ and that it is infinite-dimensional if $\lambda \neq 0$. The local Weyl modules are indexed by dominant integral weights and maximal ideals in the corresponding algebra $\mathbf{A}_\lambda$ and are known to be finite-dimensional. The work of [8,12,22] shows that the dimension of the local Weyl module depends only on the weight, and not on the choice of maximal ideal in $\mathbf{A}_\lambda$, and so the global Weyl module is a free $\mathbf{A}_\lambda$-module of finite rank.

In this paper we develop the theory of global and local Weyl modules for an arbitrary maximal parabolic. The modules are indexed by dominant integral weights of a semisimple Lie subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ which is of maximal rank; a particular example that we use to illustrate all our results is the pair (B_n, D_n) which is also an example of a Borel–de Siebenthal pair. We determine a presentation of $\mathbf{A}_\lambda$ (Theorem 1) and show that in general $\mathbf{A}_\lambda$ is not a polynomial algebra and that the corresponding algebraic variety is not irreducible. In fact we give necessary and sufficient conditions on λ for $\mathbf{A}_\lambda$ to be finite-dimensional (Proposition 5.5). In particular, when this is the case, the associated global Weyl module is finite-dimensional and under further restrictions on λ the global Weyl module is also irreducible. We also show that under suitable conditions on the maximal parabolic the algebra $\mathbf{A}_\lambda$ is a Stanley–Reisner ring which is both Koszul and Cohen–Macaulay (Proposition 5.4).

Finally we study the local Weyl modules associated with a multiple of a fundamental weight. In this case $\mathbf{A}_\lambda$ is either one-dimensional or a polynomial algebra. We determine the dimension of the local Weyl modules and prove that it is independent of the choice of a maximal ideal in $\mathbf{A}_\lambda$ (Sect. 7). This proves also that in this case the global Weyl module is a free $\mathbf{A}_\lambda$-module of finite rank. This fact is false for general λ and we give an example of this in Sect. 7. However, we will show in this example that the global Weyl module is a free module for a suitable quotient algebra of $\mathbf{A}_\lambda$, namely the coordinate ring of one of the irreducible subvarieties of $\mathbf{A}_\lambda$.

This paper is organized as follows: In Sect. 2, we recall a result of Borel and de Siebenthal which realizes all maximal proper semisimple subalgebras, $\mathfrak{g}_0$, of maximal rank, of a fixed simple Lie algebra $\mathfrak{g}$ as the set of fixed points of an automorphism of $\mathfrak{g}$. We prove some results on root systems that we will need later in the paper, and discuss the running example of the paper, which is the case where $\mathfrak{g}$ is of type B_n, and $\mathfrak{g}_0$ is of type D_n.

In Sect. 3 we extend the automorphism of $\mathfrak{g}$ to an automorphism of $\mathfrak{g}[t]$. We then study the corresponding equivariant map algebra, which is the set of fixed points of this automorphism. We discuss ideals of this equivariant map algebra, and show that in this case, the equivariant map algebra is not isomorphic to an equivariant map algebra where the action of the group is free (Proposition 3.3), which makes the representation theory much different from that of the map algebra $\mathfrak{g}[t]$. We conclude the section by making the connection between these equivariant map algebras and maximal parabolic subalgebras of the affine Kac–Moody algebra (Proposition 3.5).

In Sect. 4 we develop the representation theory of $\mathfrak{g}[t]^\tau$. Following [3,5], we define the notion of global Weyl modules, the associated commutative algebra and the local Weyl modules associated to maximal ideals in this algebra. In the case of $\mathfrak{g}[t]$ it was shown in [8] that the commutative algebra associated with a global Weyl module is a polynomial ring in finitely many variables. This is no longer true for $\mathfrak{g}[t]^\tau$; however in Sect. 5 we see that modulo the Jacobson radical, the algebra is a quotient of a finitely generated polynomial ring by a squarefree monomial ideal. By making the connection to Stanley–Reisner theory, we are able to determine the Hilbert series. In certain cases we also determine the Krull dimension, and we give a sufficient condition for the commutative algebra to be Koszul and Cohen–Macaulay (Sect. 5.2).

In Sect. 6 we examine an interesting consequence of determining this presentation of the commutative algebra which differs from the case of the current algebra greatly. More specifically we see that under suitable conditions a global Weyl module can be finite-dimensional and irreducible, and we give necessary and sufficient conditions for this to be the case (Theorem 2).

We conclude this paper by determining the dimension of the local Weyl module in the case of our running example (B_n, D_n) for multiples of fundamental weights and a few other cases. We also discuss other features not seen in the case of the current algebra. Namely we give an example of a weight where the dimension of the local Weyl module depends on the choice of maximal ideal in $\mathbf{A}_\lambda$ showing that the global Weyl module is not projective and hence not a free $\mathbf{A}_\lambda$-module.

2 The Lie algebras $(\mathfrak{g}, \mathfrak{g}_0)$

2.1 We denote the set of complex numbers, the set of integers, non-negative integers, and positive integers by $\mathbb{C}, \mathbb{Z}, \mathbb{Z}_+$ and $\mathbb{N}$ respectively. Unless otherwise stated, all the vector spaces considered in this paper are $\mathbb{C}$-vector spaces and $\otimes$ stands for $\otimes_{\mathbb{C}}$. Given any Lie algebra $\mathfrak{a}$ we let $\mathbf{U}(\mathfrak{a})$ be the universal enveloping algebra of $\mathfrak{a}$. We also fix an indeterminate t and let $\mathbb{C}[t]$ and $\mathbb{C}[t, t^{-1}]$ be the corresponding polynomial ring, respectively Laurent polynomial ring with complex coefficients.

2.2

Let $\mathfrak{g}$ be a complex simple finite-dimensional Lie algebra of rank n with a fixed Cartan subalgebra $\mathfrak{h}$. Let $I = \{1, \ldots, n\}$ and fix a set $\Delta = \{\alpha_i : i \in I\}$ of simple roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let R, R^+ be the corresponding set of roots and positive roots respectively. Given $\alpha \in R$ let $\mathfrak{g}_\alpha$ be the corresponding root space and $a_i, i \in I$ be the labels of the Dynkin diagram of $\mathfrak{g}$; equivalently the highest root of R^+ is $\theta = \sum_{i=1}^n a_i\alpha_i$. Fix a Chevalley basis $\{x_\alpha^\pm, h_i : \alpha \in R^+, i \in I\}$ of $\mathfrak{g}$, and set $x_i^\pm = x_{\alpha_i}^\pm$. Let $(\ ,\ )$ be the non-degenerate bilinear form on $\mathfrak{h}^*$ with $(\theta, \theta) = 2$ induced by the restriction of the (suitably normalized) Killing form of $\mathfrak{g}$ to $\mathfrak{h}$.

Let Q be the root lattice with basis $\alpha_i, i \in I$. Define $\mathbf{a}_i : Q \to \mathbb{Z}, i \in I$ by requiring $\eta = \sum_{i=1}^n \mathbf{a}_i(\eta)\alpha_i$, and set $\mathrm{ht}(\eta) = \sum_{i=1}^n \mathbf{a}_i(\eta)$. For $\alpha \in R$ set $d_\alpha = 2/(\alpha, \alpha)$, $\mathbf{a}_i^\vee(\alpha) = \mathbf{a}_i(\alpha)d_\alpha d_{\alpha_i}^{-1}$ and $h_\alpha = \sum_{i=1}^n \mathbf{a}_i^\vee(\alpha)h_i$. Let W be the Weyl group of $\mathfrak{g}$ generated by a set of simple reflections $s_i, i \in I$ and fix a set of fundamental weights $\{\omega_i : 1 \le i \le n\}$ for $\mathfrak{g}$ with respect to Δ.

2.3

The following is well-known (see for instance [17, Chapter X, §5]). Set $I(j) = I \setminus \{j\}$ and let ζ be a fixed primitive a_j-th root of unity.

Proposition *The assignment*

$$x_i^\pm \to x_i^\pm, \quad i \in I(j), \quad x_j^\pm = \zeta^{\pm 1}x_j^\pm,$$

defines an automorphism $\tau : \mathfrak{g} \to \mathfrak{g}$ *of order* a_j. *Moreover, the set of fixed points* $\mathfrak{g}_0$ *is a semisimple subalgebra with Cartan subalgebra* $\mathfrak{h}$ *and*

$$R_0 = \{\alpha \in R : \mathbf{a}_j(\alpha) \in \{0, \pm a_j\}\},$$

is the set of roots of the pair $(\mathfrak{g}_0, \mathfrak{h})$. *The set* $\{\alpha_i : i \in I(j)\} \cup \{-\theta\}$ *is a simple system for* R_0. □

Remark Clearly when $a_j = 1$ the automorphism τ is just the identity and hence $\mathfrak{g}_0 = \mathfrak{g}$. In the case when a_j is prime, the pair $(\mathfrak{g}, \mathfrak{g}_0)$ is an example of a semisimple Borel–de Siebenthal pair. In other words, $\mathfrak{g}_0$ is a maximal proper semisimple subalgebra of $\mathfrak{g}$ of rank n. If a_j is not prime we can find a chain of semisimple subalgebras

$$\mathfrak{g}_0 \subset \mathbf{a}_1 \subset \cdots \subset \mathbf{a}_\ell \subset \mathfrak{g},$$

such that the successive inclusions are Borel–de Siebenthal pairs. We shall be interested in infinite-dimensional analogues of these. *From now on and usually without mention we shall assume that* $a_j \ge 2$.

2.4

For our purposes we will need a different simple system for R_0 which we choose as follows. The subgroup of W generated by the simple reflections $s_i, i \in I(j)$ is the Weyl group of the semisimple Lie algebra generated by $\{x_i^\pm : i \in I(j)\}$. Let $w_\circ$ be the longest element of this group.

Lemma *The set*

$$\Delta_0 = \{\alpha_i : i \in I(j)\} \cup \{w_\circ^{-1}\theta\},$$

is a set of simple roots for $(\mathfrak{g}_0, \mathfrak{h})$ *and the corresponding set* R_0^+ *of positive roots is contained in* R^+.

Proof Since $w_\circ$ is the longest element of the Weyl group generated by s_i, $i \in I(j)$, it follows that for $i \in I(j)$,

$$w_\circ \alpha_i \in \{-\alpha_p : p \in I(j)\}.$$

Hence

$$\Delta_0 = -\, w_\circ^{-1} \left(\{\alpha_i : i \in I(j)\} \cup \{-\theta\}\right).$$

Since $w_\circ$ is an element of the Weyl group of $\mathfrak{g}_0$ it follows from Proposition 2.3 that Δ_0 is a simple system for R_0. Moreover $w_\circ^{-1}\theta \in R^+$ since $w_\circ \alpha_j \in R^+$ and $\mathbf{a}_j(\theta) = a_j$. Hence $\Delta_0 \subset R^+$ thus proving the lemma.

Let Q_0 be the root lattice of $\mathfrak{g}_0$ determined by Δ_0; clearly $Q_0 \subset Q$ and set $Q_0^+ = Q_0 \cap Q^+$, $R_0^+ = R_0 \cap Q_0^+$. Then Q_0^+ is properly contained in Q^+ and we see an example of this at the end of this section.

Remark We isolate some immediate consequences of the lemma which we will use repeatedly. From now on we set $\alpha_0 = w_\circ^{-1}\theta$, $x_0^\pm = x_{\alpha_0}^\pm$ and $h_0 = h_{\alpha_0}$. The discussion so far shows that:

(i) α_0 is a long root,
(ii) $(\alpha_0, \alpha_i) \le 0$ if $i \in I(j)$ and $(\alpha_0, \alpha_j) > 0$,
(iii) $\mathbf{a}_j(\alpha_0) = a_j$, and
(iv) $\operatorname{ht}\alpha \ge \operatorname{ht}\alpha_0$ for all $\alpha \in R_0^+$ with $\mathbf{a}_j(\alpha) = a_j$.

Example Consider the example of the Borel–de Siebenthal pair (B_n, D_n), so $j = n$. Recall that the positive roots of B_n are of the form

$$\alpha_{r,s} := \alpha_r + \cdots + \alpha_s, \quad 1 \le r \le s \le n,$$
$$\alpha_{r,\bar{s}} := \alpha_r + \cdots + \alpha_{s-1} + 2\alpha_s + \cdots + 2\alpha_n, \quad 1 \le r < s \le n.$$

Moreover, $\theta = \alpha_{1,\bar{2}}$ and so $a_n = 2$. In this case, $\mathfrak{g}_0$ is of type D_n and $\alpha_0 = \alpha_{n-1} + 2\alpha_n$. The simple system for D_n described in Lemma 2.4 is given by $\Delta_0 = \{\alpha_1, \ldots, \alpha_{n-2}, \alpha_{n-1}, \alpha_0\}$ (α_0 and α_{n-1} correspond to the spin nodes) and the root system for D_n described in Proposition 2.3 is the set of all long roots of B_n. We note that $\alpha_n \in Q^+ \setminus Q_0^+$ as mentioned earlier in this section.

2.5 For $1 \le k < a_j$ set

$$R_k = \{\alpha \in R : \mathbf{a}_j(\alpha) \in \{k, -a_j + k\}\},$$
$$\mathfrak{g}_k = \bigoplus_{\alpha \in R_k} \mathfrak{g}_\alpha.$$

Equivalently

$$\mathfrak{g}_k = \{x \in \mathfrak{g} : \tau(x) = \zeta^k x\}.$$

Setting $R_k^+ = R_k \cap R^+$, we observe that

$$[x_0^+, R_k^+] = 0, \quad 1 \le k < a_j. \tag{2.1}$$

Proposition *We have,*

(i) $\mathfrak{g}_0 = [\mathfrak{g}_1, \mathfrak{g}_{a_j-1}]$.

(ii) *For all* $1 \le k < a_j$ *the subspace* $\mathfrak{g}_k$ *is an irreducible* $\mathfrak{g}_0$*-module.*
(iii) *For all* $0 \le m < k < a_j$*, we have* $\mathfrak{g}_k = [\mathfrak{g}_{k-m}, \mathfrak{g}_m]$.

Proof Write the semisimple algebra $\mathfrak{g}_0$ as a direct sum of simple ideals. Then Δ_0 is a disjoint union of the set of a simple systems for these ideals. Each of these contain a simple root contains some simple root α_i with $\alpha_i(h_j) < 0$. Since $0 \ne h_j = [x_j^+, x_j^-] \in [\mathfrak{g}_1, \mathfrak{g}_{a_j-1}]$ it follows that $[\mathfrak{g}_1, \mathfrak{g}_{a_j-1}]$ intersects each simple ideal of $\mathfrak{g}_0$ non-trivially and part (i) is proved.

If $a_j = 2$, the proof of the irreducibility in part (ii) of the proposition can be found in [18, Proposition 8.6]. If $a_j \ge 3$ then $\mathfrak{g}$ is of exceptional type and the proof is done in a case by case fashion. One inspects the set of roots to notice that for $1 \le k < a_j$ there exists a unique root $\theta_k \in R_k^+$ such that ht θ_k is maximal. This means that $x_{\theta_k}^+$ generates an irreducible $\mathfrak{g}_0$-module and a calculation proves that the dimension of this module is precisely dim $\mathfrak{g}_k$ and establishes part (ii). Part (iii) is now immediate if we prove that the $\mathfrak{g}_0$-module $[\mathfrak{g}_{k-m}, \mathfrak{g}_m]$ is non-zero and this is again proved by inspection. We omit the details. □

Part (ii) of the proposition implies that R_k^+ has a unique element θ_k such that the following holds:

$$(\theta_k, \alpha_i) \ge 0 \;\text{ and } [x_i^+, \mathfrak{g}_{\theta_k}] = 0, \quad i \in I(j) \cup \{0\}. \tag{2.2}$$

Since $\theta_k \ne \theta$ it is immediate that

$$[x_j^+, \mathfrak{g}_{\theta_k}] \ne 0, \quad i.e., \quad \theta_k + \alpha_j \in R^+.$$

Notice that $x_{\theta_k}^- \in \mathfrak{g}_{a_j-k}$ and $[x_i^-, x_{\theta_k}^-] = 0$ for all $i \in I(j) \cup \{0\}$. Moreover

$$\mathbf{a}_i(\theta_k) > 0, \quad i \in I, \quad 1 \le k < a_j. \tag{2.3}$$

To see this note that the set $\{i : \mathbf{a}_i(\theta_k) = 0\}$ is contained in $I(j)$. Since R is irreducible there must exist $i, p \in I$ with $\mathbf{a}_i(\theta_k) = 0$ and $\mathbf{a}_p(\theta_k) > 0$ and $(\alpha_i, \alpha_p) < 0$. It follows that $(\theta_k, \alpha_i) < 0$ which contradicts (2.2). As a consequence of (2.3) we get,

$$(\theta, \theta_k) > 0, \quad 1 \le k < a_j, \quad \text{and hence } \; \theta - \theta_k \in R^+_{a_j-k}. \tag{2.4}$$

Finally, we note that since $(\theta_k + \alpha_j, \alpha_0) = (\theta_k, \alpha_0) + (\alpha_j, \alpha_0) > 0$ (see the Remark in Sect. 2.4) we now have

$$\theta_k + \alpha_j - \alpha_0 \in R, \quad k \ne a_j - 1, \quad \theta_{a_j-1} + \alpha_j - \alpha_0 \in R_0^+ \cup \{0\}. \tag{2.5}$$

Example In the case of (B_n, D_n) the set R_1 consists of all short roots of B_n and $\theta_1 = \alpha_1 + \cdots + \alpha_n$. When $n \ge 4$, $\mathfrak{g}_1$ is the natural representation of D_n. When $n = 3$, $\mathfrak{g}_1$ is the second fundamental representation of A_3.

3 The algebras $(\mathfrak{g}[t], \mathfrak{g}[t]^\tau)$

In this section we define the current algebra version of the pair $(\mathfrak{g}, \mathfrak{g}_0)$; namely we extend the automorphism τ to the current algebra and study its fixed points. The fixed point algebra is an example of an equivariant map algebra studied in [23]. We show that our examples are particularly interesting since they can also be realized as maximal parabolic subalgebras of affine Lie algebras. We also show that our examples never arise from a free action of a finite abelian group on $\mathbb{C}$. This fact makes the study of its representation theory quite different from that of the usual current algebra.

3.1 Let $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ be the Lie algebra with the Lie bracket given by extending scalars. Recall the automorphism $\tau : \mathfrak{g} \to \mathfrak{g}$ defined in Sect. 2. It extends to an automorphism of $\mathfrak{g}[t]$ (also denoted as τ) by

$$\tau(x \otimes t^r) = \tau(x) \otimes \zeta^{-r} t^r, \quad x \in \mathfrak{g}, \quad r \in \mathbb{Z}_+.$$

Let $\mathfrak{g}[t]^\tau$ be the subalgebra of fixed points of τ; clearly

$$\mathfrak{g}[t]^\tau = \bigoplus_{k=0}^{a_j - 1} \mathfrak{g}_k \otimes t^k \mathbb{C}[t^{a_j}].$$

Further, if we regard $\mathfrak{g}[t]$ as a $\mathbb{Z}_+$-graded Lie algebra by requiring the grade of $x \otimes t^r$ to be r then $\mathfrak{g}[t]^\tau$ is also a $\mathbb{Z}_+$-graded Lie algebra, i.e.,

$$\mathfrak{g}[t]^\tau = \bigoplus_{s \in \mathbb{Z}_+} \mathfrak{g}[t]^\tau[s].$$

A graded representation of $\mathfrak{g}[t]^\tau$ is a $\mathbb{Z}_+$-graded vector space V which admits a compatible Lie algebra action of $\mathfrak{g}[t]^\tau$, i.e.,

$$V = \bigoplus_{s \in \mathbb{Z}_+} V[s], \;\; \mathfrak{g}[t]^\tau[s] V[r] \subset V[r+s], \;\; r, s \in \mathbb{Z}_+.$$

3.2
Given $z \in \mathbb{C}$, let $\mathrm{ev}_z : \mathfrak{g}[t] \to \mathfrak{g}$ be defined by $\mathrm{ev}_z(x \otimes t^r) = z^r x$, $x \in \mathfrak{g}$, $r \in \mathbb{Z}_+$. It is easy to see that

$$\mathrm{ev}_0(\mathfrak{g}[t]^\tau) = \mathfrak{g}_0, \;\; \mathrm{ev}_z(\mathfrak{g}[t]^\tau) = \mathfrak{g}, \;\; z \neq 0. \tag{3.1}$$

More generally, one can construct ideals of finite codimension in $\mathfrak{g}[t]^\tau$ as follows. Let $f \in \mathbb{C}[t^{a_j}]$ and $0 \le k < a_j$. The ideal $\mathfrak{g} \otimes t^k f \mathbb{C}[t]$ of $\mathfrak{g}[t]$ is of finite codimension and preserved by τ. Hence, $\mathfrak{i}_{k,f} = (\mathfrak{g} \otimes t^k f \mathbb{C}[t^{a_j}])^\tau$ is an ideal of finite codimension in $\mathfrak{g}[t]^\tau$. Notice that

$$\ker \mathrm{ev}_0 \cap \; \mathfrak{g}[t]^\tau = \mathfrak{i}_{1,1}, \;\; \ker \mathrm{ev}_z \cap \; \mathfrak{g}[t]^\tau = \mathfrak{i}_{0,(t^{a_j} - z^{a_j})}.$$

Proposition *Let $\mathfrak{i}$ be a non-zero ideal in $\mathfrak{g}[t]^\tau$. Then there exists $0 \le k < a_j$ and $f \in \mathbb{C}[t^{a_j}]$ such that $\mathfrak{i}_{k,f} \subset \mathfrak{i}$. In particular, any non-zero ideal in $\mathfrak{g}[t]^\tau$ is of finite codimension.*

Proof We claim that $\mathfrak{g}_k \otimes t^k g \subset \mathfrak{i}$ for some $g \in \mathbb{C}[t^{a_j}]$ and $k > 0$. To prove the claim note that since $\mathfrak{i}$ is preserved by the adjoint action of $\mathfrak{h}$ one of the following holds: either (i) there is a non-zero element $H \in \mathfrak{i} \cap \mathfrak{h} \otimes \mathbb{C}[t^{a_j}]$ or, (ii) $\mathfrak{i}$ contains an element of the form $x_\alpha^+ \otimes t^{\mathbf{a}_j(\alpha)} f$ for some $f \in \mathbb{C}[t^{a_j}]$ and $\alpha \in R^+$. In the first case we write

$$0 \neq H = \sum_{i \in I(j) \cup \{0\}} h_i \otimes f_i \in \mathfrak{i} \cap \mathfrak{h} \otimes \mathbb{C}[t^{a_j}],$$

and we then have

$$[H, x_p^+] = x_p^+ \otimes \sum_{i \in I(j) \cup \{0\}} \alpha_p(h_i) f_i \in \mathfrak{i}, \quad p \in I(j) \cup \{0\}.$$

Since the Cartan matrix of $\mathfrak{g}_0$ is invertible it follows that $\sum_{i \in I(j) \cup \{0\}} \alpha_p(h_i) f_i$ is non-zero for some $p \in I(j) \cup \{0\}$ and hence we see that $\mathfrak{i}$ contains an element of the form $x_\alpha^+ \otimes t^{\mathbf{a}_j(\alpha)} g$ for some $g \in \mathbb{C}[t^{a_j}]$ and $\alpha \in R_0^+$. Let $\mathfrak{a}$ be the simple summand of $\mathfrak{g}_0$ containing x_α^+. Taking repeated commutators with elements of $\mathfrak{a}$ we see that $\mathfrak{a}\otimes g\mathbb{C}[t^{a_j}]\subset\mathfrak{i}$. Moreover recalling that $\alpha_j(\mathfrak{h}\cap\mathfrak{a})\neq 0$ we choose $h\in\mathfrak{h}\cap\mathfrak{a}$ with $\alpha_j(h)\neq 0$ and hence

$$\alpha(h)^{-1}[x_j^+\otimes t, h\otimes g\mathbb{C}[t^{a_j}]=x_j^+\otimes tg\mathbb{C}[t^{a_j}]\in\mathfrak{i}.$$

Since $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module it follows that $\mathfrak{g}_1\otimes tg\mathbb{C}[t^{a_j}]\subset\mathfrak{i}$ and the claim is proved in the first case. The preceding argument also proves the claim in case (ii) if $k=0$ and if $k>0$, the irreducibility of $\mathfrak{g}_k$ as a $\mathfrak{g}_0$-module establishes the claim.

As a consequence of the claim, we see that if we set

$$S_k=\{g\in\mathbb{C}[t^{a_j}]: x\otimes t^k g\in\mathfrak{i}\text{ for all }x\in\mathfrak{g}_k\},\quad 0\le k\le a_j-1,$$

then $S_k\neq 0$ for some $k>0$. We now prove that S_k is an ideal in $\mathbb{C}[t^{a_j}]$ and also that

$$t^{a_j}S_{a_j-1}\subset S_0\subset S_1\subset\cdots\subset S_{a_j-1}. \tag{3.2}$$

In particular this shows that S_k is non-zero for all $0\le k\le a_j-1$. Using Proposition 2.5(i) we write an element $x\in\mathfrak{g}_k$ as a sum $x=\sum_{s=1}^r[z_s,y_s]$ with $z_s\in\mathfrak{g}_0$ and $y_s\in\mathfrak{g}_k$ for $1\le s\le r$. This means that,

$$x\otimes t^k fg=\sum_{s=1}^r[z_s\otimes f, y_s\otimes t^k g],\quad f,g\in\mathbb{C}[t].$$

If $g\in S_k$ then $y_s\otimes t^k g\in\mathfrak{i}$ by definition of S_k and so the right hand side of the preceding equation is an element of $\mathfrak{i}$. Hence $x\otimes t^k fg\in\mathfrak{i}$ for all $f\in\mathbb{C}[t^{a_j}]$ and $g\in S_k$ proving that S_k is an ideal for all $0\le k\le a_j-1$. A similar argument using $[\mathfrak{g}_m,\mathfrak{g}_{k-m}]=\mathfrak{g}_k$ proves the inclusions in (3.2).

For $0\le k\le a_j-1$ let $f_k\in\mathbb{C}[t^{a_j}]$ be a non-zero generator for the ideal S_k. By (3.2) there exist $g_0,\dots,g_{a_j-1}\in\mathbb{C}[t^{a_j}]$ such that

$$f_r=g_rf_{r+1},\ 0\le r\le a_j-2,\quad t^{a_j}f_{a_j-1}=g_{a_j-1}f_0.$$

This implies

$$g_{a_j-1}f_0=g_0\cdots g_{a_j-1}f_{a_j-1}=t^{a_j}f_{a_j-1}.$$

Hence there exists a unique $m\in\{0,\dots,a_j-1\}$ such that $g_m=t^{a_j}$ and $g_p=1$ if $p\neq m$. Taking $f=f_{m+1}$, where we understand $f_{a_j}=f_0$, we see that

$$\mathfrak{i}_{k,f}\subset\mathfrak{i},\quad k=m+1-a_j\delta_{m,a_j-1}.$$

□

3.3

We now show that $\mathfrak{g}[t]^\tau$ is never a current algebra or more generally an equivariant map algebra with free action. For this, we recall from [23] the definition of an equivariant map algebra. Thus, let $\mathfrak{a}$ be any finite-dimensional complex Lie algebra and A a finitely generated commutative associative algebra. Assume also that Γ is a finite abelian group acting on $\mathfrak{a}$ by Lie algebra automorphisms and on A by algebra automorphisms. Then we have an induced action on the Lie algebra $(\mathfrak{a}\otimes A)$ (the commutator is given in the obvious way) such that $\gamma(x\otimes f)=\gamma x\otimes\gamma f$. An equivariant map algebra is defined to be the fixed point subalgebra:

$$(\mathfrak{a}\otimes A)^\Gamma:=\{z\in(\mathfrak{a}\otimes A)\mid\gamma(z)=z\ \ \forall\gamma\in\Gamma\}.$$

The finite-dimensional irreducible representations of such algebras (and hence for $\mathfrak{g}[t]^\tau$) were given in [23] and generalized earlier work on affine Lie algebras.

In the case when Γ acts freely on the set of maximal ideals of A, many aspects of the representation theory of the equivariant map algebra are the same as the representation theory of $\mathfrak{a} \otimes A$ (see for instance [11]). The importance of the following proposition is now clear.

Proposition *The Lie algebra* $\mathfrak{g}[t]^\tau$ *is not isomorphic to an equivariant map algebra* $(\mathfrak{a} \otimes A)^\Gamma$ *with* $\mathfrak{a}$ *semisimple and* Γ *acting freely on the set of maximal ideals of* A.

Proof Recall our assumption that $a_j > 1$ and assume for a contradiction that

$$\mathfrak{g}[t]^\tau \cong (\mathfrak{a} \otimes A)^\Gamma$$

where $\mathfrak{a}$ is semi-simple. Write $\mathfrak{a} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_k$ where each $\mathfrak{a}_s$ is isomorphic to a direct sum of copies of a simple Lie algebra $\mathfrak{g}_s$ and $\mathfrak{g}_s \ncong \mathfrak{g}_m$ if $m \neq s$. Clearly Γ preserves $\mathfrak{a}_s$ for all $1 \leq s \leq k$ and hence

$$\mathfrak{g}[t]^\tau \cong (\mathfrak{a} \otimes A)^\Gamma \cong \oplus_{s=1}^k (\mathfrak{a}_s \otimes A)^\Gamma.$$

Since $\mathfrak{g}[t]^\tau$ is infinite-dimensional at least one of the summands $(\mathfrak{a}_s \otimes A)^\Gamma$ is infinite-dimensional, say $s = 1$ without loss of generality. But this means that $\oplus_{s=2}^k (\mathfrak{a}_s \oplus A)^\Gamma$ is an ideal which is not of finite codimension which contradicts Proposition 3.2. Hence we must have $k = 1$, i.e. $\mathfrak{a} = \mathfrak{a}_1$. It was proven in [23, Proposition 5.2] that if Γ acts freely on the set of maximal ideals of A then any finite-dimensional simple quotient of $(\mathfrak{a} \otimes A)^\Gamma$ is a quotient of $\mathfrak{a}$; in particular in our situation it follows that all the finite-dimensional simple quotients of $(\mathfrak{a} \otimes A)^\Gamma$ are isomorphic. On the other hand, (3.1) shows that $\mathfrak{g}[t]^\tau$ has both $\mathfrak{g}_0$ and $\mathfrak{g}$ as quotients. Since $\mathfrak{g}_0$ is not isomorphic to $\mathfrak{g}$ we have the desired contradiction.

3.4

The untwisted affine Lie algebra $\widehat{\mathfrak{g}}$ associated to $\mathfrak{g}$ is defined as follows: as a vector space

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

with the commutator given by requiring c to be central, and

$$[d, x \otimes f] = x \otimes t(\partial f/\partial t), \quad [x \otimes f, y \otimes g] = [x, y] \otimes fg + \operatorname{Res}((\partial f/\partial t)g)\kappa(x, y)c.$$

Here κ is the Killing form of $\mathfrak{g}$ and $\operatorname{Res} : \mathbb{C}[t, t^{-1}] \to \mathbb{C}$ is the residue function which picks out the coefficient of t^{-1}. The Cartan subalgebra is

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

and let $\delta \in \widehat{\mathfrak{h}}^*$ be given by $\delta(d) = 1$ and $\delta(\mathfrak{h} \oplus \mathbb{C}c) = 0$. Extend $\alpha \in \mathfrak{h}^*$ to an element of $\widehat{\mathfrak{h}}^*$ by $\alpha(c) = \alpha(d) = 0$. Then the set of roots, respectively set of simple roots of the pair $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{h}})$ is

$$\widehat{R} = \{\alpha + r\delta : \alpha \in R, r \in \mathbb{Z}\} \cup \{s\delta : s \in \mathbb{Z}, s \neq 0\}, \quad \widehat{\Delta} = \{\alpha_i : i \in I\} \cup \{\delta - \theta\}.$$

The Borel subalgebra defined by this simple system is

$$\widehat{\mathfrak{b}} = ((\mathfrak{h} \oplus \mathfrak{n}^+) \otimes \mathbb{C}[t]) \oplus (\mathfrak{n}^- \otimes t\mathbb{C}[t]) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

We shall say that a parabolic subalgebra of $\widehat{\mathfrak{g}}$ is one that contains $\widehat{\mathfrak{b}}$; this is analogous to the definition for simple Lie algebras although there are some differences. In the case of simple Lie algebras any two Borel sublgebras are conjugate but this is false for affine Lie algebras. The Borel subalgebra that we are working with is called the standard Borel subalgebra and the restriction to this case is very natural. Given any subset Δ' of $\widehat{\Delta}$ let $\widehat{R}(\Delta')$ be the subset of $\widehat{R}$ consisting of elements which are in the $\mathbb{Z}$-span of Δ' and for $\alpha \in \widehat{R}$ let $\widehat{\mathfrak{g}}_\alpha$ be the corresponding root space. The proof of the next result is very similar to the one for simple Lie algebras, we include a sketch of the proof for the reader's convenience.

Lemma *Suppose that $\widehat{\mathfrak{p}}$ is a proper parabolic subalgebra of $\widehat{\mathfrak{g}}$ and assume that $\widehat{\mathfrak{b}} \neq \widehat{\mathfrak{p}}$. Then there exists a proper subset Δ' of $\widehat{\Delta}$ such that*

$$\widehat{\mathfrak{p}} = \widehat{\mathfrak{b}} + \sum_{\alpha \in \widehat{R}(\Delta')} \widehat{\mathfrak{g}}_\alpha.$$

Moreover $\widehat{\mathfrak{p}}$ is maximal iff $|\Delta'| = |I|$.

Proof It is a simple exercise to see that the Lie algebra $\widehat{\mathfrak{g}}$ is generated by $\widehat{\mathfrak{b}}$ and a non-zero element $h \otimes t^{-s-1}$ for some $h \in \mathfrak{h}$, $s \in \mathbb{Z}_+$. Since $\widehat{\mathfrak{p}}$ is a proper subalgebra it follows that $(h \otimes t^{-s}) \notin \widehat{\mathfrak{p}}$ for any $s < 0$ and any $h \in \mathfrak{h}$. Since $(h_\alpha \otimes t^{-1}) = [x_\alpha^- \otimes t^r, x_\alpha^+ \otimes t^{-r-1}]$ it now follows that $x_\alpha^+ \otimes t^{-r-1}$ is not in $\widehat{\mathfrak{p}}$ if $r > 0$. Similarly we show that $x_\alpha^- \otimes t^{-r}$ is not in $\widehat{\mathfrak{p}}$ for any $\alpha \in R^+$ if $r > 0$.

Suppose that $x_\alpha^+ \otimes t^{-1} \in \widehat{\mathfrak{p}}$ for some $\alpha \in R^+$. Taking commutators with elements $x_i^+ \otimes 1$ shows that $x_\theta^+ \otimes t^{-1} \in \widehat{\mathfrak{p}}$. A similar argument proves that if $x_\alpha^- \otimes 1 \in \widehat{\mathfrak{p}}$ for some $r \geq 0$ then $x_i^- \otimes 1 \in \widehat{\mathfrak{p}}$ for some $i \in I$. In particular, it follows that if we set

$$\Delta' = \begin{cases} \{\alpha_i : i \in I : x_i^- \in \widehat{\mathfrak{p}}\}, & \text{if} \quad x_\theta^+ \otimes t^{-1} \notin \widehat{\mathfrak{p}}, \\ \{\delta - \theta, \alpha_i : i \in I, x_i^- \in \widehat{\mathfrak{p}}\}, & \text{if} \quad x_\theta^+ \otimes t^{-1} \in \widehat{\mathfrak{p}}, \end{cases}$$

then $\Delta' \neq \emptyset$. Clearly

$$\widehat{\mathfrak{p}} \supseteq \widehat{\mathfrak{b}} + \sum_{\alpha \in \widehat{R}(\Delta')} \widehat{\mathfrak{g}}_\alpha.$$

The reverse inclusion follows if we prove that $x_\alpha^+ \otimes t^{-1}$, (resp. $x_\alpha^- \otimes 1$) is in $\widehat{\mathfrak{p}}$ only if $-\alpha + \delta$ (resp. α) is in the span of $\mathbb{Z}_+$-span of Δ'.

Suppose first that $x_\alpha^+ \otimes t^{-1} \in \widehat{\mathfrak{p}}$. We proceed by a downward induction on $\operatorname{ht}\alpha$. To see that induction begins assume that $\alpha = \theta$. By the above discussion we have $x_\theta^+ \otimes t^{-1} \in \widehat{\mathfrak{p}}$ and hence $-\theta + \delta \in \Delta'$. For the inductive step choose $\alpha_i \in \Delta$ with $\alpha + \alpha_i \in R^+$. Then $x_{\alpha_i+\alpha}^+ \otimes t^{-1}$ is a non-zero scalar multiple of $[x_i^+, x_\alpha^+ \otimes t^{-1}]$ and hence is in $\widehat{\mathfrak{p}}$. Therefore, $-(\alpha + \alpha_i) + \delta$ is in the $\mathbb{Z}_+$-span of Δ'. We also have that $(x_i^- \otimes 1)$ is a non-zero scalar multiple of $[x_\alpha^- \otimes t, x_{\alpha+\alpha_i}^+ \otimes t^{-1}]$ and hence $\alpha_i \in \Delta'$. It now follows that $-\alpha + \delta$ is in the $\mathbb{Z}_+$-span of Δ' which proves the inductive step.

To prove the result when $x_\alpha^- \otimes 1 \in \widehat{\mathfrak{p}}$ we proceed by an upward induction on $\operatorname{ht}\alpha$. If $\alpha \in \Delta$ then by definition $\alpha \in \Delta'$ and so induction begins. For the inductive step, choose $\alpha_i \in \Delta$ such that $\beta = \alpha - \alpha_i \in R^+$. Since $x_\beta^- \otimes 1$ is a non-zero scalar multiple of $[x_i^+, x_\alpha^-] \otimes 1$ we see that $x_\beta^- \otimes 1 \in \widehat{\mathfrak{p}}$. By the inductive hypothesis we have β is in the $\mathbb{Z}_+$-span of Δ'. On the other hand $[x_\beta^+, x_\alpha^- \otimes 1]$ is a non-zero scalar multiple of $x_i^- \otimes 1$ and hence $\alpha_i \in \Delta'$ and the inductive step is proved. The second statement of the lemma is obvious. □

3.5

We now make the connection between $\mathfrak{g}[t]^\tau$ and a maximal parabolic subalgebra of $\widehat{\mathfrak{g}}$ associated to $\mathfrak{g}$. We take

$$\Delta' = \{\alpha_i : i \in I(j)\} \cup \{\delta - \theta\},$$

and let $\widehat{\mathfrak{p}}$ be the associated parabolic subalgebra. Then it is easy to see that

$$\mathbb{Z}_+\Delta' \cap \widehat{R} = \{\delta - \alpha :\ \alpha \in R^+,\ \mathbf{a}_j(\alpha) = a_j\} \cup \{\alpha \in R^+,\ \mathbf{a}_j(\alpha) = 0.\}.$$

Hence $\widehat{\mathfrak{p}}$ is spanned by $\widehat{\mathfrak{b}}$ and the elements of the set

$$\{x_\alpha^+ \otimes t^{-1} : \alpha \in R^+, \mathbf{a}_j(\alpha) = a_j\} \cup \{x_\alpha^- \otimes 1 : \alpha \in R^+, \mathbf{a}_j(\alpha) = 0\}.$$

Let $\widetilde{\mathfrak{p}}$ be the quotient of the derived subalgebra of $\widehat{\mathfrak{p}}$ by the subspace $\mathbb{C}c$. Then $\widetilde{\mathfrak{p}}$ is isomorphic to a subalgebra of $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. Define a grading gr on $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ by

$$\mathrm{gr}(x_\alpha^\pm \otimes t^r) = ra_j \pm \mathbf{a}_j(\alpha),\ \ \alpha \in R^+,\ \ r \in \mathbb{Z}.$$

Observe that $\widetilde{\mathfrak{p}}$ is a graded subalgebra. The following is now trivially checked .

Proposition *The map $\phi : \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \to \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ of Lie algebras given on graded elements by $\phi(x \otimes t^r) = x \otimes t^{\mathrm{gr}(x \otimes t^r)}$ is a graded isomorphism $\widetilde{\mathfrak{p}} \cong \mathfrak{g}[t]^\tau$.* □

4 The category $\widetilde{\mathcal{I}}$

In this section we develop the representation theory of $\mathfrak{g}[t]^\tau$. Following [3,5], we define the notion of global Weyl modules, the associated commutative algebra and the local Weyl modules associated to maximal ideals in this algebra. In the case of $\mathfrak{g}[t]$ it was shown in [8] that the commutative algebra associated with a global Weyl module is a polynomial ring in finitely many variables. This is no longer true for $\mathfrak{g}[t]^\tau$; however we shall see that modulo the Jacobson radical, the algebra is a quotient of a finitely generated polynomial ring by a squarefree monomial ideal. As a consequence we see that under suitable conditions a global Weyl module can be finite-dimensional and irreducible. More precise statements can be found in Sect. 6.

4.1

Fix a set of fundamental weights $\{\lambda_i : i \in I(j) \cup \{0\}\}$ for $\mathfrak{g}_0$ with respect to Δ_0 and let P_0, P_0^+ be their $\mathbb{Z}$ and $\mathbb{Z}_+$-span respectively. Note that the subset

$$P^+ = \{\lambda \in P_0^+ : \lambda(h_j) \in \mathbb{Z}_+\}$$

is precisely the set of dominant integral weights for $\mathfrak{g}$ with respect to Δ. Also note that P^+ is properly contained in P_0^+. For example, in the B_n case, $\lambda_{n-1} \in P_0^+$, and $\lambda_{n-1}(h_n) = -1$. It is the existence of these types of weights that causes the representation theory of $\mathfrak{g}[t]^\tau$ to be different from that of $\mathfrak{g}[t]$.

For $\lambda \in P_0^+$ let $V_{\mathfrak{g}_0}(\lambda)$ be the irreducible finite-dimensional $\mathfrak{g}_0$-module with highest weight λ and highest weight vector v_λ; if $\lambda \in P^+$ the module $V_{\mathfrak{g}}(\lambda)$ and the vector v_λ are defined in the same way.

4.2

Let $\widetilde{\mathcal{I}}$ be the category whose objects are $\mathfrak{g}[t]^\tau$-modules with the property that they are $\mathfrak{g}_0$ integrable and where the morphisms are $\mathfrak{g}[t]^\tau$-module maps. In other words an object V of $\widetilde{\mathcal{I}}$ is a $\mathfrak{g}[t]^\tau$-module which is isomorphic to a direct sum of finite-dimensional $\mathfrak{g}_0$-modules. It follows that V admits a weight space decomposition

$$V = \bigoplus_{\mu \in P_0} V_\mu,\ \ V_\mu = \{v \in V : hv = \mu(h)v,\ h \in \mathfrak{h}\},$$

and we set $\mathrm{wt}\, V = \{\mu \in P_0 : V_\mu \neq 0\}$. Note that

$$w\, \mathrm{wt}\, V \subset \mathrm{wt}\, V,\ \ w \in W_0,$$

where W_0 is the Weyl group of $\mathfrak{g}_0$.

For $\lambda \in P_0^+$ we let $\widetilde{\mathcal{I}}^\lambda$ be the full subcategory of $\widetilde{\mathcal{I}}$ whose objects V satisfy the condition that $\operatorname{wt} V \subset \lambda - Q^+$; note that this is a weaker condition than requiring the set of weights be contained in $\lambda - Q_0^+$ (see Sect. 2.4).

Lemma *Suppose that V is an object of $\widetilde{\mathcal{I}}^\lambda$ and let $\mu \in \operatorname{wt} V$ and $\alpha \in R^+$. Then $\mu - s\alpha \in \operatorname{wt} V$ for only finitely many $s \in \mathbb{Z}$.*

Proof Since $\mu \in \operatorname{wt} V$ we write $\lambda - \mu = \sum_{i\in I} s_i\alpha_i$ for some $s_i \in \mathbb{Z}_+$, $i \in I$. If $s < 0$ and $p \in I$ is such that $\mathbf{a}_p(\alpha) > 0$ then $-s\mathbf{a}_p(\alpha) - s_p < 0$ or equivalently $-s_p < s\mathbf{a}_p(\alpha) < 0$ for only finitely many negative values of s. It follows that the set of negative integers for which $\mu - s\alpha \in \operatorname{wt} V$ is finite.

Suppose that $s > 0$. Since $\alpha \in P_0$ we can choose $w \in W_0$ such that $w\alpha$ is in the anti-dominant chamber for the action of W_0 on $\mathfrak{h}$. This implies that $w\alpha = -r_0\alpha_0 - \sum_{i\in I(j)} r_i\alpha_i$ where the r_i are non-negative rational numbers. Since W_0 is a subgroup of W it follows that $-w\alpha \in R^+$. Since $w\mu - (-s)(-w\alpha) = w\mu - sw\alpha \in \operatorname{wt} V$, it follows by applying the argument in the case $s < 0$ to the elements $w\mu \in \operatorname{wt} V$ and $-w\alpha \in R^+$ that $-s$ is bounded below and hence that s is bounded above. This completes the proof the lemma. □

4.3

Let

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathfrak{n}^\pm = \bigoplus_{\alpha\in R^+} \mathfrak{g}_{\pm\alpha},$$

be the triangular decomposition of $\mathfrak{g}$. Since τ preserves the subalgebras $\mathfrak{n}^\pm$ and $\mathfrak{h}$ we have

$$\mathfrak{g}[t]^\tau = \mathfrak{n}^-[t]^\tau \oplus \mathfrak{h}[t]^\tau \oplus \mathfrak{n}^+[t]^\tau.$$

Further $\mathfrak{h}[t]^\tau \cong \mathfrak{h} \otimes \mathbb{C}[t^{a_j}]$ is a commutative subalgebra of $\mathfrak{g}[t]^\tau$.

For $\lambda \in P_0^+$ the global Weyl module $W(\lambda)$ is the cyclic $\mathfrak{g}[t]^\tau$-module generated by an element w_λ with defining relations: for $h \in \mathfrak{h}$ and $i \in I(j) \cup \{0\}$,

$$hw_\lambda = \lambda(h)w_\lambda, \quad \mathfrak{n}^+[t]^\tau w_\lambda = 0, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1} w_\lambda = 0. \tag{4.1}$$

It is elementary to check that $W(\lambda)$ is an object of $\widetilde{\mathcal{I}}_j^\lambda$, one just needs to observe that the elements $x_i^\pm$, $i \in I(j) \cup \{0\}$ act locally nilpotently on $W(\lambda)$. Moreover, if we declare the grade of w_λ to be zero then $W(\lambda)$ acquires the structure of a $\mathbb{Z}_+$ graded $\mathfrak{g}[t]^\tau$-module.

Remark The definition of global and local Weyl modules goes back to [8] in the case of affine algebras, to [3] for the map algebras and to [11,13] for the equivariant map algebras.

4.4

As in [3, Section 3.4] (see also [13, Lemma 4.1]) one checks easily that the following formula defines a right action of $\mathfrak{h}[t]^\tau$ on $W(\lambda)$:

$$(uw_\lambda)a = uaw_\lambda, \quad u \in \mathbf{U}(\mathfrak{g}[t]^\tau), \quad a \in \mathfrak{h}[t]^\tau.$$

Moreover this action commutes with the left action of $\mathfrak{g}[t]^\tau$. In particular, if we set

$$\operatorname{Ann}_{\mathfrak{h}[t]^\tau}(w_\lambda) = \{a \in \mathbf{U}(\mathfrak{h}[t]^\tau) : aw_\lambda = 0\}, \quad \mathbf{A}_\lambda = \mathbf{U}(\mathfrak{h}[t]^\tau)/\operatorname{Ann}_{\mathfrak{h}[t]^\tau}(w_\lambda),$$

we get that $\operatorname{Ann}_{\mathfrak{h}[t]^\tau}(w_\lambda)$ is an ideal in $\mathbf{U}(\mathfrak{h}[t]^\tau)$ and that $W(\lambda)$ is a bi-module for $(\mathfrak{g}[t]^\tau, \mathbf{A}_\lambda)$. It is clear that $\operatorname{Ann}_{\mathfrak{h}[t]^\tau}(w_\lambda)$ is a graded ideal of $\mathbf{U}(\mathfrak{h}[t]^\tau)$ and hence the algebra $\mathbf{A}_\lambda$ is a

$\mathbb{Z}_+$-graded algebra with a unique graded maximal ideal $\mathbf{I}_0$. It is also obvious that we have an isomorphism of right $\mathbf{A}_\lambda$-modules

$$W(\lambda)_\lambda \cong \mathbf{A}_\lambda. \tag{4.2}$$

4.5

We need some additional results to further study the structure of $W(\lambda)$ as a $\mathbf{A}_\lambda$-module. For $\alpha \in R^+$ and $r \in \mathbb{Z}_+$, define elements $P_{\alpha,r} \in \mathbf{U}(\mathfrak{h}[t]^\tau)$ recursively by

$$P_{\alpha,0} = 1, \quad P_{\alpha,r} = -\frac{1}{r}\sum_{p=1}^{r}(h_\alpha \otimes t^{a_j p})P_{\alpha,r-p}, \quad r \geq 1.$$

Equivalently $P_{\alpha,r}$ is the coefficient of u^r in the formal power series

$$P_\alpha(u) = \exp\left(-\sum_{r\geq 1}\frac{h_\alpha \otimes t^{a_j r}}{r}u^r\right).$$

Writing $h_\alpha = \sum_{i=1}^n \mathbf{a}_i^\vee(\alpha)h_i$, we see that

$$P_\alpha(u) = \prod_{i=1}^n P_{\alpha_i}(u)^{\mathbf{a}_i^\vee(\alpha)}, \ \ \alpha \in R^+.$$

Set $P_{\alpha_i,r} = P_{i,r}$, $i \in I \cup \{0\}$. The following is now trivial from the Poincaré–Birkhoff–Witt theorem.

Lemma *The algebra $\mathbf{U}(\mathfrak{h}[t]^\tau)$ is the polynomial algebra in the variables*

$$\{P_{i,r} : i \in I(j) \cup \{0\}, \ r \in \mathbb{N}\},$$

and also in the variables

$$\{P_{i,r} : i \in I, \ r \in \mathbb{N}\}.$$

□

The comultiplication $\tilde{\Delta} : \mathbf{U}(\mathfrak{g}[t]^\tau) \to \mathbf{U}(\mathfrak{g}[t]^\tau) \otimes \mathbf{U}(\mathfrak{g}[t]^\tau)$ satisfies

$$\tilde{\Delta}(P_\alpha(u)) = P_\alpha(u) \otimes P_\alpha(u), \ \ \alpha \in R^+. \tag{4.3}$$

For $x \in \mathbf{U}(\mathfrak{g}[t]^\tau)$, $r \in \mathbb{Z}_+$, set

$$x^{(r)} = \frac{1}{r!}x^r.$$

4.6

The following can be found in [8, Lemma 1.3] and is a reformulation of a result of Garland [16, Lemma 7.1].

Lemma *Let $x^\pm, h$ be the standard basis of $\mathfrak{sl}_2$ and let V be a representation of the Lie subalgebra of $\mathfrak{sl}_2[t]$ generated by $(x^+ \otimes 1)$ and $(x^- \otimes t)$. Assume that $0 \neq v \in V$ is such that $(x_\alpha^+ \otimes t^r)v = 0$ for all $r \in \mathbb{Z}_+$. For all $r \in \mathbb{Z}_+$ we have*

$$(x^+ \otimes 1)^{(r)}(x^- \otimes t)^{(r)}v = (x^+ \otimes t)^{(r)}(x^- \otimes 1)^{(r)}v = (-1)^r P_r v, \tag{4.4}$$

where

$$\sum_{r\geq 0} P_r u^r = \exp\left(-\sum_{r\geq 1}\frac{h\otimes t^r}{r}u^r\right).$$

Furthermore,

$$(x^+\otimes 1)^{(r)}(x^-\otimes t)^{(r+1)}v = (-1)^r\sum_{s=0}^{r}(x^-\otimes t^{s+1})P_{r-s}v. \tag{4.5}$$

□

4.7

Proposition *For all $\lambda \in P_0^+$ the algebra $\mathbf{A}_\lambda$ is finitely generated and $W(\lambda)$ is a finitely generated $\mathbf{A}_\lambda$-module.*

Proof The proof of the proposition is very similar to the one given in [3, Theorem 2] but we sketch the proof below for the reader's convenience and also to set up some further necessary notation. Given $\alpha \in R^+$, it is easily seen that the elements $(x_\alpha^+\otimes t^{\mathbf{a}_j(\alpha)})$ and $(x_\alpha^-\otimes t^{a_j-\mathbf{a}_j(\alpha)})$ generate a subalgebra of $\mathfrak{g}[t]^\tau$ which is isomorphic to the subalgebra of $\mathfrak{sl}_2[t]$ generated by $(x^+\otimes 1)$ and $(x^-\otimes t)$. Using the defining relations of $W(\lambda)$ and Eq. (4.4) we get that

$$P_{\alpha,r}w_\lambda = 0,\quad r\geq \lambda(h_\alpha)+1,\quad \alpha\in R_0^+. \tag{4.6}$$

It also follows from Lemma 4.2 that $P_{j,r}w_\lambda = 0$ for all $r\gg 0$. Using Lemma 4.2 we see that $\mathbf{A}_\lambda$ is finitely generated by the images of the elements

$$\{P_{i,r} : i\in I(j)\cup\{0\},\quad r\leq \lambda(h_i)\}.$$

Fix an enumeration $\beta_1,\ldots,\beta_M$ of R^+. Using the Poincaré–Birkhof–Witt theorem it is clear that $W(\lambda)$ is spanned by elements of the form $X_1X_2\cdots X_M\mathbf{U}(\mathfrak{h}[t]^\tau)w_\lambda$ where each X_p is either a constant or a monomial in the elements $\{(x_{\beta_p}^-\otimes t^s) : s\in a_j\mathbb{Z}_+ - \mathbf{a}_j(\beta_p)\}$. The length of each X_r is bounded by Lemma 4.2 and equation (4.5) proves that for any $\gamma\in R^+$ and $r\in\mathbb{Z}_+$, the element $(x_\gamma^-\otimes t^{ra_j-\mathbf{a}_j(\gamma)})\mathbf{U}(\mathfrak{h}[t]^\tau)w_\lambda$ is in the span of elements $\{(x_\gamma^-\otimes t^{sa_j-\mathbf{a}_j(\gamma)})\mathbf{U}(\mathfrak{h}[t]^\tau)w_\lambda : 0\leq s\leq N\}$ for some N sufficiently large. An obvious induction on the length of the product of monomials shows that the values of s are bounded for each β and the proof is complete.

Remark Notice that the preceding argument proves that the set wt $W(\lambda)$ is finite. This is not obvious since wt $W(\lambda)$ is not a subset of $\lambda - Q_0^+$.

4.8

Let $\lambda\in P_0^+$. Given any maximal ideal $\mathbf{I}$ of $\mathbf{A}_\lambda$ we define the local Weyl module,

$$W(\lambda,\mathbf{I}) = W(\lambda)\otimes_{\mathbf{A}_\lambda}\mathbf{A}_\lambda/\mathbf{I}.$$

It follows from Proposition 4.7 that $W(\lambda,\mathbf{I})$ is a finite-dimensional $\mathfrak{g}[t]^\tau$-module in $\widetilde{\mathcal{I}}$ and $\dim W(\lambda,\mathbf{I})_\lambda = 1$. A standard argument now proves that $W(\lambda,\mathbf{I})$ has a unique irreducible quotient which we denote as $V(\lambda,\mathbf{I})$. Moreover, $W(\lambda,\mathbf{I}_0)$ is a $\mathbb{Z}_+$-graded $\mathfrak{g}[t]^\tau$-module and

$$V(\lambda,\mathbf{I}_0)\cong \mathrm{ev}_0^* V_{\mathfrak{g}_0}(\lambda), \tag{4.7}$$

where $\mathrm{ev}_0^* V$ is the representation of $\mathfrak{g}[t]^\tau$ obtained by pulling back a representation V of $\mathfrak{g}_0$.

4.9
We now construct an explicit family of representations of $\mathfrak{g}[t]^\tau$ which will be needed for our further study of $\mathbf{A}_\lambda$. Given non-zero scalars $z_1, \dots, z_k$ such that $z_r^{a_j} \neq z_s^{a_j}$ for all $1 \le r \neq s \le k$ it is easy to see that the morphism $\mathrm{ev}_0 \oplus_{s=1}^k \mathrm{ev}_{z_s} : \mathfrak{g}[t]^\tau \to \mathfrak{g}_0 \oplus \mathfrak{g}^{\oplus k}$ is a surjective morphism of Lie algebras.

Given a representation V of $\mathfrak{g}$ and $z \neq 0$, we let $\mathrm{ev}_z^* V$ be the corresponding pull-back representation of $\mathfrak{g}[t]^\tau$; note that these representations are cyclic $\mathfrak{g}[t]^\tau$-modules. Using the recursive formulae for $P_{\alpha,r}$ it is not hard to see that the following hold in the module $\mathrm{ev}_z^* V_{\mathfrak{g}}(\lambda)$, $\lambda \in P^+$ and $\mathrm{ev}_0^* V_{\mathfrak{g}_0}(\mu)$, $\mu \in P_0^+$:

$$\mathfrak{n}^+[t]v_\lambda = 0 \quad P_{i,r}v_\lambda = \binom{\lambda(h_i)}{r}(-1)^r z^{a_j r} v_\lambda, \quad i \in I, \quad r \in \mathbb{N}$$

$$\mathfrak{n}^+[t]^\tau v_\mu = 0, \quad P_{i,r}v_\mu = 0, \quad i \in I, \quad r \in \mathbb{N}.$$

The preceding discussion together with Eq. (4.3) now proves the following result. The first part of the next proposition can also be deduced from [23, Proposition 4.9].

Proposition *Suppose that $\lambda_1, \dots, \lambda_k \in P^+$ and $\mu \in P_0^+$. Let $z_1, \dots, z_k$ be non-zero complex numbers such that $z_r^{a_j} \neq z_s^{a_j}$ for all $1 \le r \neq s \le k$. Then*

$$\mathrm{ev}_0^* V_{\mathfrak{g}_0}(\mu) \otimes \mathrm{ev}_{z_1}^* V_{\mathfrak{g}}(\lambda_1) \otimes \cdots \otimes \mathrm{ev}_{z_k}^* V_{\mathfrak{g}}(\lambda_k)$$

is an irreducible $\mathfrak{g}[t]^\tau$-module. Moreover,

$$\mathfrak{n}^+[t]^\tau(v_\mu \otimes v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_k}) = 0,$$
$$\left(P_{i,r} - \pi_{i,r}\right)(v_\mu \otimes v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_k}) = 0, \quad i \in I, \quad r \in \mathbb{Z}_+,$$

where

$$\sum_{r \in \mathbb{Z}_+} \pi_{i,r} u^r = \prod_{s=1}^k (1 - z_s^{a_j} u)^{\lambda_s(h_i)}, \quad i \in I.$$

□

Remark In particular, the modules constructed in the preceding proposition are modules of the form $V(\lambda, \mathbf{I})$ where $\lambda = \mu + \lambda_1 + \cdots + \lambda_k$. The converse statement is also true; this follows from [23, Theorem 5.5]. An independent proof can be deduced once we complete our study of $\mathbf{A}_\lambda$.

5 The algebra $\mathbf{A}_\lambda$ as a Stanley–Reisner ring

For the rest of this section we denote by $\mathrm{Jac}(\mathbf{A}_\lambda)$ the Jacobson radical of $\mathbf{A}_\lambda$, and use freely the fact that the Jacobson radical of a finitely-generated commutative algebra coincides with its nilradical.

5.1
The main result of this section is the following.

Theorem 1 *The algebra $\mathbf{A}_\lambda/\mathrm{Jac}(\mathbf{A}_\lambda)$ is isomorphic to the algebra $\tilde{\mathbf{A}}_\lambda$ which is the quotient of $\mathbf{U}(\mathfrak{h}[t]^\tau)$ by the ideal generated by the elements*

$$P_{i,s}, \quad i \in I(j), \quad s \ge \lambda(h_i) + 1, \tag{5.1}$$

and

$$P_{1,r_1}\cdots P_{n,r_n},\ \ \sum_{i=1}^{n}\mathbf{a}_i^{\vee}(\alpha_0)r_i > \lambda(h_0). \tag{5.2}$$

Moreover, $\mathrm{Jac}(\mathbf{A}_\lambda)$ *is generated by the images of the elements in* (5.2) *and* $\mathrm{Jac}(\mathbf{A}_\lambda) = 0$ *if* $\mathbf{a}_j^{\vee}(\alpha_0) = 1$.

Example In the case of (B_n, D_n) we have $h_0 = h_{n-1} + h_n$ and so $\mathbf{a}_j^{\vee}(\alpha_0) = 1$. Thus, $\mathrm{Jac}(\mathbf{A}_\lambda) = 0$ and (5.2) becomes

$$P_{n-1,r_{n-1}}P_{n,r_n},\ r_{n-1}+r_n > \lambda(h_0).$$

The proof of Theorem 1 can be found in Sect. 5.6 through Sect. 5.10, For now we discuss interesting consequences of the theorem.

5.2

We recall the definition of a Stanley–Reisner ring, and the correspondence between Stanley–Reisner rings and abstract simplicial complexes (for more details, see [14]).

Let $X = \{x_1, \ldots, x_k\}$ be a set of indeterminates. A monomial $m = x_{i_1}\cdots x_{i_\ell}$ is said to be squarefree if $i_1 < \cdots < i_\ell$. An ideal of $\mathbb{C}[x_1, \ldots, x_k]$ is called a squarefree monomial ideal if it is generated by squarefree monomials. A quotient of a polynomial ring by a squarefree monomial ideal is called a Stanley–Reisner ring.

An abstract simplicial complex Σ on the set X is a collection of subsets of X such that if $A \in \Sigma$ and if $B \subset A$, then $B \in \Sigma$. There is a well known correspondence between abstract simplicial complexes, and ideals in $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_k]$ generated by squarefree monomials which is given as follows: if Σ is an abstract simplicial complex, let $\mathbf{J}_\Sigma \subset \mathbb{C}[X]$ be the ideal generated by the elements of the set

$$\{x_{i_1}\cdots x_{i_r} \mid 1 \le r \le k, \{x_{i_1}, \ldots, x_{i_r}\} \notin \Sigma\}.$$

The following proposition can be found in [14, Section 2.3].

Proposition *Given any abstract simplicial complex* Σ *on* X *the ideal* $\mathbf{J}_\Sigma \subset \mathbb{C}[X]$ *is a squarefree monomial ideal and hence the ring* $\mathbb{C}[X]/\mathbf{J}_\Sigma$ *is a Stanley–Reisner ring. Conversely, any squarefree monomial ideal* $I \subset \mathbb{C}[X]$ *is of the form* $I = \mathbf{J}_\Sigma$ *for some abstract simplicial complex* Σ *on* X. *This correspondence defines a bijection.* □

5.3

If $A \in \Sigma$, we call A a simplex, and a simplex of Σ not properly contained in another simplex of Σ is called a facet. Let $\mathcal{F}(\Sigma)$ denote the set of facets of Σ. For sets $B \subset A$, we have the Boolean interval $[B, A] = \{C : B \subset C \subset A\}$ and let $\bar{A} = [\emptyset, A]$. The dimension of Σ is the largest of the dimension of its simplexes, i.e.

$$\dim\Sigma = \max\{|A| : A \in \Sigma\} - 1.$$

The simplicial complex Σ is said to be pure if all elements of $\mathcal{F}(\Sigma)$ have the same cardinality. An enumeration $F_0, F_1, \ldots, F_p$ of $\mathcal{F}(\Sigma)$ is called a shelling if for all $1 \le r \le p$ the subcomplex

$$\left(\bigcup_{i=0}^{r-1}\bar{F}_i\right)\cap\bar{F}_r$$

is a pure abstract simplicial complex and $(\dim F_r - 1)$-dimensional.

The following can be found in [14, Theorem 5.5].

Proposition *If Σ is pure and shellable, then the Stanley–Reisner ring of Σ is Cohen–Macaulay.* □

5.4

We now prove the following consequence of Theorem 1.

Proposition *The algebra $\mathbf{A}_\lambda/\mathrm{Jac}(\mathbf{A}_\lambda)$ is a Stanley–Reisner ring with Hilbert series*

$$\mathbb{H}(\mathbf{A}_\lambda/\mathrm{Jac}(\mathbf{A}_\lambda)) = \sum_{\sigma\in\Sigma_\lambda}\prod_{P_{i,r}\in\sigma}\frac{t^{a_j r}}{1-t^{a_j r}},$$

where Σ_λ denotes the abstract simplicial complex determined uniquely by the squarefree monomial ideal generated by (5.1) and (5.2). Moreover, if $\mathbf{a}_j^\vee(\alpha_0) = 1$, the Krull dimension of $\mathbf{A}_\lambda$ is given by

$$d_\lambda = \lambda(h_0) + \sum_{i:\mathbf{a}_i(\alpha_0)=0}\lambda(h_i).$$

If in addition we have $|\{i : \mathbf{a}_i(\alpha_0) > 0\}| = 2$, then the algebra $\mathbf{A}_\lambda$ is Koszul and Cohen–Macaulay.

Example In the case (B_n, D_n), we have since $\alpha_0 = \alpha_{n-1} + 2\alpha_n$ and $h_0 = h_{n-1} + h_n$ that $\mathbf{A}_\lambda$ is Koszul and Cohen–Macaulay.

Proof It is immediate from Theorem 1 that $\mathbf{A}_\lambda/\mathrm{Jac}(\mathbf{A}_\lambda)$ is a Stanley–Reisner ring. The formula for the Hilbert series as well as the result on the Krull dimension are immediate consequences of [14, Section 2.3]. Finally, suppose that $\mathbf{a}_j^\vee(\alpha_0) = 1$ and $\{i : \mathbf{a}_i(\alpha_0) > 0\} = \{s, j\}$. Then $\mathbf{A}_\lambda$ is a quotient of a polynomial algebra by a quadratic monomial ideal, and hence Koszul (see [15]). The fact that it is Cohen–Macaulay follows from Proposition 5.3 if we prove that the simplicial complex Σ_λ is pure and that $\{F_0, \dots, F_{\min\{\lambda(h_0),\lambda(h_s)\}}\}$ defines a shelling, where

$$F_r = \bigcup_{\substack{i:\mathbf{a}_i(\alpha_0)=0\\ 1\le r_i\le\lambda(h_i)}}\{P_{i,r_i}\}\cup\{P_{j,1},\dots,P_{j,\lambda(h_0)-r}, P_{s,1},\dots,P_{s,r}\},\quad 0\le r\le\min\{\lambda(h_0),\lambda(h_s)\}.$$

For this, let F a facet of Σ_λ, i.e., F is not contained properly in another simplex of Σ_λ. It is clear that the cardinality of F is less or equal to d_λ. If it is strictly less, then $\{P_{i,r}\}\cup F$ is a face of Σ_λ for some i and r, which is a contradiction. Hence all facets have the same cardinality. The shelling property is straightforward to check. □

5.5

In this section, we note another interesting consequence of Theorem 1.

Proposition *Let $\lambda \in P_0^+$. Then $\mathbf{A}_\lambda/\mathrm{Jac}(\mathbf{A}_\lambda)$ is either infinite-dimensional or isomorphic to $\mathbb{C}$. Moreover, the latter is true iff the following two conditions hold:*

(i) *for $i \in I(j)$, we have $\lambda(h_i) > 0$ only if $\mathbf{a}_i^\vee(\alpha_0) > 0$,*
(ii) *$\lambda(h_0) < \mathbf{a}_i^\vee(\alpha_0)$ if $i = j$ or if $i \in I(j)$ and $\lambda(h_i) > 0$.*

Proof Suppose that λ satisfies the conditions in (i) and (ii). To prove that $\dim \mathbf{A}_\lambda/\mathrm{Jac}(\mathbf{A}_\lambda) = 1$ it suffices to prove that the elements $P_{i,s} \in \mathrm{Jac}(\mathbf{A}_\lambda)$ for all $i \in I$ and $s \ge 1$. Assume first that $i \ne j$. If $\lambda(h_i) = 0$ then Eq. (4.6) gives $P_{i,s}w_\lambda = 0$ for all $s \ge 1$. If $\lambda(h_i) > 0$ then the conditions imply that $\lambda(h_0) < \mathbf{a}_i^\vee(\alpha_0)$ and hence Eq. (5.2) shows that $P_{i,s} \in \mathrm{Jac}(\mathbf{A}_\lambda)$ for all $s \geq 1$. If $i = j$ then again the result follows from (5.2) and condition (ii).

We now prove the converse direction. Suppose that (i) does not hold. Then, there exists $i \neq j$ with $\mathbf{a}_i(\alpha_0) = 0$ and $\lambda(h_i) > 0$. Equation (5.2) implies that the preimage of $\mathrm{Jac}(\mathbf{A}_\lambda)$ is contained in the ideal of $\mathbf{U}(\mathfrak{h}[t]^\tau)$ generated by the elements $\{P_{i,s} : i \in I, \mathbf{a}_i^\vee(\alpha_0) > 0\}$. Hence, using Lemma 4.5 we see that the image of the elements $\{P_{i,1}^r : r \in \mathbb{N}\}$ in $\mathbf{A}_\lambda/\mathrm{Jac}(\mathbf{A}_\lambda)$ must remain linearly independent showing that the algebra is infinite-dimensional.

Suppose that (ii) does not hold. Then either $\lambda(h_0) \geq \mathbf{a}_j^\vee(\alpha_0)$ or $\lambda(h_0) \geq \mathbf{a}_i^\vee(\alpha_0)$ for some $i \in I(j)$ with $\lambda(h_i) > 0$. In either case (5.2) and Lemma 4.5 show that the image of the set $\{P_{i,1}^r : r \in \mathbb{N}\}$ in $\mathbf{A}_\lambda/\mathrm{Jac}(\mathbf{A}_\lambda)$ must remain linearly independent showing that the algebra is infinite-dimensional. ◻

Corollary *The algebra $\mathbf{A}_\lambda$ is finite-dimensional iff it is a local ring. It follows that $W(\lambda)$ is finite-dimensional iff $\mathbf{A}_\lambda$ is a local ring.*

Proof If $\mathbf{A}_\lambda$ is finite-dimensional then so is $\mathbf{A}_\lambda/\mathrm{Jac}(\mathbf{A}_\lambda)$ and the corollary is immediate from the proposition. Conversely suppose that $\mathbf{A}_\lambda$ is a local ring. By the proposition and equation (4.6), we have

$$P_{i,s} w_\lambda = 0, \quad \text{if} \quad \mathbf{a}_i^\vee(\alpha_0) = 0, \quad s \in \mathbb{N}.$$

If $\mathbf{a}_i^\vee(\alpha_0) \neq 0$ we still have from (4.6) that $P_{i,s} w_\lambda = 0$ if s is sufficiently large. Otherwise, Eq. (5.2) shows that there exists $N \in \mathbb{Z}_+$ such that

$$P_{i,s}^N w_\lambda = 0, \quad \text{for all} \quad i \in I, \quad s \in \mathbb{N}.$$

This proves that $\mathbf{A}_\lambda$ is generated by finitely many nilpotent elements and since it is a commutative algebra it is finite-dimensional. The second statement of the corollary is now immediate from Proposition 4.7. ◻

5.6

We turn to the proof of Theorem 1. It follows from Eq. (4.6) that the elements in (5.1) map to zero in $\mathbf{A}_\lambda$. Until further notice, we shall prove results which are needed to show that the elements in (5.2) are in $\mathrm{Jac}(\mathbf{A}_\lambda)$.

Given $\alpha, \beta \in R$, with $\ell\alpha + \beta \in R$, let $c(\ell, \alpha, \beta) \in \mathbb{Z}\backslash\{0\}$ be such that

$$\mathrm{ad}_{x_\alpha}^\ell(x_\beta) = c(\ell, \alpha, \beta) x_{\ell\alpha+\beta}.$$

The following is trivially checked by induction.

Lemma *Let $\gamma \in \Delta$ and $\beta \in R^+\backslash\Delta$ be such that $\beta + \gamma \notin R$ and $(\beta, \gamma) > 0$. Given $m, n, s, p, q \in \mathbb{Z}_+$ we have*

$$\begin{aligned}&(x_\gamma^+ \otimes t^p)^{(s+d_\gamma q)} (x_{\beta-\gamma}^+ \otimes t^m)^{(s)} (x_\beta^- \otimes t^n)^{(q+s)} \\ &\quad = C (x_{s_\gamma(\beta)}^- \otimes t^{n+d_\gamma p})^{(q)} (x_\gamma^+ \otimes t^p)^{(s)} (x_\gamma^- \otimes t^{m+n})^{(s)} + X\end{aligned}$$

where $X \in \mathbf{U}(\mathfrak{g}[t]^\tau)\mathfrak{n}^+[t]^\tau$ and $(d_\gamma!)^q C = c(d_\gamma, \gamma, -\beta)^q c(1, \beta - \gamma, -\beta)^s$. ◻

It is immediate that under the hypothesis of the lemma we have for all $P \in \mathbf{U}(\mathfrak{h}[t]^\tau)$ that

$$\begin{aligned}&(x_\gamma^+ \otimes t^p)^{(s+d_\gamma q)} (x_{\beta-\gamma}^+ \otimes t^m)^{(s)} (x_\beta^- \otimes t^n)^{(q+s)} P w_\lambda \\ &\quad = C (x_{s_\gamma(\beta)}^- \otimes t^{n+d_\gamma p})^{(q)} (x_\gamma^+ \otimes t^p)^{(s)} (x_\gamma^- \otimes t^{m+n})^{(s)} P w_\lambda,\end{aligned} \tag{5.3}$$

for some $C \neq 0$.

5.7
Recall that given any root $\beta \in R^+$ we can choose $\alpha \in \Delta$ with $(\beta, \alpha) > 0$. Moreover if $\beta \notin \Delta$ and β is long then $\beta + \alpha \notin R$. Setting $\alpha_{i_0} = \alpha_j$, $\beta_0 = \alpha_0$, we set $\beta_1 = s_{i_0}\beta_0$ and note that $\beta_1 \in R^+$. If $\beta_1 \notin \Delta$ then we choose $\alpha_{i_1} \in \Delta$ with $(\beta_1, \alpha_{i_1}) > 0$ and set $\beta_2 = s_{i_1}\beta_1$. Repeating this if necessary we reach a stage when $k \geq 1$ and $\beta_k \in \Delta$. In this case we set $\alpha_{i_k} = \beta_k$. We claim that

$$|\{0 \leq r \leq k : i_r = i\}| = \mathbf{a}_i^\vee(\alpha_0), \quad 1 \leq i \leq n. \tag{5.4}$$

To see this, notice that since the β_p are long roots, we have $h_{\beta_p} = h_{\beta_{p-1}} - h_{i_{p-1}}$. Hence,

$$h_0 = \sum_{s=0}^{k} h_{i_s} = \sum_{i=1}^{n} \mathbf{a}_i^\vee(\alpha_0) h_i.$$

Equating coefficients gives (5.4).

5.8
Retain the notation of Sect. 5.7. We now prove that

$$P_{i_k,s_k} \cdots P_{i_0,s_0} w_\lambda = 0, \text{ if } (s_0 + \cdots + s_k) \geq \lambda(h_0) + 1. \tag{5.5}$$

We begin with the equality

$$w = (x_0^- \otimes 1)^{(s_0+\cdots+s_k)} w_\lambda = 0, \quad (s_0 + \cdots + s_k) \geq \lambda(h_0) + 1,$$

which is a defining relation for $W(\lambda)$. Recalling that $j = i_0$ and setting

$$X_1 = (x_j^+ \otimes t)^{(s_0 + d_{\alpha_j}(s_1+\cdots+s_k))} (x_{\alpha_0-\alpha_j}^+ \otimes t^{a_j-1})^{(s_0)}$$

we get by applying (5.3)

$$0 = X_1 w = (x_{\beta_1}^- \otimes t^{d_{\alpha_j}})^{(s_1+\cdots+s_k)} P_{i_0,s_0} w_\lambda.$$

More generally, if we set

$$X_{r+1} = (x_{\alpha_{i_r}}^+ \otimes t^{\delta_{i_r,j}})^{(s_r + d_{\alpha_{i_r}}(s_{r+1}+\cdots+r_k))} (x_{\beta_r - \alpha_{i_r}}^+ \otimes t^{m_r})^{(s_r)},$$

where $m_r = a_j - \delta_{i_r,j} - d_{\alpha_j} |\{0 \leq q < r \mid i_q = j\}|$ we find after repeatedly applying (5.3) that

$$0 = (x_{\beta_k}^+ \otimes t^{\delta_{i_k,j}})^{(s_k)} X_k \cdots X_1 w = P_{i_k,s_k} \cdots P_{i_0,s_0} w_\lambda = 0.$$

This proves the assertion.

5.9
We can now prove that

$$P_{1,r_1} \cdots P_{n,r_n} \in \mathrm{Jac}(\mathbf{A}_\lambda) \text{ if } \sum_{i=1}^{n} \mathbf{a}_i^\vee(\alpha_0) r_i > \lambda(h_0).$$

Taking $s_p = r_m$ whenever $i_p = m$ in (5.5) and using (5.4) we see that

$$P_{1,r_1}^{\mathbf{a}_1^\vee(\alpha_0)} \cdots P_{n,r_n}^{\mathbf{a}_n^\vee(\alpha_0)} w_\lambda = 0 \text{ if } \sum_{i=1}^{n} \mathbf{a}_i^\vee(\alpha_0) r_i > \lambda(h_0). \tag{5.6}$$

Multiplying through by appropriate powers of P_{i,r_i}, $1 \le i \le n$ we get that for some $s \ge 0$ we have

$$P^s_{1,r_1} \cdots P^s_{n,r_n} w_\lambda = 0, \quad \text{if} \quad \sum_{i=1}^{n} \mathbf{a}_i^\vee(\alpha_0) r_i > \lambda(h_0).$$

Hence $P^s_{1,r_1} \cdots P^s_{n,r_n} = 0$ in $\mathbf{A}_\lambda$ proving that $P_{1,r_1} \cdots P_{n,r_n} \in \operatorname{Jac}(\mathbf{A}_\lambda)$. This argument proves that there exists a well-defined morphism of algebras

$$\varphi : \tilde{\mathbf{A}}_\lambda \twoheadrightarrow \mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda). \tag{5.7}$$

Lemma *If* $\mathbf{a}_j^\vee(\alpha_0) = 1$ *the map* φ *factors through* $\mathbf{A}_\lambda$, *i.e., we have a commutative diagram*

$$\begin{tikzcd}
\tilde{\mathbf{A}}_\lambda \arrow[r, two heads] \arrow[dr, two heads] & \mathbf{A}_\lambda/\operatorname{Jac}(\mathbf{A}_\lambda) \\
& \mathbf{A}_\lambda \arrow[u, two heads]
\end{tikzcd}$$

Proof Using (5.6) it suffices to prove that if $\mathbf{a}_j^\vee(\alpha_0) = 1$ then

$$\mathbf{a}_i^\vee(\alpha_0) \le 1 \quad \forall i \in I.$$

Since $\mathbf{a}_j(\alpha_0) = a_j \ge 2 > \mathbf{a}_j^\vee(\alpha_0) = 1$ we see that $\mathfrak{g}$ cannot be of simply laced type and hence α_j is short. It follows that $s_{\alpha_0}\alpha_j = \alpha_j - \alpha_0$ is also short and so $h_{\alpha_0 - \alpha_j} = d_j h_0 - h_j$. If $\mathbf{a}_i^\vee(\alpha_0) > 1$ for some $i \ne j$, then we would have

$$\mathbf{a}_i^\vee(\alpha_0 - \alpha_j) = d_j \mathbf{a}_i^\vee(\alpha_0) \ge 2d_j.$$

Since α_j is short this is impossible unless $\mathfrak{g}$ is of type F_4 and $j = 4$. This case can be handled by an inspection. $\square$

5.10

Using Lemma 5.9 and (5.7) we see that the proof of Theorem 1 is complete if we show that the map (5.7) is injective. Since $\tilde{\mathbf{A}}_\lambda$ is a quotient of $\mathbf{U}(\mathfrak{h}[t]^\tau)$ by a squarefree ideal, it has no nilpotent elements and thus $\operatorname{Jac}(\tilde{\mathbf{A}}_\lambda) = 0$. So if f is a nonzero element in $\tilde{\mathbf{A}}_\lambda$, there exists a maximal ideal $\tilde{\mathbf{I}}_f$ of $\tilde{\mathbf{A}}_\lambda$ so that $f \notin \tilde{\mathbf{I}}_f$. Therefore, by Lemma 4.5 we can choose a tuple $(\pi_{i,r})$, $i \in I$, $r \in \mathbb{N}$ satisfying the relations (5.1) and (5.2) such that under the evaluation map sending $P_{i,r}$ to $\pi_{i,r}$ the element f is mapped to a non-zero scalar. Define $z_1, \dots, z_k$ and $\lambda_1, \dots, \lambda_k \in P^+$ by

$$\pi_i(u) = 1 + \sum_{r \in \mathbb{N}} \pi_{i,r} u^r = \prod_{s=1}^{k} (1 - z_s^{a_j} u)^{\lambda_s(h_i)}, \quad i \in I$$

and set $\mu = \lambda - (\lambda_1 + \cdots + \lambda_k) \in P_0$. In what follows we show that $\mu \in P_0^+$. Since $(\pi_{i,r})$ satisfies the relations in (5.1) we have that $\mu(h_i) \in \mathbb{Z}_+$ for $i \in I(j)$. Moreover, since $(\pi_{i,r})$ satisfies (5.2) we get $\mu(h_0) \in \mathbb{Z}_+$. To see this, note that the coefficient of u^r in $\prod_{i \in I} \pi_i(u)^{\mathbf{a}_i^\vee(\alpha_0)}$ is given by

$$\sum_{(r_{i_k})} \prod_{i \in I} \prod_{k=1}^{\mathbf{a}_i^\vee(\alpha_0)} \pi_{i, r_{i_k}}, \tag{5.8}$$

where the sum runs over all tuples (r_{i_k}) such that $\sum_{i\in I}\sum_{k=1}^{\mathbf{a}_i^\vee(\alpha_0)} r_{i_k} = r$. Set $r_i = \max\{r_{i_k},\ 1 \le k \le \mathbf{a}_i^\vee(\alpha_0)\}$, $i \in I$ and observe that if $r > \lambda(h_0)$, then

$$\sum_{i\in I} \mathbf{a}_i^\vee(\alpha_0) r_i \ge r > \lambda(h_0)$$

and hence (5.8) vanishes. It follows that

$$\mu(h_0) = \lambda(h_0) - \deg\left(\prod_{i\in I} \pi_i(u)^{\mathbf{a}_i^\vee(\alpha_0)}\right) \in \mathbb{Z}_+.$$

Now using Proposition 4.9 we have a quotient of $W(\lambda)$ where f acts by a non-zero scalar on the highest weight vector. Hence $f^N \notin \mathrm{Ann}_{\mathfrak{h}[t]^\tau}(w_\lambda)$ for all $N \ge 1$, i.e. the image of f under the map (5.7) is non-zero. This proves the map (5.7) is injective, and so Theorem 1 is established.

6 Irreducible global Weyl modules

In this section we give necessary and sufficient conditions for a global Weyl module to be irreducible.

6.1
Recall from (4.7) that $\mathrm{ev}_0^* V_{\mathfrak{g}_0}(\lambda) \cong V(\lambda, \mathbf{I}_0)$ is a quotient of $W(\lambda)$ for all $\lambda \in P_0^+$.

Proposition *Let $\lambda \in P_0^+$ and $\iota : V_{\mathfrak{g}_0}(\lambda) \hookrightarrow W(\lambda)$ be the inclusion of $\mathfrak{g}_0$-modules with $\iota(v_\lambda) = w_\lambda$. Define $\Phi : \mathfrak{g}_1 \otimes V_{\mathfrak{g}_0}(\lambda) \to W(\lambda)$ by*

$$\Phi(x \otimes v) = (x \otimes t)\iota(v), \quad x \in \mathfrak{g}_1, \quad v \in V_{\mathfrak{g}_0}(\lambda).$$

The following are equivalent.

(a) *The module $W(\lambda)$ is irreducible.*
(b) *The canonical map of $\mathfrak{g}[t]^\tau$-modules $W(\lambda) \to V(\lambda, \mathbf{I}_0) \to 0$ is an isomorphism.*
(c) $(\mathfrak{g}_1 \otimes t)w_\lambda = 0$.
(d) $\Phi = 0$.
(e) *For all $\mu \in P_0^+$ with $\lambda - \mu \in Q^+$ we have* $\mathrm{Hom}_{\mathfrak{g}_0}(\mathfrak{g}_1 \otimes V_{\mathfrak{g}_0}(\lambda), V_{\mathfrak{g}_0}(\mu)) = 0$.

Proof It is clear from the remark preceding the proposition that (a) implies (b) and that (b) implies (c). We now prove that (c) implies (a). Using Proposition 2.5 (iii), we see that

$$\mathfrak{g}_k \otimes t^k = [\mathfrak{g}_{k-1} \otimes t^{k-1}, \mathfrak{g}_1 \otimes t], \quad 1 < k < a_j.$$

An obvious induction now proves that $(\mathfrak{g}_k \otimes t^k)w_\lambda = 0$ for all $1 \le k < a_j$. Next using Proposition 2.5(i) we get

$$(\mathfrak{g}_0 \otimes t^{a_j})w_\lambda = [\mathfrak{g}_1 \otimes t, \mathfrak{g}_{a_j-1} \otimes t^{a_j-1}]w_\lambda = 0.$$

Since $[\mathfrak{g}_0, \mathfrak{g}_k] = \mathfrak{g}_k$ for all $0 \le k \le a_j - 1$ a similar argument gives

$$(\mathfrak{g}_k \otimes t^{k+\delta_{k,0}a_j}\mathbb{C}[t^{a_j}])w_\lambda = 0.$$

It is immediate from the PBW theorem that $W(\lambda) = \mathbf{U}(\mathfrak{g}_0)w_\lambda$ and hence irreducible as a $\mathfrak{g}_0$-module and so, also as a $\mathfrak{g}[t]^\tau$-module.

If $\Phi = 0$ it is clear that $(\mathfrak{g}_1 \otimes t)w_\lambda = 0$. A simple checking shows that Φ is a map of $\mathfrak{g}_0$-modules. Since $\mathfrak{g}_1 = [\mathfrak{g}_0, \mathfrak{g}_1]$ it is trivially seen that $(\mathfrak{g}_1 \otimes t)w_\lambda = 0$ implies $\Phi = 0$. This proves that (c) and (d) are equivalent.

Finally we prove that (a) and (e) are equivalent. Suppose that $W(\lambda)$ is reducible in which case we have by the equivalence of (a) and (d) that $\Phi \neq 0$. Since $\mathfrak{h} \subset \mathfrak{g}_0$ we have $0 \notin \operatorname{wt} \mathfrak{g}_1$. It follows that the image of Φ does not contain an element of $W(\lambda)_\lambda$. Hence there exists $\mu \in P_0^+$ with $\lambda - \mu \in Q^+ \setminus \{0\}$ such that $\operatorname{Im}\Phi$ has a non-zero projection onto $V_{\mathfrak{g}_0}(\mu)$ which proves the forward direction. For the converse assume that $\Psi : \mathfrak{g}_1 \otimes V_{\mathfrak{g}_0}(\lambda) \to V_{\mathfrak{g}_0}(\mu)$ is a non-zero map of $\mathfrak{g}_0$-modules. Set $V = V_{\mathfrak{g}_0}(\lambda) \oplus V_{\mathfrak{g}_0}(\mu)$. The following formulae defines a $\mathfrak{g}[t]^\tau$–structure which extends the canonical $\mathfrak{g}_0$-structure:

$$(x \otimes 1)(v_1, v_2) = (xv_1, xv_2),$$
$$(y \otimes t)(v_1, v_2) = (0, \Phi(y \otimes v_1)), \quad \mathfrak{g}[t]^\tau[s](v_1, v_2) = 0, \quad s \geq 2,$$

where $(v_1, v_2) \in V$, $x \in \mathfrak{g}_0$ and $y \in \mathfrak{g}_1$. Since $\lambda - \mu \in Q^+$ it is trivially seen that V is a quotient of the global Weyl module $W(\lambda)$. □

Corollary *Suppose that $\lambda \in P_0^+$ is such that $W(\lambda)$ is reducible. Then $W(\lambda + \nu)$ is reducible for all $\nu \in P_0^+$.*

Proof A standard argument shows that we have a map of $\mathfrak{g}[t]^\tau$-modules $W(\lambda + \nu) \to W(\lambda) \otimes W(\nu)$ which sends $w_{\lambda+\nu} \to w_\lambda \otimes w_\nu$. If $W(\lambda + \nu)$ is irreducible then by part (c) of Proposition 6.1 we would have

$$(x_\alpha^- \otimes t)(w_\lambda \otimes w_\nu) = 0, \quad \alpha \in R_1^+.$$

Since this implies that $(x_\alpha^- \otimes t)w_\lambda = 0$ we would get $(\mathfrak{g}_1 \otimes t)w_\lambda = 0$. Then Proposition 6.1 implies that $W(\lambda)$ is irreducible which is a contradiction. □

6.2

Proposition *The global Weyl module is infinite dimensional if and only if* $\dim \mathbf{A}_\lambda = \infty$ *and in this case $W(\lambda)$ is reducible.*

Proof By Proposition 4.7 and (4.2) we know that $\dim W(\lambda) = \infty$ if and only if $\dim \mathbf{A}_\lambda = \infty$. Corollary 5 shows that in this case $\mathbf{A}_\lambda$ is not a local ring. Hence, there exists a maximal ideal $\mathbf{I}_1 \neq \mathbf{I}_0$ which means $W(\lambda)$ has two non-isomorphic irreducible quotients $V(\lambda, \mathbf{I}_0)$ and $V(\lambda, \mathbf{I}_1)$. This proves the proposition. □

Corollary *Suppose that $\lambda(h_i) > 0$ for some $i \in I(j)$. Then $W(\lambda)$ is a reducible $\mathfrak{g}[t]^\tau$-module if $\lambda(h_0) \geq \mathbf{a}_i^\vee(\alpha_0)$.*

Proof By Proposition 6.2 it suffices to prove that $\mathbf{A}_\lambda/\operatorname{Jac}(\mathbf{A}_\lambda)$ is infinite dimensional. If $\mathbf{a}_i^\vee(\alpha_0) = 0$ then condition (i) of Proposition 5.5 is not satisfied and so $\mathbf{A}_\lambda/\operatorname{Jac}(\mathbf{A}_\lambda)$ is infinite dimensional. If $\mathbf{a}_i^\vee(\alpha_0) > 0$ then condition (ii) is violated and we again see that $\mathbf{A}_\lambda/\operatorname{Jac}(\mathbf{A}_\lambda)$ is infinite dimensional. □

6.3

The following remarks will be useful in what follows. Suppose that $\beta \in R_0$ is such that $\mathbf{a}_j(\beta) = a_j$. If $\beta \neq \alpha_0$ then $\mathbf{a}_i(\beta) > 0$ for some $i \in I(j)$. Recall the elements $\theta_k \in R_k^+$ defined in Sect. 2.5. These can be characterized as follows: if $\alpha \in R_k$ and $\alpha \neq \theta_k$ then there exists $i \in I(j)$ such that $\mathbf{a}_i(\theta_k) > \mathbf{a}_i(\alpha)$. Further the element $-\theta_k \in R_{a_j-k}$ and if $\alpha \in R_{a_j-k}$

with $\alpha \neq -\theta_k$ then there exists $i \in I(j)$ with $\mathbf{a}_i(\alpha) > \mathbf{a}_i(-\theta_k)$. In particular $-\theta_{a_j-k}$ is the lowest weight of $\mathfrak{g}_k$ regarded as a $\mathfrak{g}_0$-module. A straightforward inspection now shows that the pair $(\alpha_0, \theta_{a_j-1})$ are given when $\mathfrak{g}$ is of classical type as follows.

If $\mathfrak{g}$ is of type B_n, then

$$\alpha_0 = \alpha_{j-1} + 2\sum_{p=j}^{n} \alpha_j, \quad \theta_1 = \alpha_1 + \cdots + \alpha_{j-1} + (\alpha_0 - \alpha_j).$$

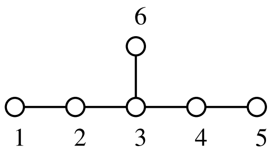
If $\mathfrak{g}$ is of type C_n, then

$$\alpha_0 = 2\left(\sum_{p=j}^{n-1} \alpha_p\right) + \alpha_n, \quad \theta_1 = \alpha_1 + \cdots + \alpha_{j-1} + (\alpha_0 - \alpha_j).$$

If $\mathfrak{g}$ is of type D_n, then

$$\alpha_0 = \alpha_{j-1} + 2\left(\sum_{p=j}^{n-2} \alpha_p\right) + \alpha_{n-1} + \alpha_n, \quad \theta_1 = \alpha_1 + \cdots + \alpha_{j-1} + (\alpha_0 - \alpha_j).$$

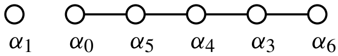
For the exceptional algebras we shall illustrate the examples and proofs that follow only in the case of E_6 and the following labeling of the Dynkin diagram.



Then $\theta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$. If $j = 2$ we have

$$\alpha_0 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6, \quad \theta_1 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$$

and the Dynkin diagram of $\mathfrak{g}_0$ is given by



If $j = 3$ then

$$\alpha_0 = \theta - \alpha_6, \quad \theta_2 = \alpha_0 - \alpha_3, \quad i.e., \quad \mathbf{a}_i(\alpha_0) = \mathbf{a}_i(\theta_2), \quad i \in I(3).$$

The case $j = 4$ is obtained from $j = 2$ by applying the non–trivial diagram automorphism of E_6.

6.4

Lemma *Let $i \in I(j) \cup \{0\}$. Then $W(\lambda_i)$ is reducible if*

(i) $i \in I(j)$ *and* $\mathbf{a}_i(\alpha_0) = 0$,
(ii) $i = 0$ *or* $i \in I(j)$ *with* $\mathbf{a}_i(\theta_{a_j-1}) \neq \mathbf{a}_i(\alpha_0)$.

Proof Part (i) is immediate from Corollary 6.2. Recall the element $w_\circ$ defined in Sect. 2.4. To prove that $W(\lambda_0)$ is reducible, it suffices by Proposition 6.1 to show that

$$w_\circ\theta_{a_j-1} \in R_1^+, \quad \mu_0 = \lambda_0 - w_\circ\theta_{a_j-1} \in P_0^+, \quad \mathrm{Hom}_{\mathfrak{g}_0}(\mathfrak{g}_1 \otimes V_{\mathfrak{g}_0}(\lambda_0), V_{\mathfrak{g}_0}(\mu_0)) \neq 0.$$

The first assertion is clear since $w_\circ\alpha_j \in R^+$. If $i \in I(j)$ then $-w_\circ^{-1}\alpha_i \in R^+$ and hence $-w_\circ^{-1}(h_i)$ is in the $\mathbb{Z}_+$-span of $h_i, i \in I(j)$. It follows that

$$\mu_0(h_i) = -w_\circ\theta_{a_j-1}(h_i) = \theta_{a_j-1}(-w_\circ^{-1}(h_i)) \geq 0, \quad i \in I(j).$$

Moreover since α_0 is a long root and $w_0^{-1}(\alpha_0) \in R^+$ we also have $w_\circ^{-1}\theta_{a_j-1}(h_0) \leq 1$. It follows again that $\mu_0(h_0) \geq 0$ and the second assertion is proved. The last assertion is a consequence of the PRV theorem (see [19, Theorem 2.10] and [21, Corollary 3]).

If $i \in I(j)$ with $\mathbf{a}_i(\theta_{a_j-1}) \neq \mathbf{a}_i(\alpha_0) > 0$ then Sect. 6.3 shows that $\mathfrak{g}$ must be of exceptional type. A case by case inspection shows that we can always find $\mu \in P_0^+$ violating the condition in Proposition 6.1(e) showing that $W(\lambda_i)$ is reducible. As an example suppose that we are in the case of E_6. If $j = 2$ we have to prove that $W(\lambda_4)$ and $W(\lambda_5)$ are reducible. For this, we note that

$$V_{\mathfrak{g}_0}(\lambda_1 + \mu) \cong V_{\mathfrak{g}_0}(\lambda_1) \otimes V_{\mathfrak{g}_0}(\mu), \quad \mu \in P_0^+, \quad \mu(h_1) = 0.$$

Hence, it follows from the representation theory of A_5 that

$$\begin{aligned} &V_{\mathfrak{g}_0}(\lambda_1 + \lambda_4) \otimes V_{\mathfrak{g}_0}(\lambda_4) \\ &\quad\cong V_{\mathfrak{g}_0}(\lambda_1 + 2\lambda_4) \oplus V_{\mathfrak{g}_0}(\lambda_1 + \lambda_3 + \lambda_5) \oplus V_{\mathfrak{g}_0}(\lambda_0 + \lambda_1 + \lambda_6) \oplus V_{\mathfrak{g}_0}(\lambda_1) \end{aligned}$$

and

$$\begin{aligned} &V_{\mathfrak{g}_0}(\lambda_1 + \lambda_4) \otimes V_{\mathfrak{g}_0}(\lambda_5) \\ &\quad\cong V_{\mathfrak{g}_0}(\lambda_1 + \lambda_4 + \lambda_5) \oplus V_{\mathfrak{g}_0}(\lambda_0 + \lambda_1 + \lambda_3) \oplus V_{\mathfrak{g}_0}(\lambda_1 + \lambda_6). \end{aligned}$$

Setting $\mu_4 = \lambda_1 \in P_0^+$ and $\mu_5 = \lambda_1 + \lambda_6 \in P_0^+$ we have

$$\lambda_4 - \mu_4 = \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 \in Q^+, \quad \lambda_5 - \mu_5 = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \in Q^+.$$

This proves that the condition in Proposition 6.1(e) is violated and so the corresponding global Weyl modules are reducible. If $j = 3$ then we have already observed that $\mathbf{a}_i(\alpha_0) = \mathbf{a}_i(\theta_2)$ for $i \in I(3)$ and so there is nothing to prove.

6.5

Theorem 2 *Let $\lambda \in P_0^+$. Then $W(\lambda)$ is an irreducible $\mathfrak{g}[t]^\tau$-module iff the following holds:*

$$\{i \in I(j) \cup \{0\} : \lambda(h_i) > 0\} \subset \{i \in I(j) : \mathbf{a}_i(\alpha_0) = \mathbf{a}_i(\theta_{a_j-1})\}. \tag{6.1}$$

Proof Suppose that λ satisfies the conditions in (6.1). By Proposition 6.1 it suffices to prove that $(\mathfrak{g}_1 \otimes t)w_\lambda = 0$. Using the irreducibility of $\mathfrak{g}_1$ it suffices to prove that

$$(x^-_{\theta_{a_j-1}} \otimes t)w_\lambda = 0. \tag{6.2}$$

By (2.5) we can write

$$\theta_{a_j-1} - \alpha_0 + \alpha_j = \sum_{i \in I(j)} p_i\alpha_i, \quad p_i \in \mathbb{Z}_+, \quad i \in I(j).$$

Since $\lambda(h_0) = 0$ and $\lambda(h_i) = 0$ for all $i \in I(j)$ with $p_i > 0$ we have the defining relations

$$(x_0^- \otimes 1)w_\lambda = 0, \quad (x_i^- \otimes 1)w_\lambda = 0, \quad i \in I(j), \quad p_i > 0.$$

Hence

$$(x_j^+ \otimes t)(x_0^- \otimes 1)w_\lambda = A(x^-_{\alpha_0-\alpha_j} \otimes t)w_\lambda = 0,$$

for some $0 \neq A \in \mathbb{C}$. Equation (6.2) follows by noting that there exists $0 \neq B \in \mathbb{C}$ such that $x^-_{\theta_{a_j}-1} = B[x^-_{i_1}, \cdots [x^-_{i_{s-1}}, [x^-_{i_s}, x^-_{\alpha_0-\alpha_j}]] \cdots]$ where $i_1, \ldots, i_s$ are elements of the set $\{i \in I(j) : p_i > 0\}$.

For the converse suppose that $\lambda(h_0) \neq 0$ and let $\mu = \lambda - \lambda_0$. Since $W(\lambda_0)$ is reducible by Lemma 6 we use Corollary 6.1 to conclude that $W(\lambda)$ is reducible. The proof if $\lambda(h_i) > 0$ for some $i \in I(j)$ with $\mathbf{a}_i(\theta_{a_j-1}) \neq \mathbf{a}_i(\alpha_0)$ is identical.

Remark Using the formulae given in Sect. 6.3 for the classical Lie algebras we see that for any j with $a_j = 2$, the preceding theorem can be formulated as follows. The global Weyl module $W(\lambda)$ is irreducible iff $\lambda(h_0) = 0$ and $\lambda(h_i) = 0$ for all $i \in I(j)$ with $\mathbf{a}_i(\alpha_0) = 0$. Unfortunately this can be false for exceptional algebras. As we saw in the proof of Lemma 6 in the case $j = 2$ for E_6 we have $\mathbf{a}_4(\theta_1) > \mathbf{a}_4(\alpha_0) > 0$. An explicit computation does show however that even for the exceptional algebras for any j with $a_j \geq 2$ we always have irreducible global Weyl modules.

7 Structure of local Weyl modules

Recall from Sect. 3 that the equivariant map algebra $\mathfrak{g}[t]^\tau$ is not isomorphic to an equivariant map algebra where the group Γ acts freely on the set of maximal ideals of A. When Γ acts freely, the finite dimensional representation theory of the equivariant map algebra is closely related to that of the map algebra $\mathfrak{g} \otimes A$ (see for instance [11]). We have already seen a major difference between the finite dimensional representation theory of $\mathfrak{g}[t]^\tau$ and that of $\mathfrak{g}[t]$. Specifically, in Sect. 6 we showed that unlike in the case of the current algebra, the global Weyl module for $\mathfrak{g}[t]^\tau$ can be finite-dimensional and irreducible for nontrivial dominant integral weights.

In this section we discuss the structure of local Weyl modules for the case of (B_n, D_n) where λ is a multiple of a fundamental weight, in which case $\mathbf{A}_\lambda$ is a polynomial algebra. We finish the section by discussing the complications in determining the structure of local Weyl modules for an arbitrary weight $\lambda \in P_0^+$. Such complications already occur for $\omega_{n-1} = \lambda_0 + \lambda_{n-1}$ when $\mathbf{A}_{\omega_{n-1}}$ is not a polynomial algebra.

7.1

Recall that we have a well established theory of local Weyl modules for the current algebra $\mathfrak{g}[t]$. Given $\lambda \in P^+$ we denote by $W^{\mathfrak{g}}_{\text{loc}}(\lambda)$, $\lambda \in P^+$ the $\mathfrak{g}[t]$-module generated by an element w_λ and defining relations

$$\mathfrak{n}^+[t]w_\lambda = 0, \quad (h \otimes t^r)w_\lambda = \delta_{r,0}\lambda(h)w_\lambda = 0, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}w_\lambda = 0.$$

We remind the reader that $\{\omega_i : 1 \leq i \leq n\}$ is a set of fundamental weights for $\mathfrak{g}$ with respect to Δ. The following was proved in [12, Corollary 2] and [22, Corollary 9.5].

$$\dim W^{\mathfrak{g}}_{\text{loc}}(\lambda) = \prod_{i=1}^{n} \dim \left(W^{\mathfrak{g}}_{\text{loc}}(\omega_i)\right)^{m_i}, \quad \lambda = \sum_{i=1}^{n} m_i\omega_i \in P^+. \tag{7.1}$$

We can clearly regard $W^{\mathfrak{g}}_{\text{loc}}(\lambda)$, $\lambda \in P^+$ as a graded $\mathfrak{g}[t]^\tau$ module by restriction, however it is not the case that this restriction gives a local Weyl module for $\mathfrak{g}[t]^\tau$. The relationship between local Weyl modules for $\mathfrak{g}[t]^\tau$ and the restriction of local Weyl modules for $\mathfrak{g}[t]$ is more complicated, as we now explain.

7.2

Given $z \in \mathbb{C}^\times$ we have an isomorphism of Lie algebras $\eta_z : \mathfrak{g}[t] \to \mathfrak{g}[t]$ given by $(x \otimes t^r) \to (x \otimes (t+z)^r)$ and let $\eta_z^* V$ be the pull-back through this homomorphism of a representation V of $\mathfrak{g}[t]$. Suppose that V is such that there exists $N \in \mathbb{Z}_+$ with $(\mathfrak{g} \otimes t^m)V = 0$ for all $m \geq N$. Then $(\mathfrak{g} \otimes (t-z)^m)\eta_z^* V = 0$ for all $m \geq N$. In particular we can regard the module $\eta_z^* V$ as a module for the finite-dimensional Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t]/(t-z)^N$. Following [11, Proposition 2.2], since $z \in \mathbb{C}^\times$ we have

$$\mathfrak{g}[t]/\mathfrak{g} \otimes (t-z)^N \mathbb{C}[t] \cong \mathfrak{g}[t]^\tau / (\mathfrak{g} \otimes (t-z)^N \mathbb{C}[t])^\tau,$$

so if V is a cyclic module for $\mathfrak{g}[t]$ then $\eta_z^* V$ is a cyclic module for $\mathfrak{g}[t]^\tau$.

We now need a general construction. Given any finite-dimensional cyclic $\mathfrak{g}[t]^\tau$-module V with cyclic vector v define an increasing filtration of $\mathfrak{g}_0$-modules

$$0 \subset V_0 = \mathbf{U}(\mathfrak{g}[t]^\tau)[0]v \subset \cdots \subset V_r = \sum_{s=0}^{r} \mathbf{U}(\mathfrak{g}[t])^\tau[s]v \subset \cdots \subset V.$$

The associated graded space gr V is naturally a graded module for $\mathfrak{g}[t]^\tau$ via the action

$$(x \otimes t^s)\overline{w} = \overline{(x \otimes t^s)w}, \quad \overline{w} \in V_r/V_{r-1}.$$

Suppose that v satisfies the relations

$$\mathfrak{n}^+[t]^\tau v = 0, \quad (h \otimes t^{2k})v = d_k(h)v, \quad d_k(h) \in \mathbb{C}, \quad k \in \mathbb{Z}_+, \quad h \in \mathfrak{h}.$$

Then since $\dim V < \infty$ it follows that $d_0(h) \in \mathbb{Z}_+$; in particular there exists $\lambda \in P_0^+$ such that $d_0(h) = \lambda(h)$ and a simple checking shows that gr V is a quotient of $W_{\mathrm{loc}}(\lambda) := W(\lambda, \mathbf{I}_0)$.

The following is now immediate.

Lemma *Let $\lambda \in P^+$ and $z \in \mathbb{C}^\times$. The $\mathfrak{g}[t]^\tau$-module $\mathrm{gr}(\eta_z^* W_{\mathrm{loc}}^{\mathfrak{g}}(\lambda))$ is a quotient of $W_{\mathrm{loc}}(\lambda)$ and hence*

$$\dim W_{\mathrm{loc}}(\lambda) \geq \dim W_{\mathrm{loc}}^{\mathfrak{g}}(\lambda).$$

□

7.3

For the rest of this section, we consider the case of (B_n, D_n), and study local Weyl modules corresponding to weights $r\lambda_i \in P_0^+$, where $r \in \mathbb{Z}_+$, and $0 \leq i \leq n-2$ (the $i = n-1$ case is discussed in Sect. 6, where these local Weyl modules are shown to be finite-dimensional and irreducible). We remind the reader that $\lambda_0 = \omega_n$, $\lambda_i = \omega_i$, $1 \leq i \leq n-2$ and $\lambda_{n-1} = \omega_{n-1} - \omega_n$. In particular, we show the reverse of the inequality in Lemma 7.2, which proves the following proposition.

Proposition *Assume that $(\mathfrak{g}, \mathfrak{g}_0)$ if of type (B_n, D_n). For $0 \leq i \leq n-2$ and $r \in \mathbb{Z}_+$ we have an isomorphism of $\mathfrak{g}[t]^\tau$-modules*

$$W_{\mathrm{loc}}(r\lambda_i) \cong \mathrm{gr}(\eta_z^* W_{\mathrm{loc}}^{\mathfrak{g}}(r\lambda_i)).$$

7.4

The next proposition summarizes some results on local Weyl modules which are needed for our study. Part (i) was proved in [8, Lemma 6.4, Proposition 6.1]. Parts (ii), (iii) can be found in [7, Theorem 1], where we remind the reader that the fundamental Kirillov–Reshetikhin modules are the same as the local Weyl modules associated to a fundamental weight.

Proposition (i) *Let x, y, h be the standard basis for $\mathfrak{sl}_2$ and set $y \otimes t^r = y_r$, For $\lambda \in P^+$ the local Weyl module $W_{\rm loc}^{\mathfrak{sl}_2}(\lambda)$ has basis*

$$\{w_\lambda,\ \ y_{r_1} \cdots y_{r_k} w_\lambda : 1 \le k \le \lambda(h),\ \ 0 \le r_1 \le \cdots \le r_k \le \lambda(h) - k\}.$$

Moreover, $y_s w_\lambda = 0$ for all $s \ge \lambda(h)$.

(ii) *Assume that $\mathfrak g$ is of type B_n (resp. D_n) and assume that $i \neq n$ (resp. $i \neq n-1, n$). Then*

$$W_{\rm loc}^{\mathfrak g}(\omega_i) \cong_{\mathfrak g} V_{\mathfrak g}(\omega_i) \oplus V_{\mathfrak g}(\omega_{i-2}) \oplus \cdots \oplus V_{\mathfrak g}(\omega_{\bar i}),$$

where

$$V_{\mathfrak g}(\omega_{\bar i}) = V_{\mathfrak g}(\omega_1),\ \ i \text{ odd},\ \ V_{\mathfrak g}(\omega_{\bar i}) = \mathbb{C},\ \ i\ \text{ even}.$$

(iii) *Assume that $\mathfrak g$ is of type B_n (resp. D_n), and let $i = n$ (resp. $i \in \{n-1, n\}$). Then*

$$W_{\rm loc}^{\mathfrak g}(\omega_i) \cong_{\mathfrak g} V_{\mathfrak g}(\omega_i).$$

□

We remind the reader of the following elementary facts on the dimension of the spin representations for B_n and D_n,

$$\dim V_{\mathfrak g}(\omega_i) = \begin{cases} \binom{2n+1}{i}, & \mathfrak g = B_n \quad i \neq n, \\[2ex] \binom{2n}{i}, & \mathfrak g = D_n,\ i \neq n-1, n. \end{cases}$$

Moreover, if $\mathfrak g$ is of type B_n,

$$\dim V_{\mathfrak g}(\omega_n) = 2^n,$$

and if $\mathfrak g$ is of type D_n and $i \in \{n-1, n\}$, then

$$\dim V_{\mathfrak g}(\omega_i) = 2^{n-1}.$$

7.5

Our goal is to prove that

$$\dim W_{\rm loc}^{\mathfrak g}(r\lambda_i) \ge \dim W_{\rm loc}(r\lambda_i),\ \ r \in \mathbb{N}.$$

The proof needs several additional results, and we consider the cases $1 \le i \le n-2$ and $i = 0$ separately.

Recall that $\mathfrak g_0[t^2] \subset \mathfrak g[t]^\tau$, and so $W_{\rm loc}(r\lambda_i)$ can be regarded as a $\mathfrak g_0[t^2]$-module by pulling back along the inclusion map $\mathfrak g_0[t^2] \hookrightarrow \mathfrak g[t]^\tau$. For ease of notation we denote the element $w_{r\lambda_i}$ by w_r.

Lemma (i) *For $1 \le i \le n-2$, $W_{\rm loc}(r\lambda_i)$ is generated as a $\mathfrak g_0[t^2]$-module by w_r and $Y w_r$ where Y is a monomial in the elements*

$$(x_{p,n}^- \otimes t^{2s+1}) w_r,\quad p \le i,\quad 0 \le s < r.$$

(ii) *$W_{\rm loc}(r\lambda_0)$ is generated as a $\mathfrak g_0[t^2]$-module by w_r and $Y w_r$ where Y is a monomial in the elements*

$$(x_{p,n}^- \otimes t^{2s+1}) w_r,\quad p \le n,\quad 0 \le s < r.$$

Proof First, for $1 \le i \le n-2$ the defining relation $x_0^- w_r = 0$ implies that

$$(x_0^- \otimes t^{2s})w_r = (x_{n-1}^- \otimes t^{2s})w_r = (x_n^- \otimes t^{2s+1})w_r = 0, \quad s \ge 0.$$

Since $x_p^- w_r = 0$ if $p \ne i$ it follows that

$$(x_{p,n}^- \otimes t^{2s+1})w_r = 0, \quad s \ge 0, \quad p > i. \tag{7.2}$$

Observe also that

$$(x_i^-)^{r+1} w_r = 0 \implies (x_i^- \otimes t^{2s})w_r = 0, \quad s \ge r,$$

and hence we also have that

$$(x_{p,n}^- \otimes t^{2s+1})w_r = 0, \quad s \ge r, \quad p \le i.$$

A simple application of the PBW theorem now gives (i).

For the case $i = 0$, we have

$$(x_{k,p}^- \otimes t^{2s})w_r = 0, \quad 1 \le k \le p \le n-1, \ s \ge 0.$$

The relation $(x_0^-)^{s+1} w_r = 0$ for $s \ge r$ implies that

$$(x_0^- \otimes t^{2s})w_r = 0, \quad s \ge r$$

and so

$$(x_n^- \otimes t^{2s+1})w_r = 0, \quad s \ge r.$$

Hence

$$(x_{p,n}^- \otimes t^{2s+1})w_r = 0, \quad 1 \le p \le n, \quad s \ge r$$

and (ii) is now clear. □

7.6

Lemma (i) *For $1 \le i \le n-2$, suppose that $Y = (x_{p_1,n}^- \otimes t^{2s_1+1}) \cdots (x_{p_k,n}^- \otimes t^{2s_k+1})$ where $p_1 \le \cdots \le p_k \le i$. Then Yw_r is in the $\mathfrak{g}_0[t^2]$-module generated by elements Zw_r where Z is a monomial in the elements $(x_{i,n}^- \otimes t^{2s+1})$ with $s \in \mathbb{Z}_+$.*

(ii) *For $i = 0$, suppose that $Y = (x_{p_1,n}^- \otimes t^{2s_1+1}) \cdots (x_{p_k,n}^- \otimes t^{2s_k+1})$ where $p_1 \le \cdots \le p_k \le n$. Then Yw_r is in the $\mathfrak{g}_0[t^2]$-module generated by elements Zw_r where Z is a monomial in the elements $(x_n^- \otimes t^{2s+1})$ with $s \in \mathbb{Z}_+$.*

Proof First, let $1 \le i \le n-2$. The proof proceeds by an induction on k. If $k = 1$ and $p_1 < i$ then by setting

$$x_{p_1,n}^- = [x_{p_1,i-1}^-, x_{i,n}^-]$$

we have

$$x_{p_1,i-1}^- (x_{i,n}^- \otimes t^{2s_1+1})w_r = (x_{p_1,n}^- \otimes t^{2s_1+1})w_r,$$

hence induction begins. For the inductive step, we observe that

$$(x_{p_1,n}^- \otimes t^{2s_1+1})\mathbf{U}(\mathfrak{g}_0[t^2]) \subset \mathbf{U}(\mathfrak{g}_0[t^2]) \oplus \sum_{m \ge 0} \sum_{p=1}^{n} \mathbf{U}(\mathfrak{g}_0[t^2])(x_{p,n}^\pm \otimes t^{2m+1}),$$

and hence it suffices to prove that for all $1 \leq p \leq n$ and Z a monomial in $(x_{i,n}^- \otimes t^{2s+1})$ we have that $(x_{p,n}^{\pm} \otimes t^{2m+1})Zw_r$ is in the $\mathfrak{g}_0[t^2]$-submodule generated by elements $Z'w_r$ where Z' is a monomial in $(x_{i,n}^- \otimes t^{2s+1})$. Denote this submodule by M. We give the proof only for $(x_{p,n}^- \otimes t^{2m+1})Zw_r$, since the other case is proven similarly. If $p = i$, there is nothing to prove and if $p > i$ we get

$$(x_{p,n}^- \otimes t^{2m+1})Zw_r = X + Z(x_{p,n}^- \otimes t^{2m+1})w_r,$$

for some element $X \in M$. Since $(x_{p,n}^- \otimes t^{2m+1})w_r = 0$ by (7.2), we are done. If $p < i$, we consider

$$\begin{aligned}(x_{p,i-1}^- \otimes t^{2m})(x_{i,n}^- \otimes t)^{\ell+1}w_r &= A(x_{p,n}^- \otimes t^{2m+1})(x_{i,n}^- \otimes t)^{\ell}w_r \\ &\quad + B(x_{p,\bar{i}}^- \otimes t^{2m+2})(x_{i,n}^- \otimes t)^{\ell-1}w_r,\end{aligned}$$

for some non-zero constants A and B. Since

$$(x_{p,i-1}^- \otimes t^{2m})(x_{i,n}^- \otimes t)^{\ell+1}w_r \in M,$$

and

$$(x_{p,\bar{i}}^- \otimes t^{2m+2})(x_{i,n}^- \otimes t)^{\ell-1}w_r \in M,$$

we have,

$$(x_{p,n}^- \otimes t^{2m+1})(x_{i,n}^- \otimes t)^{\ell}w_r \in M.$$

In order to show

$$(x_{p,n}^- \otimes t^{2m+1})(x_{i,n}^- \otimes t^{2r_1+1}) \cdots (x_{i,n}^- \otimes t^{2r_\ell+1})w_r \in M$$

we let $h \in \mathfrak{h}$ with $[h, x_{p,n}^-] = 0$ and $[h, x_{i,n}^-] \neq 0$. Then

$$(h \otimes t^{2s})(x_{p,n}^- \otimes t^{2m+1})(x_{i,n}^- \otimes t) \cdots (x_{i,n}^- \otimes t)w_r \in M$$

for all $s \geq 0$. An induction on $|\{1 \leq s \leq \ell \ : \ r_s \neq 0\}|$ finishes the proof for $1 \leq i \leq n-2$. The $i = 0$ case is identical.

7.7

Observe that the Lie subalgebra $\mathfrak{a}[t^2]$ generated by the elements $x_i^{\pm} \otimes t^{2s}$, $s \in \mathbb{Z}_+$ is isomorphic to the current algebra $\mathfrak{sl}_2[t^2]$. Hence $\mathbf{U}(\mathfrak{a}[t^2])w_r \subset W_{\text{loc}}(r\lambda_i)$ is a quotient of the local Weyl module for $\mathfrak{a}[t^2]$ with highest weight r and we can use the results of Proposition 7.4(i).

Lemma (i) *For* $1 \leq i \leq n-2$, *as a* $\mathfrak{g}_0[t^2]$-*module* $W_{\text{loc}}(r\lambda_i)$ *is spanned by* w_r *and elements*

$$\begin{aligned}Y(i, \mathbf{s})w_r := (x_{i,n}^- \otimes t^{2s_1+1}) \cdots (x_{i,n}^- \otimes t^{2s_k+1})w_r,& \\ k \geq 1, \ \ \mathbf{s} \in \mathbb{Z}_+^k, \ \ 0 \leq s_1 \leq \cdots \leq s_k \leq r-k.&\end{aligned}$$

(ii) *For* $i = 0$, *as a* $\mathfrak{g}_0[t^2]$-*module* $W_{\text{loc}}(r\lambda_i)$ *is spanned by* w_r *and elements*

$$\begin{aligned}&Y(n, \mathbf{s})w_r := (x_n^- \otimes t^{2s_1+1}) \cdots (x_n^- \otimes t^{2s_k+1})w_r, \ \ k \geq 1, \\ &\mathbf{s} \in \mathbb{Z}_+^k, \ \ 0 \leq s_1 \leq \cdots \leq s_k \leq r-k.\end{aligned}$$

Proof First, we consider the case $1 \le i \le n-2$. By Lemma 7.5 and Lemma 7.6 we can suppose that Y is an arbitrary monomial in the elements $(x_{i,n}^- \otimes t^{2s+1})$, $s \in \mathbb{Z}_+$. We proceed by induction on the length k of Y. If $k=1$, then we have

$$(x_{i,n}^- \otimes t^{2s+1})w_r = (x_{i+1,n}^- \otimes t)(x_i^- \otimes t^{2s})w_r = 0, \quad s \ge r,$$

by Proposition 7.4(i). This shows that induction begins. Suppose now that k is arbitrary and $\mathbf{s} \in \mathbb{Z}_+^k$. Then, by induction on k

$$\begin{aligned}&(x_{i+1,n}^- \otimes t)^k (x_i^- \otimes t^{2s_1}) \cdots (x_i^- \otimes t^{2s_k})\\ &\quad = A(x_{i,n}^- \otimes t^{2s_1+1}) \cdots (x_{i,n}^- \otimes t^{2s_k+1}) + X + Z,\end{aligned} \tag{7.3}$$

where A is a non-zero complex number and $X \in \sum_{m<k} \sum_{\mathbf{p}\in\mathbb{Z}_+^m} \mathbf{U}(\mathfrak{g}_0[t^2])Y(i,\mathbf{p})$, and $Z \in \mathbf{U}(\mathfrak{g}[t]^\tau)Y(i+1,\mathbf{s}')$ and so $Zw_r = 0$.

To see (7.3) we proceed by induction on k. For the base case, we have

$$(x_{i+1,n}^- \otimes t)(x_i^- \otimes t^{2s_1}) = (x_{i,n}^- \otimes t^{2s_1+1}) + (x_i^- \otimes t^{2s_1})(x_{i+1,n}^- \otimes t),$$

so induction begins. For the inductive step, we have

$$\begin{aligned}&(x_{i+1,n}^- \otimes t)^k (x_i^- \otimes t^{2s_1}) \cdots (x_i^- \otimes t^{2s_k})\\ &= (x_{i+1,n}^- \otimes t)^{k-1}(x_{i+1,n}^- \otimes t)(x_i^- \otimes t^{2s_1}) \cdots (x_i^- \otimes t^{2s_k})\\ &= (x_{i+1,n}^- \otimes t)^{k-1} \sum_{m=1}^{k} (x_i^- \otimes t^{2s_1}) \cdots (\widehat{x_i^- \otimes t^{2s_m}}) \cdots (x_i^- \otimes t^{2s_k})(x_{i,n}^- \otimes t^{2s_m+1}).\end{aligned}$$

Applying the inductive hypothesis finishes the proof of (7.3).

To finish the proof of the lemma for $1 \le i \le n-2$, we use (7.3) to write

$$\begin{aligned}&(x_{i,n}^- \otimes t^{2s_1+1}) \cdots (x_{i,n}^- \otimes t^{2s_k+1})w_r\\ &\quad = (x_{i+1,n}^- \otimes t)^k (x_i^- \otimes t^{2s_1}) \cdots (x_i^- \otimes t^{2s_k})w_r - Xw_r.\end{aligned}$$

The inductive hypothesis applies to Xw_r. By Proposition 7.4 we can write

$$(x_{i+1,n}^- \otimes t)^k (x_i^- \otimes t^{2s_1}) \cdots (x_i^- \otimes t^{2s_k})w_r$$

as a linear combination of elements where $s_p \le r-k$. Applying (7.3) once again to each summand finishes the proof for $1 \le i \le n-2$.

The case $i=0$, is similar, using the identity

$$(x_n^- \otimes t^{2s+1})w_r = (x_{n-1,n}^+ \otimes t)(x_0^- \otimes t^{2s})w_r = 0, \quad s \ge r,$$

for the induction to begin, and

$$(x_{n-1,n}^+ \otimes t)^k (x_0^- \otimes t^{2s_1}) \cdots (x_0^- \otimes t^{2s_k})w_r = A(x_n^- \otimes t^{2s_1+1}) \cdots (x_n^- \otimes t^{2s_k+1})w_r$$

for the inductive step. □

7.8

We now prove Proposition 7.3, first for $1 \le i \le n-2$. Fix an ordering on the elements $Y(i,\mathbf{s})w_r$, $\mathbf{s} \in \mathbb{Z}_+^k$ and $s_p \le r-k$ as follows: the first element is w_r and an element $Y(i,\mathbf{s})$ precedes $Y(i,\mathbf{s}')$ if $\mathbf{s} \in \mathbb{Z}_+^k$ and $\mathbf{s}' \in \mathbb{Z}_+^m$ if either $k<m$ or $k=m$ and $s_1+\cdots+s_k > s_1'+\cdots+s_k'$ and let $u_1,\ldots,u_\ell$ be an ordered enumeration of this set. Denote by U_p the

$\mathfrak{g}_0[t^2]$-submodule of $W_{\mathrm{loc}}(r\lambda_i)$ generated by the elements u_m, $m \le p$. It is straightforward to see that we have an increasing filtration of $\mathfrak{g}_0[t^2]$-modules:

$$0 = U_0 \subset U_1 \subset \cdots \subset U_\ell = W_{\mathrm{loc}}(r\lambda_i).$$

Moreover U_p/U_{p-1} is a quotient of the local Weyl module for $\mathfrak{g}_0[t^2]$ with highest weight $(r-i_p)\omega_i + i_p\omega_{i-1}$ (we understand $\omega_0 = 0$), if $u_p = Y(i, \mathbf{s})$, $\mathbf{s} \in \mathbb{Z}_+^{i_p}$. Using Eq. (7.1) and Proposition 7.4(ii) we get

$$\dim U_p/U_{p-1} \le \left(\sum_{s=0}^{i}\binom{2n-1}{s}\right)^{r-i_p}\left(\sum_{s=0}^{i-1}\binom{2n-1}{s}\right)^{i_p}.$$

Summing we get

$$\begin{aligned}\dim W_{\mathrm{loc}}(r\lambda_i) &\le \sum_{s=0}^{r}\binom{r}{s}\left(\sum_{s=0}^{i}\binom{2n-1}{s}\right)^{r-s}\left(\sum_{s=0}^{i-1}\binom{2n-1}{s}\right)^{s}\\ &= \left(\binom{2n}{i}+\binom{2n}{i-1}+\cdots+\binom{2n}{1}\right)^{r}.\end{aligned}$$

For the $i = 0$ case, U_p/U_{p-1} is a submodule of the local Weyl module for $\mathfrak{g}_0[t^2]$ with highest weight $(r-2i_p)\omega_n + i_p\omega_{n-1} = (r-i_p)\lambda_0 + i_p\lambda_{n-1}$, if $u_p = Y(n, \mathbf{s})$, $\mathbf{s} \in \mathbb{Z}_+^{i_p}$. Using equation (7.1) and Proposition 7.4(iii) we get

$$\dim U_p/U_{p-1} \le (2^{n-1})^{r-i_p}(2^{n-1})^{i_p}.$$

Summing we get

$$\dim W_{\mathrm{loc}}(r\lambda_i) \le \sum_{s=0}^{r}\binom{r}{s}(2^{n-1})^{r-s}(2^{n-1})^{s} = (2^{n-1}+2^{n-1})^r = (2^n)^r.$$

Since we have already proved that the reverse equality holds the proof of Proposition 7.3 is complete.

7.1 Concluding remarks

We discuss briefly the structure of the local Weyl modules when $\lambda \in P_0^+$ is not a multiple of a fundamental weight and such that $\mathbf{A}_\lambda$ is a proper quotient of a polynomial algebra. The simplest example is the case of (B_3, D_3) and $\lambda = \lambda_0 + \lambda_2$, where we have

$$\mathbf{A}_\lambda = \mathbb{C}[P_{2,1}, P_{3,1}]/(P_{2,1}P_{3,1}).$$

Given $a \in \mathbb{C}^\times$ let $\mathbf{I}_{(a,0)}$ denote the maximal ideal corresponding to $(P_{2,1}-a, P_{3,1})$ and for $b \in \mathbb{C}$ $\mathbf{I}_{(0,b)}$ denote the maximal ideal corresponding to $(P_{2,1}, P_{3,1}-b)$. In the first case, the local Weyl module $W(\lambda, \mathbf{I}_{(a,0)})$ is a pullback of a local Weyl module for the current algebra $\mathfrak{g}[t]$ and so

$$\dim W(\lambda, \mathbf{I}_{(a,0)}) = 22.$$

In the second case the local Weyl module $W(\lambda, \mathbf{I}_{(0,b)})$ is an extension of the pullback of a local Weyl module for the current algebra by an irreducible $\mathfrak{g}_0$-module, and it can be shown that

$$\dim W(\lambda, \mathbf{I}_{(0,b)}) = 32.$$

(see [24, Section 6.11] for details).

In particular the dimension of the local Weyl module depends on the choice of the ideal and hence the global Weyl module is not projective and hence not free as an $\mathbf{A}_\lambda$-module. However, we observe the following: If we decompose the variety corresponding to $\mathbf{A}_\lambda$ into irreducible components $X_1 \cup X_2$, where

$$X_1 = \{(a, 0) : a \in \mathbb{C}\}, \quad X_2 = \{(0, b) : b \in \mathbb{C}\},$$

we see that the dimension of the local Weyl module is constant along X_2. So pulling back $W(\lambda)$ via the algebra map

$$\varphi : \mathbf{A}_\lambda \to \mathbf{A}_\lambda, \; P_{2,1} \mapsto 0, \; P_{3,1} \mapsto P_{3,1}$$

we see that $\varphi^* W(\lambda)$ is a free $\mathbb{C}[P_{3,1}]$-module, where we view $\mathbb{C}[P_{3,1}]$ as the coordinate ring of X_2. In general, preliminary calculations do show that in the case when $\mathbf{A}_\lambda$ is a Stanley–Reisner ring there are only finitely many possible dimensions and that the dimension is constant along a suitable irreducible subvariety, i.e. the global Weyl module is free considered as a module for the coordinate ring $\mathcal{O}(X)$ of a suitable irreducible subvariety X.

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